2008

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Citation Details

Preston, S. and Vargo, J. (2008). The Indefinite Metric of R. Mrugala and the Geometry of the Thermodynamical Phase Space. Atti della Accademia Peloritana dei Pericolanti – Classe di Scienze Fisiche, Matematiche e Naturali. Vol. LXXXVI, C1S0801019 - Suppl. 1

DOI:10.1478/C1S0801019

Atti dell'Accademia Peloritana dei Pericolanti
Classe di Scienze Fisiche, Matematiche e Naturali
Vol. LXXXVI, C1S0801019 (2008) - Suppl. 1

INDEFINITE METRIC OF R. MRUGALA AND THE GEOMETRY OF THERMODYNAMICAL PHASE SPACE

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ABSTRACT. We study an indefinite metric G which was introduced by R. Mrugala and is defined on the contact phase space (P, θ) of a homogeneous thermodynamical system. We describe the curvature properties and the isometry group of the metric G. We established an isomorphism of the space (P, θ, G) with the Heisenberg Lie group H_n, endowed with the right invariant contact structure and the right invariant indefinite metric. The lift $\tilde{G}$ of the metric G to the symplectization $\tilde{P}$ of contact space (P, θ) and its properties are studied. Finally we introduce the "hyperbolic projectivization" of the space $(\tilde{P}, \tilde{\theta}, \tilde{G})$ that can be considered as the natural **compactification** of the thermodynamical phase space (P, θ, G).

1. Introduction.

Geometrical methods in the study of homogeneous thermodynamical systems pioneered by J. Gibbs ([1]) and C. Caratheodory ([2]) were further developed in the works of R. Hermann ([3]), R. Mrugala, P. Salamon and their collaborators (see ref in [4, 5, 6]), in the dissertations of H. Heemeyer ([7]) and L. Benayoun ([8]). Thermodynamical metrics (TD-metrics) in the form of the Hessian of a thermodynamical potential were explicitly introduced by F. Weinhold ([9]) and, from a different point of view, by G. Ruppeiner ([14]).

In his work [5] (see also the review paper [6]) R. Mrugala introduced the pseudo-Riemannian (indefinite) metric G of signature $(n + 1, n)$ in the thermodynamical contact space (P, θ) inducing TD metrics on the constitutive surfaces defined by different thermodynamical potentials (see below).

In this paper we study the properties of the indefinite metric G in the contact phase space (P, θ), the symplectification of (P, θ, G), and the projective compactification of the space (P, θ, G).

2. The contact structure of homogeneous thermodynamics.

The phase space of Homogeneous Thermodynamics (thermodynamical phase space, or TPS) is the (2n+1)-dimensional vector space $P = \mathbb{R}^{2n+1}$ endowed with the standard contact structure ([11, 12])

$$\theta = dx^0 + \sum_{i=1}^{n} p_l dx^l. \tag{2.1}$$

The horizontal distribution D_m of this structure is generated by two families of vector fields: $D_m =< P_i = \partial_{p_l},\ X_i = \partial_{x^i} - p_i\partial_{x^0} >$.

The 2-form

$$\omega = d\theta = \sum_{i=1}^{n} dp_l \wedge dx^l$$

is a nondegenerate, symplectic form on the distribution D.

Reeb vector field ξ - generator of $ker(d\theta)$ satisfying $\theta(\xi) = 1$, is $\xi = \partial_{x^0}$.

3. Gibbs space. Legendre surfaces of equilibrium.

Constitutive surfaces of concrete thermodynamical systems are determined by their "constitutive equations", which, in their fundamental form determine the value of a **thermodynamical potential** $x^0 = E(x^i)$ as a function of n extensive variables x^i. Dual intensive variables are then determined as the partial derivatives of the thermodynamical potential by the extensive variables: $p_i = \frac{\partial E}{\partial x^i}$.

Thus, a constitutive surface represents a **Legendre submanifold** (maximal integral submanifold) Σ_E of the contact form θ projected diffeomorphically onto the space X of variables x^i. The space Y of variables $x^0, x^i,\ i = 1, \dots, n$ is sometimes named the **Gibbs space** of the thermodynamical potential $E(x^i)$. Thermodynamical phase space (P, θ) (or, more precisely, its open subset) appears as the **first jet space** $J^1(Y \to X)$ of the (trivial) line bundle $\pi : Y \to X$. Projection of Σ_E to the Gibbs space Y is the **graph** Γ_E of the constitutive law $E = E(x^i)$.

We will be using the following local description of Legendre submanifolds. Let P^{2n+1} be a contact manifold. Choose (local) Darboux coordinates (x^0, x^i, p_j) in which $\theta = dx^0 + p_k dx^k$. Let I, J be a partition of the set of indices $[1, \dots, n]$, and consider any function $\phi(p_i, x^j),\ i \in I, j \in J$. Then the following equations define a Legendre submanifold Σ_ϕ:

(3.1)
$$\begin{cases} x^0 &= \phi - \sum_{i\in I} p_i \frac{\partial \phi}{\partial p_i}, \\ p_j &= -\frac{\partial \phi}{\partial x^j},\ j \in J, \\ x^i &= \frac{\partial \phi}{\partial p_i},\ i \in I, \end{cases}$$

and every Legendre submanifold is locally given by some choice of a splitting I, J and of a function $\phi(p_i, i \in I, x^j, j \in J)$.

