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THE INDENTATION OF AN ELASTIC LAYER BY A RIGID STAMP UNDER CONDITIONS OF COMPLETE ADHESION

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The indentation of an elastic layer by a rigid stamp under conditions of complete adhesion*)

by

J.P. Pauwelussen & J.B. Alblas**)

ABSTRACT

In this paper the indentation of an elastic layer by a rigid stamp is treated under conditions of complete adhesion beneath the stamp, where the ratio of the half-width of the contact region and the thickness of the layer is assumed to be small. The cases of a flat stamp and a polynomial shaped one are considered successively and two applications are treated.

KEY WORDS & PHRASES: *Contact problem, elastic fixed layer, rigid stamp, complete adhesion, flat stamp, polynomial shaped stamp, wedge, parabolic stamp, Fourier transform, singular integral equation.*

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*) This report will be submitted for publication elsewhere

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1. INTRODUCTION

Recently, the two-dimensional contact problem of a rigid stamp and an elastic layer attracts some special attention. In the cases of a flat smooth stamp and a cylindrical stamp where a constant coefficient of friction is assumed, results were obtained by ALBLAS and KUIPERS [1], [2]. They considered both the thick layer and the thin layer. Under conditions of complete adhesion work was done by MOSSAKOWSKI [3] and SPENCE [4]. They treated the indentation of a half-space.

In this paper, an isotropic elastic layer will be considered, attached to an undeformable base and loaded in plane strain by a rigid stamp under action of a force P per unit of length. It will be assumed that the friction between the layer and the indentor is large enough to prevent any slip underneath the stamp and that the ratio of the half-width of the contact region, and the thickness of the layer, a/b is small. Moreover, all discussions will be held within the framework of linear elasticity.

Two cases are considered successively: the stamp has a straight horizontal base and, secondly, the stamp is polynomial shaped. In the latter case two applications are treated: The indentation of the layer by a wedge and by a parabolic stamp, respectively. Throughout this paper use will be made of integral representations in the applied stresses, of the displacements in the layer and its upperboundary.
They will be referred to as the representation formulas.

In the case of a flat stamp, application of these formulas leads to a coupled pair of integral equations for the stresses, working in the contact region. Writing the equations in non-dimensional form and expanding to powers of the reciprocal thickness of the layer in these equations, the resultant ones are solved by transforming them into a set of Hilbert problems.

The found expressions for the contact stresses are compared with the solutions of the corresponding contact problems where the lower side of the stamp is allowed to slide freely along the upper boundary of the layer (ALBLAS and KUIPERS [1]) and where a flat stamp is pressed into an elastic half-space under conditions of adhesion (MUSKHELISHVILI [5]), respectively.

For a polynomial shaped stamp the difficulty arises of the increasing of the half-width of the contact region if we let the penetration of the stamp into the layer grow. MOSSAKOWSKI [3] argued that this kind of problems may be attacked incrementally, which means that the parameter a is treated as a timelike variable with the assumption of static equilibrium during the indentation. SPENCE [4] showed that for the axisymmetric case, dealing with a half-space, the horizontal displacement in the contact region (which is an unknown for this problem) necessarily is given by a polynomial in the dimensionless coordinate along the layer, having the same degree as the one which describes the shape of the stamp.

After solving the problem by means of Mossakowski's techniques this displacement deserves therefore some special attention. It appears to be of the form of a series of powers of the reciprocal thickness of the plate, the coefficients of which are polynomials of linearly increasing degree.

Finally, the solutions are worked out numerically for the wedge and the parabolic stamp and the results are compared with the solutions of the corresponding problems for a rigid smooth wedge acting on a half-space (SNEDDON [6]) and for a smooth parabolic stamp which is pressed into a layer (ALBLAS and KUIPERS [2]), respectively. For this last case, the solution is derived from the general results obtained in [2] by taking the coefficient of friction equal to zero.

2. THE REPRESENTATION FORMULAS

We assume a coordinate system Oxyz with the origin O and the x- and z-axis in the upper plane of the layer and the y-axis pointing out of the layer. The stresses at the upper boundary of the layer (i.e. $y = 0$) are functions of x and consist of a normal pressure $P(x)$ and a shear stress in the x-direction, $T(x)$. The case of plane strain will be considered.

