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The importance of strictly local martingales; applications to radial Ornstein–Uhlenbeck processes

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Abstract. For a wide class of local martingales (M_t) there is a default function, which is not identically zero only when (M_t) is strictly local, i.e. not a true martingale. This 'default' in the martingale property allows us to characterize the integrability of functions of $\sup_{s\le t} M_s$ in terms of the integrability of the function itself. We describe some (paradoxical) mean-decreasing local sub-martingales, and the default functions for Bessel processes and radial Ornstein–Uhlenbeck processes in relation to their first hitting and last exit times.

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1. Introduction

In this paper we encounter a number of examples of *strictly local martingales*, i.e. local martingales which are not martingales. A local martingale (M_t) is a true martingale if and only if $\mathbb{E}M_T = \mathbb{E}M_0$ for every bounded stopping time T. The quantity $\gamma_M(T) = \mathbb{E}M_0 - \mathbb{E}M_T$ quantifies the 'strict' local property of the local martingale and shall be called the 'default'. The corresponding γ_M shall be called the default function. This 'default' in the martingale property allows us to characterize the exponents p for which

$\mathbb{E}\left[\sup_{s\le t}|M_s|^p\right] < \infty$ and more generally, those p for which certain families of semi-martingales $(X_t^{(\alpha)} : t \ge 0)$ parameterized by $\alpha \in I$ satisfy:

$$\sup_{\alpha\in I}\mathbb{E}\left[\sup_{s\le t}|X_s^{(\alpha)}|^p\right] < \infty \ .$$

In §2.1 a representation for the default $\gamma_M(T)$ is given in terms of the weak tail of $\sup_{s\le T} M_s$, which shall be called the default formula. From this we also see that a local martingale whose negative part (M^-) belongs to class DL is a true martingale if and only if $\gamma_M(t) \equiv 0$. In §3 the default formula is generalized to perturbations of local martingales, especially to semi-martingales. In particular we discuss under which conditions on a local martingale (M_t), $\gamma_M(T)$ equals the weak tail of $\sup_{s\le T}|M_s|$.

Strictly local martingales appear naturally in applications: for example, the local martingale components found in many applications of Itô's formula are often strictly local martingales, especially when we are working on noncompact spaces. On the other hand, the expectation of a stochastic integral with respect to a martingale is often assumed to be zero, e.g. in a number of probabilistic discussions related to partial differential equations. A natural question is, then, whether the mean value function of a strictly local martingale has any special property? In §3.2, local martingales with mean value function $m(t)$ given by an arbitrary continuous positive non-increasing function $m : \mathbb{R}^+ \to \mathbb{R}^+$ are constructed. Similarly we give examples of strictly local sub-martingales, i.e. the sum of a strictly local martingale and an increasing process, which are mean decreasing. (A local sub-martingale (X_t) is said to be *mean decreasing* if $\mathbb{E}X_t \le \mathbb{E}X_s$ whenever $t > s$, and $\mathbb{E}X_{t_*} < \mathbb{E}X_{s_*}$ for at least one pair of numbers $t_* > s_*$.)

These results are applied to study the integrability properties of functionals of Bessel processes and radial Ornstein–Uhlenbeck (O.–U.) processes. First we calculate the default function γ for radial O.–U. processes, in terms of the law of T_0, the first time the process hits zero. Using the Girsanov theorem for last passage times, the law of T_0, for a $(4-\delta)$ dimensional O.–U. process starting from a is in turn given by the last hitting time of a by a δ-dimensional O.–U. process starting from 0. There is a brief analogous discussion for O.–U. processes with non-linear drift. In §3.4 we give a related non-integrability result for general diffusions using techniques from stochastic flows.

We would like to mention other topics where strictly local martingales play an essential role, namely; asymptotics for the Wiener sausage (Spitzer [18], Le Gall [11], see also section 2), and probabilistic proofs of the two main theorems in Nevanlinna theory (K. Carne [3], A. Atsuji [1]). Strictly local martingales have also come to play a role in mathematical finance as discussed in Delbaen and Schachermayer [4]. See also Sin [17] who gives a class of stock price models with stochastic volatility for which the most natural candidates for martingale measures yield only strictly local martingales, and Ornstein–Uhlenbeck processes are given as volatility processes in, e.g., Stein-Stein [19] and Heston [10]. See also Elworthy-Li-Yor [5], Galtchouk-Novikov[7] and Takaoka [20] for related work. In fact Takaoka [20] relates the weak tails of $\sup_{t\le T}|M_T|$ and $\langle M\rangle_T^{\frac{1}{2}}$ under 'almost' minimal assumptions.

Finally, in order to avoid some possible confusion, let us stress that our *strictly local martingales* do not bear any relation with *Le Jan's strict martingales*.

2. Strictly local martingales

Here we define a *strictly local martingale* to be a local martingale which is not a true martingale. Here are a couple of examples:

1. Let $\{R_t\}$ be a δ-dimensional Bessel process, $\delta > 2$. Then $\{R_t^{2-\delta}\}$ is a strictly local martingale. In fact $\mathbb{E}R_t^{2-\delta}$ is a strictly decreasing function in t by direct calculation. Alternatively see (34).
2. More generally, let $(X_t, t \le \zeta)$ be a regular transient diffusion on $(0,\infty)$ and s its speed measure such that (i) $\zeta = \inf\{t > 0 : X_{t-} = 0, \text{ or } \infty\}$ (ii) $s(0) = -\infty, s(\infty) < \infty$. (iii). 0 is an entrance point for the diffusion X. Then $\{s(X_t)\}$ is a strictly local martingale. See Elworthy-Li-Yor [5].

2.1. *The default formula*

A. Let X_t be a stochastic process adapted to an underlying filtration $(\mathscr{F}_t)$. Recall that X belongs to class D if the family of random variables X_S where S ranges through all stopping times S is uniformly integrable. It is in class DL if for each $a > 0$, $\{X_S\}$ is uniformly integrable over all bounded stopping times $S \le a$. If T is a stopping time, we write $X_\cdot^T = X_{T\wedge\cdot}$. Below it will be convenient to decompose X_t into its positive and negative parts, $X_t = X_t^+ - X_t^-$.

We have the following maximal and limiting maximal equalities.

Lemma 2.1. *Let M_t be a continuous local martingale with $\mathbb{E}|M_0| < \infty$. Let T be a finite stopping time such that $(M^T)^-$ is of class D. Then $M_{T\wedge\cdot}$ is bounded in L^1 and*

$$\mathbb{E}\left[M_T \mathbf{1}_{(\sup_{t\le T} M_t < x)}\right] + xP\left(\sup_{t\le T} M_t \ge x\right) + \mathbb{E}[M_0 - x]^+ = \mathbb{E}M_0 \ . \quad (1)$$

Furthermore,

$$\lim_{x\to\infty} xP\left(\sup_{t\le T} M_t \ge x\right) = \mathbb{E}M_0 - \mathbb{E}M_T \quad . \tag{2}$$

If $\lim_{t\to\infty} M_t$ *exists and is finite, e.g. for a positive local martingale* (M_t), *the stopping time* T *does not need to be finite.*

Note. A special case of the default formula appeared in Carne [3]. See also Atsuji [1]. A complement to the default formula in terms of $\langle X\rangle_T$ is presented in Elworthy-Li-Yor [5].

Proof. Let $(S_n, n \in \mathbb{N})$ be a reducing sequence of stopping times for $M_\cdot$, i.e. (S_n) increases to ∞ as $n \to \infty$ and is such that each $\{M_{t\wedge S_n}\}$ is a uniformly integrable martingale. Set $T_x = \inf\{t \ge 0 : M_t \ge x\}$. First, assume M_0 is a constant. If $x > M_0$ then $T_x = \inf\{t > 0 : M_t = x\}$ and

$$\mathbb{E}\left[M_{T_x\wedge T\wedge S_n}\right] = \mathbb{E}M_0 \quad . \tag{3}$$

Letting $n \to \infty$, we obtain:

$$\mathbb{E}\left[M_{T_x\wedge T}\right] = \mathbb{E}M_0 \quad . \tag{4}$$

This is (1):

$$\mathbb{E}\left[M_T \mathbf{1}_{(\sup_{t\le T} M_t < x)}\right] + xP\left(\sup_{t\le T} M_t \ge x\right) = \mathbb{E}M_0 \quad .$$

On the other hand (1) is clearly true for $x \le M_0$. In general for M_0 an integrable random variable, we take conditional expectations with respect to $\mathscr{F}_0$ to get (1). Now M_T is integrable by (4):

$$\mathbb{E}|M_T| \le \varliminf_{x\to\infty} \mathbb{E}\left(M_{T\wedge T_x} + 2M^-_{T\wedge T_x}\right) \le \mathbb{E}M_0 + 2\mathbb{E}M_T^- \quad . \tag{5}$$

Write

$$\mathbb{E}\left[M_T \mathbf{1}_{(\sup_{t\le T} M_t < x)}\right] = \mathbb{E}\left[M_T^+ \mathbf{1}_{(\sup_{t\le T} M_t < x)}\right] - \mathbb{E}\left[M_T^- \mathbf{1}_{(\sup_{t\le T} M_t < x)}\right]$$

and take the limit as $x \to \infty$ in (1) to get

$$\mathbb{E}M_T + \lim_{x\to\infty} xP\left(\sup_{t\le T} M_t \ge x\right) = \mathbb{E}M_0 \quad . \tag{6}$$

If $\lim_{t\to\infty} M_t$ exists and is finite, the argument above holds for non-finite T. □

B. For a stopping time T, set

$$\gamma_M(T) = \mathbb{E}M_0 - \mathbb{E}M_T \quad . \tag{7}$$

We call the quantity defined in (7) the *default* of the process (M^T) and the limiting maximal equality (2) the *default formula*.

Proposition 2.2. *A local martingale* $\{M_t\}$ *such that* $\mathbb{E}|M_0| < \infty$ *and its negative part belongs to class DL is a super-martingale. It is a martingale if and only if* $\mathbb{E}M_t = \mathbb{E}M_0$, *i.e.* $\gamma_M(t) = 0,$ *for all* $t > 0$.

Proof. The second statement is well known and follows from the above discussions: $\{M_t\}$ is a martingale if and only if $\gamma_M(T) = 0$ for all bounded stopping times T, the latter is equivalent to $\gamma_M(t) = 0$ for all $t > 0$. The equivalence of the latter comes from formula (2):

$$\gamma_M(T) = \lim_{x\to\infty} x P\left(\sup_{t\le T} M_t \ge x\right) \le \lim_{x\to\infty} x P\left(\sup_{t\le t_0} M_t \ge x\right) \quad ,$$

where t_0 is an upper bound for T.

To prove $\{M_t\}$ is a super-martingale, take two bounded stopping times $S \le T$. We only need to prove $\mathbb{E}M_S \ge \mathbb{E}M_T$. Let $N_t = M_{t+S}$, then $\{N_t\}$ is a local martingale with respect to the filtration $\{\mathscr{F}_{t+S}\}$. So (2) gives:

$$\lim_{x\to\infty} x P\left(\sup_{t\le T-S} N_t \ge x\right) = \mathbb{E}N_0 - \mathbb{E}N_{T-S} = \mathbb{E}M_S - \mathbb{E}M_T \quad .$$

Thus $\mathbb{E}M_S \ge \mathbb{E}M_T$ and $\{M_\cdot\}$ is a super-martingale. □

2.2. *Integrability of functionals*

A. Let $F : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $F(x) = \int_0^x dy f(y)$, for f a nonnegative Borel function.