In physics, the most commonly used thermodynamical potentials ϕ are: internal energy U, entropy S, free energy of Helmholtz F, enthalpy and the free Gibbs energy. On the intersection of the domains of these representations, corresponding points are related by a **Legendre transformation** (see [13]).

4. Thermodynamical metrics (Σ_ϕ, η_ϕ).

Let $\phi(x^i)$ be a TD potential (S,U,F, etc.) - a function of chosen extensive variables x^i. And let Σ_ϕ be the corresponding constitutive surface (Legendre submanifold). The Thermodynamical metric on the submanifold $\Sigma_\phi \subset P$ is defined as follows

$$\eta_\phi = Hess(\phi),\ \eta_{\phi\ ij} = \frac{\partial^2 \phi}{\partial x^i \partial x^j}.$$

The Thermodynamical metric η_U (the Weinhold metric) in the space X of extensive variables corresponding to the choice of internal energy U as the thermodynamical potential E was explicitly introduced by F.Weinhold (see [9]). G. Ruppeiner's metric η_S corresponding to the choice of entropy S as the thermodynamical potential E was defined and intensively studied by Ruppeiner ([14]) in the framework of the fluctuational theory of thermodynamical systems. It was suggested that the curvature of these metrics is related to the interactions in the (microscopical) system, and that singularities of the scalar curvature of these metrics were related to the properties of system near the phase transition and the triple point of the thermodynamical system ([14, 10]).

Interest in these metrics is partly due to the fact that the definiteness of η_U (positive or negative) at a point $x \in X$ of the constitutive surface delivers the local criteria of stability of the equilibria given by the corresponding point of the surface Γ_E or Σ_E (see [13]).

We mention also: the Statistical Distance η_S (Fisher, 1922, P.Salamon et. al.,1983) - a TD length related to the dissipation and efficiency of processes; the relative information metric η_I (F.Schlogl, 1969, K.Ingarden et. al, 1982, R.Mrugala, H.Janyszek, 1989); and finally the non-equilibrium TD metrics (J.Casas-Vazquez, D.Jou, 1985; S.Sieniutycz,S.Berry, 1991), see ref. in [15].

Example 1. Van der Waals gas. As an example, consider 1 mole of a Van der Waals gas. Here $P = \mathbb{R}^5$, $\theta = du - Tds + pdv$, where v is the volume, s is the entropy, and u is the internal energy of 1 mole of gas. The fundamental constitutive relation giving the internal energy is $u = u_0 + (v-b)^{-\frac{R}{c_v}} e^{\frac{s}{c_v}} - \frac{a}{v}$ (see [13]). Weinhold metric η_U has the form

$$\eta_U = \begin{pmatrix} \frac{T}{c_v} & -\frac{TR}{(v-b)c_v} \\ -\frac{TR}{(v-b)c_v} & \left(\frac{TR}{(v-b)^2}(1+\frac{R}{c_v}) - \frac{2a}{v^3}\right) \end{pmatrix},$$

with the determinant $det(\eta_{ij}) = \frac{T}{v^3(v-b)c_v}(pv^3 - av + 2ab)$ and the scalar curvature

$$R(\eta_U) = \frac{aRv^3}{c_v(pv^3 - av + 2ab)^2}.$$

5. The indefinite thermodynamical metric G of R. Mrugala.

In the paper [5], R. Mrugala defined a pseudo-Riemannian (indefinite) metric G of signature $(n+1, n)$ in the thermodynamical contact space (P, θ). It is given by the formula:

$$G = 2dp_k \odot dx^k + \theta \otimes \theta, \tag{5.1}$$

where $dp_k \odot dx^k = \frac{1}{2}(dp_k \otimes dx^k + dx^k \otimes dp_k)$ is the symmetrical product of 1-forms.

Its physical motivation is two-fold. First, it appeared naturally in the framework of statistical mechanics. Second, its reduction to the Legendre submanifolds Σ_ϕ corresponding to the choice of entropy or internal energy as the TD potential coincides with the previously studied Ruppeiner and Weinhold metrics η_ϕ. More specifically, one has

Theorem 1 (R.Mrugala, [5]). *Let $i_\phi : \Sigma_\phi \to P$ be a constitutive (Legendre) submanifold of (P, θ) corresponding to the TD potential $\phi(x^i)$. Then*

$$i^*_\phi(G) = \eta_\phi.$$

Returning to the space P, we see that in the coordinates $x^0; p_s; x^i$, the matrix of the metric G is

$$G = (G_{ij}) = \begin{pmatrix} 1 & 0 & p_j \\ 0 & 0 & I_n \\ p_i & I_n & p_i p_j \end{pmatrix}. \tag{5.2}$$

It is easy to see that $det(G) = (-1)^n$ and, therefore, the metric G is non-degenerate.

Introduce the following canonical, non-holonomic frame in the tangent bundle $T(P)$

$$\xi = \partial_{x^0},\ P_l = \partial_{p_l},\ X_i = \partial_{x^i} - p_i \partial_{x^0} \tag{5.3}$$

and note that the only nonzero commutator relation is $[P_i, X_j] = -\delta_{ij}\xi$. In this frame, the matrix of the metric G has the form $G = (G_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{pmatrix}$.

