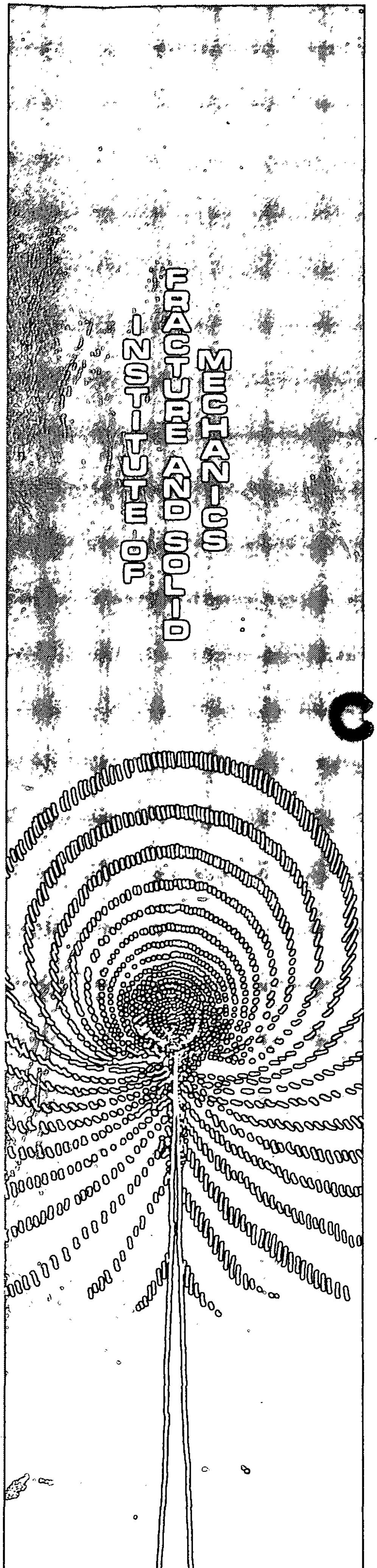

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THE INCLUSION PROBLEM WITH A CRACK CROSSING THE BOUNDARY*

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Abstract

The problem of an elastic plane containing an elastic inclusion is considered. It is assumed that both the plane and the inclusion contain a radial crack and the two cracks are collinear. The problem is formulated in terms of a system of singular integral equations. In the interesting limiting cases in which the crack tips approach the interface from either one or both sides, the dominant parts of the kernels become generalized Cauchy kernels giving rise to stress singularities of other than -1/2 power. For these unusual cases of a crack terminating at or crossing the interface stress intensity factors are defined and some detailed results are given for various crack-inclusion geometries and material combinations.

1. INTRODUCTION

In studying the fracture of composite materials which consist of more than one perfectly bonded homogeneous elastic phase with different mechanical properties, it was shown that the singular behavior of the stress state in the close neighborhood of a crack tip does not remain "self-similar" as it enters and crosses an interface separating two phases of the composite [1,2]. If the crack tip remains in the same homogeneous medium as it propagates, during the crack propagation the characteristic

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square root singularity and the related angular distribution of the stresses at and around the crack tip remain unchanged, the only possible change taking place in the multiplicative constant known as the stress intensity factor. This makes it possible to apply any one of the conventional fracture criteria to this phase of the fracture propagation. On the other hand, since the singular behavior of the stress field around the crack tip terminating at a bimaterial interface is drastically different than that of a crack tip imbedded into a homogeneous medium [1], as the crack enters and crosses the interface an abrupt change takes place in the crack tip stress field. Thus, since the stress field does not remain similar to itself during this phase of fracture propagation, for studying the related fracture phenomenon a closer examination of the crack tip stress field and some modifications of the existing theories or possibly a new fracture criterion are needed.

A detailed treatment of this problem was given in [1] and [2] for the case in which the interface is a plane and the crack length and its distance to the interface are sufficiently small so that the perturbed stress field can be approximated by that of a crack in two bonded elastic half planes. However, in materials such as ceramics and fiber reinforced composites, the crack length is usually of the order of inclusion or fiber diameter and the stress state in the uncracked medium is quite different than that of two bonded half planes. Hence, for this type of problems clearly the assumption of bonded half planes will not be valid. In this paper we will consider the plane elastostatic problem of a crack terminating at and crossing the bimaterial interface in an elastic matrix containing a circular elastic inclusion. The special case of the problem in which the crack is imbedded in the elastic matrix was recently discussed in [3]. Since [3] contains sufficiently detailed results of the single crack problem, in this paper, aside from a sample solution for the purpose of verification, we will not discuss this problem. Instead, we will give the solution of the problems of a crack in the inclusion with one or both ends approaching and terminating at the interface, of two collinear cracks one in the inclusion and one in the matrix, and of a crack crossing the interface. The analysis and the results given in [3] for the limiting case of the crack tip terminating at the inclusion boundary appear to be incorrect. Therefore some results for this case will also be given.

2. THE INTEGRAL EQUATIONS FOR THE GENERAL PROBLEM

Consider the plane elastostatic problem for an elastic matrix with constants κ_1, μ_1 containing a perfectly bonded circular elastic inclusion of radius a and with constants κ_2, μ_2 where μ_i is the shear modulus, and $\kappa_i = 3-4\nu_i$ for plane strain and $\kappa_i = (3-\nu_i)/(1+\nu_i)$ for plane stress, ν_i being the Poisson's ratio (i=1,2). Let the medium contain two (radial) collinear cracks with end points at y=0 and $x = a_1, b_1, a_2, b_2$ (Figure 1). In addition to the geometry, let the external loads also be symmetric with respect to the plane of the cracks, y=0. The

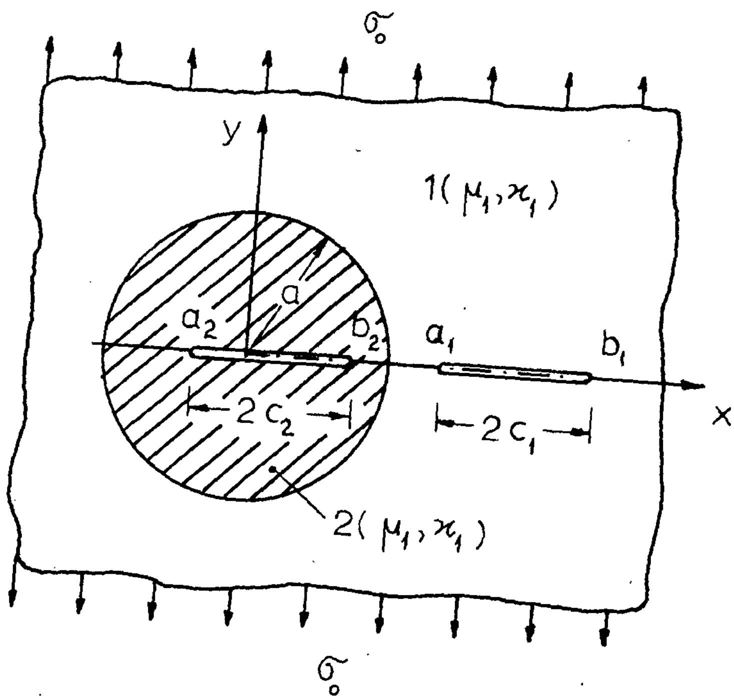

Figure 1. Inclusion-crack geometry.

integral equations for this problem can easily be written down by using the dislocation solutions given in [4] and [5] as the Green's functions. In the usual manner the solution of the problem can be expressed as the sum of two sets of stresses: (a) the stresses in the medium without the cracks and under the given external loads, and (b) the perturbed stresses for the cracked medium where equal and opposite of the stresses found in (a) and applied to the crack surfaces are the only external loads. It is clear that the solution (b) alone will contain the singularities.

