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# The impact of negative interest rates on the pricing of options written on equity: a technical study for a suitable estimate of early termination

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# The impact of negative interest rates on the pricing of options written on equity: a technical study for a suitable estimate of early termination

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## Abstract

This work aims to investigate the main problems that impact the pricing models and the sensitivity measures of American options written on shares without a pay-out, in the presence of negative interest rates with a specific focus on the Monte Carlo method. The first paragraph carries out a review of the anomalies caused by such an odd condition and focuses thereafter on the core topic of the research by treating a wide range of numerical models suitable for unbiased evaluation of the early exercise, thus expanding the existing literature. The two following paragraphs are dedicated to describing the models used for the correct estimation of fair value: binomial lattice models (Cox-Ross-Rubinstein - CRR Tree, Leisen Reimer - LR Tree, Jarrow-Rudd - JR Tree and Tian Tree), trinomial stochastic trees, Finite Difference Method (FDM) scheme and the Longstaff-Schwartz Monte Carlo. Particular attention is paid to this last approach which allows to combine the flexibility of traditional numerical integration schemes for stochastic processes on equity with the estimation of the convenience of exercising the American option ahead of time. After conducting quantitative tests both on pricing and on the estimation of sensitivity measures, the LR Tree was selected as the most performing deterministic algorithm to be compared with the Monte Carlo stochastic technique. The final part of the work focuses on quantifying the valuation gap introduced by negative interest rates in the valuation of American options written on an unprofitable underlying comparing the traditional valuation approach and the deterministic Leisen Reimer model and the Longstaff-Schwartz stochastic model.

**Key Words:**

Negative Interest rates, American Option pricing, early-exercise valuation, extreme market conditions, sensitivity measures, lattice models, Cox-Ross-Rubinstein (CRR) Tree, Leisen Reimer (LR) Tree, Jarrow-Rudd (JR) Tree, Tian Tree, CRR Trinomial Tree, Finite Difference Method (FDM), Stochastic Differential Equation (SDE), Longstaff-Schwartz Monte Carlo

## 1) The main problems caused by negative interest rates

The first paragraph introduces the main problems related to the presence of negative nominal interest rates. After a brief introduction, in which the main historical facts that led central banks to reduce interest rates to negative values will be discussed, more specific issues will be addressed, linked to the technical issues caused by the zero lower bound of rates. The last of these issues, namely the effect that negative interest rates have on options written on equity that do not pay dividends, will be the core topic of this work.

### *1.1) Negative interest rates and historical-economic context*

It all began in the United States of America in 2006, the year in which the so-called subprime mortgage crisis broke out: the term subprime refers to those loans with high financial risk, mainly mortgage loans, that many American banks disbursed in favor of customers with a high risk of default. Between 2000 and 2003, the Fed drastically reduced interest rates, from 6.5% to 1%. This decreasing trend in rates, together with other factors, triggered an increase in the demand for mortgage loans, fueled in turn by a growing real estate market, characterized by speculative practices and also by a parallel financial market based on the securitization of the same mortgage loans, in which large banks, retail banks and institutional investors held and traded very complex financial instruments such as MBS - Mortgage-Backed Securities or CDO - Collateralized Debt Obligation (Holt, 2009). The crisis started to appear in the second half of 2006 when the US housing bubble began to deflate in the face of a rate hike by the FED, from 1% to 5.25%, which occurred in a rather short period of time, from 2004 to 2006. The graph in Figure 1 traces the trend of the Federal Fund Rate. The sudden rise in rates caused a high percentage of default on subprime loans. Consequently, the prices of financial instruments built on these loans, which constituted a market whose capitalization vastly exceeded the total value of the underlying loans, collapsed along with the real estate market (Bernanke, 2010). The consequences were rather disastrous, involving the main US Investment Banks, which suffered very heavy losses caused by the sharp depreciation of MBS and CDOs, and collaterally by CDS (Credit Default Swaps). The market values of the banks also dropped significantly. The most striking and emblematic case occurred in September 2008, when Lehman Brothers, one of the most important American financial institutions, founded in 1850, filed for bankruptcy (Lehman Brothers Holding Inc, 2008). The American Government decided not to save the bank to give a strong signal to the financial system, and to prevent future moral hazard behavior by other banks or financial institutions typically considered "Too Big To Fail". This complex series of circumstances led to a strong mistrust in the financial system and to a contagion effect that caused the so-called credit crunch: with this term, which literally means a "tightening of credit", we mean a significant and sudden contraction in the supply of credit, at the end of a prolonged period of expansion. The banks initially froze the interbank credit market out of mutual distrust, wary of the solvency of their respective counterparties. This led to a general contraction of credit even in the retail market and corporate market, triggering a process of economic recession that brought a strong impact on the real economy. The credit crisis globally spread like wildfire, also affecting the Eurozone, where many institutions were saved with public funds or with bail-in procedures. The strong distrust in the financial system generated panic in certain cases, as in the case of Northern Rock, a British institution specializing in real estate loans, which in 2007 suffered an out-and-out bank run (Dunkley, 2017). To face those circumstances, the ECB adopted a series of measures aimed at implementing an expansive monetary policy, with the aim of stimulating the economy and ensuring price stability in the Eurozone. The ECB therefore cut interest rates until they reached negative levels in 2014 (ECB, 2014). The goal was to disincentivize banks from depositing liquidity with the central bank, whose negative interest rate would have led to a loss on deposits, in favor of the use of such liquidity for the provision of loans to businesses and consumers. The graph in Figure 2 shows the evolution of the Deposit Facility Rate, i.e., the interest rate that banks receive on deposits with the European Central Bank. The graph clearly shows the systematic cut of the DFR, aggressively from 2008 until the first half of 2009, then more gradually, reaching the zero level in 2012, and finally assuming negative values from 2014.

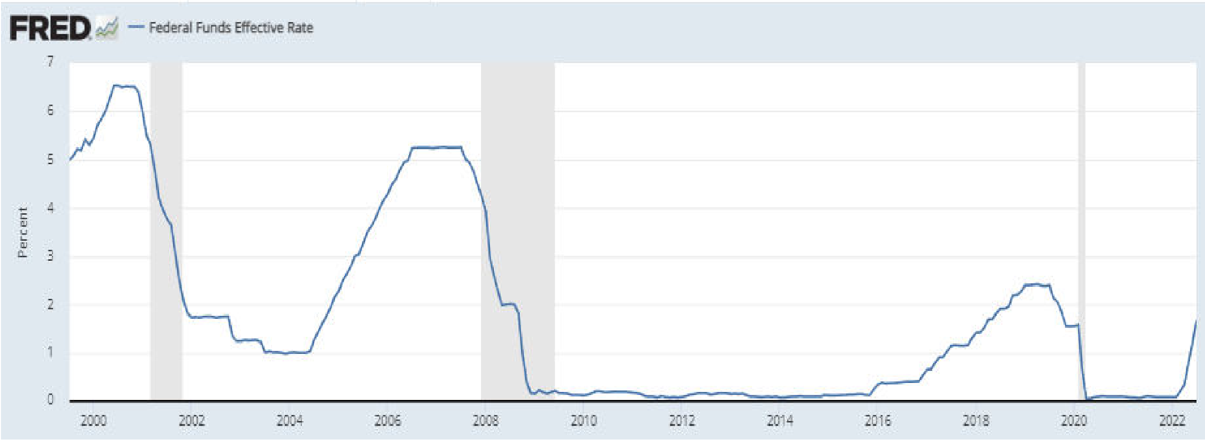


*Figure 1 - Federal Fund Rate between 2000 and 2022 (Source: FRED Economic Data)*


*Figure 2 - ECB Deposit Facility Rate years 2000-2022 (Source: FRED Economic Data).*

After briefly discussing the economic and financial causes that led to negative nominal interest rates, the main effects caused by this anomalous condition are discussed hereafter.

*1.2) Problems related to negative interest rates*

For the first time in history, apart from a brief period in Japan in the 90s, a scenario of negative nominal interest rates occurs (Ansa, 2014). This peculiar circumstance brings a series of consequences within the financial system. The unexpected impacts that never occurred before are numerous and are accompanied by a dense literature which tries to analyze the potential repercussions on markets and financial instruments, whose issues and exchanges usually took place under the assumption of the presence of positive levels of interest rates.

*1.2.1) Potential effects on the investment choices of investors*

In the presence of negative interest rates, Government bonds with floating coupons linked to Euribor (for example CCTs) could theoretically yield negative coupons, which will be set to zero (floor) and this could lead investors to select riskier investments, but with a positive return. In this way, the investor might invest his money in financial instruments characterized by greater volatility, therefore potentially unsuitable to his risk appetite.

*1.2.2) Application of negative rates on deposits*

There is also the possibility that banks apply negative interest rates on the deposits of their account holders, inducing the latter to withdraw their money and deposit it with other institutions which, on the contrary, do not charge their customers on deposits. This issue can trigger a competition mechanism within the banking system, which results in the bank taking over this cost, with obvious repercussions on the financial statements.

*1.2.3) Anomalies in the interest rates term structure*

Moving on to more quantitative problems, one of them is certainly linked to the anomalies found in the interest rate curve (Cafferata et al., 2019). Let us consider the 6-month Euribor, one of the most relevant interest rates as it is typically used as a parameter for indexing mortgages, bonds and derivatives. The historical and prospective evolution of the 6-month Euribor has been analyzed by financial analysts and traders from all over the world. However, most of the models for representing the dynamics of this short-term rate are affected by the issue of negative interest rates, since their structure does not allow negative values within the formulas of the stochastic differential equations (Giribone, 2020).