The metric G is compatible with the contact structure θ in the following natural sense (see [16] for definitions, [17] or [18] for more details)

Theorem 2. (1) *The indefinite metric G is compatible with the contact structure θ in the following sense. For a natural almost contact structure (ϕ, θ, ξ) on (P, θ) (see* [5]*) where $\phi = \begin{pmatrix} 0 & p_1, \dots, p_n & 0 \\ 0 & 0_n & I_n \\ 0 & -I_n & 0_n \end{pmatrix}$, one has*

$$G(\phi X, \phi Y) = -G(X, Y) - \theta(X)\theta(Y),\ X, Y \in T(P),$$

(comp. with Riemannian case, [16]*).*

(2) *(P, θ, G) is an indefinite Sasaki manifold: the Nijenhuis Tensor of the almost complex structure on the symplectization $\tilde{P}$ of (P, θ) is zero.*

$$J(X, f\partial_t) = (\phi(X) - f\xi, \eta(X))\partial_t).$$

6. Levi-Civita Connection and the curvature properties of metric G.

In this section, we present the Levi-Civita connection $\Gamma^\alpha_{\beta\gamma}$ of the metric G ([21]), and we calculate the covariant derivatives of vector fields of the frame (5.3) with respect to the vector fields of the same frame.

We have (for the proof, which is a standard exercise in calculations, see [18])

Proposition 1. *The nonzero Christoffel coefficients $\Gamma^\alpha_{\beta\gamma}$ of the metric G are the following ones*

$$\Gamma^0_{0x^i} = \frac{1}{2}p_i;\ \Gamma^0_{x^i p_j} = \frac{1}{2}\delta_{ij};\ \Gamma^0_{x^i x^j} = p_i p_j;\ \Gamma^{p_i}_{0 p_j} = \frac{1}{2}\delta^i_j;$$
$$\Gamma^{x^i}_{0x^j} = -\frac{1}{2}\delta^i_j;\ \Gamma^{p_i}_{x^j p_k} = \frac{1}{2}\delta^i_k p_j;\ \Gamma^{x^i}_{x^j x^k} = -\frac{1}{2}\left(\delta^i_j p_k + \delta^i_k p_j\right). \tag{6.1}$$

The covariant derivative of the connection Γ takes a particularly simple form when expressed in the frame (5.3):

Proposition 2. *In the canonical frame (ξ, P_i, X_i) the covariant derivatives of the metric G are*

$$
\textbf{(6.2)} \qquad \begin{aligned} &\nabla_\xi \xi = 0, \nabla_{P_i} P_j = 0, \nabla_{X_i} X_j = 0;\ \nabla_\xi P_i = \nabla_{P_i} \xi = \frac{1}{2} P_i; \\ &\nabla_\xi X_j = \nabla_{X_j} \xi = -\frac{1}{2} X_j;\ -\nabla_{P_i} X_j = \frac{1}{2} \delta_{ij} \xi = \nabla_{X_i} P_j. \end{aligned}
$$

Proof. Let G be any pseudo-Riemannian metric. Then, for any vector fields X, Y, Z, the Levi-Civita connection of metric G satisfies:

$$
YG(X, Z) = G(\nabla_Y X, Z) + G(X, \nabla_Y Z),
$$

$$
ZG(X, Y) = G(\nabla_Z X, Y) + G(X, \nabla_Z Y),
$$

$$
XG(Y, Z) = G(\nabla_X Y, Z) + G(Y, \nabla_X Z).
$$

If we add the first two equations and subtract the third, the result is:

$$
\textbf{(6.3)} \quad \begin{aligned} 2G(X, \nabla_Y Z) = YG(X, Z) + ZG(X, Y) - XG(Y, Z) + \\ + G(Y, [X, Z]) + G(Z, [X, Y]) + G(X, [Y, Z]). \end{aligned}
$$

Here we have used the fact that the Levi-Civita connection is symmetric; that is, $\nabla_X Y - \nabla_Y X = [X, Y]$ for all X, Y.

In taking X, Y, Z from the vectors of the frame $\{\xi, P_i, X_j\}$, scalar products $G(X, Y)$, $G(Y, Z), G(Z, X)$ will all be constant; therefore, the first three terms on the right side of (6.2) will vanish, leaving:

$$
\textbf{(6.4)} \qquad 2G(X, \nabla_Y Z) = G(Y, [X, Z]) + G(Z, [X, Y]) + G(X, [Y, Z]).
$$

Among the basic vectors, the only pair with non-zero Lie bracket is $[P_i, X_j] = -\delta_{ij}\xi$. It follows that if we substitute basic vectors into the equation above, the right side will equal zero unless two of the vectors are P_i and X_j, respectively. Since their bracket is proportional to ξ, which is orthogonal to the contact distribution, the third vector must be ξ. In particular, we immediately obtain the following relations:

$$
\nabla_\xi \xi = 0, \nabla_{P_i} P_j = 0, \nabla_{X_i} X_j = 0.
$$

Additionally, we see that the only nonzero component of $\nabla_{P_i} X_j$ is the ξ component, which is found by: $2G(\xi, \nabla_{P_i} X_j) = G(\xi, \delta_{ij}\xi) = -\delta_{ij}$.