Let $u(x,y)$ and $v(x,y)$ denote the displacements in the layer in x- and y-direction, respectively, and $t_{ij}(x,y)$ the stresses then, recalling that complete adhesion between the layer and the rigid base at $y = -b$ is assumed, the following boundary conditions have to be satisfied

$$
\begin{aligned}
&t_{xy}(x,0) = T(x),\\
(2.1)\qquad &t_{yy}(x,0) = -P(x),\\
&u(x,-b) = v(x,-b) = 0, \qquad -\infty < x < \infty.
\end{aligned}
$$

It is well known that, for the case of plane strain, the stresses can be described by means of a so-called Airy function $\chi(x,y)$ in the following form

$$(2.2) \qquad t_{xx} = \frac{\partial^2 \chi}{\partial y^2}, \quad t_{xy} = -\frac{\partial^2 \chi}{\partial x \partial y}, \quad t_{yy} = \frac{\partial^2 \chi}{\partial x^2},$$

where $\chi(x,y)$ satisfies the biharmonic equation

$$(2.3) \qquad \Delta^2 \chi(x,y) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 \chi(x,y) = 0.$$

The solution of the boundary-value problem (2.1)-(2.3) may be obtained by means of Fourier transforms.

Denoting the Fourier transform of a function $f(x)$ by $\bar{f}(\xi)$, assuming that the occuring functions are suitable regular and that the transform $\bar{\chi}(\xi,y)$ has vanishing derivatives up to fourth order as $|x| \to \infty$, we arrive after transformation of the equation (2.3) at an ordinary differential equation for $\bar{\chi}(\xi,y)$, the solution of which may be written in the form

$$(2.4) \qquad \bar{\chi}(\xi,y) = (A+B\xi y) \cosh(\xi y) + (C+D\xi y) \sinh(\xi y),$$

where A-D are to be determined by the boundary conditions (2.1). We shall restrict ourselves to the case that the applied stresses $P(x)$, $T(x)$ act on a finite interval $[-a,a]$, $a > 0$, while the remaining part of the upper surface is unloaded. After determining the Fourier transforms of the stresses, $\bar{t}_{ij}(\xi,y)$ and by application of Hooke's law in its transformed form, taking the inverse Fourier transforms of the results and applying the convolution theorem for Fourier integrals we obtain the following expressions for the displacements in the layer

$$(2.5) \qquad \begin{aligned} u(x,y) &= \frac{1-\nu}{2\pi G} \int_{-a}^{a} \left[\frac{\alpha_1(x-\xi;y)}{2(1-\nu)} P(\xi) + \beta_1(x-\xi;y) T(\xi)\right] d\xi, \\ v(x,y) &= \frac{1-\nu}{2\pi G} \int_{-a}^{a} \left[\alpha_2(x-\xi;y) P(\xi) + \frac{\beta_2(x-\xi;y)}{2(1-\nu)} T(\xi)\right] d\xi, \end{aligned}$$

where ν and G denote the Poisson's ratio and the shear modulus, respectively,

$\alpha_i(x;y)$ and $\beta_i(x;y)$ $(i = 1,2)$ are given by

$$\left.\begin{matrix}\alpha_1(x;y)\\ \beta_2(x;y)\end{matrix}\right\} = 2\int_0^\infty \Big\{[\xi b\cosh(\xi y)\mp(1-2\nu)\sinh(\xi y)]\xi(b+y) +$$

$$\mp(3-4\nu)\sinh(\xi(b+y))[\xi y\cosh(\xi b)\mp(1-2\nu)\sinh(\xi b)]\Big\}\frac{\sin(\xi x)}{\xi\Delta(\xi b)}\,d\xi,$$

(2.6)

$$\left.\begin{matrix}\alpha_2(x;y)\\ \beta_1(x;y)\end{matrix}\right\} = \frac{1}{1-\nu}\int_0^\infty \Big\{[2(1-\nu)\cosh(\xi y)\mp\xi b\sinh(\xi y)]\xi(b+y) +$$

$$\mp(3-4\nu)\sinh(\xi(b+y))[2(1-\nu)\cosh(\xi b)\mp\xi y\sinh(\xi b)]\Big\}\frac{\cos(\xi x)}{\xi\Delta(\xi b)}\,d\xi,$$

and $\Delta(t)$ is given by

$$\Delta(t) = \tfrac{1}{2}[2t^2+(5-12\nu+8\nu^2) + (3-4\nu)\cosh(2t)] \tag{2.7}$$