*1.2.4) Anomalies in the surface of the implied volatilities in caps and floors*

Implied volatility provides an estimate of the expected volatility of the underlying made by the market maker, during the residual life of an option. It is one of the most important parameters in evaluating the price of an option. Before the advent of negative interest rates, the main framework used for determining the implied volatility was the Black framework, i.e., the log-normal model (Black, 1976). In a context with negative interest rates, the log-normal volatility surfaces observed in financial markets are incomplete. To address this problem, market makers have replaced Black's log-normal model with Bachelier's normal model (Haug, 2007). This model guarantees the integrity of the volatility surfaces when interest rates are negative. Normal volatility is expressed in basis points and is obtained by numerically inverting the Bachelier formula starting from the premiums of actively listed options on the market. The problem of moving from the log-normal model to the normal model exposes the counterparties to model risk, i.e., the risk associated with carrying out transactions with a different model, therefore at different prices (Giribone, Ligato & Mulas, 2017). Furthermore, the problem of changing the model has caused various issues to existing contracts, since, if the reference pricing model is contractually specified, such calculation method cannot be changed. To solve these data missing problems, rather complex techniques have been applied to rebuild the incomplete volatility surfaces of the Black model, also related to Machine Learning techniques (Caligaris, Giribone & Neffelli, 2017).

*1.2.5) Valuation of options written on interest rates*

One of the most debated issues in the sector's literature is the impact that negative interest rates have on options written on interest rates (Burro et al., 2017). Interest Rate options are financial instruments that have been widely used in recent years. They are generally embedded, i.e., incorporated within bonds or in bank assets such as mortgages. Among the most common we can mention Caps, Floors and Collars. Those derivative contracts are called yield-based options and are characterized by a cash settlement which amount is the difference between the value of the underlying and a strike price. Before the advent of negative interest rates, the main framework for pricing this type of contract has always been Black's log-normal model. With the advent of negative rates, this model can no longer work due to the negative input (i.e., the forward rate $F$) which should be inserted within the auxiliary variables in the well-known Black closed formula. This issue arises because the logarithm of a negative value does not exist in real numbers. This makes it impossible to use pricing techniques based on the log-normal model for valuating options written on interest rates. (Giribone & Ligato, 2016). A very similar phenomenon occurs for swaptions: negative interest rates do not allow for a correct estimate of the fair value and of the sensitivity measures even for this type of interest rate derivative.

*1.3) Effect of negative interest rates on the pricing of options written on equity that pays no dividend*

A decidedly less debated issue is that relating to the effects that negative interest rates have on options written on non-profitable shares, that is, those that pay no dividends. According to the theory of options, there are fourteen fundamental properties that options must satisfy, regardless of the pricing model used (Hull, 2015). In particular, one of them states:

"Under the assumption that the underlying share pays no dividend, it will never be worth exercising an American call option prematurely, so it will be priced like its European analogue".

In mathematical terms, this can be written as:

$f_A(S, K, T, r, 0, \sigma) = f_E(S, K, T, r, 0, \sigma)$ (1)

$f_A(S, K, T, r, 0, \sigma) \geq f_E(S, K, T, r, 0, \sigma) \geq S_t - K\, e^{-rT}$ (2)

Where: $S$ is the spot level of the underlying, $K$ is the strike price of the option, $T$ is the time to maturity, $r$ is the risk-free rate, $q = 0$ is the continuous dividend yield and $\sigma$ is the volatility. Furthermore:

$f_A(S, K, T, r, 0, \sigma) \geq \max[0\, ; S_t - K]$ (3)

If we combine (2) with (3) we obtain:

$S_t - K < St - K\, e^{-rT}$ (4)

Well, this property is no longer valid in the presence of negative interest rates, and a bias can therefore be observed between the price of the European call and of the American call option, with resulting effects also on the sensitivity measures of the options, i.e., on the Greeks (Cafferata, Giribone, & Resta, 2017). A considerable problem therefore arises, namely that of identifying robust alternative routines that allow the valuation of this type of option in the presence of negative rates. The most widely used techniques for pricing American options, especially short-term ones, are the so-called quasi-closed formulas, such as the Bjerksund-Stensland formula (Bjerksund & Stensland, 2002). The issue with those techniques is that their algorithms are characterized by an if-condition which, in the absence of a dividend yield, returns the price of the European option. However, the extreme situation of a market characterized by negative interest rates is not considered. It is therefore necessary to implement techniques that consider the possibility of having an early exercise, even if the underlying has no pay-out. In (Cafferata, Giribone & Resta, 2017), the authors analyze the issues mentioned above and integrate their work with a numerical experiment in which certain pricing techniques applied to an American call option are implemented in different market scenarios: a first case in which interest rates are positive and the underlying pays a dividend, a second case in which the rates are again positive but the underlying share is not profitable, and finally a third case where market conditions are the most extreme since, in addition to a zero dividend yield, negative interest rates are observed. The models used for pricing the options were essentially the following:

- The Barone-Adesi-Whaley (BAW) model (Barone-Adesi & Whaley, 1987).

- The 1993 Bjerksund-Stensland model (Bjerksund & Stensland, 1993).

- The 2002 Bjerksund-Stensland model (Bjerksund & Stensland, 2002).

- The trinomial model (Boyle, 1986).

The first three models consist of the so-called quasi-closed formulas, while the fourth is a lattice model. The Authors infer from the results of the experiment that the bias generated by the quasi-closed formulas, used for calculating the price of an American call and of a European call, is due to approximation errors, and that such bias can be reduced through the implementation of a trinomial stochastic tree (Cafferata, Giribone, & Resta, 2017). The purpose of the present article is to integrate that work with further deterministic numerical methods and to compare them to those based on the Monte Carlo methodology. In fact, the latter generally allows greater flexibility in the definition of payoffs which can be useful for modeling structured products.

## 2) The implemented deterministic numerical pricing methodologies

The main lattice techniques and the finite difference method implemented for pricing and determining the most common options sensitivity measures will be described in the following subparagraphs.

*2.1) Binomial stochastic trees*

The binomial method of Cox-Ross-Rubinstein represents one of the most widely used deterministic algorithms to evaluate options characterized by non-standard payoffs. The first formulation of the binomial method dates back to 1979 by John C. Cox, Stephen A. Ross and Mark E. Rubinstein (Cox J. C., Ross S. A. & Rubinstein M., 1979). They demonstrated how to build a binomial tree which could discretize and approximate a geometric Brownian motion, in such a way that, if a large number of time intervals were considered, the use of the binomial method to evaluate European options would be equivalent to using the continuously defined Black-Scholes-Merton formula (Hull, 2015). The most interesting aspect, however, is that the binomial model allows to evaluate American options and many exotic options, for which there is often no exact pricing formula. The analytical formulas of Black-Scholes-Merton are, in fact, almost always unsuitable for providing a fair value for options with non-standard characteristics, such as the possibility of exercising them before maturity (Bermuda /American options) or with particularly complex payoffs (exotic options). In all of these cases, therefore, a numerical methodology has to be used for the valuation of the derivative.
The technique essentially involves dividing the time between the option valuation date and its expiry date into numerous time intervals, assuming that during the intervals two possible changes may occur in the value of the underlying of the derivative.
By way of example, it will be assumed that the underlying is a share but, this valuation methodology may be extended in a generalized form to numerous types of underlying (GBS - Generalized Black-Scholes pricing framework) (Haug, 2007).
The value of the share in a binomial tree, after a time interval $\Delta t$, can increase by a fixed amount $u$ with a probability of $p$ or it can decrease by a fixed amount $d$ with a probability equal to $1-p$. $N$ corresponds to the number of time intervals into which the time between the option valuation date and its expiry has been divided.
In order to distinctly identify each node of the binomial tree, the reference time interval is defined with the index $j$ and the possible value assumed by the financial instrument, moving from one node to the next, is defined with the index $i$. The first node in the tree is identified with the values ($j=0$, $i=0$). If the price of the asset increases by the amount $u$ (such that $S \cdot u > S$), the second node will be identified with the values ($j=1$, $i=1$). If, on the other hand, the price of the asset decreases by the amount $d$ (such that $S \cdot d < S$), the position of the tree will be identified with the values ($j=1$, $i=0$). This is shown in Figure 3, which shows, by way of example, a binomial tree with five time-intervals.

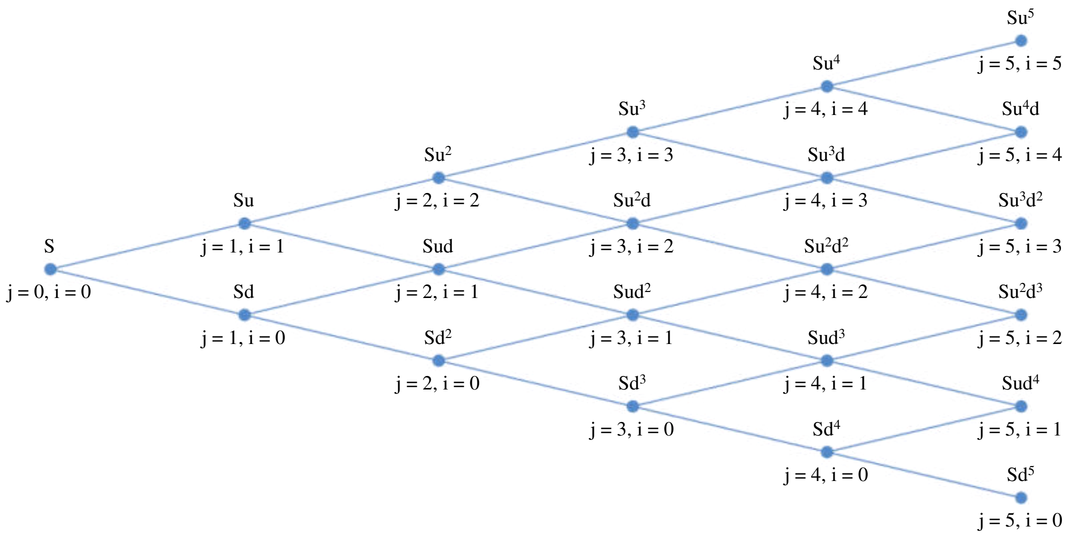


*Figure 3. Example of a stochastic tree with arborescence $N=5$*

The number of paths leading to a general node ($j$, $i$) is equal to: $\frac{j!}{i!(j-i)!}$. Starting from this discretization scheme of the underlying, the pay-off $g(\cdot)$ is applied in the terminal nodes, and, proceeding backwards, the recombination of the tree is performed until the starting node ($j=0$, $i=0$) is reached, in which the price of the derivative is determined. In order to obtain a matching with the stochastic dynamics postulated by the Black-Scholes framework, Cox, Ross and Rubinstein suggested to select the parameters $u$ and $d$ so that, for each discrete time interval $\Delta t$, the assumed future values of the asset be consistent with the mean and the theoretical variance of the continuous model (Di Franco, Polimeni & Proietti, 2002). To this end, Cox, Ross and Rubinstein set the parameters $u$ and $d$ as follows: $u = \exp(\sigma\sqrt{\Delta t})$ and $d = 1/u = \exp(-\sigma\sqrt{\Delta t})$, where $\Delta t = T/N$ is the length of each time interval (i.e. the time interval

between price movements), $T$ is the time to expiry of the option expressed in years, $\sigma$ is the annualized volatility of the share price and $N$ is the number of time intervals. Under those hypotheses, the probability that the share price increases between one interval and the next is defined as risk-neutral probability and its value is equal to: $p = (\exp(b\Delta t) - d)/(u - d)$, where $b$ is the parameter known as cost-of-carry.
Depending on the value assumed by this parameter, we reach a pricing framework that can be used for a large number of option underlyings. In particular (Haug, 2007):