Therefore, $\nabla_{P_i} X_j = -\frac{1}{2}\delta_{ij}\xi$. Note that interchanging P_i with X_j changes the sign in the right side: $\nabla_{X_j} P_i = \frac{1}{2}\delta_{ij}\xi$.

The next equation to consider is:

$$
2G(P_i, \nabla_\xi X_j) = G(\xi, [P_i, X_j]) = -\delta_{ij}.
$$

It follows that $\nabla_\xi X_j = -\frac{1}{2} X_j$. Interchanging the roles of P_i, X_j yields $\nabla_\xi P_i = \frac{1}{2} P_i$. By the symmetry of the connection, $\nabla_{X_j} \xi = \nabla_\xi X_j$ and $\nabla_{P_i} \xi = \nabla_\xi P_i$ since ξ commutes with both X_j and P_i. This ends the proof. □

7. Curvature properties of G

The curvature properties of G are collected in the following statement:

Theorem 3. (1) *The Ricci Tensor of G in the frame* $\xi, \partial_{p_j}, \partial_{x^i}$ *is*

$$Ric(G) = (R_{ij}) = \begin{pmatrix} -\frac{n}{2} & 0 & -\frac{n}{2}p_j \\ 0 & 0 & \frac{1}{2}\delta_{ij} \\ -\frac{n}{2}p_i & \frac{1}{2}\delta_{ij} & -\frac{n}{2}p_ip_j \end{pmatrix},$$

(2) *The scalar curvature of G is* $R(G) = \frac{n}{2}$,
(3) *The sectional curvatures* $K(X,Y),\ X,Y = \partial_{x^0}, \partial_{x^i}, \partial_{p_j}$ *for nondegenerate planes* (X,Y) *are* $K(X,Y) = 0,$ *except* $K(\partial_{x^i}, \partial_{p_i}) = \frac{3}{4}$.

For the proof of these results and for some additional results, see [18].

Proposition 3. Killing vector fields: $\mathfrak{iso}_{\mathbf{G}}$. *The Lie algebra* $\mathfrak{iso}_G$ *of the isometry group* $Iso(G)$ *of the metric G is the Lie algebra* $\mathfrak{g}l(n,\mathbb{R}) \times h_n$ *- the semidirect product of the linear Lie algebra* $\mathfrak{g}l(n,R)$ *(with generators* $\{Q^k_l = p_l\partial_{p_k} - x^k\partial_{x^l}\}$*) with the Heisenberg Lie algebra* $\mathfrak{h}_n$ *(with generators* $\{\xi = \partial_{x^0}, A_i = \partial_{p_i} + x^i\partial_{x^0}, B_j = -\partial_{x^j}\}$*). All these vector fields are* θ*-contact with the contact Hamiltonians* $H_\xi = 1, H_{A_j} = p_j, H_{B_i} = x^i, H_{Q^k_l} = x^k p_l$*, see* [11].

8. The Heisenberg Group as the thermodynamical phase space.

The commutator relations for the vector fields of the canonical frame (5.3) show that the Heisenberg group H_n acts at least locally on the space P. Actually, much more is true.

Recall ([19]) that the Heisenberg group H_n is the nilpotent Lie group of $n \times n$ real matrices

$$g = g(\bar{a}, \bar{b}, c) = \begin{pmatrix} 1 & \bar{a} & c \\ 0 & I_n & \bar{b} \\ 0 & 0 & 1 \end{pmatrix} \tag{8.1}$$

with the matrix product (see, for instance, [19]).

The Lie algebra $\mathfrak{h}_n$ of the Heisenberg group is formed by the matrices $X(\bar{a}, \bar{b}, z) = \left(\begin{smallmatrix} 0 & \bar{a} & z \\ 0 & 0 & \bar{b} \\ 0 & 0 & 0 \end{smallmatrix}\right)$ with the conventional matrix bracket as the Lie algebra operation. The Lie algebra $\mathfrak{h}_n$ is mapped diffeomorphically onto H_n by the exponential mapping

$$exp(X(\bar{a}, \bar{b}, z)) = g(\bar{a}, \bar{b}, c = z + \frac{1}{2}\langle \bar{a}, \bar{b}\rangle).$$

We construct the diffeomorphic mapping $\chi : H_n \Longleftrightarrow P$ by defining

$$\chi :\ g = \begin{pmatrix} 1 & \bar{x} & x^0 \\ 0 & I_n & \bar{p} \\ 0 & 0 & 1 \end{pmatrix} \to m = \begin{pmatrix} -x^0 \\ \bar{p} \\ \bar{x} \end{pmatrix}. \tag{8.2}$$

The action of the group H_n on itself by left translation: $L_g : g_1 \to gg_1$ defines the corresponding left action of H_n on the space P as $T_g : m \to \chi(L_g\chi^{-1}(m)))$, or

$$T_g \begin{pmatrix} x^0 \\ \bar{p} \\ \bar{x} \end{pmatrix} = \chi\left(\begin{pmatrix} 1 & \bar{a} & z \\ 0 & I_n & \bar{b} \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \bar{x} & -x^0 \\ 0 & I_n & \bar{p} \\ 0 & 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} x^0 - c - \bar{a}\cdot\bar{p} \\ \bar{p} + \bar{b} \\ \bar{x} + \bar{a} \end{pmatrix}. \tag{8.3}$$