Consider now the crack-inclusion problem shown in Figure 1. Let the crack surface tractions in the pertubation problem (b) be

$$\sigma_{1yy}(x,0) = p_1(x) , \qquad (a_1<x<b_1) ,$$

$$\sigma_{2yy}(x,0) = p_2(x) , \qquad (a_2<x<b_2) . \qquad (1.a,b)$$

For example, for the uniaxial tension at infinity $\sigma_{1yy}^{\infty} = \sigma_0$ shown in the figure, the solution to problem (a), and hence, the tractions p_1 and p_2 are given by

$$\sigma_{1yy}^{a}(x,0) = \sigma_0[1 - \frac{a^2}{2x^2}\frac{m(\kappa_1-1) - (\kappa_2-1)}{2m + (\kappa_2-1)} - \frac{a^4}{x^4}\frac{3(m-1)}{2(1+m\kappa_1)}] ,$$

$$\sigma_{2yy}^{a}(x,0) = \sigma_0\frac{m(\kappa_1+1)}{2}(\frac{1}{2m+\kappa_2-1} + \frac{1}{1+m\kappa_1}),$$

$$p_1(x) = -\sigma_{1yy}^{a}(x,0) , \qquad p_2(x) = -\sigma_{2yy}^{a}(x,0) , \qquad (2.a-d)$$

where $m = \mu_2/\mu_1$. Define

$$f_1(x) = \frac{\partial}{\partial x}[v_1(x,+0) - v_1(x,-0)] , \qquad (a_1<x<b_1)$$

$$f_2(x) = \frac{\partial}{\partial x}\,[v_2(x,+0) - v_2(x,-0)]\ , \qquad (a_2<x<b_2), \qquad (3.a,b)$$

where v_1 and v_2 are the y-component of the displacement vectors in the matrix and in the inclusion, respectively. f_1 and f_2 can be looked upon as the unknown functions of the problem defined in the intervals (a_1,b_1) and (a_2,b_2), respectively. The considerations of displacement continuity require that

$$\int_{a_k}^{b_k} f_k(x)dx = 0\ , \qquad (k=1,2)\ . \qquad (4)$$

Using the dislocation solutions given in [4] and [5] as the Green's functions, after some manipulations the following system of integral equations may easily be obtained to determine f_1 and f_2:

$$\int_{a_1}^{b_1} \frac{f_1(t)}{t-x}\,dt + \int_{a_1}^{b_1} [k_{11s}(x,t) + k_{11f}(x,t)]f_1(t)dt$$

$$+ \gamma \int_{a_2}^{b_2} [k_{12s}(x,t) + k_{12f}(x,t)]f_2(t)dt = \frac{\pi(\kappa_1+1)}{2\mu_1}\,p_1(x)\ ,$$

$$(a_1<x<b_1)\ ,$$

$$\int_{a_2}^{b_2} \frac{f_2(t)}{t-x}\,dt + \frac{1}{\gamma}\int_{a_1}^{b_1} [k_{21s}(x,t) + k_{21f}(x,t)]f_1(t)dt$$

$$+ \int_{a_2}^{b_2} [k_{22s}(x,t) + k_{22f}(x,t)]f_2(t)dt = \frac{\pi(\kappa_2+1)}{2\mu_2}\,p_2(x)\ ,$$

$$(a_2<x<b_2) \qquad (5.a,b)$$

where

$$k_{11s}(x,t) = \frac{1}{t-s}\,[(A_1 + A_2)\,\frac{s}{2x} + \frac{A_1}{x^2}\,(3s^2 - a^2)(1 - \frac{2s}{t})]$$

$$+ A_1[(1 - \frac{4s}{t})\,\frac{s(s^2 - a^2)}{x^2(t - s)^2} - \frac{s^3(s^2 - a^2)^2}{a^4 t(t-s)^3}\,] ,$$

$$k_{11f}(x,t) = \frac{A_1 a^2}{x^2}\,(\frac{2}{x} + \frac{1}{2t} - \frac{3a^2}{tx^2}) - [M(\kappa_2+1) - 1]\frac{a^2}{2tx^2} ,$$

$$k_{12s}(x,t) = (1 - \frac{A_3+A_4}{2})\frac{1}{t-x} + (A_3-A_4)\frac{x}{t(t-x)} + \frac{A_3-A_4}{2}\,\frac{x^2-a^2}{t(t-x)^2} ,$$

$$k_{12f}(x,t) = \frac{A_3 - A_4}{2t} + (A_5 - \frac{A_3 + A_4}{2})\frac{1}{x} + (A_5 - A_4)\frac{a^2}{x^3}$$

$$- \frac{A_4 - A_3}{2}\,\frac{a^2}{tx^2} - \frac{(1 - A_3)A_6}{2}\,\frac{t}{x} ,$$

$$k_{21s}(x,t) = [1 - \frac{A_1 + B_1}{2} + (A_1 - B_1)\frac{x}{t}]\frac{1}{t-x} - \frac{A_1 - B_1}{2}\,\frac{a^2-x^2}{t(t-x)^2}$$

$$k_{21f}(x,t) = [(1-m)M + \frac{A_1 - B_1}{2}]\frac{1}{t} ,$$

$$k_{22s}(x,t) = [\frac{A_3 + A_4}{2}\,\frac{s}{x} - A_3\,\frac{a^2 - 3s^2}{x^2}\,(1 - \frac{2s}{t})]\frac{1}{t-s}$$

$$+ A_3\,\frac{s}{x^2}\,(1 - \frac{4s}{t})\,\frac{s^2 - a^2}{(t-s)^2} - A_3\,\frac{s^3(s^2 - a^2)^2}{a^4 t(t-s)^3} ,$$

$$k_{22f}(x,t) = \frac{A_3 + B_3}{2x} + A_3[\frac{t+2s}{x^2} - \frac{a^2(t+3s)}{tx^3} - \frac{1}{t} - A_6\,\frac{t}{a^2}] ;$$

(6.a-h)

$$\gamma = \frac{\mu_2}{\mu_1}\,\frac{1+\kappa_1}{1+\kappa_2} , \qquad s = a^2/x , \qquad m = \mu_2/\mu_1 ,$$

$$M = \frac{m(\kappa_1+1)}{(\kappa_2+m)(\kappa_2-1+2m)} , \qquad A_1 = \frac{1-m}{1+m\kappa_1} , \qquad A_2 = \frac{\kappa_2-m\kappa_1}{\kappa_2+m} ,$$

$$A_3 = \frac{m-1}{m+\kappa_2}\ , \qquad A_4 = \frac{m\kappa_1-\kappa_2}{1+m\kappa_1}\ , \qquad A_5 = 1 - \frac{\kappa_2+1}{m(\kappa_1+1)}$$

$$A_6 = \frac{2(1-m)}{2m+\kappa_2-1}\ . \tag{7}$$

We note that for $-a<a_2$, $b_2<a$, $a<a_1$, *(i.e., if none of the* crack tips is on an interface) (see Figure 1) the kernels $(k_{ijs} + k_{ijf})$, $(i,j = 1,2)$ are bounded functions of x and t in the intervals given by (5). Thus, in this case the set of equations (5) is an ordinary system of singular integral equations *with simple Cauchy type singular kernels. Since the displacement* derivatives f_1 and f_2 have integrable singularities at the end points of the corresponding intervals, the index of the equations is $\kappa=1$. Consequently, the general solution of the system will contain two arbitrary constants [6], which are determined from *the continuity conditions (4). On the other hand,* a close examination of the "Fredholm kernels" $k_{ijs} + k_{ijf}$, $(i,j = 1,2)$ would indicate that, if one or more crack tips terminate at the bimaterial interface (i.e., if any one or a combination of the three cases $a_2 = -a_1$, $b_2 = a$, and $a_1 = a$ is valid) (see Figure 1), certain parts of these kernels become unbounded as both of the arguments x and t approach the end point on the interface. These parts of the kernels which go to infinity as $(x,t) \to \mp a$ are indicated by $k_{ijs}(x,t)$, $(i,j = 1,2)$ in (6). It is easy to see that k_{ijs} become infinite as $(x-a)^{-1}$, hence, together with the simple Cauchy kernels, $(t-x)^{-1}$, they constitute a set of generalized Cauchy kernels.

Singular behavior of the solution of integral equations having similar generalized Cauchy kernels was studied in detail in [1] and [2]. Following the complex function technique outlined in [6] and using the procedure described in [1] and [2], if we define the unknown functions f_1 and f_2 in terms of unknown bounded functions g_1 and g_2 and unknown powers α_1, β_1, α_2, and β_2 as follows:

$$f_1(x) = g_1(x)(b_1-x)^{\alpha_1}(x-a_1)^{\beta_1} ,$$

$$f_2(x) = g_2(x)(b_2-x)^{\alpha_2}(x-a_2)^{\beta_2} ,$$

$$(-1 < \mathrm{Re}(\alpha_j,\beta_j) < 0, \quad j=1,2), \qquad (8.a,b)$$

for various typical crack geometries, from (5) the characteristic equations giving α_i and β_i may be obtained as:

(a) $-a < a_2 < b_2 < a < a_1 < b_1$:

$$\cot\pi\alpha_j = 0 , \quad \cot\pi\beta_j = 0 , \quad (j=1,2); \qquad (9)$$

(b) $-a < a_2 < b_2 < a , \quad a = a_1 < b_1$:

$$\cot\pi\alpha_1 = 0 , \quad \cot\pi\alpha_2 = 0 , \quad \cot\pi\beta_2 = 0 ,$$

$$2\cos\pi\beta_1 + (A_1 + A_2) - 4A_1(\beta_1 + 1)^2 = 0 ; \qquad (10.a\text{-}d)$$