- if $b = r$ the definition is suitable for pricing options written on shares that pay no dividend.
- if $b = r - q$ the definition is suitable for pricing options written on shares or indices with a continuous dividend yield $q$.
- if $b = 0$ the definition is suitable for pricing options on futures.
- if $b = r - r_{FOR}$ the definition is suitable for pricing currency options.

The general formulation for pricing a European option is therefore:

$$Price = \exp(-rT)\sum_{i=0}^{N}\frac{N!}{i!(N-i)!}p^i(1-p)^{N-i}g(Su^id^{N-i},K) \quad (5)$$

The up and down jump factors $(u, d)$ and the respective probabilities $(p)$ of increasing/decreasing the price level of the underlying in the next step, $\Delta t = T/N$, depend on the model used.

In the CRR (Cox-Ross-Rubinstein) Tree $u$, $d$, $\Pi$ are chosen to match the first two moments of the price level distribution, as discussed before (Cox J. C., Ross S. A. & Rubinstein M., 1979):

$$u = \exp\left(\sigma\sqrt{\Delta t}\right) \quad (6)$$
$$d = \exp\left(-\sigma\sqrt{\Delta t}\right) \quad (7)$$
$$\Pi = \frac{\exp(b\Delta t) - d}{u - d} \quad (8)$$

There are other binomial methodologies in the literature, which allow the matching with the mean and the theoretical variance of the continuous Black-Scholes-Merton model. These are called alternative stochastic binomial trees. The most popular are:

In the JR (Jarrow-Rudd) Tree $u$ and $d$ are chosen in order to have a probability of $\frac{1}{2}$ (Jarrow & Rudd, 1993):

$$u = \exp\left[(b - \sigma^2/2)\Delta t + \sigma\sqrt{\Delta T}\right] \quad (9)$$
$$d = \exp\left[(b - \sigma^2/2)\Delta t - \sigma\sqrt{\Delta T}\right] \quad (10)$$
$$\Pi = \frac{1}{2} \quad (11)$$

The model proposed by Tian equals the first three moments of the log-normal distribution followed by the underlying (Tian, 1993):

$$u = \frac{1}{2}\exp(b\Delta t)v\left(v + 1 + \sqrt{v^2 + 2v - 3}\right) \quad (12)$$
$$d = \frac{1}{2}\exp(b\Delta t)v\left(v + 1 - \sqrt{v^2 + 2v - 3}\right) \quad (13)$$
$$\Pi = \frac{\exp(b\Delta t) - d}{u - d} \quad (14)$$
$$v = \exp(\sigma^2\Delta t) \quad (15)$$

The Leisen and Reimer tree sets the $u$ and $d$ factors so that the tree is centered around the strike price. This makes the convergence tend to the option value more smoothly and with a better performance (Leisen & Reimer, 1996). The parameters characterizing the chain are:

$$\Pi = h_{PP}(d_2) \quad (16)$$
$$u = \exp(b\Delta t)\frac{h_{PP}(d_1)}{h_{PP}(d_2)} \quad (17)$$
$$d = \frac{\exp(b\Delta t) - pu}{1 - p} \quad (18)$$
$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + (b + \sigma^2/2)T}{\sigma\sqrt{T}} \quad (19)$$
$$d_2 = d_1 - \sigma\sqrt{T} \quad (20)$$

Preizer-Pratt suggest two methods for the $h_{PP}(x)$ calculation (Giribone & Ligato, 2016).

The first inversion method (LR1 Tree) sets:

$$h_{PP1}(x) = \frac{1}{2} + \eta\left\{\frac{1}{4} - \frac{1}{4}\exp\left[-\left(\frac{x}{N+\frac{1}{3}}\right)^2\left(N + \frac{1}{6}\right)\right]\right\}^{\frac{1}{2}} \quad (21)$$

While the second inversion method (LR2 Tree) estimates:

$h_{PP2}(x) = \frac{1}{2} + \eta \left\{ \frac{1}{4} - \frac{1}{4} \exp\left[ -\left( \frac{x}{N+\frac{1}{3}+\frac{0.1}{N+1}} \right)^2 \left( N + \frac{1}{6} \right) \right] \right\}^{\frac{1}{2}}$ (22)

Where: $\begin{cases} \eta = +1, x \geq 0 \\ \eta = -1, x < 0 \end{cases}$ (23)

Up to now we have shown the procedure to be used to value a European option, characterized by the fact that it can only and exclusively be exercised at maturity. It is therefore necessary to introduce the so-called early exercise feature into the model, which is the characteristic that distinguishes an American option: the possibility to exercise it at any time, from instant 0 to expiry $T$. Such peculiarity translates into the fact that, instead of valuating the pay-off only at maturity, and proceeding backwards through the backwardation algorithm, the pay-off also has to be determined in each discrete time interval, in order to verify whether it is convenient to exercise the option early or whether to bring it to maturity: the option price will be the higher of the values defined in the two respective scenarios (Hull, 2015). We can therefore infer that, in each node of the binomial tree, the value of the American call option is:

$$C_t = \max[C_{Dead}; C_{Alive}] = \max\left[S_t - K; \frac{C_u \Pi + C_d(1-\Pi)}{1+r}\right] \text{ (24)}$$

While for an American put option it is:

$$P_t = \max[P_{Dead}; P_{Alive}] = \max\left[K - S_t; \frac{P_u \Pi + P_d(1-\Pi)}{1+r}\right] \text{ (25)}$$

In order to generalize the procedure for a multi-step tree, two distinct indices have to be introduced: $i$ which identifies the time-step and $j$ which identifies the expected price: the equation which considers the right to early exercise in a general node of the binomial tree is the following (Haug, 2007):

- for the call option:

$$C_{i,j} = \max\left[S \cdot u_i \cdot d_{j-1} - K; \frac{C_{j+1,i+1}\Pi + C_{j+1,i}(1-\Pi)}{1+r}\right] \text{ (26)}$$

- for the put option:

$$P_{i,j} = \max\left[K - S \cdot u_i \cdot d_{j-1}; \frac{P_{j+1,i+1}\Pi + P_{j+1,i}(1-\Pi)}{1+r}\right] \text{ (27)}$$

The principle to be applied to valuate an option, taking the early exercise into account, is the same in all types of lattice models (Giribone & Raviola, 2019). In implementing the backwardation algorithm, the potential convenience of bringing the option to maturity has to be considered.

*2.2) Trinomial stochastic trees*

The construction of a trinomial tree is very similar to the procedure followed for developing a multi-step binomial tree (Boyle, 1986). Generally, the construction of the trinomial tree that represents the evolution of the price of the underlying occurs by using stochastic differential equations (SDE). The first step is to build the price chain of the underlying until maturity. The following step is to calculate the option price, starting from the pay-off function at maturity, and discounting the future expected values (backwards induction phase). Firstly, it is necessary to introduce the stochastic differential equation of a Brownian geometric motion, which describes the evolution of the price of the underlying (Hull, 2015):

$$dS = (r-q)Sdt + \sigma S dW_t \text{ (28)}$$

- $\mu = r - q$ is the annualized expected return earned by an investor over the time period $dt$.
- $\sigma$ is the annualized volatility of the asset.
- $dW_t$ is a Wiener process.

The variable $x = \ln(S)$ is defined by obtaining the modified stochastic differential equation:

$dx = \nu dt + \sigma dW$ with $\nu = r - q - \frac{1}{2}\sigma^2$ (29)

Now let us consider what happens to the variable $x$ in a time interval $\Delta t$. The model supposes that it can assume three different values: it can increase (up) or decrease (down) by an amount equal to $\Delta x$ or remain unchanged (no change). A probability is associated with each of the potential changes in the price of the underlying. To find the values of such probabilities, and obtain convergence with the Black-Scholes model, it is necessary to equal the mean and the variance in the interval $\Delta x$, and also impose the sum of the three probabilities equal to 1:

$E[\Delta x] = p_u(\Delta x) + p_m(0) + p_d(-\Delta x) = \nu \Delta t$ (30)
$E[\Delta x^2] = p_u(\Delta x^2) + p_m(0) + p_d(+\Delta x^2) = \sigma^2 \Delta t + \nu^2 \Delta t^2$ (31)
$p_u + p_m + p_d = 1$ (32)

Defining $\alpha = \frac{v\Delta t}{\Delta x}$ e $\beta = \frac{\sigma^2 \Delta t + v^2 \Delta t}{\Delta x^2}$, the values of the probabilities become as follows: $p_u = \frac{\alpha+\beta}{2}$, $p_d = \frac{\beta-\alpha}{2}$ e $p_m = 1 - \beta$.