We have in coordinates $(\bar{a},\bar{b},c)$ generators of the Lie algebra of right invariant vector fields on H_n : $\xi_Z = \partial_c,\ \xi_{A_i} = \partial_{a_i} + b_i\partial_c,\ \xi_{B_j} = \partial_{b_j}$. Applying diffeomorphism χ to these vector fields we get the correspondence

(8.4) $$\chi_{*m}(\xi_Z) = -\partial_{x^0}, \chi_{*m}(\xi_{A_i}) = \partial_{x^i} - p_i\partial_{x^0} = X_i, \chi_{*m}(\xi_{B_j}) = \partial_{p_j} = P_j$$

The pullback of the contact form $\theta = dx^0 + p_l dx^l$ from P to H_n defines the 1-form θ_H on H_n

$$\theta_H = \chi^*(\theta) = -dc + b_i da^i.$$

The Reeb vector field of this form is $\xi_H = -\xi_c = -\partial_c$, and we have $\chi_*(\xi_H) = \xi$ for the Reeb vectors of contact manifolds (H_n,θ_H) and (P,θ).

The kernel, D_H, of this 1-form (a distribution of codimension 1 on H_n) is, at each point g, generated by the values of vector fields ξ_{A_i},ξ_{B_j} of left translations, and is therefore **right invariant**. As a result, the distribution D_H (or, what is the same, the right invariant form θ_H) defines the **right invariant contact structure on** H_n (given as the kernel of the form θ_H).

The metric G is transferred under the diffeomorphism χ into the metric G_H on the Heisenberg group. This metric is constant in the right invariant (non-holonomic) frame $(\xi_C,\xi_{A_i},\xi_{B_j})$ and is, therefore, right invariant by itself.

As a result we've proved the following

Theorem 4. *The diffeomorphism χ defined by $\chi:\ g = \begin{pmatrix} 1 & \bar{x} & x^0 \\ 0 & I_n & \bar{p} \\ 0 & 0 & 1 \end{pmatrix} \to m = \begin{pmatrix} -x^0 \\ \bar{p} \\ \bar{x} \end{pmatrix}$ determines an isomorphism of the "thermodynamical metric contact manifold" (P,θ,G) with the Heisenberg group H_n endowed with the right invariant contact from θ_H and the right invariant metric G_H of signature $(n+1,n)$.*

Remark 1. The following description of the action of the automorphism group $Aut(H)$ on the set of left invariant contact structures of Heisenberg group H_n was sent to the authors by M. Goze ([20]). We reformulate this result for the right invariant contact structures.

Let $\{X_i,Y_j,Z\}$ be a standard basis of $\mathfrak{h}_n$ with the only nontrivial brackets being $[X_i,Y_i] = Z,\ i = 1,\ldots,n$. Let $\{\alpha_i,\beta_j,\omega\}$ be the dual basis in $\mathfrak{h}_n^*$. Then, extending this basis to the coframe of right invariant vector fields, we get the relations $d\alpha_i = d\beta_j = 0,\ d\omega = \sum_i \alpha_i\wedge\beta_i$. Then $\omega\wedge(d\omega)^n \neq 0$ and, therefore, the one-form ω defines the right invariant contact structure on H_n.

Proposition 4. *Let $\omega_1 = a\omega + \sum_i a_i\alpha_i + \sum_j b_j\beta_j$ be a contact form in $\mathfrak{h}_n^*$. The group $Aut(H_n)$ acts transitively on the set of right invariant contact structures with the isotropy group of ω being the intersection of the group $Aut(H_n)$ with the group of ω-conformally contact diffeomorphisms of H_n.*

It follows from this that the contact structure θ_H of the thermodynamical phase space is the typical representative of the $Aut(H_n)$-conjugacy class of right invariant contact structures on the Heisenberg Group, defined by a choice of the canonical basis $\{X_i,Y_j,Z\}$ of the Lie algebra $\mathfrak{h}_n$ and, therefore, unique, up to an automorphism of the group H_n.

Remark 2. With the isomorphism of the TPS (P,θ,G) with (H_n,θ_H,G_H) established, many properties of the metric G can be obtained from the corresponding results for invariant metrics on Lie groups.