For $|x| + |y| < \rho < 2b$, the functions (2.6) can be expanded as uniformly convergent series

$$\alpha_1(x;y) = \sum_{n,m=0}^\infty A^+_{nm}b^{-(2n+1+m)}x^{2n+1}y^m - \frac{2xy}{x^2+y^2} + 2(1-2\nu)\arctan\left(\frac{x}{y}\right),$$

$$\beta_1(x;y) = \sum_{n,m=0}^\infty B^+_{nm}b^{-(2n+m)}x^{2n}y^m - \log\left(\frac{x^2+y^2}{b^2}\right) + \frac{1}{1-\nu}\frac{y^2}{x^2+y^2},$$

(2.8)

$$\alpha_2(x;y) = \sum_{n,m=0}^\infty B^-_{nm}b^{-(2n+m)}x^{2n}y^m + \log\left(\frac{x^2+y^2}{b^2}\right) + \frac{1}{1-\nu}\frac{y^2}{x^2+y^2},$$

$$\beta_2(x;y) = \sum_{n,m=0}^\infty A^-_{nm}b^{-(2n+1+m)}x^{2n+1}y^m + \frac{2xy}{x^2+y^2} + 2(1-2\nu)\arctan\left(\frac{x}{y}\right),$$

Next, we shall derive expressions for the displacements at the upper boundary of the layer. They are the representation formulas which will be used in the following sections. To this end we take $y = 0$ in (2.5), (2.6) and (2.8), thus obtaining

$$u(x,0) = \frac{1-\nu}{2\pi G}\int_{-a}^{a}\left[P(\xi)\frac{\alpha_1(x-\xi)}{2(1-\nu)} + T(\xi)\beta_1(x-\xi)\right]d\xi,$$

(2.9)

$$v(x,0) = \frac{1-\nu}{2\pi G}\int_{-a}^{a}\left[P(\xi)\alpha_2(x-\xi) + T(\xi)\frac{\beta_2(x-\xi)}{2(1-\nu)}\right]d\xi,$$

where for $|x| < 2b$

$$
(2.10)\quad
\begin{aligned}
\alpha_1(x) &:= \alpha_1(x;0) = \sum_{k=0}^{\infty} A_k \left(\frac{x}{b}\right)^{2k+1} - \pi(1-2\nu)\,\mathrm{sign}(x),\\
\beta_1(x) &:= \beta_1(x;0) = \sum_{k=0}^{\infty} B_k^{+} \left(\frac{x}{b}\right)^{2k} - 2\log\left(\frac{|x|}{b}\right),\\
\alpha_2(x) &:= \alpha_2(x;0) = \sum_{k=0}^{\infty} B_k^{-} \left(\frac{x}{b}\right)^{2k} + 2\log\left(\frac{|x|}{b}\right),\\
\beta_2(x) &:= \beta_2(x;0) = \alpha_1(x).
\end{aligned}
$$

Here, the coefficients A_k, $B_k^{\pm}$ are given by

$$
(2.11)\quad
\begin{aligned}
A_k &= \frac{8(1-\nu)(-1)^k}{(2k+1)!} \int_0^{\infty} [t^2+2(1-\nu)(1-2\nu)]\, \frac{t^{2k}}{2\Delta(t)}\, dt,\\
B_k^{\pm} &= \frac{(-1)^k}{(2k)!} \int_0^{\infty} \{4t \mp (1-e^{-4t})[4t^2+2(5-12\nu+8\nu^2) +\\
&\quad + (3-4\nu)e^{-2t}]\}\, \frac{t^{2k-1}}{2\Delta(t)}\, dt \pm \begin{cases} 4\log 2 & ;\ k=0,\\ \frac{(-1)^{k-1}}{k} 2^{-4k}; & k=1,2,3,\dots \end{cases}
\end{aligned}
$$

and they were calculated by numerical integration for several values of Poisson's ratio. The results are presented in Table 1,2 below.

ν	A_0	A_1	A_2	A_3
0.0	2.996	-0.888	0.295	-0.093
0.1	2.571	-0.832	0.288	-0.093
0.2	2.176	-0.789	0.287	-0.095
0.3	1.820	-0.762	0.292	-0.098
0.4	1.513	-0.760	0.305	-0.105
0.5	1.275	-0.799	0.339	-0.120

- Table 1 -

ν	B_0^+	B_1^+	B_2^+	B_3^+	B_0^-	B_1^-	B_2^-	B_3^-
0.0	0.288	0.683	-0.204	0.067	0.745	-1.148	0.349	-0.106
0.1	0.445	0.667	-0.213	0.073	0.790	-1.206	0.379	-0.118
0.2	0.626	0.658	-0.228	0.081	0.882	-1.294	0.424	-0.135
0.3	0.838	0.662	-0.254	0.095	1.053	-1.432	0.490	-0.160
0.4	1.090	0.690	-0.299	0.117	1.366	-1.656	0.596	-0.199
0.5	1.419	0.772	-0.384	0.157	1.953	-2.047	0.783	-0.270

- Table 2 -

NOTE 1. For $-b \le y < 0$, the functions $\alpha_i(x,y)$, $\beta_i(x,y)$ are twice differentiable due to the exponential behaviour of the integrands in (2.6). Therefore, the expressions (2.5) for the displacements are twice differentiable with respect to x or to y, and it is easily shown that they satisfy the Navier equations.