We then proceed with the creation of the trinomial tree. Two indices are defined, $n$, which represents time, and $j$, which represents the price of the underlying. If $S$ is the price of the underlying at $n = 0$, then the $j$-th price level is equal to $S_j^n = S \exp(j\Delta x)$.
We can therefore define the vector $\vec{S}$ as:

$\vec{S}[-N] = S\exp(-N\Delta x)$ (33)

$\vec{S}[j] = \vec{S}[j-1] \exp(\Delta x) \quad \text{with} \quad j = -N+1, \ldots, N$ (34)

Where $N$ is the number of sub-periods in the interval $(0, T)$, $T$ is the time to maturity, or $N\Delta t = T$.
Similarly to the development of the price of the underlying, the discretized values relating to the development of the price of the call option $C$ are represented by the variable $C_j^n$.

The value of the call option at maturity is known, and its possible variants, respectively under the two different assumptions of continuous and discrete time, respectively, are given by:

$C(S,T) = \max(S - K, 0)$ (35)

$C_j^N = \max(S_j^N - K, 0)$ (36)

Finally, the price of the call option is determined at the $n$-th time interval as a discounted expectation under the hypothesis of risk neutrality, based on the value of the call option in the interval $n+1$:

$C_j^n = \exp(-r\Delta t)\,(p_u C_{j+1}^{n+1} + p_m C_j^{n+1} + p_d C_{j-1}^{n+1})$ (37)

In short, the logical sequence of the steps to follow is:

1. The structure of the trinomial tree is created.
2. The value of the call option is initialized in the tree using the default probability values.
3. The pay-off vector $C_j^N$ is calculated.
4. The values of the call options in the previous intervals are calculated $C_j^n$.

The first two steps represent the forward induction, while the second two steps implement the backward induction.
A very similar reasoning can be done to valuate a put option and to verify the convenience of early exercise during the backwardation phase.

*2.3) The finite difference method*

The finite difference method (FDM) is a numerical scheme that can be applied in quantitative finance for valuating options. The solution scheme of the explicit finite difference method for the fundamental Black-Scholes-Merton PDE (partial difference equation) is equivalent to the discounted expectations procedure of a trinomial tree (Duffy, 2006). Let us consider the PDE:

$$-\frac{\partial C}{\partial t} = \frac{1}{2}\sigma^2 \frac{\partial^2 C}{\partial x^2} + v\frac{\partial C}{\partial x} - rC \quad (38)$$

We proceed from the expiry of the option backwards until the initial instant and the approximation of the explicit finite differences is constructed as follows:

$$-\frac{C_j^{n+1} - C_j^n}{\Delta t} = \frac{1}{2}\sigma^2 \frac{C_{j+1}^{n+1} - 2C_j^{n+1} + C_{j-1}^{n+1}}{\Delta x^2} + \frac{v\left(C_{j+1}^{n+1} - C_{j-1}^{n+1}\right)}{2\Delta x} - rC_j^{n+1} \quad (39)$$

Rearranging the terms of the equation we obtain:

$$C_j^n = p_u C_{j+1}^{n+1} + p_m C_j^{n+1} + p_d C_{j-1}^{n+1} \quad (40)$$

It is interesting to underline the fact that such equation, which represents the value of the call option in the interval $n$ as the average, weighted by the probabilities of the three possible "states of the world" in the subsequent interval $n+1$, is the same which also describes the call option value in the pricing framework of a trinomial model, apart from the discount factor.

The values of the probabilities $p_u$, $p_m$ e $p_d$ are defined as follows:

$p_u = \frac{\Delta t\sigma^2}{2\Delta x^2} + \frac{\Delta t v}{2\Delta x}$ (41), $p_m = 1 - 2\frac{\Delta t\sigma^2}{2\Delta x^2} - r\Delta t$ (42) and $p_d = \frac{\Delta t\sigma^2}{2\Delta x^2} - \frac{\Delta t v}{2\Delta x}$ (43)

The probability values must be positive, and this entails certain restrictions on the amplitude of the step $\Delta$t. In general, the relationship between the time-steps and the spot price is as follows: $\Delta x = \sigma\sqrt{3\Delta t}$ (Clewelow & Strickland, 1998).

The ways in which early exercise can be taken into account for the FDM method are discussed below. An American option with a maturity $T$ and with a function $f$ for pay-offs can be exercised at any time until maturity. Let us define $P(S_t,t)$ as the option pay-off function, and $V(S_t,t)$ as the early exercise function, where $V: (0,\infty) \times [0,T] \to \mathbb{R}$.

$V(S_t,t)$ represents the price of the instrument at time $t$, which means that it values the future payoffs of the instrument, therefore $P(S_t,t)$ is the pay-off of the instrument if it were exercised at time $t$. This means that upon maturity:

$V(S,t) = P(S,t)$ with $S > 0$ (44)

Furthermore, based on the principle of non-arbitrage

$V(S,t) \geq P(S,t)$ with $(S,t) \in (0,\infty) \times [0,T]$ (45)

If the sign of the inequality would be strictly greater, it would not be convenient for the option holder to exercise it, since it would be more profitable to sell it at a price equal to $V(S,t)$, rather than exercise it for a value of $P(S,t)$. If, on the other hand, $V(S,t) = P(S,t)$, the optimal choice is to immediately exercise the option since by holding it till maturity, the holder risks losing money. Taking the above consideration into account, the general rule can be determined: the option holder must exercise it as soon as $V(S,t) = P(S,t)$. Up to the optimal exercise value, the relationship that dominates the dynamics of the option price is given by the fundamental Black-Scholes-Merton PDE (Hull, 2015):

$$rS\frac{\partial V(S,t)}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V(S,t)}{\partial S^2} - rV(S,t) + \frac{\partial V(S,t)}{\partial t} = 0 \quad (46)$$

Where $V(S,t) > P(S,t)$.

At the optimal moment for exercise, the following equation applies:

$V(S,t) = P(S,t)$ (47)

These relations can be represented by the following free-boundary problem (Duffy, 2006):

$$\begin{cases} V(S,t) \geq P(S,t) \\ V(S,t) = P(S,t) \text{ or } rS\frac{\partial V(S,t)}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V(S,t)}{\partial S^2} - rV(S,t) + \frac{\partial V(S,t)}{\partial t} = 0 \\ V(S,T) = (K-S)^+ \quad \text{Terminal Condition} \\ \lim_{S\to 0} V(S,t) = K \quad \text{Left boundary condition} \\ \lim_{S\to\infty} V(S,t) = 0 \quad \text{Right boundary condition} \end{cases} \quad (48)$$

With $S > 0$ and $t \in [0,T]$.

The system describes the locus of points where $V(S,t) = P(S,t)$. In those points the system is not governed by the partial differential equation and the option holder of the option should exercise the option. In order to calculate the price of the American put option, the problem has to be transformed into a standard scheme for PDE. To do this, the traditional coordinate transformation has to be used (Giribone & Ligato, 2015). The established parabolic PDE is thus obtained from which the canonical resolution methods can be used.

## 3) The Monte Carlo methodology and the Longstaff-Schwartz algorithm

This stochastic pricing methodology, originally introduced by Boyle in 1977 (Boyle, 1977), can be used to value most options, but above all, thanks to its flexibility, it is mostly useful for pricing highly exotic derivatives or structured products. Since the value of a derivative is closely linked to the pattern of the price of the underlying financial asset $S(t)$ in the time period between the drafting of the contract and the maturity $t \in [0,T]$, it is necessary to mathematically describe a dynamic that represents the potential future trajectories of the asset on which the option is written. In this regard, the Monte Carlo method can be used to simulate a wide range of stochastic processes. The most common stochastic process and consistent with the Black-Scholes pricing framework is called Geometric Brownian motion and it is represented by the well-known Stochastic Differential Equation (SDE): $dS(t) = \mu S(t)dt + \sigma S(t)dW_t$. The SDE which identifies the Brownian geometric motion can be integrated through the Eulero-Maruyama numerical scheme and then implemented in a numerical processing software as follows (Kloeden & Platen, 1992):

$$dS(t) = \mu S(t)dt + \sigma S(t)dW_t \to \Delta S = \mu S\Delta t + \sigma S\Delta W \to S_t = S_{t-1} + \mu S_{t-1}\Delta t + \sigma S_{t-1}\varepsilon\sqrt{\Delta t} \quad (49)$$

Stochastic calculus allows to formulate an analytical expression for the simulation of $S(t=T)$. Such result is considered extremely important for practical purposes, since it allows direct simulations of the asset to be performed at a general future time $t = T$, without needing to know the values assumed by the asset in the previous times $S(t < T)$. Starting from the hypothesis that the variable follows a stochastic process such as: $dS(t) = \mu S(t)dt + \sigma S(t)dW_t$, given Ito's lemma, we can state that there is a function $G(S(t))$ that follows the dynamics (Hull, 2015):

$$dG(S,t) = \left(\frac{\partial G(S,t)}{\partial S}\mu S + \frac{\partial G(S,t)}{\partial t} + \frac{1}{2}\frac{\partial^2 G(S,t)}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial G(S,t)}{\partial S}\sigma S dW_t \quad (50)$$

We define $G = \ln(S)$: $dG = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dW_t$, $d\ln(S(t)) = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dW_t$, and by integrating the expression over time, we obtain:

$$\int_0^T d\ln(S(t)) = \int_0^T \left(\mu - \frac{\sigma^2}{2}\right)dt + \int_0^T \sigma dW_t \rightarrow \ln\left(\frac{S(T)}{S(0)}\right) = \left(\mu - \frac{\sigma^2}{2}\right)T + \sigma dW_T \rightarrow$$

$$S(T) = S(0)\exp\left[\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma dW_T\right] \text{ (51)}$$

The above expression can be easily implemented in a vectorized way in a numerical processing software, such as R, for example.

$$S(T) = S(0)\exp\left[\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma dW_T\right] \rightarrow S(T) = S(0)\exp\left[\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\varepsilon\sqrt{\Delta T}\right] \text{ (52)}$$

