THE IMMERSION OF MANIFOLDS

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1. M. Hirsch [3] has shown that the immersion problem for manifolds is just a cross section problem for the stable normal bundle. Our object here is to find conditions under which sections of the tangent bundle will imply sections in the normal bundle (and conversely). First we need some notation.

Given an integer t, let $j(t)$ be the maximum integer such that the 2^t-fold Whitney sum of the Hopf bundle over $RP^{j(t)-1}$ is trivial. If ξ is a stable bundle, let $\mathrm{gd}(\xi)$ denote the geometric dimension of ξ.

THEOREM A. *Let M^m be an m-dimensional manifold, $m \leqq 2^t - 1$, whose stable tangent bundle τ_0 is trivial over the $(j(t)-1)$-skeleton. If $m-j(t)+1$ is odd or if $H^q(M; Z_p)=0$ for all $p \neq 2$ and $q \neq 0$, m, then* $\mathrm{gd}(\tau_0) \leqq m-j(t)+1$ *implies* $\mathrm{gd}(-\tau_0) \leqq m-j(t)+1$.

To illustrate the strength of Theorem A we offer

THEOREM B. *Let $m=2^t$, then RP^{m-1} immerses in $R^{2m-j(t)+1}$ but not in $R^{2m-j(t)}$.*

The negative result in Theorem B is due to James [4]. Milgram in [8] has obtained linear immersions of RP^m which agree with those of Theorem B only if $m=15$ and 31.

2. **Outline of the proofs.** Let X be a space, then by $X[k]$ we denote the kth-Eilenberg subcomplex of the space X, i.e., $X[k]$ is $(k-1)$-connected and there is a map $f: X[k] \to X$ such that $f_*: \pi_q(X[k]) \cong \pi_q(X)$ for $q \geqq k$. Let BO_n and BO denote, respectively, the classifying spaces of n-plane bundles and stable bundles. The natural map $BO_n \to BO$ induces maps $p: BO_n[k] \to BO[k]$ for all k.

The key step in the proof of Theorem B is

THEOREM C. *For each $m < 2^t$ there exists an H-space E, an H-map $\phi: E \to BO[j(t)]$ and a fiber map $\psi: BO_{n(t,m)}[j(t)] \to E$ such that*

$$\begin{array}{ccc} BO_{n(t,m)}[j(t)] & \xrightarrow{p} & BO[j(t)] \\ & \psi \searrow \quad \nearrow \phi & \\ & E & \end{array}$$

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is commutative, where $n(t, m)=m-j(t)+1$. *Let* F *be the fiber of* ψ. *If* $n(t, m)$ *is odd, then* F *is* m*-connected. If* $n(t, m)$ *is even, then* F *is* $(n(t, m)-1)$*-connected and* $\pi_k(F)$, *for* $n(t, m)\leqq k\leqq m$ *is either zero or a finite group of odd order.*

We will now outline the proof of Theorem C. We will assume t and m are fixed integers.

As in [2] or [5], we can construct a fiber space $\phi_0\colon X\to BO$ as a composite of principal fiber spaces

$$X = X_s \to X_{s-1} \to \cdots \to X_0 = BO \tag{2.1}$$

where $X_k\to X_{k-1}$ has fiber a product of Eilenberg-Mac Lane spaces of type (Z, q) or (Z_2, q). There is a fiber map $\psi_0\colon BO_n\to X$ such that

$$\begin{array}{ccc} BO_n & \xrightarrow{p_0} & BO \\ & {\scriptstyle\psi_0}\searrow \quad \nearrow{\scriptstyle\phi_0} & \\ & X & \end{array} \tag{2.2}$$

is commutative. If F is the fiber of ψ_0, then F satisfies the conditions of Theorem C and we have,

PROPOSITION 2.3. *Let* $t>4$, *then the diagram of Theorem* C *is induced by diagram* (2.2) *under the natural map* $BO[j(t)]\to BO$.

Because of (2.3), it suffices to prove that E is an H-space and that ϕ is an H-map.

Now the fiber space $E\to BO[j(t)]$ is a composite of principal fibrations,

$$E = E_s \to E_{s-1} \to \cdots \to E_0 = BO[j(t)] \tag{2.4}$$

induced by (2.1). Let F_k be the fiber of $E_k\to E_{k-1}$. Then $E_k\to E_{k-1}$ is classified by a map $f_k\colon E_{k-1}\to BF_k$ where BF_k is the classifying space of F_k. Assume that E_{k-1} is an H-space and that $\phi_{k-1}\colon E_{k-1}\to E_0$ is an H-map. Then E_k will be an H-space if the k-invariants, i.e. the images of the fundamental classes of BF_k are primitive in $H^*(E_{k-1})$. We have a principal fiber space

$$K(J_t, j(t-1)-1) \to BO[j(t)] \to BO[j(t-1)]$$

where $J_t=Z$ or Z_2, accordingly with [1]. Let

$$\overline{E} = \overline{E}_s \to \cdots \to \overline{E}_0 = K(J_t, j(t-1)-1) \tag{2.5}$$

be the induced fibrations over $\overline{E}_0$ of (2.4).