9. Symplectization of manifold (P, θ, G).

Let $\tilde{P}$ be the standard (2n+2)-dim real vector space $\mathbb{R}^{2n+2}$ with the coordinates $(p_i, x^j),\ i, j = 0, \dots n$, endowed with the 1-form $\tilde{\theta} = \sum_{i=0}^{n} p_i dx^i$, and the standard symplectic structure

(9.1)
$$\omega = d\theta = \sum_i dp_i \wedge dx^i.$$

We consider the **embedding** of the space $(P, \theta = dx^0 + p_l dx^l)$ into $\tilde{P}$

(9.2)
$$J : (x^0, x^i, p_j) \to (x^0, x^i; p_0 = 1, p_l,\ l = 1, \dots, n)$$

as the affine subspace $p_0 = 1$. Then,

Proposition 5. (1) *Pullback by J of the 1-form $\tilde{\theta}$ coincides with the contact form θ: $J^*(\tilde{\theta}) = \theta$.*

(2) *The symplectic manifold $(\tilde{P} = \{(p, x) \in \mathbb{R}^{2n+2} | p_0 > 0\}, \omega)$ is the standard* **symplectization** *of (P, θ) (see* [12]*) and J is the section of the symplectization bundle $\pi : \tilde{P} \to P$.*

(3) *The symmetrical tensor*

(9.3)
$$\tilde{G} = (\tilde{G}_{ij}) = \begin{pmatrix} 0_{n+1 \times n+1} & I_{n+1} \\ I_{n+1} & p_i p_j \end{pmatrix}$$

determines in $\tilde{P}$ a pseudo-Riemannian metric of signature $(n + 1, n + 1)$.

(4) *The restriction of the metric $\tilde{G}$ to the image of the embedding J coincides with the metric G: $J^*\tilde{G} = G$.*

(5) *There is a bijection between the Legendre submanifolds of the contact manifold (P, θ) and the homogeneous (under the action*

$$(x^i, p_j) \to (x^0, \lambda p_0, x^1, p_1, \dots, x^n, p_n)$$

of R^+ on the manifold $\tilde{P}$) Lagrangian submanifolds of the symplectic manifold $(\tilde{P}, \omega)$). This correspondence is defined by the intersection of a homogeneous Lagrangian submanifold $\tilde{K}$ with the image of the embedding J and by the action of the dilatation group on the image of a Legendre submanifold $K \subset P$ under the embedding J.

Proof. Almost all the statements of this Proposition follow simply from the construction or are known ([11, 12]). The determinant of the matrix (13.4) of the metric $\tilde{G}$ is equal to $(-1)^{n+1}$ which proves its nondegeneracy. □

The basic properties of the metric $\tilde{G}$ in the space $\tilde{P}$ are given in the following two statements proved in [18].

Proposition 6. (1) *The Ricci Tensor of $\tilde{G}$ is*

(9.4)
$$Ric(\tilde{G}) = \begin{pmatrix} 0_{(n+1)\times(n+1)} & \frac{n+2}{2} I_{n+1} \\ \frac{n+2}{2} I_{n+1} & \frac{n+2}{2} p_i p_j \end{pmatrix} = \frac{n+2}{2}\tilde{G}.$$

(2) *The scalar curvature of $\tilde{G}$ is $R(\tilde{G}) = Tr(\tilde{G}^{-1} \frac{n+2}{2} \tilde{G}) = (n + 1)(n + 2)$.*

(3) *$\tilde{G}$ is an indefinite Einstein metric of constant scalar curvature $R(\tilde{G}) = (n+1)(n+2)$.*

Theorem 5. *The Lie algebra* $\mathfrak{iso}_{\tilde{G}} \simeq \mathfrak{sl}(n+2,\mathbb{R})$ *of Killing vector fields of the metric* $\tilde{G}$ *is (as a vector space) the linear sum*

$$\mathfrak{iso}_{\tilde{G}} = \mathfrak{q} \oplus \mathfrak{d} \oplus \mathfrak{x}$$

of Lie subalgebras: 1) $\mathfrak{q} = \langle Q^i_j = x^i \partial_{x^j} - p_j \partial_{p_i} \rangle$, *with the commutator relations* $[Q^i_j, Q^p_k] = \delta^p_j Q^i_k - \delta^i_k Q^p_j$. *The subalgebra* $\mathfrak{q}$ *is isomorphic to* $\mathfrak{gl}(n+1,\mathbb{R})$,

The abelian subalgebra 2) $\mathfrak{d} = \langle D^i = \frac{x^i}{2} Q + (1 - \frac{1}{2}(x^l p_l))\partial_{p_i}, \rangle$ *where* $Q = \sum_i Q^i_i = \sum_i (x^i \partial_{x^i} - p_i \partial_{p_i})$ *is the generator of hyperbolic rotation* $H_t : (p,x) \to (e^{-t}p, e^t x)$.

And the abelian subalgebra 3) $\mathfrak{x} = \langle X_s = \frac{\partial}{\partial x^s} \rangle$.

Generators Q^i_j, X_j, D^j *satisfy the following commutator relations*

$$[Q^i_j, X_s] = -\delta^i_s X_j;\ [Q^i_j, D^s] = \delta^s_j D^i;\ [X_s, D^i] = \frac{1}{2} Q^i_s + \frac{1}{2}\delta^i_s Q.$$

Vector fields Q^i_j, X_i, D^j *are Hamiltonian with Hamiltonian functions*

$$H_{Q^i_j} = -x^i p_j;\ H_{X_k} = -p_k;\ H_{D^s} = x^s(1 - \frac{\langle \bar{x}, \bar{p} \rangle}{2}).$$

10. Hyperbolic projectivization $\hat{P}$ of $\tilde{P}$ and the "partial orbit structure" of $\hat{P}$.

In this section we construct a natural compactification of the TPS P endowed with the extension of the contact structure θ and that of the indefinite metric G.