(c) $-a < a_2 < b_2 = a < a_1 < b_1$:

$$\cot\pi\alpha_1 = 0 , \quad \cot\pi\beta_1 = 0 , \quad \cot\pi\beta_2 = 0 ,$$

$$2\cos\pi\alpha_2 + (A_3 + A_4) - 4A_3(\alpha_2 + 1)^2 = 0 , \qquad (11.a\text{-}d)$$

(d) $-a = a_2 < b_2 = a < a_1 < b_1$:

$$\cot\pi\alpha_1 = 0 \ , \qquad \cot\pi\beta_1 = 0 \ , \qquad \alpha_2 = \beta_2 \ ,$$

$$2\cos\pi\alpha_2 + (A_3 + A_4) - 4A_3(\alpha_2 + 1)^2 = 0 \ , \qquad (12.a\text{-}d)$$

(e) $-a < a_2 < b_2 = a = a_1 < b_1$ (through crack):

$$\cot\pi\alpha_1 = 0 \ , \qquad \cot\pi\beta_2 = 0 \ , \qquad \beta_1 = \alpha_2 \ ,$$

$$[(K_1 - K_2)\cos^2\frac{\pi\alpha_2}{2} - K_1(1 + \alpha_2)^2]^2$$

$$+ K_3^2\sin^2\frac{\pi\alpha_2}{2}\cos^2\frac{\pi\alpha_2}{2} - K_2^2(1 + \alpha_2)^2 = 0 \ , \qquad (13.a\text{-}d)$$

$$K_1 = 2(1-m)/m \ , \qquad K_2 = \frac{1 + \kappa_2}{m} - (1 + \kappa_1) \ ,$$

$$K_3 = \frac{1 + \kappa_2}{m} + (1 + \kappa_1) \ . \qquad (14)$$

The equations (9), (10.a-c), (11.a-c), (12.a,b) and (13.a,b) give -0.5 as the acceptable root which is the well-known result for a crack tip surrounded by a homogeneous medium. (11.d) and (12.d) are identical to the previously obtained [1,2] characteristic equation for a crack tip terminating at a bimaterial interface. (13.d) is the same as the characteristic equation for two bonded quarter planes [2,7,8]. The characteristic equations (9 -13) are derived under the conditions that $g_j(a_j)$ and $g_j(b_j)$, (j=1,2) are finite and nonzero. (13.d) is the expression of vanishing coefficient determinant in two homogeneous linear algebraic equations in $g_1(a)$ and $g_2(a)$. Hence, in the case of a crack crossing the boundary $g_1(a)$ and $g_2(a)$ are not independent and are related by [2]

$$g_1(a)[A_1 + A_2 - 4A_1(1+\alpha_2)^2 + 2\cos\pi\alpha_2]$$

$$= g_2(a)\gamma\left(\frac{a-a_2}{b_1-a}\right)^{\frac{1}{2}} [A_3 + A_4 - 2 - 2(1+\alpha_2)(A_3-A_4)] . \qquad (15)$$

The derivation of equations (9 - 15) follows very closely the procedure outlined in [1] and [2] in great detail and therefore is omitted in this paper.

3. STRESS INTENSITY FACTORS

From the viewpoint of applications of the results in fracture studies, one of the important quantities of interest is the strength of the stress singularity at the crack tips characterized by the stress intensity factors. For the crack tips imbedded in a homogeneous medium, the stress intensity factors are defined in terms of "cleavage" stresses and are related to the density functions f_1 and f_2 as follows:

$$k(b_j) = \lim_{x\to b_j} \sqrt{2(x-b_j)}\,\sigma_{jyy}(x,0) = -\lim_{x\to b_j} \sqrt{2(b_j-x)}\,f_j(x)\omega_j ,$$

$$k(a_j) = \lim_{x\to a_j} \sqrt{2(a_j-x)}\,\sigma_{jyy}(x,0) = \lim_{x\to a_j} \sqrt{2(x-a_j)}\,f_j(x)\omega_j ,$$

$$\omega_j = 2\mu_j/(1+\kappa_j), \qquad (j=1,2). \qquad (16.a,b)$$

The asymptotic expressions (16) may easily be obtained from (5) by noting that the expressions given by (5) are valid outside as well as along the cuts (a_1,b_1) and (a_2,b_2) (i.e., $p_1(x) = \sigma_{1yy}(x,0)$, $(-\infty<x<-a,\ a<x<\infty)$ and $p_2(x) = \sigma_{2yy}(x,0)$, $(-a<x<a)$), and by directly applying the function-theoretic method to (5) [2]. Using the same procedure, from (5) the stress intensity factor for a crack tip terminating at the interface may

be obtained as follows [2]:

$-a < a_2 < b_2 < a = a_1 < b_1$:

$$k(a) = \lim_{x \to a} \sqrt{2}\, (a-x)^{-\beta_1}\, \sigma_{2yy}(x,0)$$

$$= -\sqrt{2}\, g_1(a)\, \frac{\sqrt{b_1 - a}}{\sin \pi\beta_1}\, \frac{\mu_1}{1+\kappa_1}\, [A_1 + A_2 - 2 - 2(1+\beta_1)(A_1 - A_2)] \;, \tag{17}$$

$-a < a_2 < b_2 = a < a_1 < b_1$:

$$k(a) = \lim_{x \to a} \sqrt{2}\, (x-a)^{-\alpha_2}\, \sigma_{1yy}(x,0)$$

$$= \sqrt{2}\, g_2(a)\, \frac{\sqrt{a - a_2}}{\sin \pi\alpha_2}\, \frac{\mu_2}{\kappa_2+1}\, [A_3 + A_4 - 2 - 2(1+\alpha_2)(A_3 - A_4)] \;. \tag{18}$$

In the case of a through crack (i.e., if $-a<a_2<b_2=a=a_1<b_1$), for all practical combinations of material constants the functions f_1 and f_2, and consequently, the stresses have an integrable singularity at the intersection of the crack and the interface (i.e., $-1<\alpha_2=\beta_1<0$). In fracture studies the quantities of interest here are the distribution of contact stresses along the interface. Thus, to characterize these stresses one may define the following stress intensity factors:

$$k_y(a) = \lim_{y \to 0} y^{-\alpha_2}\, \sigma_{1yy}(a,y) \;, \quad k_{xy}(a) = \lim_{y \to 0} y^{-\alpha_2}\, \sigma_{1xy}(a,y) \;. \tag{19.a,b}$$

Developing asymptotic expressions for σ_{jyy} and σ_{jxy} around the point (a,0) and using (8), it can be shown that the constants

k_y and k_{xy} are related to $g_1(a)$ and $g_2(a)$ as follows [2]:

$$k_y(a) = \frac{\mu_1}{2\sin\frac{\pi\alpha_2}{2}} \{\frac{g_1(a)}{\sqrt{b_1-a}} [(1-2\alpha_2)\frac{m}{1+m\kappa_1} + \frac{m}{m+\kappa_2}]$$

$$- \frac{g_2(a)}{\sqrt{a-a_2}} [(1-2\alpha_2)\frac{m}{m+\kappa_2} + \frac{m}{1+m\kappa_1}]\} ,$$

$$k_{xy}(a) = \frac{\mu_1}{2\cos\frac{\pi\alpha_2}{2}} \{\frac{g_1(a)}{\sqrt{b_1-a}} [(1-2\alpha_2)\frac{m}{1+m\kappa_1} - \frac{m}{m+\kappa_2}]$$

$$+ \frac{g_2(a)}{\sqrt{a-a_2}} [(1-2\alpha_2)\frac{m}{m+\kappa_2} - \frac{m}{1+m\kappa_1}]\} . \qquad (20.a,b)$$

4. EXAMPLES AND RESULTS

Referring to Figure 1, if $-a<a_2<b_2<a<a_1<b_1$ (including the special cases of single cracks, i.e., $a_2 = b_2$ or $a_1 = b_1$), the system of singular integral equations can be solved in a straightforward manner by using, for example, the technique described in [9]. In all the examples discussed in this section the external load was assumed to be the uniaxial tension $\sigma_{1yy}^{\infty} = \sigma_0$ applied to the matrix perpendicular to the plane of the cracks and away from the inclusion-crack region (see equation 2). Following sample calculations for a simple crack were made as a spot check for the results given in [3]:

(A) $\kappa_1 = \kappa_2 = 2$, $\mu_2/\mu_1 = 1/3$, $a_1/a = 1.1$, $b_1/a = 2.1$:

$$\frac{k(a_1)}{\sigma_0\sqrt{c_1}} = 1.482 , \quad \frac{k(b_1)}{\sigma_0\sqrt{c_1}} = 1.160 , \quad c_1 = (b_1-a_1)/2 ;$$

(B) $\kappa_1 = \kappa_2 = 2, \quad \mu_2/\mu_1 = 1/3, \quad a_1 = a, \quad b_1/a = 2$:

$$\frac{k(b_1)}{\sigma_0\sqrt{c_1}} = 1.233 \ , \qquad \frac{k(a)}{\sigma_0 c_1^{-\beta_1}} = 1.092 \ , \qquad c_1 = (b_1-a)/2 \ ,$$

$\beta_1 = -0.62090$ (see (10.d) and (17));

(C) $\kappa_1 = 1.8, \quad \mu_2 = 0, \quad a_1/a = 1.05, \quad b_1/a = 2.05$:

$$\frac{k(b_1)}{\sigma_0\sqrt{c_1}} = 1.515 \ , \qquad \frac{k(a_1)}{\sigma_0\sqrt{c_1}} = 2.800 \ , \qquad c_1 = (b_1-a_1)/2 \ ;$$

(D) $\kappa_1 = 1.8, \quad \mu_2 = 0, \quad a_1 = a, \quad b_1 = 2$ (edge crack) :

$$\frac{k(b_1)}{\sigma_0\sqrt{c_1}} = 2.808 \ , \qquad c_1 = (b_1-a)/2 \ .$$

These results agree with that of [3] for $a_1>a$. However, because of the change in the power of the singularity β_1 for $a_1=a$, the extrapolated results in [3] are clearly in error ($k(a_1)$ tends to zero or infinity as $a_1 \to a$).