The formula allows to simulate, in a manner consistent with the BS framework, the value of the asset underlying an option at any point in time and in an efficient way in terms of computer time. This modeling methodology allows maximum flexibility in the definition of the pay-off $g(\cdot)$ even in the presence of exotic derivatives. As mentioned before, the problem with American option pricing is that such options can be exercised at any time until maturity, unlike a European option that can only be exercised at maturity. In this section we will use the following notation for the pay-off function: $P(S(t)) = \max(K - S(t), 0)$ and $P(S(t)) = \max(S(t) - K, 0)$ respectively for a put and for a call option. According to the Black-Scholes-Merton pricing framework, the underlying is modelled using the Geometric Brownian Motion (52).
For ease of notation, we consider the price of a put option, but the results can be extended to call options as well. The value at time 0 of a European option can be described as:

$$u(S,0) = E\left[\exp(-rT)P(S(T))\right] \text{ (53)}$$

That is, the expected value of the discounted payoff at time $T$. In a similar way, the value of an American option at time 0 is given by:

$$u(S,0) = \sup_{t\,\in[0,T]} E\left[\exp(-rT)P(S(T))\right] \text{ (54)}$$

i.e., the expected value of the discounted payoff at the time of exercise that yields the greatest payoff. This corresponds to the optimization problem of finding the optimal stopping time

$$t^* = \inf\{t \geq 0 | S(t) \leq b^*(t)\} \text{ (55)}$$

for some a-priori unknown exercise boundary $b^*$ (Brandimarte, 2006). Thus, in order to price an American option, we need to find the optimal stopping time $t^*$ and then estimate the expected value:

$$u(S,0) = E\left[\exp(-rt^*)P(S(t^*))\right] \text{ (56)}$$

One of the most popular methods to solve this problem is called LSM, developed by Longstaff and Schwartz (Longstaff & Schwartz, 2001). This approach uses a dynamic programming approach to find the optimal stopping time and the Monte Carlo (i.e., numerical integration of Stochastic Differential Equation - SDE) to approximate the expected value. Dynamic programming is a general method for solving optimization problems by dividing them into smaller subproblems and combining their solution to solve the main problem (Kamien & Schwartz, 2012). In this case, this means that we divide the interval $[0, T]$ into a finite set of time points $\{0, t_1, t_2, \dots, t_N\}$ and, for each of these points, we decide if it is better to exercise than to hold on to the option. Starting from time $T$ and working backwards to time 0, we update the stopping time each time we find a time where it is better to exercise, until we find the smallest time where exercise is better. Let $C(S(t_i))$ denote the value of holding on to the option at time $t_i$ i.e., the continuation value, and let the exercise value at time $t_i$ be the payoff $P(S(t_i))$. Then the dynamic programming algorithm to find the optimal stopping time can be summarized in the following pseudo-code:

> $t^* \leftarrow t_N$
> for $t$ from $t_{N-1}$ to $t_1$ do
> > if $C(S(t)) < P(S(t))$ then
> > > $t^* \leftarrow t$
> > else
> > > $t^* \leftarrow t^*$
> > end if
> end for

Using the same argument as in Equation (54), the continuation value at time $t_i$ can be described in terms of conditional expectation:

$$C(S(t_i)) = E\left[\exp(-r(t^* - t_i))P(S(t^*))|S(t_i)\right] \text{ (57)}$$

Where $t^*$ is the optimal stopping time in $\{t_{i+1}, \dots, t_N\}$. For ease of notation, we define the current payoff $\mathcal{P}$ as:

> for $t = t_N$:
> $\mathcal{P} = P(S(t))$
> from $t = t_{N-1}$ to $t = t_1$:
> if $C\big(S(t)\big) < P\big(S(t)\big)$ then $\mathcal{P} = P\big(S(t)\big)$, otherwise $\mathcal{P} = \exp(-r\Delta t)\,\mathcal{P}$

Where $\Delta t = t_{i+1} - t_i$.

Given this notation, Equation (57) becomes:

$$C\big(S_i(t)\big) = E[\exp(-r\Delta t)\mathcal{P}|S(t_i)] \quad (58)$$

To estimate this conditional expectation, the LSM method uses regular least squares regression (Huynh, Lai & Soumare, 2008).
This can be done since the conditional expectation is an element in $L^2$ space, which has an infinite countable orthonormal basis and thus all elements can be represented as a linear combination of a suitable set of basis functions.
So, to estimate this we need to choose a finite set of orthogonal basis functions, and project the discounted payoffs onto the space spanned by them. In the proposed implementation, the basis function chosen is the Laguerre polynomials, where the first four are defined as follows (Koornwinder, 2013):

$$\begin{cases} L_0 = 1 \\ L_1 = 1 - X \\ L_2 = \frac{1}{2}(2 - 4X + X^2) \\ L_3 = \frac{1}{6}(6 - 18X + 9X^2 - X^3) \end{cases} \quad (59)$$

Given a set of realized paths $S_i(t)$, $i = 1,\dots,n$ that are in-the-money at time $t$, i.e. $P\big(S_i(t)\big) > 0$, and the payoffs $\mathcal{P}_i = \mathcal{P}\big(S_i(t)\big)$, the conditional expectation in Equation (58) can be estimated as:

$$\hat{C}\big(S_i(t)\big) = \sum_{j=0}^{k} \hat{\beta}_j\, L_j\big(S_i(t)\big) \quad (60)$$

Where $L_0,\dots,L_k$ are the first $k$ Laguerre polynomials and $\hat{\beta}_0,\dots,\hat{\beta}_k$ are the estimated regression coefficients. The regression coefficients are obtained by regressing the discounted payoffs $y_i = \exp(-r\Delta t)\mathcal{P}_i$ against the current values $x_i = S_i(t)$ by regular least squares:

$$\big(\hat{\beta}_0,\dots,\hat{\beta}_k\big)^T = (\boldsymbol{L}^T\boldsymbol{L})^{-1}\boldsymbol{L}^T(y_1,\dots,y_n)^T \quad (61)$$

Where $\boldsymbol{L}_{i,j} = L_j(x_i)$, $i = 1,\dots,n$ and $j = 0,\dots,k$.

By approximating Equation (58) with Equation (60), we introduce an error in our estimation. In (Clement, Lamberton & Protter, 2002) it is shown that $\lim_{k\to\infty} \hat{C}\big(S(t)\big) = C\big(S(t)\big)$.
Now that we have a method to estimate the continuation value, we can simulate a set of $M$ realized paths $S_i(t)$, $t = 0, t_1, t_2, \dots, t_N$ and $i = 1,2,\dots,M$ and use the previous pseudo code to find the optimal stopping times $t_i^*$ for all paths, and then estimate the expected value in Equation (56) using Monte Carlo:

$$\hat{u} = \frac{1}{M}\sum_{i=1}^{M} \exp(-rt_i^*)P\big(S(t_i^*)\big) \quad (62)$$

One way to speed up the algorithm is to use the discounted payoffs $\mathcal{P}_i$ in the Monte Carlo step instead of the optimal stopping times.
Since they are constructed and updated recursively in the same way as the stopping times, by the time we have gone from time $t = t_N$ and $t = t_1$ they will be:

$$\mathcal{P}_i = \exp\big(-r(t_i^* - t_1)\big)P\big(S(t_i^*)\big) \quad (63)$$

Which means that:

$$\exp(-r\Delta t)\mathcal{P}_i = \exp(-rt_i^*)P\big(S(t_i^*)\big) \quad (64)$$

Thus Equation (62) becomes:

$$\hat{u} = \frac{1}{M}\sum_{i=1}^{M} \exp(-r\Delta t)\mathcal{P}_i \quad (65)$$

A pseudo code for the LSM algorithm is provided.

In each step only paths that are in-the-money are used since they are the only ones where the decision to exercise or continue is relevant.

> Initiate paths $S_i(t), t = 0, t_1, t_2, \dots, t_N, i = 1,2,\dots,M$

> Set $\mathcal{P}_i \leftarrow P(S_i(t_N))$ for all $i$
> for $t$ from $t_{N-1}$ to $t_1$ do
> > Find paths $\{i_1, i_2, \dots, i_n\}$ that are in the money: $P(S_i(t)) > 0$
> > Set $ITM\ paths \leftarrow \{i_1, i_2, \dots, i_n\}$
> > Set $x_i \leftarrow S_i(t)$ and $y_i \leftarrow \exp(-r\Delta t)\,\mathcal{P}_i$ for $i \in ITM\ paths$
> > Perform regression on $x, y$ to obtain coefficients $\hat{\beta}_0, \dots, \hat{\beta}_k$
> > Estimate the continuation value $\hat{C}(S_i(t))$
> > Calculate the value of immediate exercise $P(S_i(t))$ for $i \in ITM\ paths$
> > for $i$ from 1 to $M$ do
> > > if ($i \in ITM\ paths$) and $\left(P(S_i(t)) > \hat{C}(S_i(t))\right)$ then
> > > > $\mathcal{P}_i \leftarrow P(S_i(t))$
> > > else
> > > > $\mathcal{P}_i \leftarrow \exp(-r\Delta t)\,\mathcal{P}_i$
> > > end if
> > end for
> end for
> $Price \leftarrow \frac{1}{M}\sum_{i=1}^{M} \exp(-r\Delta t)\mathcal{P}_i$

## 4) Choice of the deterministic valuation model as a benchmark for the Monte Carlo method

The purpose of this paragraph is to select the best deterministic method to be compared with the results calculated with the Monte Carlo method. In order to carry out such validation of the methodologies described in paragraph 3, with the consequent selection of the best pricing approach, two different scenarios are considered:

**CASE A – "Theoretical Case"**
The entry parameters for the theoretical case are characteristic of a non-stressed market, as the risk-free rate is positive and equal to 6%, and in addition the option underlying pays a dividend, with a continuous dividend yield, or a rate of return calculated on a continuous basis, equal to 1%. Furthermore, the put option is ITM (in the money) as the strike price is greater than the spot. The parameters are as follows: $S = 40$ → Spot price; $K = 50$ → Strike price; $T = 2$ → Time to maturity (years); $r = 6\%$ → Risk-free rate; $q = 1\%$ → Continuous dividend yield; $b = r - q$ → Cost of carry; $\sigma = 25\%$ → Annualized volatility for the underlying