Stong in [9], has determined the mod 2-cohomology of $BO[n]$. Using this result we prove,

PROPOSITION 2.6. *For $k=0, 1, \cdots, s$, the natural map $\overline{E}_k \to E_k$ induces a monomorphism $H^*(E_k) \to H^*(\overline{E}_k)$ in dimensions $\leqq 2^t - 1$.*

Again inductively, if we assume that E_{k-1} is an H-space, then $\overline{E}_{k-1}$ is an H-space and $\overline{E}_{k-1} \to E_{k-1}$ is an H-map. Therefore from (2.6) follows

PROPOSITION 2.7. *If the k-invariants of $\overline{E}_k \to \overline{E}_{k-1}$ are primitive, then E_k and $\overline{E}_k$ are H-spaces and $E_k \to \overline{E}_k$ is an H-map.*

In fact, we show that $\overline{E}_k$ is an infinite loop space. Again inductively, if $\overline{E}_{k-1}$ is an infinite loop space and $\Omega^{-r}\overline{E}_{k-1}$ denotes a space such that $\Omega^r(\Omega^{-r}\overline{E}_{k-1}) = \overline{E}_{k-1}$, it suffices to show that the k-invariants of $\overline{E}_k \to \overline{E}_{k-1}$ are in the image of

$$(2.8) \qquad \Sigma^r \, \overline{E}_{k-1} \xrightarrow{g} \Omega^{-r}\overline{E}_{k-1} \to \Omega^{-r}BF_k$$

where g is homotopic to the adjoint of the identity map $\overline{E}_{k-1} \to \overline{E}_{k-1}$.

In order to achieve this, we pass to Thom complexes. First observe that for $r \geqq 2^t$, $\overline{E}_0$ is the fiber of $BO_r[j(t)] \to BO_r[j(t-1)]$. Assume from now on, that $r \geqq 2^t$. Let

$$(2.9) \qquad E' = E'_s \to E'_{s-1} \to \cdots \to E'_0 = BO_r[j(t)]$$

be the tower induced over E'_0 from (2.4). Then the tower (2.5) is induced from (2.9) by the mapping $\overline{E}_0 \to E'_0$. Now let η_r be the canonical bundle over E'_0 and consider the induced bundles over the spaces E'_k and $\overline{E}_k$. Let ME'_k and $M\overline{E}_k$ denote the corresponding Thom complexes. We have a sequence of maps

$$(2.10) \qquad ME' = ME'_s \to ME'_{s-1} \to \cdots \to ME'_0.$$

LEMMA 2.11. *In dimensions less than or equal to $r+m$, the map $ME'_k \to ME'_{k-1}$ looks like a principal fiber map, with fiber a product of Eilenberg-Mac Lane complexes and k-invariants $\{U \cup k_i\}$, where the k_i are the k-invariants for $E'_k \to E'_{k-1}$.*

Let Y_q be the universal example of an integral cohomology class y of dimension q such that

(1) all primary cohomology operations vanish on y,

(2) all operations that raise dimension by less than $j(t)$ vanish on y.

Then we need the following

THEOREM 2.12. *There exists a tower of fiber spaces*

$$\begin{array}{ccccc} K(Z, r) \leftarrow & A_1 \leftarrow & \cdots & \leftarrow A_s \\ \nwarrow & \nwarrow & & \nearrow p_s \\ & & \sum^{r-n} Y_n. & & \end{array}$$

The fiber G_k at each stage is a product of Eilenberg-MacLane spaces. If $k>1$, then $G_k=\Omega^{-r}F_k\times S_k$ where F_k is the $(k-1)$-stage of an Adams resolution over Z_2 of V_n through dimension $m-1$ and S_k is the k-stage of an Adams resolution over Z_2 of S^r through dimension $\Gamma+j(t)-1$. Also $G_1=\prod_{i=1}^{t-1} K(Z_2,\ r+2^i-1)\times\Omega^{-r}F_1$. In addition, the fiber of p_s is $(m-1)$ connected.

The proof of (2.12) follows closely that of [6, Theorem A].

Let f_k be the composite $A_k\rightarrow BG_{k+1}\rightarrow\Omega^{-r-1}F_{k+1}$ where the first map classifies $A_{k+1}\rightarrow A_k$ and the second is the obvious projection.

THEOREM 2.13. *There is a mapping λ_{k-1}: $ME'_{k-1}\rightarrow A_{k-1}$ such that $\lambda^*_{k-1}f^*_{k-1}(\gamma_i)=U\cup k_i$, where γ_i ranges over the fundamental classes of $\Omega^{-r-1}F_k$ and the k_i over the k-invariants of $E'_k\rightarrow E'_{k-1}$.*

Now we are ready to indicate how Theorem C follows. Observe that $M\overline{E}_k=\Sigma^r(\overline{E}_k\cup pt)$, since the induced bundle over $\overline{E}_k$ is trivial, and let i_k: $\Sigma^r\overline{E}_k\rightarrow ME'_k$ be the composite $\Sigma^r\overline{E}_k\rightarrow M\overline{E}_k\rightarrow ME'_k$.

Suppose that we have proved $\overline{E}_k$ has the structure of an infinite loop space, then the following lemma implies $\overline{E}_{k+1}$ has the structure of an infinite loop space:

LEMMA 2.14. *There is a commutative diagram, up to homotopy,*

$$\begin{array}{ccccc} \Sigma^r\,\overline{E}_k & \xrightarrow{\rho_k} & \Omega^{-r}\,\overline{E}_k & & \\ \downarrow i_k & & \downarrow j_k & & \\ ME'_k & \xrightarrow{\lambda_k} & A_k & \xrightarrow{f_k} & BF_{k+1} \end{array}$$

where λ_k is given in (2.13). ρ_k is the adjoint of the identity map and f_kj_k classifies the bundle $\Omega^{-r}\overline{E}_{k+1}\rightarrow\Omega^{-r}\overline{E}_k$.

The proof of Theorem C is then completed.

Theorem A follows by establishing a mod p version of (2.13) and using the fact that the top class of the Thom complex of the normal bundle to a manifold immersed in euclidean space is stably spherical.

Finally, Theorem B follows from Theorem A by using the existence of tangent vector fields for real projective spaces due to Hurwitz-Radon.

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