For the cracks terminating at or going through the interface, the system of singular integral equations (5) (dominant parts of which have generalized Cauchy kernels) is solved by using the technique described in [10]. The results obtained for various material combinations and crack geometries are given in Tables 1 - 6 and Figures 2 - 8 (see (16 - 19) for definitions of stress intensity factors). Table 1 shows the effect of μ_2/μ_1 on the power of stress singularity β_1 and on the stress intensity factors for a crack in the matrix with one end touching the interface (the limiting case of the results given in [3]). Figure 2

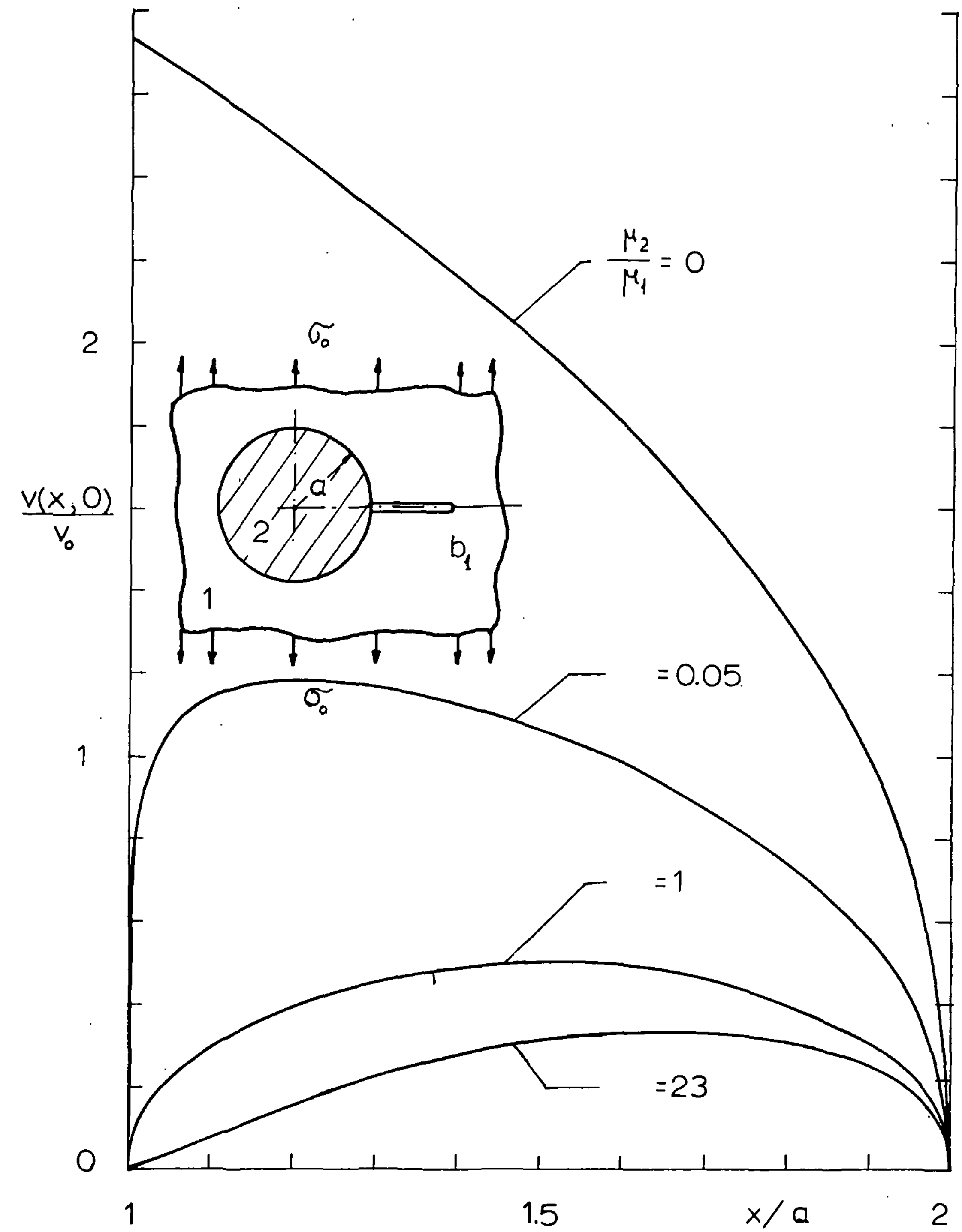

Figure 2. Crack surface displacement for a crack in the matrix with one tip on the interface ($\kappa_1 = \kappa_2 = 1.8$, $b_1/a = 2$, $v_0 = (1+\kappa_1)a\sigma_0/\mu_1$).

Table 1. The effect of modulus ratio on the stress intensity factors for a crack terminating at the interface ($a_1 = a$, $b_1/a = 2$, $\kappa_1 = \kappa_2 = 1.8$, $c_1 = (b_1 - a)/2$).

$m = \frac{\mu_2}{\mu_1}$	$-\beta_1$	$\frac{k(b_1)}{\sigma_0 \sqrt{c_1}}$	$\frac{k(a)}{\sigma_0 c_1^{-\beta_1}}$
0		2.808	
0.05	0.81730	1.615	1.053
1/3	0.62049	1.229	0.5836
1.0	0.5	1.000	1.000
3.0	0.40074	0.8610	1.299
10.0	0.33277	0.7969	1.389
23.0	0.30959	0.7796	1.375
100	0.29387	0.7691	1.345
300	0.28883	0.7667	1.348

shows the crack surface displacement $v(x,0)$ for four typical values of μ_2/μ_1, (0, 0.05, 1 and 23) which is obtained from (see (3))

$$v(x,0) = -\frac{1}{2}\int_x^{b_1} f_1(x)dx \ , \tag{21}$$

where the normalizing factor is

$$v_0 = (1+\kappa_1)a\sigma_0/\mu_1 \ . \tag{22}$$

Tables 2 - 4 and Figures 3 and 4 show the results for a single crack located in the inclusion. The limiting stress intensity factors 0 and ∞ shown by an arrow in Table 2 is the trend based on the square root singularity. The correct stress intensity factors and the related α_2 or β_2 are given in Table 3. Some of the results given in Table 2 are also shown in Figures 3 and 4. The limiting values of the stress intensity factors shown in these figures for crack length $2c_2$ approaching zero are obtained from the uniformly loaded infinite plane solution with the stress state away from the crack given by (2.b), namely

$$\lim_{c_2\to 0} \frac{k}{\sigma_0\sqrt{c_2}} = \frac{m(\kappa_1+1)}{2}\left(\frac{1}{2m+\kappa_2-1} + \frac{1}{1+m\kappa_1}\right) \ . \tag{23}$$

Table 4 shows the results for a completely cracked inclusion (i.e., a_2=-a, b_2=a). Table 5 and Figure 5 show the results for the case where both the matrix and the inclusion contain a crack (see the insert in Figure 5). The material constants used in this problem correspond to an epoxy matrix and an aluminum inclusion. In Table 5 the values of $k(a_1)$ corresponding to a_1=a (the numbers in parentheses) are evaluated from (17) with β_1 = -0.33811