**CASE B – "Market Case"**
The second case, which we call "market case" for simplicity, deals with an American put option written on the S&P500 index. As in the previous case, the parameters for pricing the option are characteristic of a normal market condition, in which the risk-free rate is positive and equal to 1.949%, and the rate of return calculated on a continuous basis is equal to 1.466%. The ATM put option (At The Money) is valued with the market data as of 30/03/2022 (Source: Bloomberg®): $S = 4617.09$, $K = 4617.09$, $T = 1$, $r = 1.949\%$, $q = 1.466\%$, and $\sigma = 20.526\%$.
Before testing the Monte Carlo method in a stressed market scenario, a benchmark deterministic model has to be identified to compare the results obtained between the deterministic and the stochastic techniques. To do this, it is necessary to determine which of the deterministic models has returned the best performance in terms of convergence level to the Black-Scholes model. The adopted approach for this purpose was to implement the closed Black-Scholes formula for pricing the European put option in the theoretical case, determining its price and sensitivity measures (Hull, 2015). The second step consisted in building a ranking model based on the magnitude of the percentage error regarding the prices, and on the mean and the standard deviation of the Greeks error, all of this compared to the results obtained with the B&S closed formula, for each of the deterministic models that have been implemented. The results obtained are reported in Table 1:

| Price B&S | Delta B&S | Vega B&S | Rho B&S | Theta B&S | Gamma B&S |
|---|---|---|---|---|---|
| 8.778847 | -0.5568458 | 21.79769 | -62.10536 | 0.2780668 | 0.02724711 |

*Table 1. Price and Greeks of the European put option obtained with the Black-Scholes formula in the theoretical case*

Where the price of the put option was calculated with the closed Generalized Black-Scholes formula (Black & Scholes, 1973), (Haug, 2007):

$$P = K\,e^{-rT}\,N(-d_2) - Se^{(b-r)T}N(-d_1) \quad (66)$$

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(b + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}};\ d_2 = d_1 - \sigma\sqrt{T} \quad (67)$$

While the sensitivity measures were calculated with the exact formulas illustrated below (Haug, 2007):

$$Delta = e^{(b-r)T}(N(d_1) - 1) \ (68)\ Vega = Se^{(b-r)T}n(d_1)\sqrt{T} \ (69)\ Rho = -TKe^{-rT}N(-d_2) \ (70)$$

$$Theta = -\frac{(Se^{(b-r)T})n(d_1)\sigma}{2\sqrt{T}} + (b-r)Se^{(b-r)T}N(-d_1) + rKe^{-rT}N(-d_2) \quad (71) \quad Gamma = \frac{n(d_1)e^{(b-r)T}}{S\sigma\sqrt{T}} \quad (72)$$

Once the price and the Greeks values of the European put option were determined with the Black-Scholes pricing framework, a dataset was drawn up with the differences, in absolute value and percentage value, between the price values calculated with the different deterministic methods. All the scenarios presented thereafter were conducted using 5,000 steps for the lattice models, the grid for the FDM is 5,000 by 5,000 in the underlying value/time to maturity discretization dimensions.

| Ranking | Pricing Model | Price | Absolute Error | Error % |
|---|---|---|---|---|
| 1° | Leisen-Reimer (LR) | 8.778846 | 8.418940e-07 | 9.590029e-08 |
| 2° | Explict Finite Difference (FDM) | 8.778863 | 1.612288e-05 | 1.836560e-06 |
| 3° | Tian (TIAN) | 8.778895 | 4.806051e-05 | 5.474582e-06 |
| 4° | Trinomial (TRI) | 8.778932 | 8.528244e-05 | 9.714538e-06 |
| 5° | Cox-Ross-Rubinstein (CRR) | 8.778950 | 1.027063e-04 | 1.169929e-05 |
| 6° | Jarrow-Rudd (JR) | 8.779062 | 2.153926e-04 | 2.453541e-05 |

*Table 2. Ranking of deterministic models for the pricing of the European put option in the theoretical case*

Subsequently, a further dataset was drawn up which contains the values of the mean and the standard deviation of the differences between the results obtained for the sensitivity measurements, calculated with the exact B&S formulas and those obtained with deterministic models, using the following numerical formulas: 2-sided finite difference for Delta, Vega and Rho, 1-sided finite difference for Theta, and Finite Central Difference for Gamma (Duffy, 2006).

| Rank | Model | Delta Error | Vega Error | Rho Error | Theta Error | Gamma Error |
|---|---|---|---|---|---|---|
| 1° | LR | 0.16791 % | 1.0335e-04 % | 1.2587 e-06 % | 0.025862 % | 0.006015 % |
| 2° | TRI | 2.47082 % | 1.9697e-04 % | 1.0927 e-03 % | 0.453552 % | 8.595987 % |
| 3° | JR | 0. 13030 % | 3.1874e-04 % | 9.07064 e-02 % | 0.795114 % | 0.077334 % |
| 4° | CRR | 0.11538 % | 2.6884e-04 % | 2.27259 e-03 % | 1.522928 % | 0.174636 % |
| 5° | FDM | 6.07082 % | 6.1439e-05 % | 2.02134 e-03 % | 2.11055 % | 1.949889 % |
| 6° | TIAN | 0.223363% | 1.5530e-04 % | 4.41009 e-01 % | 5.817020 % | 0.327921% |

*Table 3. Ranking of deterministic models for estimating the Greeks of the European put option in the theoretical case*

The results obtained from the comparison of the deterministic models with the Black-Scholes model have pinpointed the Leisen-Reimer binomial model as the most performing model, since the discrepancies generated by this pricing technique are significantly lower compared to the other implemented models. Given the outcome of the ranking model, the following tables show the comparison between the benchmark model and the LSM method for pricing the American put option in the theoretical case and in the market case. The number of paths simulated through the Monte Carlo technique is equal to 50,000 for the theoretical case and 100,000 for the market case. Price convergence was tested with 200 replications for both cases.

| Theoretical Case | Price | Delta | Vega |
|---|---|---|---|
| **LR** | 10.55684 | -0.7526719 | 14.4264 |
| **LSM** | 10.54429 | -0.7780889 | 13.9426 |

*Table 4. Comparison of the price and the main Greeks between the LR Tree and the MC method - theoretical case.*

| Market Case | Price | Delta | Vega |
|---|---|---|---|
| LR | 362.8773 | -0.4482972 | 1805.850 |
| LSM | 362.2143 | -0.4511456 | 1798.989 |

*Table 5. Comparison of the price and the main Greeks between the LR Tree and the MC method - market case.*

The standard deviation of the expected fair value for LSM is 0.005555 in the theoretical case and 0.5237 in the market case.

## 5) Definition of the price surfaces of a call option and of the Greeks under stressed market conditions

This paragraph shows the experimental results that define the valuation gap between a European call option and the corresponding American option written on equity with a zero pay-out and in the presence of negative interest rates, thus demonstrating the violation of the property of options according to which, under the assumption that the underlying equity pays no dividend, it will never be convenient to exercise an American call option prematurely, so such option will be priced like the European one (Hull, 2015). As already mentioned in the previous paragraph, the two pricing models involved in the stress-test are the binomial Leisen-Reimer technique, as a deterministic technique that has proved to be the most performing, and the Monte Carlo method of Longstaff-Schwartz, a stochastic technique whose simulation error is monitored based on the results of the LR model. This experimental phase was carried out according to the following scheme:

- Change of the option type in the two case studies, from put option to call option.
- Change in the scenario: the dividend yield parameter is equalized to zero and the risk-free interest rate is set as a parameter, covering values ranging from strongly negative to above zero values.
- Definition of the valuation gap surfaces and of the error of the estimation of the Greeks between the European call option and the corresponding American option with the Leisen-Reimer model.
- Definition of the price surfaces of the American call option with the Longstaff-Schwartz Monte Carlo method.
- Definition of the surfaces of the Greeks of the American call option with the Longstaff-Schwartz Monte Carlo method.
- Comparison between the methodological error introduced by the LR model and the experimental error of the LSM stochastic method.

The surfaces were calculated on the one hand, by setting the risk-free interest rate as a parameter, and on the other hand by setting the four fundamental input parameters for pricing the option. In particular, the following ranges of variation have been defined:

Risk-free rate $r \in [-10\% \,;\, 2\%]$
Spot price $S \in [S - 50\%\, S \,;\, S + 50\%\, S]$
Strike price $K \in [K - 50\%\, K \,;\, K + 50\%\, K]$
Annualized volatility $\sigma \in [1\% \,;\, 70\%]$
Time to maturity $T \in \left[\frac{1}{360} \,;\, T\right]$

Regarding the granularity applied to the ranges of parameters used for calculating the different surfaces, the scheme was as follows:

For price surfaces:
- $r$ step $25\ basis\ point = \frac{25}{10000}$
- $S$ step $\frac{S}{100}$
- $K$ step $\frac{K}{100}$
- σ step 1%
- $T$ step $T * \frac{7}{360}$ for the theoretical case and $T * \frac{3.5}{360}$ for the market case.

For Greeks surfaces:
- $r$ step $50\ basis\ point = \frac{50}{10000}$
- $S$ step $\frac{S}{50}$
- $K$ step $\frac{K}{50}$
- σ step 2%
- $T$ step $T * \frac{14}{360}$ for the theoretical case and $T * \frac{7}{360}$ for the market case.

The granularity applied to the ranges of parameters for calculating the Greeks surfaces is reduced compared to that relating to the price surfaces. This choice is aimed at obtaining a suitable trade-off between the experimental test grid and the computational time, which is systematically greater for estimating the sensitivity measures, since it is calculated with numerical formulas, and the individual option pricing procedure is implemented at least twice for each measure.