Proposition 2.3. *Let* $\{M_t\}$ *be a local martingale and* T *a stopping time such that the default formula* (2) *holds with* $\gamma_M(T) = \mathbb{E}M_0 - \mathbb{E}M_T$ *finite. If* $\int^\infty \frac{f(y)}{y} dy < \infty$, *then*

$$\mathbb{E}\left[F(\sup_{s\le T} M_s)\right] < \infty \quad .$$

Furthermore if also $\gamma_M(T) > 0$, *then* $\int^\infty \frac{f(y)}{y} dy = \infty$ *if and only if*

$$\mathbb{E}\left[F(\sup_{s\le T} M_s)\right] = \infty \quad .$$

Proof. This follows from

$$\mathbb{E}\left[F(\sup_{s\le T} M_s)\right] = \int_0^\infty f(y)P\left(\sup_{s\le T} M_s \ge y\right)dy$$

and (2). □

B. As an example we look at the integrability of certain standard Brownian and Bessel functionals. For $\delta \in \mathbb{R}^+$, let $\{R_t\}$ be a δ-dimensional Bessel process. The expectation of a δ-dimensional Bessel process starting from a will be denoted by $E_a^{(\delta)}$. The subscript may be omitted if no initial value is specified. For $\delta \ge 2$, $\{R_t\}$ is the unique non-negative solution to the stochastic differential equation:

$$d\rho_t = d\beta_t + \frac{\delta-1}{2\rho_t}dt \quad ,$$

where $\{\beta_t\}$ is a 1-dimensional Brownian motion. For $\delta < 2$ the situation is different because the Bessel process $(R_t : t \ge 0)$ has a non-trivial set of zeros. Moreover there is an increasing process $L_t(R)$ whose support is precisely this set of zeros, and such that $R_t^{2-\delta} - L_t(R), t \ge 0$, is a local martingale, in fact a martingale with moments of all orders. Thus for our purposes the case $\delta < 2$ is uninteresting and we shall restrict discussions to $\delta \ge 2$. Furthermore $\{\frac{1}{R_t^{\delta-2}}\}$ is a local martingale for $\delta > 2$ and when $\delta = 2$, $\log r$ is a scale function for R_t and $(\log R_t)$ is therefore again a local martingale. This leads to:

Corollary 2.4. *Let $I = [r_1, r_2]$ be an interval with $r_1 > 0$. Let $t > 0$ and $p > 0$ we have*

1. *For $\delta > 2$,*

$$\sup_{a\in I} E_a^{(\delta)} \sup_{s\le t}\left(\frac{1}{R_s^{\delta-2}}\right)^p < \infty. \qquad \textit{if and only if } p < 1 \ . \tag{8}$$

   This also holds for $t \equiv \infty$.

2. *For $\delta > 2$,*

$$\sup_{a\in I} E_a^{(\delta)}\left[\int_0^t \frac{1}{R_s^{2(\delta-1)}}ds\right]^{\frac{p}{2}} < \infty, \quad \textit{if and only if } p < 1 \ .$$

3. *For $a \ne 0$,*

$$E_a^{(2)}\left[\sup_{s\le t}|\log\frac{1}{R_s}|^p\right] < \infty, \qquad \textit{if and only if } p < 1$$

   and

$$E_a^{(2)}\left[\int_0^t \frac{ds}{R_s^2}\right]^{\frac{p}{2}} < \infty, \qquad \textit{if and only if } p < 1 \ .$$

Proof. Since $\{\frac{1}{R_t^{\delta-2}}\}$ is a strictly local martingale, (8) is a direct consequence of Proposition 2.3 if I contains a single point. It also holds for $t = \infty$ by the last sentence of Lemma 2.1, since $\lim_{t\to\infty} \frac{1}{R_t} = 0$. For genuine intervals I using a corresponding result for a family of local martingales, c.f. Proposition 3.5. For part 2, note that for $\delta \neq 2$, $(R_t)^{-(\delta-2)}$ satisfies:

$$(R_t)^{-(\delta-2)} = (R_0)^{-(\delta-2)} - (\delta-2)\int_0^t \frac{d\beta_s}{R_s^{\delta-1}} \tag{9}$$

The required result follows from the Burkholder-Davis-Gundy inequality and (8) for each $0 \le t \le \infty$.

For the case of $\delta = 2$, we apply Proposition 2.3 to $\log \frac{1}{R_t}$. First note that the negative part of $\log \frac{1}{R_t}$ is uniformly integrable over finite times:

$$\left[\log \frac{1}{R_t}\right]^- \equiv \log R_t \mathbf{1}_{(R_t \ge 1)} \le (R_t - 1)^+ \le R_t \quad .$$

Next we observe that $\{\log \frac{1}{R_t}\}$ is a strictly local martingale, since $E_a^{(2)} R_t$ is strictly increasing in t and so

$$E_a^{(2)}\left[\log\left(\frac{1}{R_t}\right)\right] \le -\log E_a^{(2)}[R_t] < \log \frac{1}{a} \quad .$$

The required result follows. □

From Lemma 2.1 we also have, for a 2-dimensional Bessel process,

$$\lim_{r\to\infty} rP\left\{\inf_{s\le t} \log R_s \le -r\right\} = E^{(2)} \log R_t - \log a \quad .$$

This is closely related to Spitzer's asymptotics [18], although Spitzer presents a different quantity on the right hand side; see also the estimate (0a) for Wiener sausages in Le Gall[11], and (12) in Elworthy, Li and Yor[5].

Using Girsanov transform, the result of Corollary 2.4 extends to radial Ornstein–Uhlenbeck processes of integer dimensions, as below.

Let $n \ge 2$ be an integer. An n-dimensional radial Ornstein–Uhlenbeck process $\{R_t\}$ with parameter λ can be realized as the radial part of an n-dimensional Ornstein–Uhlenbeck process $\{U_t\}$ with parameter λ as the solution to the stochastic differential equation:

$$U_t = x + B_t - \lambda \int_0^t U_s\, ds, \quad x \in \mathbb{R}^n, \lambda \in \mathbb{R}^1 \quad , \tag{10}$$

which can be solved explicitly to give: $U_t = e^{-\lambda t}\left(x + \int_0^t e^{\lambda s} dB_s\right)$. For $\lambda \ge 0$, the reciprocal $\{\frac{1}{R_t^{n-2}}\}$ is a local sub-martingale as seen by Itô's formula:

$$\frac{1}{|U_t|^{n-2}} = \frac{1}{|x|^{n-2}} - (n-2)\int_0^t \frac{\langle U_s, dB_s\rangle}{|U_s|^n} + (n-2)\int_0^t \frac{\lambda}{|U_s|^{n-2}} ds \quad (11)$$

for $x \neq 0$ (when $x = 0$, this is true with x replaced by U_ϵ and the lower limit 0 in the integrals replaced by ϵ, for any $\epsilon > 0$ and so we have a local sub-martingale for $t > 0$).

Proposition 2.5. *Let $n > 2$, $\lambda > 0$, and $R_t = |U_t|$ be the radial part of an n-dimensional O.–U. process, then for K a compact subset of $\mathbb{R}^n - \{0\}$,*

$$\sup_{x\in K} {}^{-\lambda}E^{(n)}_{|x|}\left[\sup_{s\le t}\left(\frac{1}{R_s}\right)^p\right] < \infty \qquad \textit{if and only if } p < n-2 \ . \qquad (12)$$

Here we denote by ${}^{-\lambda}E^n_a$ the expectation of a n-dimensional radial O.–U. process with parameter λ, starting from a.

Proof. First note that when $\lambda = 0$, this is the Bessel case and (12) is true as shown earlier. For $\lambda > 0$, the Girsanov transform gives:

$$\begin{aligned}{}^{-\lambda}E^{(n)}_{|x|}\sup_{s\le t}\left(\frac{1}{R_s}\right)^p &= \mathbb{E}\left(e^{-\lambda\int_0^t\langle x+B_s,dB_s\rangle - \frac{\lambda^2}{2}\int_0^t|x+B_s|^2ds}\sup_{s\le t}\frac{1}{|x+B_s|^p}\right)\\ &= \mathbb{E}\left(e^{-\frac{\lambda}{2}\left[|x+B_t|^2-|x|^2-nt\right]-\frac{\lambda^2}{2}\int_0^t|x+B_s|^2ds}\sup_{s\le t}\frac{1}{|x+B_s|^p}\right)\\ &\le e^{\frac{\lambda}{2}\left(|x|^2+nt\right)}\mathbb{E}\left(\sup_{s\le t}\frac{1}{|x+B_s|^p}\right)\end{aligned}$$

which is bounded on K for $p < n-2$ by (8). So when $p < n-2$,

$$\sup_{x\in K} {}^{-\lambda}E^{(n)}_{|x|}\left[\sup_{s\le t}\left(\frac{1}{R_s}\right)^p\right] < \infty \ .$$

On the other hand note that (R_t^{2-n}) satisfies the stochastic differential equation

$$d\rho_t = -(n-2)\rho_t^{(1-n)/(2-n)}\, d\beta_t + (n-2)\lambda\rho_t\, dt. \qquad (e_\lambda)$$

and $\frac{1}{|x+B_t|^{n-2}}$ solves

$$d\rho_t = -(n-2)\rho_t^{(1-n)/(2-n)}\, d\beta_t. \qquad (e_0)$$

By the comparison theorem $\frac{1}{R_t} \ge \frac{1}{|x+B_t|}$ almost surely. Thus, if $p \ge n-2$,

$${}^{-\lambda}E^{(n)}_{|x|}\sup_{s\le t}\left(\frac{1}{R_s}\right)^p \ge \mathbb{E}\sup_{s\le t}\left(\frac{1}{|x+B_s|}\right)^p = \infty \ ,$$

again by (8).

Remark. We leave to the reader the rather easy modification of the statement and the proof of Proposition 2.5 in the case $n > 2$, n not an integer. See also Section 3.3.

3. The default formula for general processes

In complete generality note that if ξ, η, ρ are real valued random variables with $\xi \le \eta + \rho$ and $\mathbb{E}\rho < \infty$, then

$$\overline{\lim_{x\to\infty}}\, xP(\xi \ge x) \le \overline{\lim_{x\to\infty}}\, xP(\eta \ge x) \ .$$

This will give us a perturbation result:

Lemma 3.1. *Let $X_t = M_t + D_t$ be the sum of two stochastic processes. Then if $\mathbb{E}\ \sup_{s\le t} |D_s| < \infty$,*

$$\overline{\lim_{x\to\infty}}\, xP(\sup_{s\le t} X_s \ge x) = \overline{\lim_{x\to\infty}}\, xP(\sup_{s\le t} M_s \ge x)$$

and

$$\underline{\lim}_{x\to\infty}\, xP(\sup_{s\le t} X_s \ge x) = \underline{\lim}_{x\to\infty}\, xP(\sup_{s\le t} M_s \ge x) \ .$$

□

Consequently the weak tails of $\sup_{s\le t} X_s$ and $\sup_{s\le t} M_s$ are equal if they exist and they exist at the same time. (By the *weak tail*, of a random variable ξ, we mean $\lim_{x\to\infty} xP\ (\xi \ge x)$ if it exists.) In particular if (M_t) is a local martingale, the weak tail of $\sup_{s\le t} X_s$ is related to the default function γ_M. However the assumption on D_t can be much weakened when (D_t) is of bounded variation.

Recall that a continuous *semi-martingale* $(X_t : t \ge 0)$ is a stochastic process with canonical decomposition $X_t = M_t + A_t$ into the sum of a continuous local martingale M_t and a continuous adapted process A_t of bounded variation (starting from 0). It is a *local sub(super)martingale* if A_t is increasing (decreasing) in t. It is *integrable* if X_t is integrable for each t.