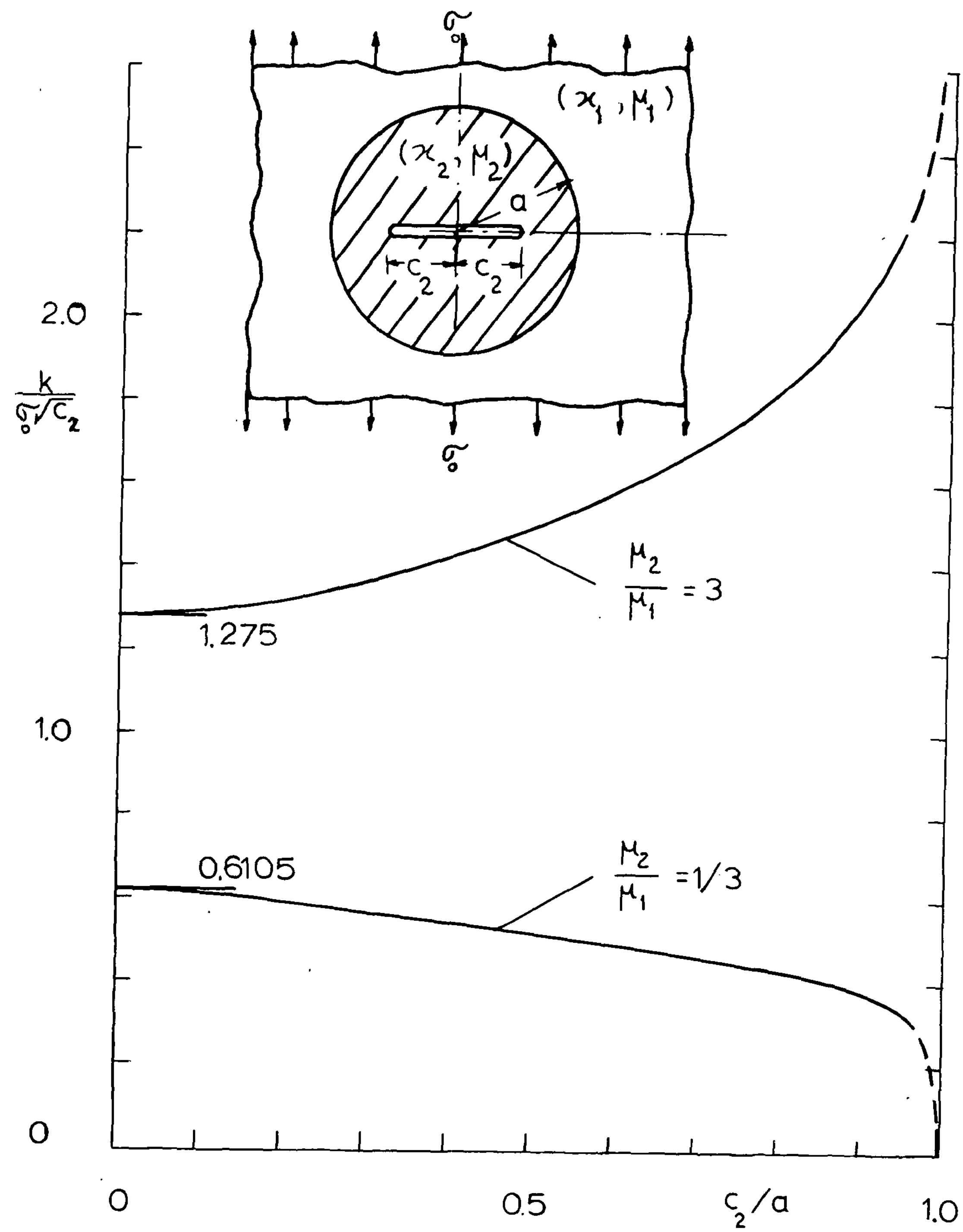

Figure 3. Stress intensity factor for a symmetrically located crack in the inclusion ($\kappa_1 = \kappa_2 = 1.8$).

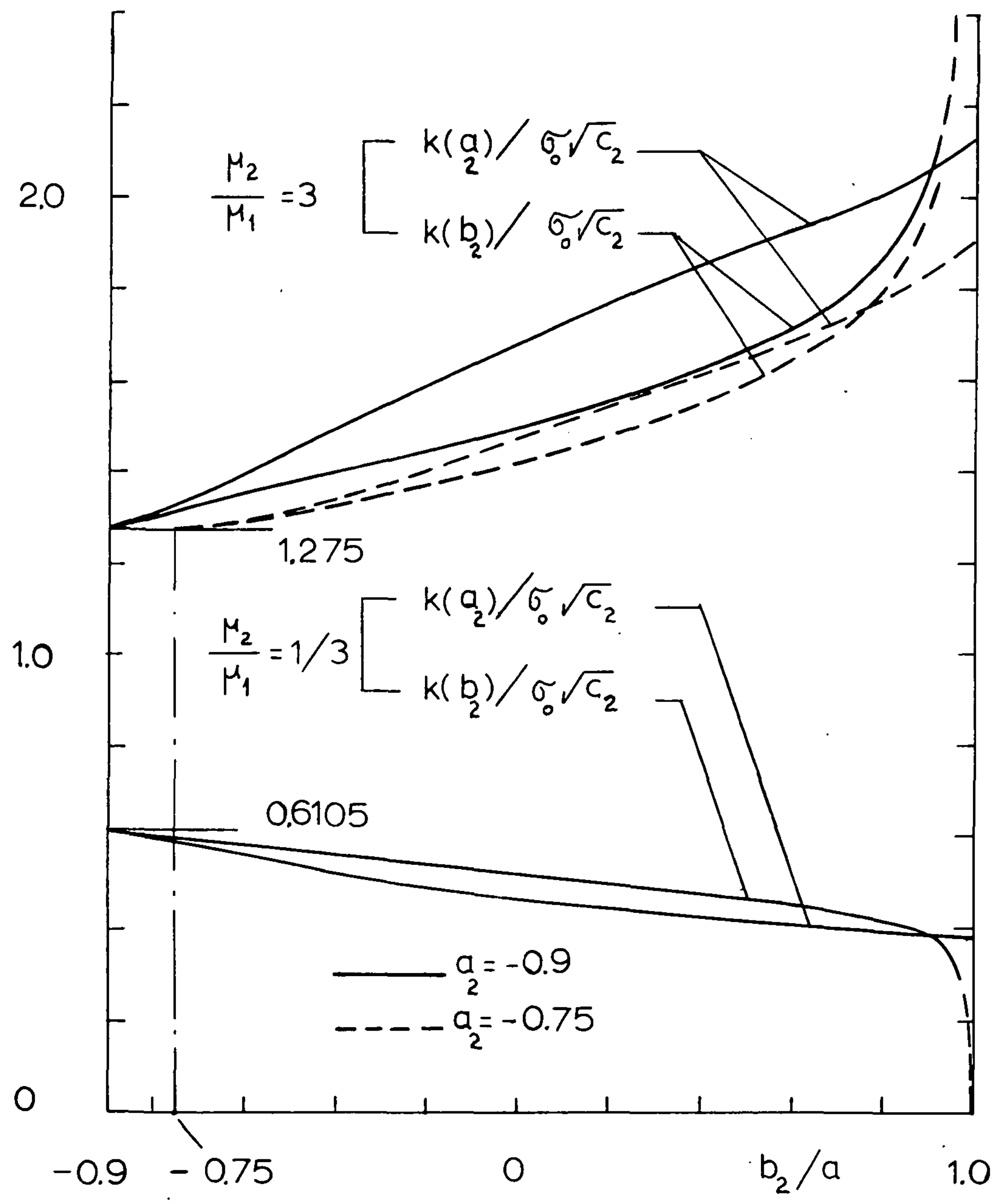

Figure 4. Stress intensity factors for a crack located in the inclusion ($\kappa_1 = \kappa_2 = 1.8$, one tip fixed at $a_2 = -0.9a$ or $a_2 = -0.75a$, b_2 variable, $c_2 = (b_2 - a_2)/2$).