### *5.1) Surfaces of the valuation gap between the European and American call option with the Leisen-Reimer model*

In this sub-paragraph, the surfaces that determine the valuation gap between the European and the American call option are presented respectively, and those relating to the error in the estimation of sensitivity measures, both for the theoretical case and for the market case. The error is measured as a percentage according to the following formulas:

For the price surfaces → $\%\ Price\ Error = 100 * \frac{PAm - PEu}{PAm}$

For the Greeks surfaces → $\% \ Greek \ Error = 100 * \frac{GreekAm - GreekEu}{GreekAm}$

The surfaces are shown in a three-dimension box, based on the following orientation of the axes:
Horizontal axis 1 (x) → parameter $r$ (risk-free rate)
Horizontal axis 2 (y) → variable parameter $(S, K, \sigma, T)$
Vertical axis (z) → Error ($\% \ Price \ Error, \% \ Greek \ Error$)
Furthermore, a surface coloring scheme is respected, based on the variable parameter used in the calculation:
$Spot \ Price \ (S)$ → $Red$
$Strike \ Price \ (K)$ → $LightBlue$
$Volatility \ (\sigma)$ → $Yellow$
$Time \ to \ maturity \ (T)$ → $Green$
In order not to burden the dissertation, all cases for the price of the considered derivative are reported, while for the surfaces of the Greeks, only those of the theoretical case are displayed.

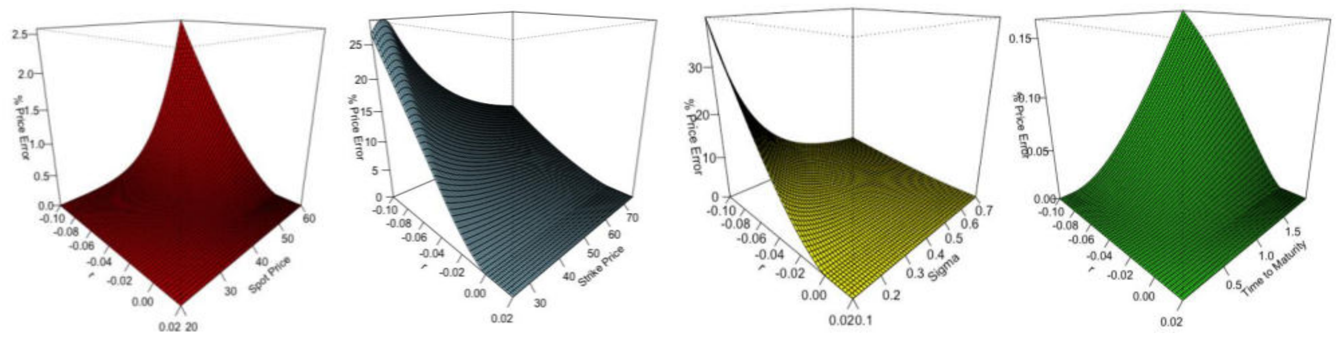


*Figure 4. Gap in the fair value of the American-European LR in the theoretical case*

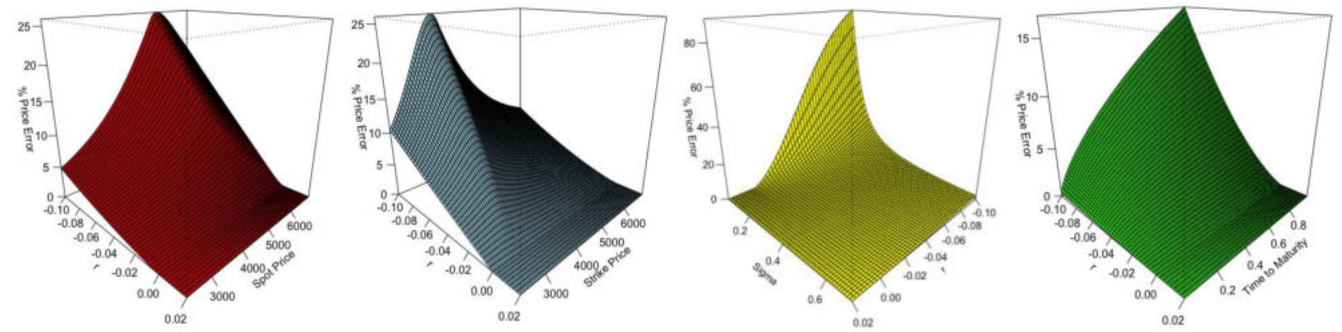


*Figure 5. Gap in the fair value of the American-European LR in the market case*

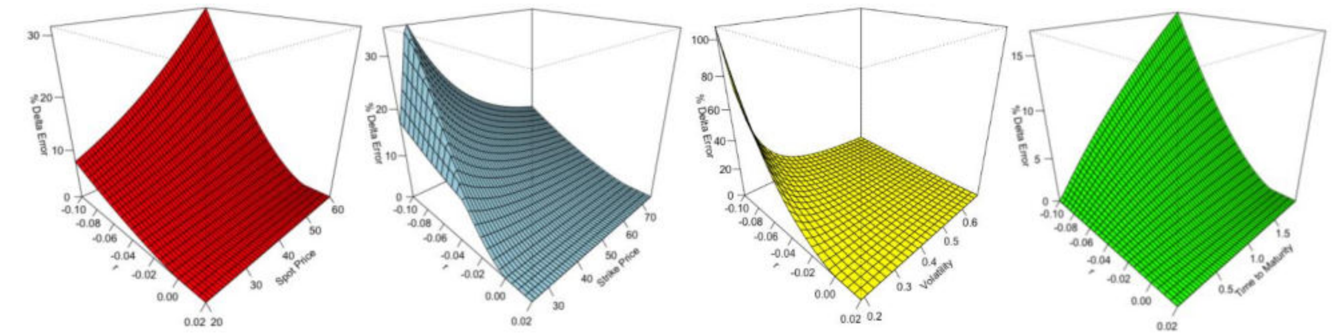


*Figure 6. Gap in the estimate of the Delta of the American-European LR in the theoretical case*

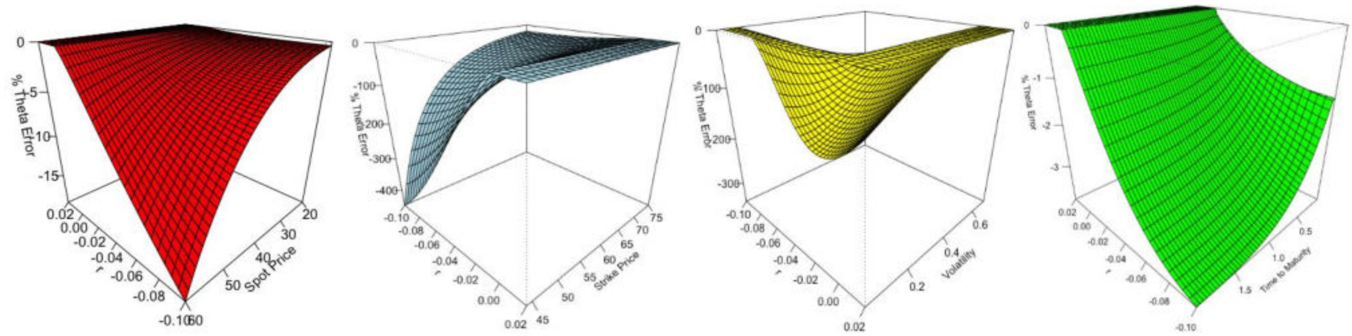


*Figure 7. Gap in the estimate of the Theta of the American-European LR in the theoretical case*

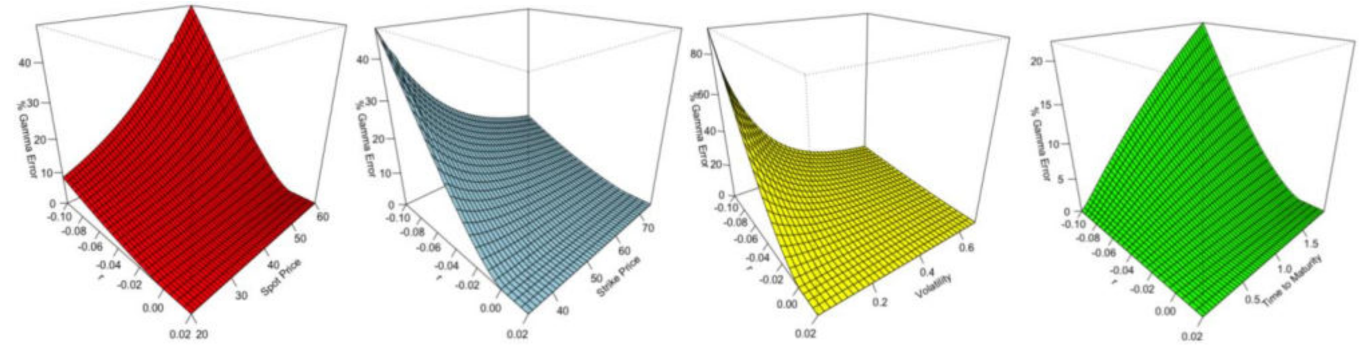


*Figure 8. Gap in the estimate of the Gamma of the American-European LR in the theoretical case*

*5.2) Surfaces of the valuation gap between the European and the American call option with the Monte Carlo model.*

Following the grid and color conventions used for the LR case, only the surfaces related to the theoretical and the market case for pricing are shown, in order not to burden the dissertation.