Lemma 3.2. *Let $X_t = M_t + A_t$ be a continuous semi-martingale with $\mathbb{E}|X_0| < \infty$ and $A_0 = 0$. Let T be a finite stopping time such that $(X^T)^-$ is of class D.*

1. *Suppose A_t is a monotone process. Then*

$$\mathbb{E}\left[X_T \mathbf{1}_{(\sup_{t\le T} X_t < x)}\right] + xP\left(\sup_{t\le T} X_t \ge x\right) + \mathbb{E}[X_0 - x]^+ \\ = \mathbb{E}X_0 + \mathbb{E}\left[A_{T\wedge T_x}\right] \ , \qquad (13)$$

where $T_x = \inf\{t \ge 0 : X_t \ge x\}$. Furthermore if $\mathbb{E}A_T < +\infty$, then $X_{T\wedge\cdot}$ is bounded in L^1 and

$$\lim_{x\to\infty} xP\left(\sup_{t\le T} X_t \ge x\right) = \mathbb{E}X_0 - \mathbb{E}X_T + \mathbb{E}A_T \ . \tag{14}$$

Let $\gamma_X(T) = \mathbb{E}X_0 - \mathbb{E}X_T + \mathbb{E}A_T$. Then $\gamma_X(T)$ is finite for local super-martingales with $(X^T)^-$ in class D.

2. *For a general A of finite variation if its increasing part A^1_t satisfies $\mathbb{E}A^1_T < \infty$, then $\mathbb{E}(A_T) < \infty$ and both* (13) *and* (14) *hold.*

Proof. Let $(S_n, n \in \mathbb{N})$ be a reducing sequence of stopping times for (M_t). As in the proof of Lemma 2.1, we have:

$$\mathbb{E}\left[X_{T_x\wedge T\wedge S_n}\right] = \mathbb{E}X_0 + \mathbb{E}\left[A_{T_x\wedge T\wedge S_n}\right] \ , \tag{15}$$

first assuming M_0 is a constant. Now the assumption on X^T allows us to take $n \to \infty$ to get:

$$\mathbb{E}\left[X_{T_x\wedge T}\right] = \mathbb{E}X_0 + \mathbb{E}\left[A_{T_x\wedge T}\right] \ , \tag{16}$$

which gives (13). Both (16) and (13) hold for general M_0 by taking conditional expectation with respect to $\mathscr{F}_0$. Now, X_T is integrable if $\mathbb{E}A_T < +\infty$; indeed by (16):

$$\mathbb{E}|X_T| \le \varliminf_{x\to\infty} \mathbb{E}\left(X_{T\wedge T_x} + 2X^-_{T\wedge T_x}\right) \le \mathbb{E}X_0 + \mathbb{E}A_T + 2\mathbb{E}X^-_T \ . \tag{17}$$

Since $\mathbb{E}|X_T| < \infty$ we can take the limit as $x \to \infty$ to get

$$\mathbb{E}X_T + \lim_{x\to\infty} xP\left(\sup_{t\le T} X_t \ge x\right) = \mathbb{E}X_0 + \mathbb{E}A_T \ . \tag{18}$$

In particular, if $\{X_t\}$ is a local super-martingale with $(X^T)^-$ in class D then (18) holds and $\lim_{x\to\infty} xP\left(\sup_{t\le T} X_t \ge x\right)$ is finite.

For part 2 note that the assumption on A implies the integrability of both X_T and A_T, using (17), and thus the argument above is valid. □

Remarks.

(1). Assume A is of finite variation with $A_0 = 0$. Note that $M^- \le X^- + A^+ \le X^- + A^1$. If $(X^T_\cdot)^-$ belongs to class D, then so does $(M^T_\cdot)^-$ provided that $(A^+)^T$ is of class D or more generally A^1_T is integrable. On the other hand by Lemma 2.1 if $(M^T)^-$ is in class D, the integrability of A_T is equivalent to that of X_T.

(2). If $(X^T)^-$ is of class D and $\mathbb{E}A^1_T < \infty$, by (1) above $(M^T)^-$ is of class D and so the lemma can be applied to both X and M to yield:

$$\lim_{x\to\infty} xP\left(\sup_{t\le T} X_t \ge x\right) = \lim_{x\to\infty} xP\left(\sup_{t\le T} M_t \ge x\right) \ . \tag{19}$$

(3). If M_t is a strictly local martingale with $(M^-)^T$ in class D and if the limiting maximal equality (14) holds, then there is a bounded stopping time S such that $\gamma_X(S) > 0$. Just note that $\gamma_X(T) = \gamma_M(T) \geq 0$ for all finite stopping times T and (M_t) is not a true martingale implies that there is a bounded stopping time S such that $\mathbb{E}M_S < \mathbb{E}M_0$, i.e. $\gamma_M(S) = \mathbb{E}M_0 - \mathbb{E}M_S > 0$.

On the other hand let $X_t = M_t + A_t$ be the sum of a martingale and an integrable increasing process. By going back to the proof of Lemma 3.2 it is easily seen that (14) holds and $\gamma_X(T) \equiv 0$ for all bounded stopping times T.

We now show how the lemma applies to $X' = |M|$ and $X'' = M$, for M a local martingale. Let $\{L_t(M)\}$ be the local time at 0 of $\{M_t\}$. We first note the double implication:

$$\{(M^T)^- \text{ is of class } (D)\} \overset{(a)}{\Longrightarrow} \{\mathbb{E}(L_T(M)) < \infty\} \overset{(b)}{\Longrightarrow} \{\mathbb{E}(|M_T|) < \infty\} .$$

But there is no converse. Both (a) and (b) follow easily from Tanaka's formula and Fatou's lemma. Note also that $\{\mathbb{E}(L_T(M)) < \infty\}$ is equivalent to

$$\{M_S, S \leq T \text{ stopping times}\} \text{ being bounded in } L^1 \quad ,$$

and also to $\{M_S^-, S \leq T$ stopping times$\}$ being bounded in L^1.

We now compare $\gamma_{|M|}$ and γ_M .

Proposition 3.3. *Let* $(M_t : t \geq 0)$ *be a local martingale, with* $\mathbb{E}(|M_0|) < \infty$. *Then*

1. *if* $\mathbb{E}(L_T(M)) < \infty$, *one has*

$$\gamma_{|M|}(T) = \lim_{x\to\infty} xP\left(\sup_{t\leq T}|M_t| \geq x\right), \tag{20}$$

where

$$\gamma_{|M|}(T) := \mathbb{E}(|M_0|) - \mathbb{E}(|M_T|) + \mathbb{E}(L_T(M)) \quad .$$

2. *if* $(M^T)^-$ *is of class D, then* $\gamma_{|M|}(T) = \gamma_M(T)$.

Proof. The first assertion comes from Tanaka's formula and the lemma, while the second follows from the equality: $\mathbb{E}(M_T^-) = \mathbb{E}(M_0^-) + \frac{1}{2}\mathbb{E}(L_T(M))$. □

Corollary 3.4. [*Rao[14], Azéma, Gundy and Yor [2]*] *Consider* $(M_t : t \geq 0)$, *a local martingale, with* $\mathbb{E}(|M_0|) < \infty$. *If* $\mathbb{E}(L_T(M)) < \infty$, *then* $\gamma_{|M|}(T) = 0$ *if and only if* (M^T) *is uniformly integrable, or equivalently*

$$\lim_{x\to\infty} xP\left((\langle M\rangle_T)^{\frac{1}{2}} \geq x\right) = 0 \quad .$$

Proof. The first assertion follows easily from Scheffé's lemma, while the second condition is found in Azéma, Gundy and Yor ([2], Theorem 7). □

Comments.

1. Note that under the condition $\mathbb{E}(L_T(M)) < \infty$ we have, in complete generality: $\gamma_{|M|}(T) = \gamma_{M^+}(T) + \gamma_{M^-}(T)$ and $\gamma_M(T) = \gamma_{M^+}(T) - \gamma_{M^-}(T)$. Scheffé's lemma again shows that, under the same condition the equality $\gamma_{|M|}(T) = \gamma_M(T)$ is in fact equivalent to $\gamma_{M^-}(T) = 0$ and to $(M^T)^-$ being of class D.
2. An example where $\gamma_{|M|}(T)$ and $\gamma_M(T)$ differ is $T_1 = \inf\{t : L_t(B) > 1\}$, where $L_t(B)$ is the local time at 0, of a Brownian motion $(B_\cdot)$, and $M_\cdot = B_\cdot$. Then: $\gamma_{|B|}(T_1) = 1$ and $\gamma_B(T_1) = 0$. Another example which is even more directly related to our first comment is $T_2 = \inf\{t : B_t = 1\}$, $M = B$; then $\gamma_{|B|}(T_2) = 1$, and $\gamma_B(T_2) = -1$.
3. Assume for simplicity $M_0 = 0$. One may wonder under which 'minimal condition' on M^T the formula (20) is valid. However it is not enough to just have $L_T(M) - |M_T|$ integrable. Take indeed, $T = \inf\{t : |B_t| = 1 + L_t(B)\}$ and $M = B$. Then, $L_T(B) - |B_T| = -1$, which cannot hold together with (20).

3.1. Integrability of functionals

As for strictly local martingales, there is also an integrability criterion for a more general family of stochastic processes: Let $F : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $F(x) = \int_0^x dy f(y)$, for f a nonnegative Borel function. Then

$$\lim_{y\to\infty} yP\left(\sup_{s\le T} X_s \ge y\right) > 0$$

implies that the finiteness of $\int^\infty \frac{f(y)}{y}dy$ is equivalent to that of $\mathbb{E}\left[F(\sup_{s\le T} X_s)\right]$.

Proposition 3.5. *Let $\{X_t^\alpha\}$ be a family of semi-martingales with decomposition $X_t^\alpha = M_t^\alpha + A_t^\alpha$.*

1. *Let T be a stopping time such that for each $\{X_t^\alpha\}$ (13) holds. (e.g. if $[(X^\alpha)^T]^-$ is of class D.) Suppose $\sup_\alpha \mathbb{E}[X_0^\alpha]$, $\sup_\alpha \mathbb{E}[(X_T^\alpha)]$ and $\sup_\alpha \mathbb{E}[A_T^\alpha]$ are all finite. Then $\int^\infty \frac{f(y)}{y}dy < \infty$ implies*
$$\sup_\alpha \mathbb{E}F\left(\sup_{s\le T} X_s^\alpha\right) < \infty \ .$$
2. *Conversely if for some α_0, the default satisfies $0 < \gamma_{\alpha_0}(T) < \infty$, then $\int^\infty \frac{f(y)}{y}dy = \infty$ implies*
$$\mathbb{E}F\left(\sup_{s\le T} X_s^{\alpha_0}\right) = \infty \ .$$

Proof. First

$$\sup_\alpha \mathbb{E}F\left(\sup_{s\le T} X_s^\alpha\right) = \sup_\alpha \int_1^\infty \frac{f(y)}{y}\cdot yP\left(\sup_{s\le T} X_s^\alpha \ge y\right)dy + \sup_\alpha \int_0^1 f(y)\,P\left(\sup_{s\le T} X_s^\alpha \ge y\right)dy\ .$$

By (13),

$$\begin{aligned}\sup_\alpha \int_1^\infty \frac{f(y)}{y}\cdot yP\left(\sup_{s\le T} X_s^\alpha \ge y\right)dy &= \sup_\alpha \int_1^\infty \frac{f(y)}{y}\Big[\mathbb{E}[X_0^\alpha] - \mathbb{E}[X_0^\alpha - x]^- + \mathbb{E}A^\alpha_{T\wedge T_y^\alpha} \\ &\qquad + \mathbb{E}(X_T^\alpha)\mathbf{1}_{\{\sup_{t\le T} X_t^\alpha < y\}}\Big]dy \\ &\le \sup_\alpha \int_1^\infty \frac{f(y)}{y}\left[\mathbb{E}[X_0^\alpha] + \mathbb{E}A_T^\alpha + \mathbb{E}(X_T^\alpha)\right]dy\ ,\end{aligned}$$

which is finite if $\int^\infty \frac{f(y)}{y}dy$ is. Here T_y^α is the first time X_t^α takes value y.