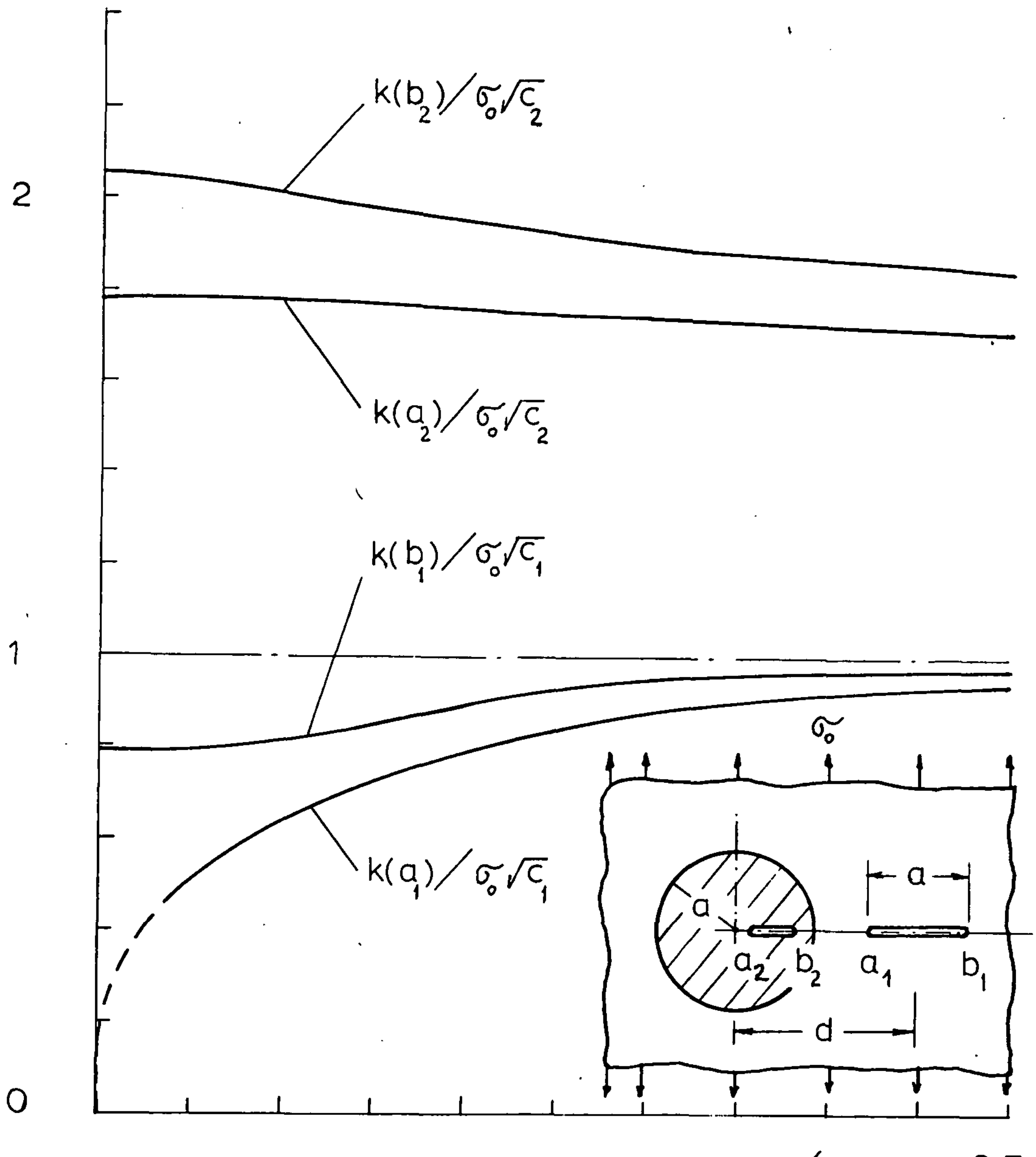

Figure 5. Stress intensity factors for a crack in the matrix (epoxy) and a crack in the inclusion (aluminum) ($\kappa_1 = 1.6$, $\kappa_2 = 1.8$, $\mu_2/\mu_1 = 23.077$; $a_2 = 0.3a$, $b_2 = 0.8a$, $2c_1 = (b_1 - a_1) = a$ fixed, $d = (b_1 + a_1)/2$ variable).

Table 2. Stress intensity factors for a crack located in the inclusion ($\kappa_1 = \kappa_2 = 1.8$, $c_2 = (b_2 - a_2)/2$).

a_2/a	b_2/a	$\frac{\mu_2}{\mu_1} = 3$		$\frac{\mu_2}{\mu_1} = 1/3$	
		$\frac{k(a_2)}{\sigma_0\sqrt{c_2}}$	$\frac{k(b_2)}{\sigma_0\sqrt{c_2}}$	$\frac{k(a_2)}{\sigma_0\sqrt{c_2}}$	$\frac{k(b_2)}{\sigma_0\sqrt{c_2}}$
-0.9	-0.75	1.324	1.309	0.5886	0.5950
-0.9	-0.5	1.451	1.376	0.5416	0.5684
-0.9	-0.25	1.572	1.438	0.5032	0.5452
-0.9	0	1.684	1.501	0.4719	0.5219
-0.9	0.25	1.790	1.572	0.4450	0.4969
-0.9	0.50	1.890	1.664	0.4220	0.4682
-0.9	0.75	1.990	1.822	0.4020	0.4300
-0.9	1.00	2.140	$\to \infty$	0.3830	$\to 0$
-0.75	-0.5	1.314	1.306	0.5917	0.5950
-0.75	-0.25	1.389	1.359	0.5596	0.5710
-0.75	0	1.475	1.419	0.5266	0.5448
-0.75	0.25	1.564	1.492	0.4958	0.5166
-0.75	0.5	1.655	1.588	0.4681	0.4847
-0.75	0.75	1.752	1.752	0.4437	0.4437
-0.75	1.0	1.907	$\to \infty$	0.4212	$\to 0$
-0.1	0.1	1.283	1.283	0.6046	0.6046
-0.25	0.25	1.332	1.332	0.5796	0.5796
-0.50	0.50	1.491	1.491	0.5144	0.5144
-0.90	0.90	2.062	2.062	0.3900	0.3900

Table 3. Stress intensity factors for a crack located in the inclusion ($\kappa_1 = \kappa_2 = 1.8$, $c_2 = (b_2 - a_2)/2$).

a_2/a	b_2/a	$-\beta_2$	$-\alpha_2$	$\frac{k(a_2)}{\sigma_0 c_2^{-\beta_2}}$	$\frac{k(b_2)}{\sigma_0 c_2^{-\alpha_2}}$
			$(\mu_2/\mu_1) = 3$		
-0.75	1.0	0.5	0.62049	1.907	0.6175
-0.90	1.0	0.5	0.62049	2.140	0.6300
-1.0	1.0	0.62049	0.62049	0.7920	0.7920
			$(\mu_2/\mu_1) = 1/3$		
-0.75	1.0	0.5	0.40074	0.4212	0.9550
-0.90	1.0	0.5	0.40074	0.3830	0.9400
-1.0	1.0	0.40074	0.40074	0.9330	0.9330

Table 4. Stress intensity factor for a completely cracked inclusion.

$\frac{\mu_2}{\mu_1}$	$\kappa_1 = \kappa_2 = 1.8$		$\kappa_1 = 2.2,\ \kappa_2 = 1.8$		$\kappa_1 = 1.8,\ \kappa_2 = 2.2$		$\kappa_1 = \kappa_2 = 2.2$	
	$-\alpha_2$	$\frac{k(a)}{\sigma_0 a^{-\alpha_2}}$	$-\alpha_2$	$\frac{k(a)}{\sigma_0 a^{-\alpha_2}}$	$-\alpha_2$	$\frac{k(a)}{\sigma_0 a^{-\alpha_2}}$	$-\alpha_2$	$\frac{k(a)}{\sigma_0 a^{-\alpha_2}}$
0.2	0.36621	0.7890	0.38087	0.7848	0.32027	1.046	0.33845	1.010
0.6	0.45025	1.014	0.47028	0.9456	0.42123	1.174	0.44466	1.068
1.0	0.5	1.0	0.51991	0.9209	0.47724	1.107	0.5	1.0
2.0	0.57451	0.8843	0.59188	0.8165	0.55687	0.9465	0.57624	0.8613
5.0	0.67885	0.6555	0.69124	0.6194	0.66380	0.6940	0.67733	0.6500

Table 5. Stress intensity factors for a cracked inclusion in a cracked matrix ($\mu_2/\mu_1 = 23.077$, $\kappa_1 = 1.6$, $\kappa_2 = 1.8$, $c_1 = (b_1 - a_1)/2$, $c_2 = (b_2 - a_2)/2$).

$\frac{a_1}{a}$	$\frac{b_1}{a}$	$\frac{a_2}{a}$	$\frac{b_2}{a}$	$\frac{k(a_1)}{\sigma_0\sqrt{c_1}}$	$\frac{k(b_1)}{\sigma_0\sqrt{c_1}}$	$\frac{k(a_2)}{\sigma_0\sqrt{c_2}}$	$\frac{k(b_2)}{\sigma_0\sqrt{c_2}}$
1.05	1.55	0.45	0.95	0.335	0.683	1.947	2.716
1.00	1.50	0.45	0.95	(0.861)	0.634	1.942	2.732
1.00	2.00	0.30	0.80	(1.091)	0.790	1.782	2.061
1.25	2.25	0.30	0.80	0.681	0.833	1.771	1.997
1.50	2.50	0.30	0.80	0.831	0.926	1.742	1.919
1.75	2.75	0.30	0.80	0.898	0.950	1.719	1.870
2.00	3.00	0.30	0.80	0.932	0.963	1.702	1.838

found from (10.d), (and with normalizing factor $\sigma_0 c_1^{-\beta_1}$ instead of $\sigma_0\sqrt{c_1}$).