Consider the action of the one-parameter group R in the space $\tilde{P}$ acting by the one-parameter group HR of **hyperbolic rotations**

(10.1) $$g^t : (p_l, x^i) \to (e^t p_l, e^{-t} x^i).$$

We have obviously

Lemma 1. (1) *The 1-form* $\tilde{\theta}$ *is invariant under this action of the group HR.*
(2) *The metric* $\tilde{G}$ *is invariant under the action of the group HR.*

Cover the space $\mathbb{R}^{2n+2}$ with the open subsets of two types:

(1) Sets of the first type are $\hat{U}_j = \{m | p_j \neq 0\}$, and associate with these sets the affine domains $U_j \equiv \mathbb{R}^{2n+1}$ of the projective space $P_{2n+1}(\mathbb{R})$ with the coordinates $(x^i p_j, \frac{p_l}{p_j})$.
(2) Sets of the second type are $\hat{V}_k = \{m | x^k \neq 0\}$, and associate with these sets the affine domains $V_k \equiv \mathbb{R}^{2n+1}$ of the projective space $P_{2n+1}(\mathbb{R})$ with the coordinates $(\frac{x^i}{x^k}, p_l x^k)$.

On the intersections $U_{j_1} \cap U_{j_2}$ we have relations between the corresponding affine coordinates $x^i p_{j_2} = x^i p_{j_1} \cdot (\frac{p_{j_2}}{p_{j_1}})$; $\frac{p_k}{p_{j_2}} = \frac{p_k}{p_{j_1}} \cdot (\frac{p_{j_1}}{p_{j_2}})$.

On the intersections $U_j \cap V_k$, we have relations between the corresponding affine coordinates $x^l p_j = \frac{x^l}{x^k} \cdot (x^l p_j)$; $\frac{p_l}{p_j} = p_l x^k \cdot (\frac{1}{x^k p_j})$.

Finally, on the intersections $V_{j_1} \cap V_{j_2}$ we have relations between the corresponding affine coordinates $\frac{x^k}{x^{j_2}} = \frac{x^k}{x^{j_1}} \cdot (\frac{x^{j_1}}{x^{j_2}})$; $p_l x^{j_2} = p_l x^{j_1} \cdot (\frac{x^{j_2}}{x^{j_1}})$.

This shows that the affine coordinates of all the affine charts are related by transition functions that are **invariant under the action of hyperbolic rotations**. Thus, they are glued into the standard projective space $P_{2n+1}(\mathbb{R})$. Using this one can easily prove the following

Proposition 7. (1) *The space $\hat{P}$ of orbits of the points $\tilde{P} \setminus 0$ under the action of the group HR is canonically diffeomorphic to the projective space $P_{2n+1}(\mathbb{R})$.*

(2) *The projections $\hat{\theta}$ of the 1-form $\tilde{\theta}$ and that of the metric $\hat{G}$ of $\tilde{G}$ endow the projective space $\hat{P}$ with a contact structure and a metric of signature $(n+1,n)$.*

(3) *The composition J of the embedding $j : P \to \tilde{P}$ and the projection $\tilde{P} \to \hat{P}$ defines the* **compactification** *$(\hat{P},\hat{\theta},\hat{G})$ of the TPS (P,θ,G) with the contact structure and the Mrugala metric G.*

Now we consider the lift to the space $\tilde{P}$ of the action of the group H_n on P discussed in Sec. 8, and we consider the action of subgroups of H_n on the cells of smaller dimension that make up the standard CW-structure of the projective space $\hat{P}$.

The differential operators X_i, P_j, ξ of the canonical frame (5.3) act also in the space $\tilde{P}$ with the same commutator relations and generate the action of the Lie group H_n on the space $\tilde{P}$ leaving hyperplanes $p_0 = const$ invariant.

Consider the sequence of subgroups $L_{n-k} \subset H_n$, defined by

$$L_{n-k} = \{g(\bar{a},\bar{b},c) | b_1 = \ldots = b_k = 0\},$$

for $k = 1,\ldots,n$. These subgroups form the series

$$H_n \supset L_{n-1} \supset L_{n-2} \supset \ldots \supset L_0. \tag{10.2}$$

It is easy to see that $L_{n-k} \simeq \mathbb{R}^k \times H_{n-k}$ is the product of the k-dim abelian group $\mathbb{R}^k$ and the Heisenberg group H_{n-k}.

The right invariant vector fields on H_n tangent to (and generated by) the subgroup L_{n-k} are (in terms of the isomorphism of Sec. 8) $\xi, X_i, i = 1,\ldots,n$, and $P_j, j = k+1,\ldots,n$.

In the space $\tilde{P}$, consider the affine planes V_k defining the cells of the standard cell structure of the projective space $\hat{P} = P_{2n+1}(\mathbb{R})$ with respect to the (hyperbolically) homogeneous coordinates of $\hat{P}$: $V_k = \{(x^i, p_j | p_0 = p_1, \ldots = p_{k-1} = 0,\ p_k = 1\}$ with $k = 0,1,\ldots n$. The projective space $\hat{P}$ is obtained by gluing to the cell $V_0 = j(P)$ the smaller cells V_1, V_2, V_3, consecutively and, finally, by gluing in n-dim projective space $P_n(\mathbb{R})$ obtained by the action of hyperbolic rotations (usual dilations here) on the subspace $V_{n+1} = \{(x^i, p_j = 0 | j = 0,\ldots,n\}$.