For the converse part just notice that

$$\mathbb{E}F\left(\sup_{s\le T} X_s^{\alpha_0}\right) = \int_0^\infty \frac{f(y)}{y}\cdot yP\left(\sup_{s\le T} X_s^{\alpha_0} \ge y\right)dy\ .$$

□

Example. Let $\alpha \in \mathbb{R}$ and $\delta > 2$. Observe that,

$$(R_t)^{-\alpha} = (R_0)^{-\alpha} - \alpha\int_0^t \frac{d\beta_s}{R_s^{1+\alpha}} + \frac{1}{2}\alpha(\alpha+2-\delta)\int_0^t \frac{ds}{R_s^{\alpha+2}}\ . \qquad (21)$$

So $\{\frac{1}{R_t^\alpha}\}$ is a local sub-martingale if $\alpha \ge \delta - 2$ or $\alpha < 0$. It is a super-martingale otherwise. We can now follow the discussion of Corollary 2.4 to consider integrability properties of $\int_0^\infty\{\frac{1}{R_t^\alpha}\}\,dt$ for $\alpha \ne \delta - 2$.

First recall Corollary 2, identity (2.c) in Yor [24]:

$$\int_0^\infty \frac{ds}{R_s^\alpha} \overset{\text{law}}{=} \frac{2}{(\alpha-2)^2} Z_{\frac{\delta-2}{\alpha-2}}\ ,$$

where Z_ν is a Gamma variable with parameter ν. See (39) below for the law of Z_ν. So

$$\int_0^\infty \frac{ds}{R_s^\alpha} < \infty \ \text{ a.s.} \tag{22}$$

if and only if $\alpha > 2$, and for $\alpha > 2$, $p \geq 0$

$$E_a^{(\delta)} \left[\int_0^\infty \frac{ds}{R_s^\alpha} \right]^p < \infty, \qquad a > 0 \ , \tag{23}$$

if and only if $p < \frac{\delta-2}{\alpha-2}$.

In fact (22) can be seen more directly. For $\alpha = 2$, $\frac{1}{\log t} \int_0^t \frac{ds}{R_s^2}$ converges a.s., as $t \to \infty$, to a constant, using an ergodic theorem relative to the scaling property on path space [Revuz and Yor [15]: Exercise (1.17), Chap. XIII, p. 522 and Exercise (3.20), Chap. X, p. 430]: let (W_t) be the Brownian path, then

$$T_c : W_t :\to \frac{1}{\sqrt{c}} W_{ct}$$

leaves the Wiener measure invariant and is ergodic since W and $W \circ T_c$ are asymptotically independent.

If $\alpha < 2$, by scaling,

$$\frac{1}{t^{1-\frac{\alpha}{2}}} \int_0^t \frac{ds}{R_s^\alpha} \stackrel{\text{law}}{=} \int_0^1 \frac{du}{\tilde{R}_u^\alpha} \ ,$$

where $\tilde{R}$ is a Bessel process starting from 0. And for $2 < \alpha < \delta$,

$$\mathbb{E} \left[\int_1^\infty \frac{ds}{R_s^\alpha} \right] = \left(\int_1^\infty \frac{ds}{s^{\frac{\alpha}{2}}} \right) \mathbb{E} \left(\frac{1}{R_1^\alpha} \right) < \infty \ .$$

So (22) holds in this case also. For $\alpha \geq \delta$ use the transience property of the process to see the conclusion.

The finiteness of the expectation of the stochastic integral in (22) also follows alternatively from (8) and (9).

Proposition 3.6. *Let* $\delta > 2$. *For* $\beta < 2$,

$$\sup_{a \in \mathbb{R}_+} E_a^{(\delta)} \left[\int_0^t \frac{ds}{R_s^\beta} \right]^k < \infty, \qquad \textit{for any } 0 < k < 1 \ . \tag{24}$$

For $\beta = 2$, (24) *holds with subsets bounded away from* 0:

$$\sup_{\epsilon \leq a < \infty} E_a^{(\delta)} \left[\left(\int_0^t \frac{ds}{R_s^2} \right)^k \right] < \infty, \qquad \textit{for any } k \in (0,1), \epsilon > 0 \ . \tag{25}$$

Proof. Recall that if $X_t = M_t + A_t$ is a continuous sub-martingale then for any stopping time T and $k > 0$ and some constant c_k (Corollaire 3, p. 13 in Yor [21]),

$$\mathbb{E}[(A_T)^k] \le c_k \mathbb{E}[\sup_{s \le T}(X_s)^k] \ . \tag{26}$$

This is a consequence of the Garsia-Neveu lemma discussed in the same reference. In (21), take $\alpha = \beta - 2$ and set

$$A_t^{(\beta)} = \frac{1}{2}(\beta - 2)(\beta - \delta) \int_0^t \frac{ds}{R_s^{\beta}} \ ,$$

$$M_t^{(\beta)} = (2 - \beta) \int_0^t \frac{d\beta_s}{R_s^{\beta - 1}} \ .$$

So $(R_t)^{2-\beta} = (R_0)^{2-\beta} + M_t^{(\beta)} + A_t^{(\beta)}$. We check Proposition 3.5 applies to show the right hand side of (26) is finite. First the bracket

$$\langle M^{(\beta)} \rangle_t = (2 - \beta)^2 \int_0^t \frac{ds}{R_s^{2(\beta - 1)}} < \infty \ .$$

In fact $\mathbb{E}\langle M^{(\beta)} \rangle_t < \infty$ by direct calculation using polar coordinates and the fact that $2(\beta - 1) < 2$. We see that $\{R_t^{2-\beta}\}$ is a sub-martingale satisfying the conditions of Proposition 3.5 with initial value a in a compact set. For $k < 1$ set $F(x) = x^k$ in Proposition 3.5. The required (24) now follows since by a stochastic monotonicity argument the supremum there is attained at the point $x = 0$.

For $\beta = 2$, we use Itô's formula for the logarithmic function:

$$\log R_t = \log R_0 + \int_0^t \frac{d\beta_s}{R_s} + \frac{\delta - 2}{2} \int_0^t \frac{ds}{R_s^2} \ , \tag{27}$$

where $\{\beta_t\}$ is again a one-dimensional Brownian motion. The same argument yields (25) and the supremum there is attained at $\epsilon = 0$. □

3.2. Some paradoxical examples

A process (M_t) is said to satisfy (LB) if, for any t, $\{M_T : T \le t, T$ stopping times$\}$ is bounded in L^1.

Proposition 3.7. *Let $\{M_t'\}$ and $\{M_t''\}$ be two orthogonal continuous local martingales starting from* 1 *satisfying* (*LB*). *Then $\{N_t\}$ defined by $N_t = M_t' - M_t''$ is a local martingale*, *and it is a martingale if and only if $\{M_t'\}$ and $\{M_t''\}$ are.*

Proof. Recall from Corollary 3.4 that a local martingale $\{M_t\}$ satisfying (LB) is a martingale if and only if

$$\lim_{x\to\infty} x\, P\left(\langle M\rangle_t^{1/2} \geq x\right) = 0 \ .$$

Clearly $\{N_t\}$ is a local martingale satisfying (LB) and it is a martingale if the other two are. But

$$\langle M'\rangle_t^{1/2} \ \leq \ \langle N\rangle_t^{1/2}$$

since

$$\langle N\rangle_t \ = \ \langle M'\rangle_t + \langle M''\rangle_t \ \ .$$

So $\{M_t'\}$ and $\{M_t''\}$ are martingales if $\{N_t\}$ is. □

In the above proposition if $\{M_t'\}$ and $\{M_t''\}$ are taken to be independent copies of a strictly local martingale, then $\{N_t\}$ is a strictly local martingale with mean zero. More generally given $m(\cdot)$, any non-negative continuous non-increasing function of t, we can find a strictly local martingale with $m(\cdot)$ as its expectation:

Proposition 3.8. *If* $m : \mathbb{R}^+ \to (0, 1]$ *is a continuous non-increasing non-negative function with* $m(0) = 1$, *then there exists a non-negative local martingale* $\{M_t\}$ *such that* $m(t) = \mathbb{E}M_t$.

Proof. Let $\{R_t\}$ be a 3-dimensional Bessel process starting from 1 and

$$r_3(t) = \mathbb{E}\left[\frac{1}{R_t}\right] \ .$$

Then r_3 is a strictly decreasing function, see e.g. (30) below. Define

$$M_t = \frac{1}{R_{r_3^{-1}[m(t)]}} \ .$$

Clearly $\{M_t\}$ so defined is a local martingale and $\mathbb{E}M_t = m(t)$. □

Corollary 3.9. *Given a function* $m(t)$ *of bounded variation, there is a local martingale having* $m(t)$ *as its expectation.*

Note that if $m(t)$ is of bounded variation, then there exists two non-decreasing positive functions $f^{(+)}$ and $f^{(-)}$ such that $m(t) = f^{(+)} - f^{(-)}$. And there are two non-negative local martingales $M^{(+)}$ and $M^{(-)}$ such that $f^{(+)}(t) = \mathbb{E}M^{(+)}(t)$ and $f^{(-)}(t) = \mathbb{E}M^{(-)}(t)$. In fact

Corollary 3.10. *Given $\{M_t\}$ an integrable local martingale with $M_0 = 0$ and satisfy (LB) there is a function $f_0 \geq 0$ such that for every pair of continuous non-increasing non-negative functions with $f^{(+)}, f^{(-)} \geq f_0$ and $\mathbb{E}M_t = f^{(+)}(t) - f^{(-)}(t)$ there exists a decomposition $M_t = M_t^{(+)} - M_t^{(-)}$ such that $\{M_t^{(+)}\}$ and $\{M_t^{(-)}\}$ are local non-negative martingales with $\mathbb{E}M_t^{(+)} = f^{(+)}(t)$, $\mathbb{E}M_t^{(-)} = f^{(-)}(t)$.*

Proof. Set $m(t) = \mathbb{E}M_t$. Assume first that $\{M_t\}$ is non-negative. By Proposition 3.8 we can find a non-negative local martingale $M_t^{(-)}$ with $\mathbb{E}M_t^{(-)} = f^{(-)}(t)$. Set $M_t^{(+)} = M_t + M_t^{(-)}$. For general M there is the Krickeberg decomposition $M_t = M_t^p - M_t^n$ with $\{M_t^p\}$, $\{M_t^n\}$ continuous, integrable non-negative local martingales [Revuz-Yor [15]: Exercise 1.49, Chap. IV, p. 136]. Suppose that $f^{(-)} \geq \mathbb{E}M_t^n$. From above we can decompose M_t^p as $M_t^p = M_t^{p+} - M_t^{p-}$ with $\mathbb{E}M_t^{p+} = f^{(+)}(t)$. Set $M_t^{(+)} = M_t^{p+}$ and $M_t^{(-)} = M_t^{p-} + M_t^n$. □

Finally we give some examples of paradoxical local sub-martingales.