The results for the crack crossing the interface are shown in Table 6 and Figures 6 - 8. These results are also given for an epoxy matrix containing an aluminum inclusion. It should be noted that in solving the system of singular integral equations (5) for this problem, the single-valuedness conditions (4) are no longer valid. The two conditions necessary to account for the two arbitrary constants arising from the solution of the integral equations are the continuity condition $v_1(a,0) = v_2(a,0)$ and the relation (15) which must be satisfied by the functions g_1 and g_2. The stress intensity factors k_y and k_{xy} given here are defined by (19) and are evaluated from (20). The limits 0 and $\mp\infty$ shown by an arrow in the table (and indicated by dashed lines in the figures) toward which the stress intensity factors tend as the crack tip approaches the interface are again the consequence of the change in the power of singularity. For the materials under consideration the powers α_j and β_j, $(j=1,2)$ are found to be (see (8)):

$$-a < a_2 < b_2 = a = a_1 < b_1: \quad \alpha_1 = \beta_2 = -0.5, \quad \alpha_2 = \beta_1 = -0.27326 \;;$$

$$a_1 = a = b_2, \quad a_2 \rightarrow a < b_1: \quad \alpha_1 = -0.5, \quad \beta_2 \rightarrow \alpha_2 = \beta_1 \rightarrow -0.33811 \;;$$

$$a_1 = a = b_2, \quad b_1 \rightarrow a > a_2: \quad \beta_2 = -0.5, \quad \alpha_1 \rightarrow \beta_1 = \alpha_2 \rightarrow -0.33811 \;. \tag{24.a-c}$$

Figure 8 shows some sample results for the crack surface displacements obtained from (see (3))

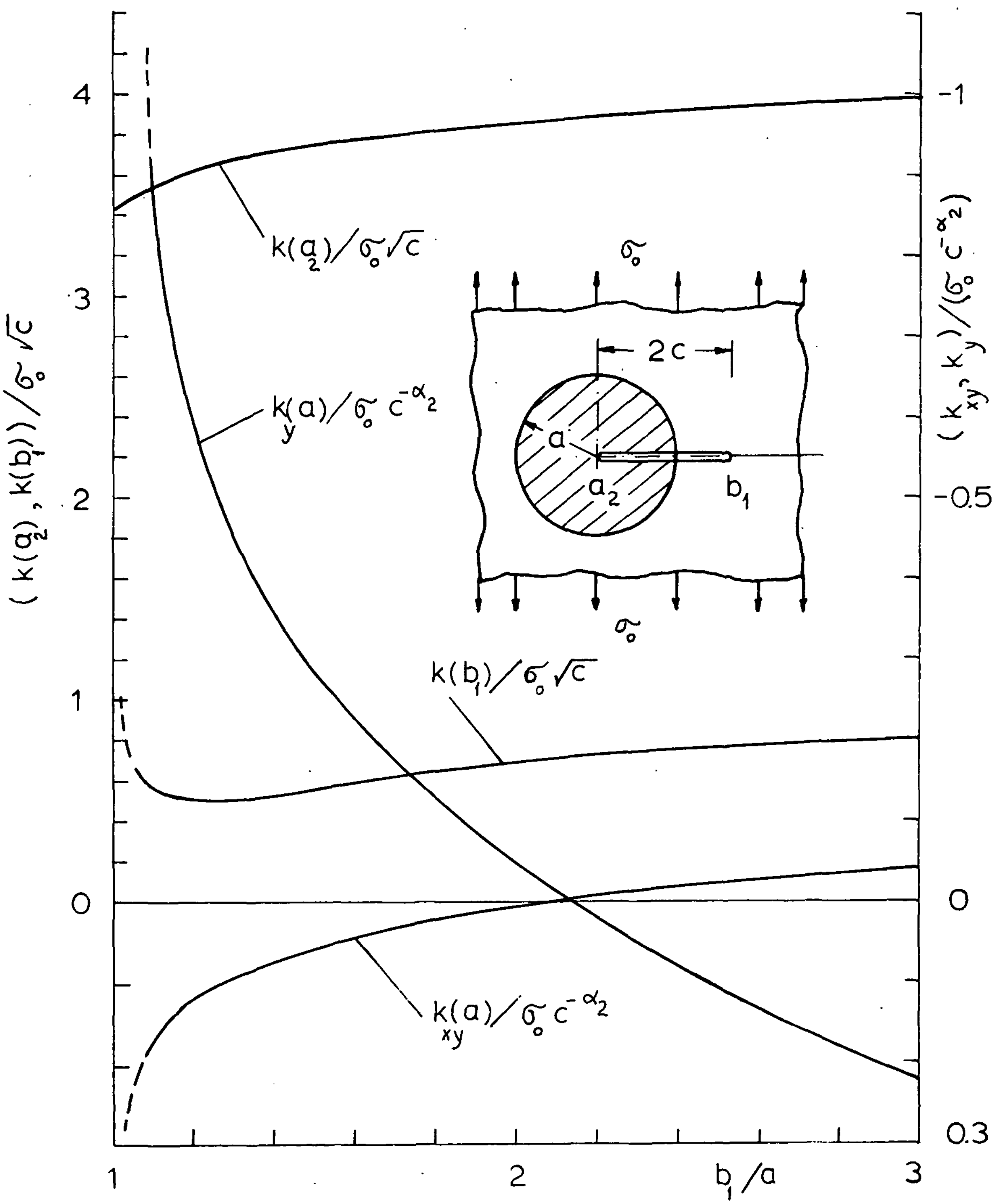

Figure 6. Stress intensity factors for a crack going through the matrix-inclusion interface ($\kappa_1 = 1.6$, $\kappa_2 = 1.8$, $\mu_2/\mu_1 = 23.077$, $\alpha_1 = \beta_2 = -0.5$, $\alpha_2 = \beta_1 = -0.27326$, $c = (b_1 - a_2)/2$, $a_2 = 0$ fixed, b_1 variable).

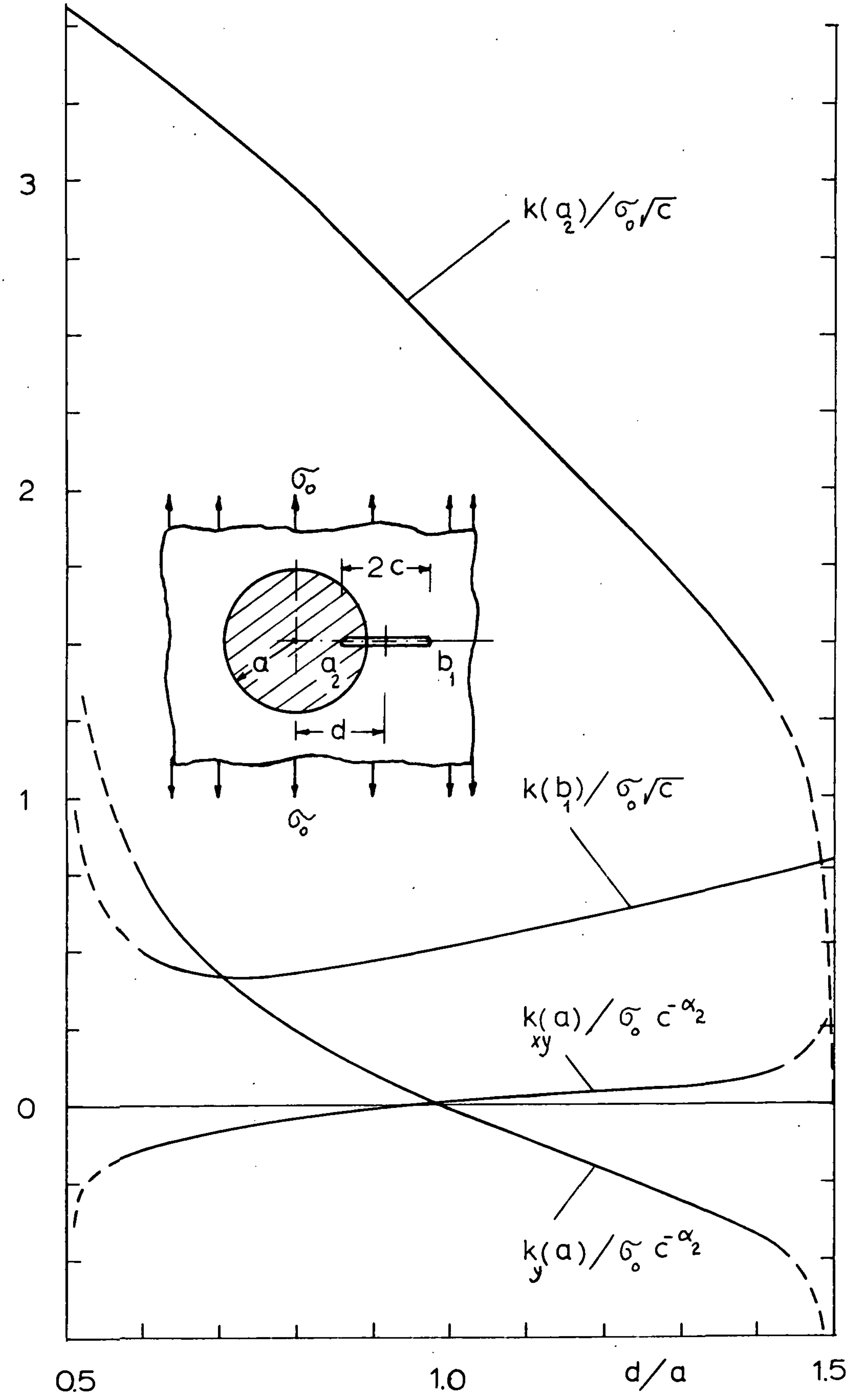

Figure 7. Stress intensity factors for a crack going through the interface ($\kappa_1 = 1.6$, $\kappa_2 = 1.8$, $\mu_2/\mu_1 = 23.077$, $\alpha_2 = \beta_1 = -0.27326$, $2c = (b_1 - a_2) = a$ fixed, $d = (b_1 + a_2)/2$ variable).