Projective space $\hat{P}$ is the union of cells $V^k,\ k = 2n+1,\ldots,0$:

$$\begin{tikzcd}
V^{2n+1} \arrow[r, "\cup"] \arrow[d, "\simeq"] & V^{2n} \arrow[r, "\cdots"] \arrow[d, "\simeq"] & V^{n+1} \arrow[r, "\cup"] \arrow[d, "\simeq"] & V^n \arrow[r, "\cup"] \arrow[d, "\simeq"] & \ldots V^0 \arrow[d, "\simeq"] \\
P \simeq H_n & R \times H_{n-1} & R^{n-2} \times H_1 & R^n & R^0
\end{tikzcd}$$

It is easy to see now that each cell V_k is **canonically diffeomorphic** to the group L_{n-k} whose action on V_k is induced by the action of the Heisenberg group H_n on the space $\tilde{P}$ considered above.

So, even though the action of H_n on P cannot be extended to the compactification $\hat{P}$, a coherent action of the subgroups of series (10.2) produces the partial cell structure of $\hat{P}$ starting with the projective subspace $V_{n+1} \simeq P_n(\mathbb{R})$.

The restriction of the 1-form $\tilde{\theta}$ to the cell V_k has the form

$$\theta_k = \tilde{\theta}|_{V_k} = dx^k + \sum_{i=k+1}^{n} p_i dx^i.$$

Therefore, this form determines the canonical contact structure on the Heisenberg factor of the cell $V_k \simeq L_{n-k} \simeq \mathbb{R}^k \times H_{n-k}$ and is zero on the first factor.

The restriction of the Mrugala metric G to the cell V_k has, in variables $(p_k, \ldots, p_n; x^0; \ldots, x^{k-1}; x^k, \ldots, x^n)$, the form

$$G_k = \tilde{G}|_{V_k} = \begin{pmatrix} 0_{(n-k)\times(n-k)} & 0_{(n-k)\times k} & I_{n-k} \\ 0_{k\times(n-k)} & 0_{k\times k} & 0_{k\times(n-k)} \\ I_{n-k} & 0_{(n-k)\times(n-k)} & p_i p_j \end{pmatrix} \tag{10.3}$$

Thus, this metric is zero on the first abelian factor of the cell $V_k \simeq L_{n-k}$ and coincides with the metric G of R. Mrugala on the Heisenberg factor H_{n-k} of the cell V_k.

Combining these arguments we get the following

Theorem 6. (1) *The restriction of the action of the Heisenberg group H_n on the space $P \simeq V_0$ (embedded in $\hat{P}$) to the subgroup $L_{n-k} \simeq R^k \times H_{n-k}$ of the form (23.1) extends to the action of this subgroup on the cell $\pi(V_k) \subset \hat{P}$ and determines the diffeomorphism of the group L_{n-k} with V_k and with its image $\pi(V_k) \subset \hat{P}$.*

(2) *The restrictions of the 1-form $\tilde{\theta}$ and metric $\tilde{G}$ to the cell V_k endow the Heisenberg factor H_{n-k} of V_k with the contact structure θ_k and the Mrugala metric G_k and they are both zero on the abelian factor $\mathbb{R}^k$.*

Remark 3. Every cell V_k represents the thermodynamical phase space of an abstract thermodynamical system with $n+1$ extensive and $n-k$ intensive variables. This corresponds to a situation where the thermodynamical potential $\phi(x^i)$ depends on $x_{k+1}, \ldots x^n$ but not on the first k extensive variables x^i. As a results the participation of factors $x^i, i = 0, \ldots, k-1$ in the processes is "switched out" and they become parameters only.

Example 2. Consider the case $n = 2$, i.e. take P^5 to be five-dimensional with the contact form $\theta = dU - SdT + pdV$ (a one-component homogeneous system, per 1 mole). The hypersurface $\mathcal{C}$ (see Sec.11) has, in this case, the well known form $U - ST + pV = 0$. Its lift to $\tilde{P}$ - $\tilde{\mathcal{C}}$ has the form $p_0U - ST + pV = 0$.

The intersection of this quadric with the plane $p_0 = 0$ is the (degenerate) quadric $\mathcal{C}_1$: $pV = ST$. Fixing the value of S at $S = R - const$ determines the cell $\simeq V_1$ that projects onto the cell V_1 of the compact space $\hat{P}$. The image of the quadric $\mathcal{C}_1$ under this projection determines in the 3-dim H_1-factor of the cell V_1 the surface $pV = RT$ given by the equation of a monatomic ideal gas.

The hypersurface $\mathcal{C}$ is the submanifold containing all the constitutive (equilibrium) surfaces of all thermodynamical systems with the TPS P^5. Closures of these surfaces in $\hat{P}$ contain points from cells of smaller dimension V_1 and V_2. Thus, the equation of a monatomic ideal gas appears here as the equation of the surface formed by the limit points in V_1 of all possible constitutive surfaces in $(P^5, \theta = dU - SdT + pdV)$.

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Presented: September 26, 2005
Published on line: February 01, 2008