1. If $\{M_t\}$ is a positive strictly local martingale with $M_0 = 1$, then X_t defined by $X_t = M_t + \frac{1}{2}(1 - \mathbb{E}M_t)$ has $\gamma(t) = 1 - \mathbb{E}M_t > 0$ for all t.
2. Another example of mean decreasing local sub-martingales is $\{\frac{1}{R_t^{n-2}}\}$ for $R_t = |U_t|$ the O-U process with parameter λ and U_t defined by (10). For $x = 0$, $t > 0$ and $n = 3$ this can be seen from the representation of U_t: $U_t = e^{-\lambda t}\hat{B}_{a(t)}$ for $a(t) = \frac{e^{2\lambda t}-1}{2\lambda}$ where $\hat{B}_.$ is a Brownian motion. So $\mathbb{E}\frac{1}{R_t} = e^{\lambda t}\mathbb{E}\frac{1}{|\hat{B}_{a(t)}|} = \frac{\sqrt{2\lambda}}{\sqrt{1-e^{-\lambda t}}}$, which is decreasing in t.
3. Take two positive increasing functions $u(t)$ and $v(t)$ on $\mathbb{R}^+$ such that $\frac{v(t)}{(\sqrt{u(t)})^{n-2}}$ is decreasing in t. Let B_t be an n-dimensional Brownian motion. Then the process defined by $X_t = \frac{v(t)}{|B_{u(t)}|^{n-2}}$ is a mean decreasing sub-martingale, since

$$\mathbb{E}X_t = \frac{v(t)}{\left(\sqrt{u(t)}\right)^{n-2}} \left(\mathbb{E}\frac{1}{|B_1|^{n-2}}\right).$$

3.3. Application to radial Ornstein–Uhlenbeck processes

Let $\delta > 1$ and $\lambda \in \mathbb{R}$. Consider a radial Ornstein–Uhlenbeck process of dimension δ and parameter λ, namely the solution to the equation:

$$dR_t = d\beta_t + \frac{\delta - 1}{2R_t}dt - \lambda R_t\, dt, \qquad R_t \geq 0 \;, \tag{28}$$

where $\{\beta_t\}$ is a one dimensional Brownian motion. Denote by ${}^{-\lambda}E_a^{\delta}$ the expectation of a δ-dimensional radial O.–U. process with parameter λ.

In part A the default function for a radial Ornstein–Uhlenbeck process for the case $\delta \in \mathbb{R}$ is calculated in terms of the first hitting time T_0, and in Part D in terms of the last exit times (using a Girsanov theorem for last passage times from part C). (For $\delta < 2$, $\gamma(t) = 0$ since $\sup_{s\le t} R_s^{2-\delta}$ has finite expectation, thus a true sub(super) martingale. In part B the law T_0 is obtained for such δ.)

A. In the following we give an explicit expression of the default function $\gamma(t)$ for radial Ornstein–Uhlenbeck processes.

Let Y_t be the canonical process on $C(\mathbb{R}^+, \mathbb{R})$ and $T_0 = \inf\{t : Y_t = 0\}$. Denote by ${}^{-\lambda}P_a^{(\delta)}$ and ${}^{-\lambda}W_a$ respectively the distribution of a δ-dimensional radial O.–U. process (R_t) and that of a 1-dimensional O.–U. process, with parameter λ and initial value a. The corresponding expectations will be respectively ${}^{-\lambda}E_a^{(\delta)}$ and ${}^{-\lambda}E_a$. And $\mathbb{E}$ denotes the usual expectation with respect to the probability measure P.

If $\lambda = 0$, all the superscripts on the left will be omitted. Then it is known, for $a > 0$ and $\delta = 3$

$$P_a^{(3)}\big|_{\mathscr{F}_t} = \frac{1}{a} Y_{t\wedge T_0} \cdot W_a\,|_{\mathscr{F}_t} \quad , \tag{29}$$

which expresses the BES(3) process as the Doob h-transform of Brownian motion for $h(x) = x$ and so

$$E_a^{(3)}\left(\frac{1}{Y_t}\right) = \frac{1}{a} W_a\,(T_0 > t) \quad . \tag{30}$$

In particular this shows how $E_a^{(3)}\left(\frac{1}{Y_t}\right)$ decreases in t.

For a general δ-dimensional Bessel process, $\delta > 2$, there is an analogous result:

$$P_a^{(\delta)}\big|_{\mathscr{F}_t} = \left[\frac{1}{a} Y_{t\wedge T_0}\right]^{\delta-2} \cdot P_a^{(4-\delta)}\,|_{\mathscr{F}_t} \quad . \tag{31}$$

For both (29) and (31), a quick proof is obtained by using Girsanov theorem. See also (2.c, p. 514 in Yor [22]). Here is a result generalizing these to the case $\lambda \neq 0$:

Lemma 3.11. *For $a > 0$, $\lambda \in \mathbb{R}$, and $2 < \delta < \infty$,*

$${}^{-\lambda}P_a^{(\delta)}\big|_{\mathscr{F}_t} = \left[\frac{e^{\lambda t}}{a}\right]^{\delta-2} \cdot \left[Y_{t\wedge T_0}\right]^{\delta-2} \cdot {}^{-\lambda}P_a^{(4-\delta)}\big|_{\mathscr{F}_t} \quad . \tag{32}$$

In particular,

$$ {}^{-\lambda}P_a^{(3)}|_{\mathscr{F}_t} = \left[\frac{e^{\lambda t}}{a}\right] Y_{t\wedge T_0} \cdot {}^{-\lambda}\, W_a|_{\mathscr{F}_t} \ . \tag{33} $$

Proof. Applying the Girsanov theorem to (28), as we did in our proof of Proposition 2.5, and using (31) we get:

$$ \begin{aligned} {}^{(-\lambda)}P_a^{(\delta)}\,|_{\mathscr{F}_t} &= \exp\left(-\lambda\int_0^t Y_s\,d\beta_s - \frac{\lambda^2}{2}\int_0^t Y_s^2\,ds\right)\cdot P_a^{(\delta)}\,|_{\mathscr{F}_t} \\ &= \exp\left(-\frac{\lambda}{2}[Y_t^2 - a^2 - \delta t] - \frac{\lambda^2}{2}\int_0^t Y_s^2\,ds\right)\cdot P_a^{(\delta)}\,|_{\mathscr{F}_t} \\ &= \exp\left(-\frac{\lambda}{2}[Y_t^2 - a^2 - \delta t] - \frac{\lambda^2}{2}\int_0^t Y_s^2\,ds\right)\cdot\left[\frac{Y_{T_0\wedge t}}{a}\right]^{\delta-2} \\ &\quad \cdot P_a^{(4-\delta)}\,|_{\mathscr{F}_t} \\ &= \left[\frac{e^{\lambda t}}{a}\right]^{\delta-2}\left[Y_{T_0\wedge t}\right]^{(\delta-2)} \cdot {}^{(-\lambda)}\,P_a^{(4-\delta)}\,|_{\mathscr{F}_t} \ . \end{aligned} $$

□

Remark. Note that $\delta' = 4 - \delta$ is negative for $\delta > 4$. This is allowed since the process is stopped when it reaches zero. Moreover there is also a well defined concept of Bessel processes of dimension $\delta' < 0$. See Revuz and Yor [15]; Exercise (1.49), Chap. X, p. 430.

From the above lemma we now calculate the default function $\gamma(t)$ for the sub(super)-martingale $\{R_t^{2-\delta}\}$ starting from $a > 0$.

Proposition 3.12. *Let* $\lambda \in \mathbb{R}$, *and* $2 < \delta < \infty$. *For the reciprocal* $\{\frac{1}{R_t^{\delta-2}}\}$ *of a* δ*-dimensional*, *radial Ornstein–Uhlenbeck process of parameter* λ, *the default function is*

$$ \gamma_\delta(t) = \left[\frac{1}{a}\right]^{\delta-2} \cdot\ {}^{-\lambda}E_a^{(4-\delta)}\left[e^{(\delta-2)\lambda T_0}\mathbf{1}_{T_0\le t}\right] \ . \tag{34} $$

Proof. Write $R_t^{2-\delta} = M_t + A_t$ as the sum of the local martingale

$$ M_t = -(\delta-2)\int_0^t \frac{d\beta_s}{R_s^{\delta-1}} $$

and the monotone process

$$ A_t = (\delta-2)\int_0^t \frac{\lambda}{R_s^{\delta-2}}ds \ . $$

First by (32)

$$^{(-\lambda)}E_a^{(\delta)}\left(\frac{1}{R_t^{\delta-2}}\right) = \left[\frac{e^{\lambda t}}{a}\right]^{\delta-2} \cdot {}^{-\lambda}P_a^{4-\delta}(T_0 > t)$$

and

$$\begin{aligned}\mathbb{E}[A_t] &\equiv (\delta-2)\mathbb{E}\int_0^t \frac{\lambda}{R_s^{\delta-2}} ds \\ &= (\delta-2)\lambda \int_0^t \left[\frac{e^{\lambda s}}{a}\right]^{\delta-2} \cdot {}^{(-\lambda)}P_a^{4-\delta}(T_0 > s) ds < \infty \ .\end{aligned}$$

Consequently the default function satisfies

$$\begin{aligned}\gamma_\delta(t) &= \left[\frac{1}{a}\right]^{\delta-2} - \mathbb{E}[R_t^{2-\delta}] + \mathbb{E}[A_t] \\ &= \left[\frac{1}{a}\right]^{\delta-2} - \left[\frac{e^{\lambda t}}{a}\right]^{\delta-2} \cdot {}^{-\lambda}P_a^{4-\delta}(T_0 > t) + \left[\frac{1}{a}\right]^{\delta-2} \\ &\quad \cdot {}^{-\lambda}E_a^{4-\delta} \int_0^{t\wedge T_0} d\left(e^{(\delta-2)\lambda s}\right) \\ &= \left[\frac{1}{a}\right]^{\delta-2} \cdot {}^{-\lambda}E_a^{4-\delta}\left[\exp((\delta-2)\lambda T_0)\mathbf{1}_{T_0\le t}\right].\end{aligned}$$

□

B. For completeness we look briefly at the case $\delta < 2$: an expression for the law of T_0 of a δ, $\delta < 2$, dimensional radial Ornstein–Uhlenbeck process shall be given.

Recall a δ-dimensional radial Ornstein–Uhlenbeck R_t can be represented as

$$R_t = e^{-\lambda t} \hat{R}_{u_\lambda(t)} \ , \tag{35}$$

for $\{\hat{R}_t\}$ a δ-dimensional Bessel process starting from the same initial point x and

$$u_\lambda(t) = \int_0^t e^{2\lambda s}\, ds = \frac{e^{2\lambda t} - 1}{2\lambda} \ .$$

As before, T_0 denotes the first hitting time of 0 by our (canonical) process. Let $\hat{T}_0 = \inf\{t \ge 0 : \hat{R}_t = 0\}$. Note for $\{\hat{R}_t\}$ the point 0 is reached almost surely and is instantaneously reflecting.