Table 6. The stress intensity factors for a crack crossing the interface ($\kappa_1 = 1.6$, $\kappa_2 = 1.8$, $\mu_2/\mu_1 = 23.077$, $\alpha_2 = -0.27326$, $c = (b_1 - a_1)/2$).

$\frac{a_2}{a}$	$\frac{b_1}{a}$	$\frac{k(b_1)}{\sigma_0\sqrt{c}}$	$\frac{k(a_2)}{\sigma_0\sqrt{c}}$	$\frac{k_y(a)}{\sigma_0 c^{-\alpha_2}}$	$\frac{k_{xy}(a)}{\sigma_0 c^{-\alpha_2}}$
0	1.0	$\to \infty$		$\to -\infty$	$\to \infty$
0	1.1	0.548	3.564	-0.847	0.170
0	1.3	0.513	3.701	-0.446	0.0894
0	1.5	0.570	3.756	-0.282	0.0565
0	1.7	0.626	3.799	-0.171	0.0342
0	1.9	0.672	3.838	-0.0835	0.0167
0	2.1	0.710	3.874	-0.0113	0.00227
0	2.5	0.765	3.935	0.105	-0.0211
0	3.0	0.811	3.996	0.219	-0.0440
-1.0	1.5		$\to \infty$	$\to -\infty$	$\to \infty$
-0.9	1.5	0.920	5.600	-1.102	0.221
-0.7	1.5	0.954	5.242	-0.954	0.191
-0.5	1.5	0.757	5.003	-0.753	0.151
-0.3	1.5	0.670	4.570	-0.547	0.110
-0.1	1.5	0.598	4.034	-0.363	0.0727
0.1	1.5	0.547	3.481	-0.209	0.0420
0.3	1.5	0.518	2.956	-0.0852	0.0171
0.5	1.5	0.510	2.465	0.0171	-0.00343
0.7	1.5	0.525	1.981	0.108	-0.0216
0.9	1.5	0.572	1.383	0.212	-0.0426
1.0	1.5		$\to 0$	$\to \infty$	$\to -\infty$
0.1	1.1	0.487	3.377	-0.730	0.146
0.3	1.3	0.425	2.978	-0.229	0.0459
0.7	1.7	0.619	1.961	0.211	-0.0424
0.9	1.9	0.731	1.401	0.412	-0.0826
1.0	2.0		$\to 0$	$\to \infty$	$\to -\infty$

$$v(x,0) = \frac{1}{2}\int_{a_2}^{x} f(x)dx \ , \qquad (a_2 \le x \le b_1) \ ,$$

$$v(x,0) = \begin{cases} v_2(x,+0) \ , & (a_2 \le x \le a) \ , \\ v_1(x,+0) \ , & (a \le x \le b_1) \ , \end{cases}$$

$$f(x) = \begin{cases} f_2(x) \ , & (a_2 \le x \le a) \ , \\ f_1(x) \ , & (a \le x \le b_1) \ . \end{cases} \qquad (25)$$

The results given in this paper show the effect of the inclusion-crack geometry and the material constants on the behavior of the stresses around the singular points. In addition to their application to fracture through conventional theories whenever valid, they may be used in connection with a simple criterion such as "a maximum cleavage strength at a characteristic distance" in studying fracture initiation from singular points where the power is not -0.5. It should also be noted that the problem of radial cracks which are not collinear may be solved without too much difficulty by using the technique described in this paper.

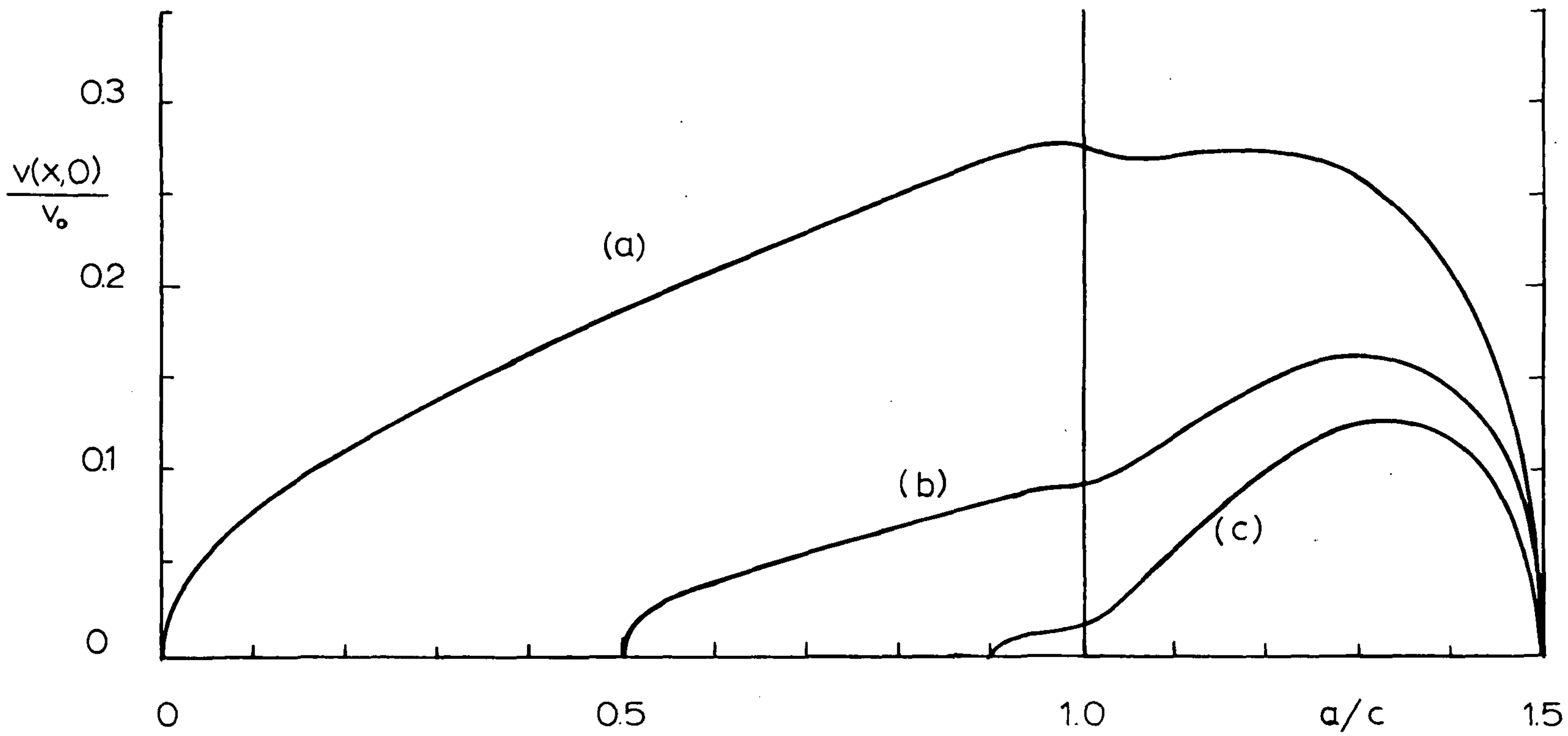

Figure 8. Crack surface displacement for cracks going through the interface ($\kappa_1 = 1.6$, $\kappa_2 = 1.8$, $\mu_2/\mu_1 = 23.077$, $v_o = (1+\kappa_1)a\sigma_o/\mu_1$, $b_2 = a$ fixed, a_2 variable: (a) $a_2 = 0$, (b) $a_2 = a/2$, (c) $a_2 = 0.9a$).

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