NOTE 2. By integration by parts of the expressions (2.6), we find for their asymptotic behaviour for large $|x|$

$$(2.12) \qquad \alpha_i(x;y),\ \beta_i(x;y) = \mathcal{O}\left(\frac{b^2}{x^2}\right), \quad \left(\frac{|x|}{b} \to \infty\right), \qquad i = 1,2.$$

Therefore we find the displacements (2.5) to behave as

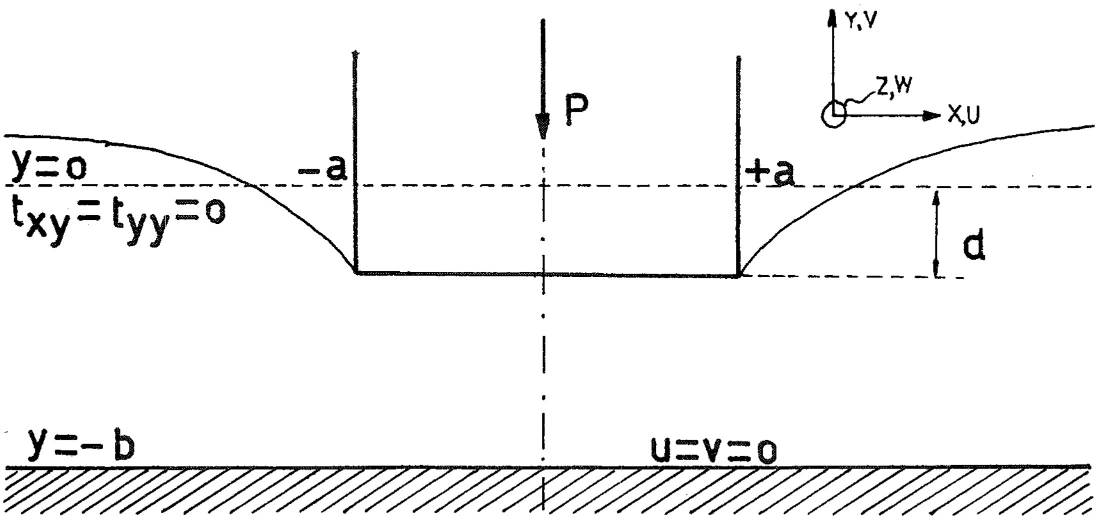
$$(2.13) \qquad u(x,y),\ v(x,y) = \mathcal{O}\left(\frac{ab^2}{x^2}\right), \quad \left(\frac{|x|}{b} \to \infty\right).$$

In the half-space case, the displacement $v(x,y)$ behaves logarithmically for large $|x|$ where the coefficient of $\log |x|$ only depends on the total normal load P. This is in correspondence with Saint-Venants Principle.

3. A FLAT STAMP

In this Section we shall consider the problem of the layer as described in the preceding Section, into which a rigid flat stamp is pressed.

The stamp which is of infinite extension in the z-direction has a width 2a and the penetration of the stamp into the layer is d (see fig. 1).



- fig. 1 -

The penetration d is due to a normal force per unit of length, P while the total shear force on the stamp is zero. Recalling that the layer is fixed at y = -b on a rigid base and that there is complete adhesion beneath the stamp at y = 0, we arrive at the following boundary conditions

$$t_{yy}(x,0) = t_{xy}(x,0) = 0, \quad |x| > a,$$
$$(3.1)$$
$$u(x,0) = 0, \; v(x,0) = -d, \quad |x| < a,$$

at y = 0 and

$$(3.2) \qquad u(x,-b) = v(x,-b) = 0$$

for all $x \in \mathbb{R}$ and at y = -b. The stresses P(x) and T(x) at the upper surface of the layer, which are unknowns of the problem are restricted to the relations

$$\text{(3.3)} \qquad \int_{-a}^{a} P(x)dx = P, \qquad \int_{-a}^{a} T(x)dx = 0.$$

There are two ways of posing the problem

(i) P is given, then the penetration d is a function of b,

(ii) We require a certain penetration d, then the total load P depends on the thickness b.