Lemma 3.13. *Let* $0 < \delta < 2$. *For a* δ*-dimensional radial O.–U. process expressed by* (35) *with parameter* λ:

$$e^{2\lambda T_0} = (1 + 2\lambda \hat{T}_0)^+, \qquad \lambda \in \mathbb{R}^1 \ . \tag{36}$$

Proof. For λ non-negative, the equality

$$u_\lambda(T_0) = \hat{T}_0 \ , \tag{37}$$

follows readily from (35). And similarly, if $\lambda < 0$, write $\mu = -\lambda$ and observe that

$$\frac{1 - e^{2\mu T_0}}{2\mu} = (\hat{T}_0 \wedge \frac{1}{2\mu}) \ . \tag{38}$$

The required result follows. □

Let $\beta = 4 - \delta$ $(0 < \delta < 2)$. Set $\nu = \frac{\beta}{2} - 1$, and Z_ν a Gamma variable with parameter ν, i.e.,

$$P(Z_\nu \in dt) = \frac{t^{\nu-1} e^{-t}}{\Gamma(\nu)} dt \tag{39}$$

and let $\{\hat{R}_t^\beta\}$ and $\{\hat{R}_t^{(4-\beta)}\}$ be respectively the β-dimensional Bessel process starting from 0 and the $(4 - \beta)$-dimensional Bessel process starting from a. Define

$$\hat{L}_a^\beta = \sup\{t \geq 0 : \hat{R}_t^\beta = a\} \ . \tag{40}$$

Proposition 3.14. *Let* $0 < \delta < 2$. *If* T_0 *is the first time a* δ*-dimensional radial O.–U. process, starting from* a *parameterized by* $\lambda \in \mathbb{R}^1$, *hits zero, then*

$$e^{2\lambda T_0} \stackrel{\text{law}}{=} \left(1 + \lambda \frac{a^2}{Z_\nu}\right)^+ \ . \tag{41}$$

In particular for λ *negative* $P(T_0 = \infty) = P(Z_\nu < -\lambda a^2)$.

Proof. By D. Williams' time reversal theorem (see e.g. Revuz-Yor [15] (Corollary (4.6), Chap. VII, p.316, and Exercise (1.23), Chap. XI, p.451), Pitman-Yor [13], Getoor-Sharpe [9], or Sharpe [16]), the distribution of $\{\hat{R}^\beta_{\hat{L}_a^\beta - t}, 0 \leq t \leq \hat{L}_a^\beta\}$ is equivalent to that of $\{\hat{R}_t^{(4-\beta)} : 0 \leq t \leq \hat{T}_0\}$. In particular $\hat{T}_0 \stackrel{\text{law}}{=} \hat{L}_a^\beta$ and $e^{2\lambda T_0} = \left(1 + 2\lambda \hat{L}_a^\beta\right)^+$. On the other hand the distribution of $\hat{L}_a^\beta$ is given by Yor [24], Le Gall [11], Getoor-Sharpe [9]:

$$\hat{L}_a^\beta \stackrel{\text{law}}{=} \frac{a^2}{2Z_\nu} \ . \tag{42}$$

So $e^{2\lambda T_0} = (1 + \frac{\lambda a^2}{Z_v})^+$ by (36). □

C. To give an expression of $\gamma_\delta(t)$ in terms of the last exit time (part D), we first prove the following theorem.

Let Q_x and P_x be two families of transient diffusions taking values in $\mathbb{R}^+$. Let σ and s be respectively the scale functions corresponding to Q_a and P_a (with $\sigma(\infty) = s(\infty) = 0$), and α_1 and α_2 the diffusion coefficients. Define L_a to be the last time the canonical process $\{Y_t\}$ hits a. Let $\mathscr{F}_{L_a}$ be the σ-field generated by $\{H_{L_a} : H \text{ previsible}\}$.

Lemma 3.15 (Girsanov theorem for last passage times). *Suppose both scale functions are regular and the transition semigroups have densities. Suppose $Q_x|_{\mathscr{F}_t} = D_t \cdot P_x|_{\mathscr{F}_t}$. Then, if $\sigma(a) \neq 0$ and $s'(a) \neq 0$*

$$Q_x|_{\mathscr{F}_{L_a}} = h(a) D_{L_a} \cdot P_x|_{\mathscr{F}_{L_a}} \quad . \tag{43}$$

Here $h(a) = \frac{\sigma'(a)}{\sigma(a)} \cdot \frac{s(a)}{s'(a)} \cdot \frac{\alpha_1(a)}{\alpha_2(a)}$.

Proof. Let $Q_t(x, dy)$ and $P_t(x, dy)$ be the corresponding transition functions. Write $Q_t(x, dy) = q_t(x, y)dy$ and $P_t(x, dy) = p_t(x, y)dy$. Then it is known that (p. 326 in Pitman and Yor [13])

$$Q_x(L_a \in dt) = -\frac{1}{2}\frac{\sigma'(a)}{\sigma(a)}\alpha_1(a) q_t(x, a)\, dt \quad .$$

Thus

$$Q_x(L_a \in dt) = h(a)\frac{q_t(x, a)}{p_t(x, a)} P_x(L_a \in dt) \quad . \tag{44}$$

On the other hand, take H previsible. Then

$$\mathbb{E}_{Q_x}\{H_t | Y_t = a\} Q_x(Y_t \in da) = \mathbb{E}_{P_x}\{H_t D_t | Y_t = a\} P_x(Y_t \in da) \quad , \tag{45}$$

since for any bounded Borel function f:

$$\mathbb{E}_{Q_x}[f(Y_t)H_t] = \int f(a)\mathbb{E}_{Q_x}\{H_t | Y_t = a\} Q_x(Y_t \in da)$$

and

$$\mathbb{E}_{Q_x}[f(Y_t)H_t] = \int f(a)\mathbb{E}_{P_x}\{H_t D_t | Y_t = a\} P_x(Y_t \in da) \quad .$$

Therefore, by the existence of regular conditional probability

$$\begin{aligned}
\mathbb{E}_{Q_x}(H_{L_a}) &= \int Q_x(L_a \in dt)\mathbb{E}_{Q_x}(H_{L_a}|L_a = t) \\
&= \int Q_x(L_a \in dt)\mathbb{E}_{Q_x}(H_t|L_a = t) \\
&= \int Q_x(L_a \in dt)\mathbb{E}_{Q_x}(H_t|Y_t = a), \quad \text{by the Markov property} \\
&= \int Q_x(L_a \in dt)\frac{p_t(x,a)}{q_t(x,a)}\mathbb{E}_{P_x}(H_t D_t|Y_t = a) \\
&= h(a)\int P_x(L_a \in dt)\mathbb{E}_{P_x}(H_t D_t|Y_t = a) \\
&= h(a)\,\mathbb{E}_{P_x}[H_{L_a} D_{L_a}]\ ,
\end{aligned}$$

by reversing the first three steps. □

D. Now assume $\delta > 2$. For a δ-dimensional radial O.–U. process with parameter $\lambda \in \mathbb{R}^+$ there is a 'time reversal' type result: let L_a be the last time a δ-dimensional radial O.–U. process, starting from 0, hits a.

Proposition 3.16. *Let $\lambda \geq 0$ and $\delta > 2$, then for any bounded measurable function F on path space,*

$$^{-\lambda}E_a^{(4-\delta)}\left[F(Y_t : t \leq T_0)\right] = h(a)\cdot {}^{\lambda}E_0^{\delta}\left[F(Y_{L_a - t}, t \leq L_a)e^{2\lambda L_a}\right]\ , \quad (46)$$

where h is a deterministic function.

Proof. First, the Girsanov-Maruyama theorem and the time reversal theorem for Bessel processes give:

$$\begin{aligned}
&{}^{-\lambda}E_a^{(4-\delta)}\left[F(Y_t : t \leq T_0)\right] \\
&\quad = E_a^{(4-\delta)}\left[F(Y_t : t \leq T_0)\exp\left(\tfrac{\lambda}{2}[a^2 + (4-\delta)T_0] - \tfrac{\lambda^2}{2}\int_0^{T_0} Y_s^2\,ds\right)\right] \\
&\quad = E_0^{\delta}\left[F(Y_{L_a - t}, t \leq L_a)\exp\left(\tfrac{\lambda}{2}[a^2 + (4-\delta)L_a] - \tfrac{\lambda^2}{2}\int_0^{L_a} Y_s^2\,ds\right)\right]\ . \qquad (47)
\end{aligned}$$

Apply Lemma 3.15 to obtain:

$$^{-\lambda}E_a^{(4-\delta)}\left[F(Y_t : t \leq T_0)\right] = h(a)\cdot {}^{\lambda}E_0^{\delta}\left[F(Y_{L_a - t}, t \leq L_a)e^{2\lambda L_a}\right]\ .$$

□

As a consequence the default function γ_δ discussed in part A also takes the following form:

Proposition 3.17. *Let $\lambda \geq 0$ and $\delta > 2$. For a δ-dimensional radial Ornstein–Uhlenbeck process with parameter λ, the default function is given by*:

$$\gamma_\delta(t) = \left[\frac{1}{a}\right]^{\delta-2} \cdot h(a) \cdot {}^{\lambda}E_0^\delta\left[e^{\delta\lambda L_a}\mathbf{1}_{(L_a\leq t)}\right], \qquad t \geq 0 \ . \tag{48}$$

Corollary 3.18. *For $\lambda \geq 0$ and $\delta > 2$, we have*:

$${}^{-\lambda}P_a^{(4-\delta)}(T_0 \in du) = \left(h(a)\, e^{2\lambda u}\right)\, {}^{\lambda}P_0^\delta\,(L_a \in du) \ \ , \tag{49}$$

and ${}^{-\lambda}E_a^{(4-\delta)}\left[e^{-2\lambda T_0}\right] = \left({}^{\lambda}E_0^\delta[e^{2\lambda L_a}]\right)^{-1} = h(a)$.

From (47) we can calculate the law of T_0 via

$$\begin{aligned}&{}^{-\lambda}P_a^{(4-\delta)}(T_0 \in du)\\ &= E_0^\delta\left[L_a \in du, \exp\left(\frac{\lambda}{2}[a^2 + (4-\delta)L_a] - \frac{\lambda^2}{2}\int_0^{L_a} Y_s^2\, ds\right)\right]\end{aligned} \tag{50}$$

and a particular case of Lévy's formula in Yor [23] (p. 18):

$$\begin{aligned}E_0^{(\delta)}\left\{e^{-\frac{\lambda^2}{2}\int_0^u Y_s^2 ds}\,|L_a = u\right\} &= E_0^{(\delta)}\left\{e^{-\frac{\lambda^2}{2}\int_0^u Y_s^2 ds}\,|Y_u = a\right\}\\ &= \left[\frac{\lambda u}{\sinh(\lambda u)}\right]^{\frac{\delta}{2}} e^{-\frac{a^2}{2u}(\lambda u \coth(\lambda u)-1)} \ .\end{aligned}$$

Recall, (42), that L_a has the same law as $\frac{a^2}{2Z_\nu}$ for Z_ν a Gamma variable with parameter $\nu = \frac{\delta}{2} - 1$. So the density $\phi_a(t)$ of L_a is given after a straightforward calculation (see Getoor [8]) by, see (39):

$$\phi_a(t) = \frac{a^{\delta-2}}{2^\nu\Gamma(\nu)}\left[\frac{1}{t}\right]^{\frac{\delta}{2}} e^{-\frac{a^2}{2t}} \ . \tag{51}$$

Thus

$$\begin{aligned}&{}^{-\lambda}P_a^{(4-\delta)}(T_0 > t)\\ &= E_0^\delta\left[\mathbf{1}_{(L_a>t)} e^{\frac{\lambda}{2}[a^2+(4-\delta)L_a]}\left[\frac{\lambda L_a}{\sinh(\lambda L_a)}\right]^{\frac{\delta}{2}} e^{-\frac{a^2}{2L_a}(\lambda L_a \coth(\lambda L_a)-1)}\right]\\ &= \int_t^\infty \frac{a^{\delta-2}}{2^\nu\Gamma(\nu)} \cdot e^{\frac{\lambda a^2}{2}} \cdot e^{\frac{\lambda}{2}[(4-\delta)t - a^2\coth(\lambda t)]} \cdot \left[\frac{\lambda}{\sinh(\lambda t)}\right]^{\frac{\delta}{2}} dt \ \ ,\end{aligned}$$

using the above formula (51) for ϕ_a.