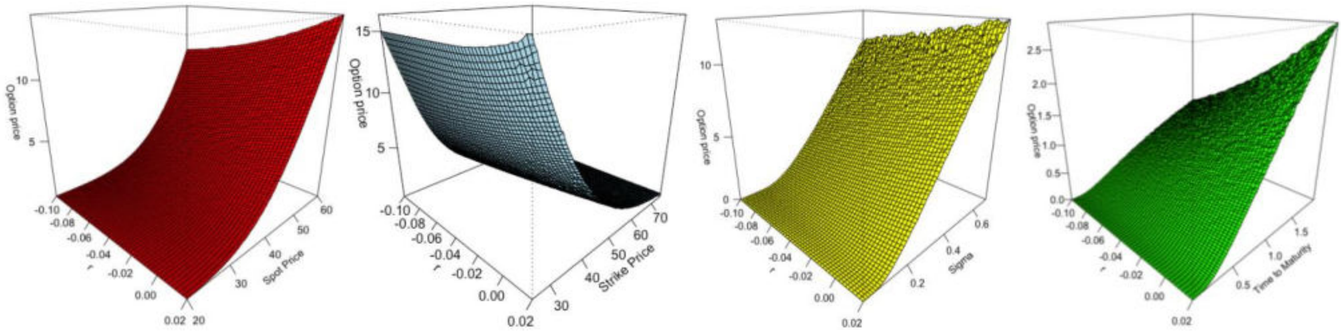


*Figure 9. Estimate of the fair value of the Monte Carlo of the American option in the theoretical case*

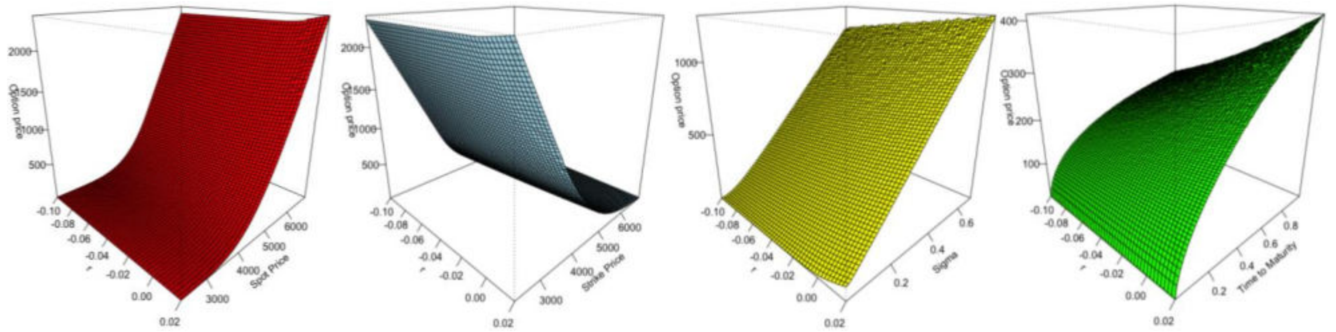


*Figure 10. Estimate of the fair value of the Monte Carlo of the American option in the market case*

*5.3) Comparison between the methodological (LR) and the experimental error (LSM) in the pricing and estimation of the Greeks*

This paragraph illustrates the comparison surfaces between the methodological error of the Leisen-Reimer model and the experimental error produced by the simulations of the Longstaff-Schwartz Monte Carlo method. The last phase of the experiment involves the comparison between the valuation gap of the prices calculated using the LR model and the size of the experimental error of the stochastic method. The purpose of this comparison is to analyze the behavior of the experimental error produced by the Monte Carlo simulations, and the extent of the discrepancy produced by such error with respect to the values of the LR benchmark, in extreme market conditions. In order to represent the extent of the Monte Carlo simulation error, with respect to the valuation gap of the Leisen-Reimer binomial model, a mask has to be applied to the error matrix containing the differences between the values of the European and the American call option: if the simulation error, resulting from the absolute value of the difference between the price of the American option calculated with the LSM and with the LR, is greater than the valuation gap produced by the deterministic model, then the resulting matrix highlights the size of the experimental error in absolute value, otherwise, that is, if the valuation gap of the LR is smaller than the experimental error, the matrix does not show such discrepancy. It should be remembered that for the Monte Carlo method only the sensitivity measures were estimated, and for them, the combination of the sensitivity of the numerical formulas and the model randomness allow to valuate with a relatively low error margin. The Greeks for which it was possible to obtain an acceptable and reasonably robust estimate were the Delta and the Vega. In order not to overly burden the dissertation, the comparison surfaces between the LR methodological error and the experimental error are shown below only for the case of pricing.

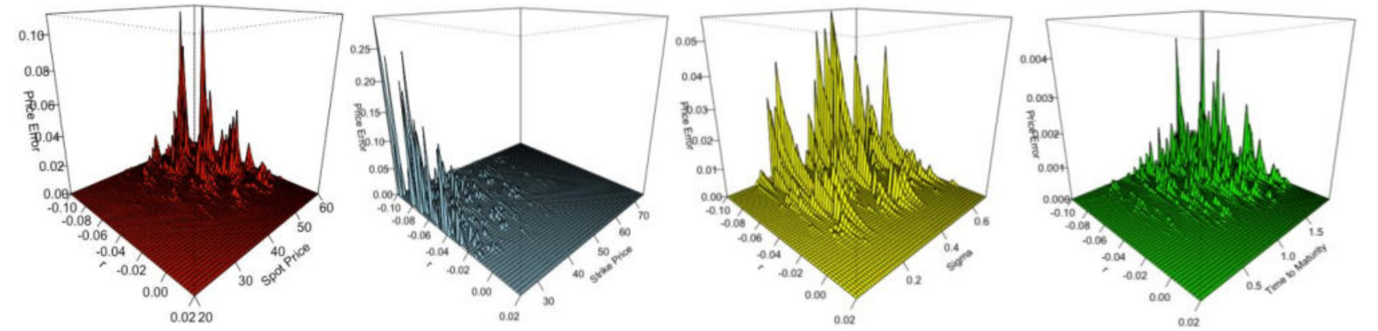


*Figure 11. Comparison surface between methodological error (LR) and experimental error (LSM) for pricing the call option in the theoretical case*

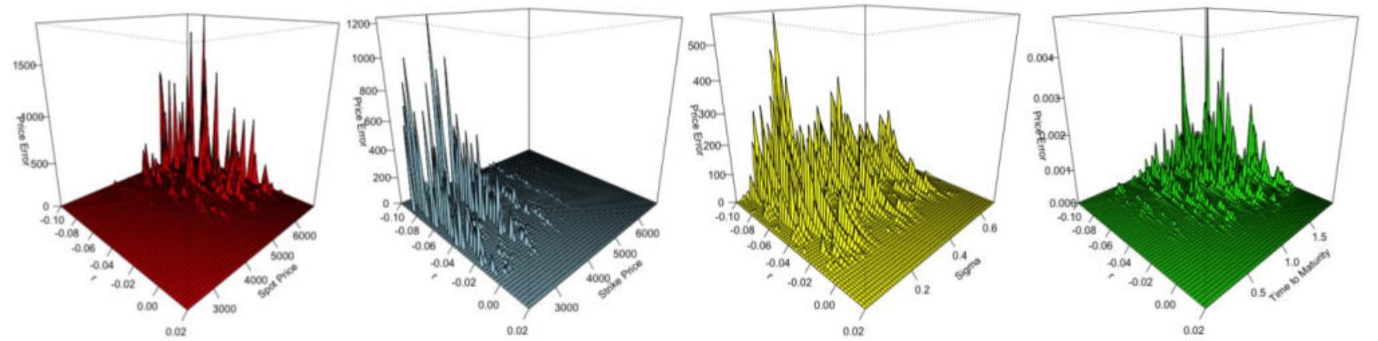


*Figure 12. Comparison surface between methodological error (LR) and experimental error (LSM) for pricing the call option in the market case*

## 6) Conclusions

The objective of the paper is to investigate the main problems that impact the pricing models and the sensitivity measures of American options written on shares that do not pay dividends, in the presence of negative interest rates. A literary review is conducted and the most popular lattice pricing methods are implemented as well as the Monte Carlo technique. The Leisen-Reimer binomial model proved to be the most performing deterministic methodology among those implemented in a non-stressed market context, this methodology was then tested in extreme market conditions, i.e., in the presence of negative interest rates and on a non-profitable underlying. Using the stress test, the valuation gap between the American and the European call option was determined. This discrepancy can be interpreted as the model risk deriving from the change in the valuation technique of the derivative, necessary it is impossible to use traditional techniques (quasi-closed formula), which are only applicable in ordinary market conditions (i.e. positive interest rates). The experimental results obtained with the Leisen-Reimer model were used as a benchmark for controlling the stability and performance of the Longstaff-Schwartz Monte Carlo in extreme scenarios. The concept underlying the Monte Carlo method is that being a stochastic methodology, by definition, it always produces different outputs, due to its random nature. The question which this work tries to answer is how the error generated by the Monte Carlo method "overlaps" the discrepancy relating to the model risk of the Leisen-Reimer.

If the error produced by the Monte Carlo method is lower than the model risk, then the same observations apply as for the LR: in a certain range of pricing parameters, the Monte Carlo method is stable and gives similar results to the LR. On the contrary, when the error generated by the stochastic methodology is higher, the model risk discrepancy is not so significant, therefore the outputs generated by Monte Carlo are considered unstable, in relative terms. Ultimately, if we observe the comparison surfaces between the methodological error and the experimental error, we observe that in many cases, except for rare out-of-scale peaks for the most extreme regions of the surfaces, the Monte Carlo method is reasonably stable, since the ranges of the error generated by the simulations are lower than the benchmark values. From the analysis of the surfaces, certain regions can be identified where a Monte Carlo instability occurs, caused by the combination of the following factors:

- for surfaces for which the spot price and the strike price are parameterized, extremely negative interest rate values, close to -10% and the deep in-the-money (ITM) option.
- for surfaces for which volatility is parameterized, extremely negative interest rate values and very low volatility values.
- for surfaces for which the time to maturity is parameterized, extremely negative interest rate values and the option close to maturity.

It should be highlighted that the Monte Carlo method, in the region of positive interest rates, always returns the results of the LR upon convergence. To conclude, we can state that the performance of the Monte Carlo method is effective for the interest rate intervals around zero, except for an extreme stress-test on the parameters $S, K, \sigma, T$. However, the further we delve into the regions of extremely negative interest rates, the greater the instability of the model. Considering the dynamic of the Euribor historical series from 2014 until today, and considering the negative values assumed by nominal interest rates, we can state that the Monte Carlo model ensures a reasonable reliability in the pricing of options written on equity, even in a context of moderately negative interest rates.

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