Corollary 3.19. *For $a > 0$, $\lambda > 0$ and $\delta > 2$,*

$$^{-\lambda}P_a^{(4-\delta)}(T_0 \in dt) = \frac{a^{\delta-2}}{2^\nu\Gamma(\nu)} \cdot \mathrm{e}^{\frac{\lambda a^2}{2}} \cdot \mathrm{e}^{\frac{\lambda}{2}[(4-\delta)t - a^2 \coth(\lambda t)]} \cdot \left[\frac{\lambda}{\sinh(\lambda t)}\right]^{\frac{\delta}{2}} dt \ .$$

In particular for $\delta = 3$ we get

$$^{-\lambda}W_a(T_0 \in dt) = a \cdot \frac{\mathrm{e}^{\frac{\lambda a^2}{2}}}{\sqrt{2\pi}} \cdot \mathrm{e}^{\frac{\lambda}{2}[t - a^2 \coth(\lambda t)]} \cdot \left[\frac{\lambda}{\sinh(\lambda t)}\right]^{\frac{3}{2}} dt \ . \tag{52}$$

Remarks. (i) Note for $\delta = 3$, (52) can be obtained directly from

$$^{-\lambda}W_a(T_0 > t) = E_a\left[\mathbf{1}_{(T_0 > t)}\mathrm{e}^{\frac{\lambda}{2}[a^2 + T_0] - \frac{\lambda^2}{2}\int_0^{T_0} Y_s^2 ds}\right]$$

and the fact that for a one-dimensional Brownian motion starting from a

$$W_a(T_0 \in dt) = \frac{a}{\sqrt{2\pi t^3}}\mathrm{e}^{-\frac{a^2}{2t}} dt \ , \tag{53}$$

e.g. see p. 107 of Revuz and Yor [15].
(ii) If the O.–U. process considered has integer dimension $\delta = n$ and with $\lambda > 0$, then $u_\lambda(T_0) \overset{\text{law}}{=} L_a$ as shown previously. There is a closed form expression for the law of T_0:

$$^{-\lambda}P_a^{(4-\delta)}(T_0 \in dt) = {}^{\lambda}P_0^{\delta}(L_a \in dt)\mathrm{e}^{2\lambda t}\frac{\phi_a(u_\lambda(t))}{\phi_a(t)} \ , \tag{54}$$

in terms of ϕ_a (see (51)), following from the observation that

$$^{-\lambda}P_a^{(4-\delta)}(T_0 \in dt) = \mathrm{e}^{2\lambda t}\phi_a(u_\lambda(t))\, dt \ .$$

(iii) For discussions on the asymptotics see the next section.

F. Generalized O.–U. processes. Let λ be a non-negative function on $\mathbb{R}^+$ and consider equation (10) with λ there replaced by the function $\lambda(|x|)$:

$$X_t = x + B_t - \int_0^t \lambda(|X_s|)X_s\, ds, \quad x \in \mathbb{R}^n \ . \tag{55}$$

This equation has a global solution, which we call a *generalized O.–U. process* and keep the notation introduced for constant λ. For $\delta > 2$ the equation

$$dR_t = d\beta_t + \frac{\delta - 1}{2R_t}dt - \lambda(R_t)R_t\, dt, \qquad R_t \geq 0 \ , \tag{56}$$

gives rise to the corresponding δ-dimensional radial O.–U. process. The next result follows from a similar proof to that of lemma 3.11.

Lemma 3.20. *For $a > 0$ and $2 < \delta < \infty$,*

$$ {}^{-\lambda}P_a^{(\delta)}\,|_{\mathscr{F}_t} = \left[\frac{Y_{t\wedge T_0}}{a}\right]^{\delta-2} \cdot \mathrm{e}^{\int_0^t (\delta-2)\lambda(Y_s)ds}\ .^{(-\lambda)}\,P_a^{4-\delta}\,|_{\mathscr{F}_t}\quad . \tag{57} $$

In particular,

$$ {}^{-\lambda}P_a^{(3)}|_{\mathscr{F}_t} = \frac{1}{a}Y_{t\wedge T_0}\cdot \mathrm{e}^{\int_0^t \lambda(Y_s)ds}\ .^{(-\lambda)}\,W_a|_{\mathscr{F}_t}\quad , \tag{58} $$

for ${}^{(-\lambda)}W_a$ the law of the solution to $dX_t = d\beta_t - \lambda(|X_t|)X_t dt$ starting from a. Here, β_t is a one-dimensional Brownian motion.

Assume λ is bounded. From the lemma we observe that for the local sub-martingale $|X_t|^{2-\delta} = N_t + A_t$ starting from $a > 0$ we have

$$ \mathbb{E}\frac{1}{|X_t|^{\delta-2}} = \left[\frac{1}{a}\right]^{\delta-2}\ .^{(-\lambda)}\,E_a^{4-\delta}\left(\mathrm{e}^{\int_0^t (\delta-2)\lambda(Y_s)ds}\mathbf{1}_{(T_0>t)}\right)\quad . $$

Here is a result analogous to proposition 3.12:

Corollary 3.21. *For $a > 0$, $2 < \delta < \infty$. The default function for the reciprocal $\{\frac{1}{R_t^{\delta-2}}\}$ of a δ-dimensional radial generalized O.–U. process is:*

$$ \gamma(t) = \left[\frac{1}{a}\right]^{\delta-2}\ .^{(-\lambda)}\,E_a^{4-\delta}\left(\mathrm{e}^{\int_0^{T_0} (\delta-2)\lambda(Y_s)ds}\mathbf{1}_{(T_0\le t)}\right)\quad . $$

We now obtain some tail asymptotics using the previous result. Note that the semigroup P_t induced by the n-dimensional Ornstein–Uhlenbeck process satisfies

$$ \int P_t f\, d\mu = \int f\, d\mu\quad , $$

for each f in L^1 where μ is the measure given by $\mu(dx) = \mathrm{e}^{-2\int_0^{|x|} r\lambda(r)dr}dx$. Suppose λ is bounded below by a positive number, then by the ergodic theorem,

$$ \lim_{t\to\infty}\mathbb{E}\frac{1}{|X_t|^{n-2}} = \frac{\int_{\mathbb{R}^n}[\frac{1}{|x|}]^{n-2}\mu(dx)}{\int_{\mathbb{R}^n}\mu(dx)}\quad . $$

But by corollary 3.21,

$$ \mathbb{E}\frac{1}{|X_t|^{n-2}} = \left[\frac{1}{a}\right]^{n-2}\ .^{-\lambda}\,E_a^{(4-n)}\mathrm{e}^{\int_0^t (n-2)\lambda(Y_s)ds}\cdot\mathbf{1}_{(T_0>t)}\quad . $$

If λ is a constant, $\lim_{t\to\infty} \mathbb{E}\left(\frac{1}{|X_t|^{n-2}}\right) = \frac{\lambda^{\frac{n}{2}-1}}{\Gamma(\frac{n}{2})}$. So ${}^{-\lambda}P_a^{(4-n)}(T_0 > t)$ is decreasing exponentially in t at rate $-(n-2)\lambda$:

$$\lim_{t\to\infty} \frac{1}{t} \log\left[{}^{-\lambda}W_a(T_0 > t)\right] = -(n-2)\lambda \ .$$

More precisely,

$$\begin{aligned} {}^{-\lambda}W_a(T_0 > t) &= P\left(Z_\nu < \frac{a^2}{2u_\lambda(t)}\right) = \frac{1}{\Gamma(\nu)} \int_0^{a^2/2u_\lambda(t)} x^{\nu-1} \mathrm{e}^{-x}\, dx \\ &\sim \frac{a^{2\nu}}{\Gamma(\nu+1)} \frac{\lambda^\nu}{\mathrm{e}^{2\lambda\nu t}} \quad \text{as } t \to \infty \ . \end{aligned}$$

3.4. *A non-integrability result for general diffusions*

The integrability properties of local martingale functionals are related to criteria for strong p-completeness of stochastic flows. Let (X_t^x) be the solution to a stochastic differential equation (s.d.e.) with C^r coefficients, $r \geq 2$, on an n-dimensional complete Riemannian manifold starting from x and $\zeta(x)$ the explosion time. The s.d.e. is said to be *strongly p-complete*, for $1 \leq p \leq n$, if $\{X_t^x\}$ is jointly continuous in time and space for all time when restricted to a smooth p-simplex. See [12]. Roughly speaking strong p-completeness means the flow sends a C^{r-1} p-dimensional submanifold to a C^{r-1} submanifold. When $p = n$, this is equivalent to saying that the solution flow has a C^{r-1} version, i.e. a version which is jointly continuous in time and space (and hence C^{r-1} in space) almost surely. Let $\{T_x F_t\}$ be the derivative flow related to the s.d.e.. The basic criterion is as follows: A s.d.e. is strongly p-complete if for each compact subset K of M,

$$\sup_{x\in K} \mathbb{E}\left[\sup_{s\leq t} |T_x F_s|^{p+\delta} \mathbf{1}_{t<\zeta(x)}\right] < \infty, \qquad \text{some } \delta > 0 \ . \tag{59}$$

Here $|-|$ denotes the Riemannian norm and δ can be taken to be zero for $p = 1$. In particular there is no explosion if (59) holds for $p = 1$ (and under somewhat weaker conditions [12]).

For example consider $dX_t = dB_t$ on $\mathbb{R}^n - \{0\}$ for $\{B_t\}$ an n-dimensional Brownian motion starting from 0. This is an example from one of the very few known classes of non-explosive stochastic differential equations which has no solution flow. See Elworthy [6]. Furthermore it was shown by a direct argument, that it is strongly $(n-2)$-complete but not strongly $(n-1)$-complete (Li [12]).

To test the criterion on this example, we first choose a conformal Riemannian metric on $\mathbb{R}^n$ such that $\mathbb{R}^n - \{0\}$ is a complete metric space. The criterion takes the following form:

$$\sup_{x\in K} \mathbb{E}\left[\sup_{s\leq t} \rho(X_s^x)^{p+\delta}\right] < \infty \tag{60}$$

for a class of functions ρ, as shall be described in detail later. Since we know this s.d.e. on $\mathbb{R}^n - \{0\}$ is not strongly $(n-1)$-complete, (60) will not hold for $p = n-1$. This foretells the non-integrability of certain functionals of diffusion processes described below.

A. Let $\alpha_0, \alpha_1, \ldots, \alpha_m$ be C^2 maps from $\mathbb{R}^n$ to $\mathbb{R}^n$, and $\{B_t^i\}_1^m$ be m independent linear Brownian motions. We consider the s.d.e. (in Itô form) on $\mathbb{R}^n$:

$$dX_t = \sum_1^m \alpha_i(X_t) dB_t^i + \alpha_0(X_t)\, dt \tag{61}$$

and make the assumption

$$\sup_{x\in K} \mathbb{E}\, \mathbf{1}_{t<\bar\zeta(x)} \sup_{s\leq t} |T_x F_s|^p < \infty, \quad \text{for all compact } K \text{ in } \mathbb{R}^n - \{0\} \text{ and } p \geq 1\ , \tag{62}$$

where $\bar\zeta(x)$ is the explosion time of (X_t^x) as a process on $\mathbb{R}^n - \{0\}$. The above condition holds if the coefficients of the s.d.e. have linear growth with their first derivatives having sub-logarithmic growth, i.e. $|D\alpha_i(x)| \leq c(1 + \ln|x|)$ for each i. See [12].

Proposition 3.22. *Let $p > n-1$ and $\{X_t^x\}$ be the solution to equation* (61). *Assume that the coefficients do not vanish identically at* 0, *and that* (62) *holds. If $\rho : (0,\infty) \to (0,\infty)$ is a C^2 function such that*

$$\int_0^1 \rho(s)\, ds = \infty\ , \tag{63}$$

then for some compact set $K \subset \mathbb{R}^n - \{0\}$

$$\sup_{x\in K} \mathbb{E}\, \mathbf{1}_{t<\bar\zeta(x)} \sup_{s\leq t} \rho(|X_s^x|)^p = \infty\ . \tag{64}$$

Proof. First note that the s.d.e. (61) is not strongly $(n-1)$-complete on $\mathbb{R}^n - \{0\}$. If it were it would be strongly complete by [[12], Theorem 2.3] and there would be a flow $\bar F_t$ of diffeomorphisms of $\mathbb{R}^n - \{0\}$ onto open subsets of $\mathbb{R}^n - \{0\}$, continuous in $t \in [0,\infty)$, with $\bar F_0$ the identity map. These will map the unit sphere S^{n-1} to a random codimension 1 submanifold S_t^{n-1} which will be the boundary of an open neighbourhood D_t of the origin.

Now (61) on $\mathbb{R}^n$ has a local flow $\{F_t(x) : t \geq 0\}$ which must restrict to $\bar{F}_t$ and consequently must also be global (c.f. the proof of Theorem 2.3 in [12]). But then F_t will map the open unit disc to D_t and this implies that $F_t(0) = 0$, for each t, contradicting the non-vanishing of the coefficients at 0.

On the other hand using the function ρ we may define a Riemannian metric on $\mathbb{R}^n - \{0\}$ by:

$$|v|_x = \rho(|x|)|v|, \quad v \in T_x\left(\mathbb{R}^n - \{0\}\right) \ . \tag{65}$$

This is a complete metric on $\mathbb{R}^n - \{0\}$ under condition (63). For this metric we have:

$$\begin{aligned}\mathbb{E}\, \mathbf{1}_{t<\bar{\zeta}(x)} \sup_{s\leq t} |T_x F_s|^p_{F_s(x)} &\leq \mathbb{E}\, \mathbf{1}_{t<\bar{\zeta}(x)} \cdot \left(\sup_{s\leq t} [\rho(|X^x_s|)]^p |T_x F_s|^p \right) \\ &\leq \left[\mathbb{E}\, \mathbf{1}_{t<\bar{\zeta}(x)} \sup_{s\leq t} [\rho(|X^x_s|)]^{p\alpha} \right]^{\frac{1}{\alpha}} \\ &\quad \times \left[\mathbb{E}\, \mathbf{1}_{t<\bar{\zeta}(x)} \sup_{s\leq t} |T_x F_s|^{p\beta} \right]^{\frac{1}{\beta}}\end{aligned}$$

for conjugate numbers $\alpha, \beta > 1$. Thus condition (62) implies that

$$\sup_{x\in K} \mathbb{E}\, \mathbf{1}_{t<\bar{\zeta}(x)} \sup_{s\leq t} |T_x F_s|^p_{F_s(x)} < \infty$$

if $\sup_{x\in K} \mathbb{E}\, \mathbf{1}_{t<\bar{\zeta}(x)} \sup_{s\leq t} [\rho(|X^x_s|)]^{p'}$ is finite for some $p' > p$. Now, from (59), the finiteness of the first quantity for compact sets K for a number $p > n-1$ implies the s.d.e. (61), when considered on $\mathbb{R}^n - \{0\}$, is strongly $(n-1)$-complete. However strong $(n-1)$-completeness implies strong completeness, by [12], and this does not hold. Consequently (64) is true for $p > n-1$. □

B. Finally we compare Proposition 3.22 and condition (59) with the results of §3.3.

Let λ be a positive number and consider the Ornstein–Uhlenbeck equation $dX_t = dB_t - \lambda X_t\, dt$ on $\mathbb{R}^n - \{0\}$, $n > 2$. It has solution

$$X^x_t = x\mathrm{e}^{-\lambda t} + \int_0^t \mathrm{e}^{-\lambda(t-s)}\, dB_s \ ,$$

and $T_x F_t(v) = \mathrm{e}^{-\lambda t} v$. This equation is known to be strongly $(n-2)$-complete but not strongly $(n-1)$-complete by a direct argument from the definition. Let $\rho : R^+ \to \mathbb{R}^+$ be a C^2 function and define a Riemannian metric on $\mathbb{R}^n - \{0\}$ by (65). Then

$$\mathbb{E}\sup_{s\le t}|T_xF_s|^p_{X_s^x} = \mathbb{E}\sup_{s\le t}\mathrm{e}^{-\lambda sp}\left(\frac{\rho(|X_s^x|)}{\rho(|x|)}\right)^p \tag{66}$$

and so

$$\mathrm{e}^{-\lambda tp}\mathbb{E}\sup_{s\le t}\left(\rho(|X_s^x|)\right)^p \le [\rho(|x|)]^p\mathbb{E}\sup_{s\le t}|T_xF_s|^p_{X_s^x} \le \mathbb{E}\sup_{s\le t}\left(\rho(|X_s^x|)\right)^p \quad ,$$

and the integrability of $\sup_{s\le t}|T_xF_s|^p$ is equivalent to that of $\sup_{s\le t}\rho(|X_s^x|^p)$. Define: $G(y)=\rho(y^{\frac{1}{2-n}})$. Then $G(|x|^{2-n})=\rho(|x|)$, and

$$\mathbb{E}\sup_{s\le t}\left[\rho(|X_s^x|)\right]^p = \mathbb{E}\sup_{s\le t}\left[G(|X_s^x|^{2-n})\right]^p \quad .$$

From Proposition 3.5, $\mathbb{E}\sup_{s\le t}\left[G(|X_s^x|^{2-n})\right]^p$ is finite if and only if $\int_1^\infty \frac{dy}{y}\left([G(y)]^p\right)'$ is finite. This is seen to be equivalent, after integration by parts, to

$$\int_0^1 \frac{\rho(z)^p}{z^{3-n}}dz < \infty \quad . \tag{67}$$

Now for $n\ge 3$, $p<n-2$ and $\rho(|x|)=\frac{1}{|x|}$, (67) holds and

$$\sup_{x\in K}\mathbb{E}\sup_{s\le t}\rho(|X_s^x|)^p < \infty \tag{68}$$

for K any compact subset of $\mathbb{R}^n - \{0\}$ by the discussions in section 3.3.

Thus for $n\ge 3$, criterion (59) with $\rho(y)=\frac{1}{y}$ is strong enough to imply the strong $(n-3)$-completeness of the Ornstein–Uhlenbeck equation on $\mathbb{R}^n - \{0\}$ but not the (known) strong $(n-2)$-completeness. Also for $n=3$ and $\rho(y)=\frac{1}{y}$ (64) holds for $p\ge n-2$, not just for $p>n-1$. This is also true for higher dimensions: for $\lambda=0$ this can be checked by Proposition 3.5 and when $\lambda\ne 0$ by applying the one-dimensional comparison theorem for s.d.e.'s to see that the radial O.–U. process $\{|X_t^x|\}$ satisfies: $|X_t^x| < |x+B_t|$ for each t. This suggests that it should be possible to extend the values of p for which Proposition 3.22 holds and also to sharpen criterion (59) for strong p-completeness.

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References

[1] Atsuji, A.: Nevanlinna theory via stochastic calculus. J. Funct. Anal. **132**(2), 473–510 (1995)

[2] Azéma, J., Gundy, R., Yor, M.: Sur I'intégrabilité uniforme des martingales continues. In Sém. Prob. XIV, Lect. Notes in Math. Vol. 784, pages 53–61. Springer, (1980)

[3] Carne, T.K.: Brownian motion and Nevanlinna theory. Proc. London Math. Soc. **52**(3), 349–368 (1986)

[4] Delbaen, F., Schachermayer, W.: Arbitrage possibilities in Bessel processes and their relations to local martingales. Probab. Theory Relat. Fields, **102**(3), 357–366 (1995)

[5] Elworthy, K.D., Xue-Mei Li, Yor, M.: On the tails of the supremum and the quadratic variation of strictly local martingales. In LNM 1655, Séminaire de probabilités XXXI, pages 113–125. Springer, (1997)

[6] Elworthy, K.D.: Stochastic dynamical systems and their flows. In A. Friedman and M. Pinsky, editors, Stochastic Analysis, pages 79– 95. London, New York: Academic press, (1978)

[7] Galtchouk, L.I., Novikov, A.: On Wald's equation. Discrete time case. In Sém. Prob. XXXI, volume 1655 of LNM, pages 126–135. Springer, Berlin, (1997)

[8] Getoor, R.K.: The Brownian escape process. Ann. Probab. **7**(5), 864–867 (1979)

[9] Getoor, R.K., Sharpe, M.J.: Excursions of Brownian motion and Bessel processes. Z. Wahrscheinlichkeithstheorie verw. Geb. **47**, 86–106 (1979)

[10] Heston, S.: A closed form solution for options with stochastic volatility with applications to bond and currency options. Rev. Financial Studies, **6**, 327–343 (1993)

[11] Le Gall, J.-F.: Sur la saucisse de Wiener et les points multiples du mouvement brownien. The Annals of Prob. **14**(4), 1219–1244 (1986)

[12] Xue-Mei Li. Strong p-completeness and the existence of smooth flows on noncompact manifolds. Probab. Theory Relat. Fields, **100**(4), 485–511 (1994)

[13] Pitman, J., Yor, M.: Bessel processes and infinitely divisible laws. In Stochastic Integrals, Durham conference 1980, LNM, volume 851, pages 285–370, (1980)

[14] Rao, M.: Quasi-martingales. Math. Scand. **24**, 79–92 (1969)

[15] Revuz, D., Yor, M.: Continuous martingales and Brownian motion, 3rd edition. Springer-Verlag, (1999)

[16] Sharpe, M.: Some transformations of diffusions by time reversal. The Annals of Prob. **6**(8), 1157–1162 (1980)

[17] Sin, C.A.: Complications with stochastic volatility models. Adv. Appl. Prob. **30**, 256–268 (1998)

[18] Spitzer, F.: Some theorems about 2-dimensional Brownian motion. Trans. Amer. Math. Soc. **87**, 187–197 (1958)

[19] Stein, E., Stein, J.: Stock price distributions with stochastic volatility: an analytic approach. Financial Studies, **4**, 727–752 (1991)

[20] Takaoka, K.: Some remarks on the uniform intégrability of continuous martingales. To appear in Sém. Prob, XXXIII LNM, Springer, (1999)

[21] Yor, M.: Les inégalités de sous-martingales comme conséquences de la relation de domination. Stochastics, **3**, 1–15 (1979)

[22] Yor, M.: On some exponential functionals of Brownian motion. Adv. App. Prob. **24**, 509–531 (1992)

[23] Yor, M.: Some Aspects of Brownian Motion. Part 1: Some Special Functionals. In Lectures in Mathematics, ETH Zürich. Birkhäuser Verlag, Basel, (1992)

[24] Yor, M.: Sur certaines fonctionnelles exponentielles du mouvement brownien réel. J. Appl. Prob. **29**(1), 202–208 (1992)