In this Section we shall confine ourselves to the first case. However, this is not a real restriction as the reversion form problem (i) to problem (ii) is trivial since there always exists, for b fixed, a linear relationship between P and d. Using the representations (2.9) and introducing

$$\text{(3.4)} \qquad \begin{gathered} x = x'a, \quad \xi = \xi'a, \quad b = b'a, \quad \frac{2\pi d}{1-\nu} = d'a, \\ P(\xi) = P'(\xi')G, \quad T(\xi) = T'(\xi')G, \quad P = P'aG, \end{gathered}$$

the conditions $(3.1)^2$ may be written in nondimensional form

$$\text{(3.5)} \qquad \begin{gathered} \int_{-1}^{1} \left[P(\xi)\,\frac{\alpha_1(x-\xi)}{2(1-\nu)} + T(\xi)\beta_1(x-\xi)\right]d\xi = 0, \\ \int_{-1}^{1} \left[P(\xi)\alpha_2(x-\xi) + T(\xi)\,\frac{\beta_2(x-\xi)}{2(1-\nu)}\right]d\xi = -d, \qquad |x| < 1. \end{gathered}$$

In (3.5), and everywhere in this paper when no confusion is possible we omit the primes.

It may be shown by arguments, similar to the ones used by ALEKSANDROV [7, section 2] that the unknowns $P(\xi)$ and $T(\xi)$ may be expanded in uniformly convergent series of the form

$$\text{(3.6)} \qquad \begin{gathered} P(\xi) = \sum_{\ell=0}^{\infty} P_\ell(\xi) b^{-\ell}, \\ T(\xi) = \sum_{\ell=0}^{\infty} T_\ell(\xi) b^{-\ell}, \end{gathered}$$

Since b does not affect the total load P it follows from (3.3) that

$$\int_{-1}^{1} P_\ell(\xi)d\xi = P\delta_{\ell 0},$$

(3.7)

$$\int_{-1}^{1} T_\ell(\xi)d\xi = 0, \quad \ell = 0,1,2,\ldots$$

where $\delta_{\ell 0}$ is the Kronecker delta.

Substituting the expansions (2.10) and (3.7) into (3.5), differentiating the resultant equations with respect to x and equating the coefficients of equal powers of b^{-1} we obtain the systems of integral equations

(3.8)

$$-\frac{\pi(1-2\nu)}{1-\nu} P_\ell(x) + \frac{1}{2(1-\nu)} \sum_{k=0}^{[(\ell-1)/2]} A_k(2k+1) \int_{-1}^{1} (x-\xi)^{2k} P_{\ell-2k-1}(\xi)d\xi +$$

$$-2\int_{-1}^{1} \frac{T_\ell(\xi)}{x-\xi} d\xi + \sum_{k=0}^{[\ell/2]} B_k^{+}(2k) \int_{-1}^{1} (x-\xi)^{2k-1} T_{\ell-2k}(\xi)d\xi = 0,$$

(3.9)

$$2\int_{-1}^{1} \frac{P_\ell(\xi)}{x-\xi} d\xi + \sum_{k=0}^{[\ell/2]} B_k^{-}(2k) \int_{-1}^{1} (x-\xi)^{2k-1} P_{\ell-2k}(\xi)d\xi +$$

$$-\frac{\pi(1-2\nu)}{1-\nu} T_\ell(x) + \frac{1}{2(1-\nu)} \sum_{k=0}^{[(\ell-1)/2]} A_k(2k+1) \int_{-1}^{1} (x-\xi)^{2k} T_{\ell-2k-1}(\xi)d\xi = 0,$$

where $\ell = 0,1,2,\ldots$ and $[r]$ denotes the entier of r. The equations (3.8) and (3.9) are of the form

$$-\frac{\pi(1-2\nu)}{1-\nu} P_\ell(x) + 2\int_{-1}^{1} \frac{T_\ell(\xi)}{\xi-x} d\xi = Q_\ell^{(1)}(x), \tag{3.10}$$

$$2\int_{-1}^{1} \frac{P_\ell(\xi)}{\xi-x} d\xi + \frac{\pi(1-2\nu)}{1-\nu} T_\ell(x) = Q_\ell^{(2)}(x), \quad \ell = 0,1,2,\ldots \tag{3.11}$$

where $Q_\ell^{(1)}(x)$ and $Q_\ell^{(2)}(x)$ are even and odd polynomials in x, respectively, of a degree $\leq \ell-1$. For the integrals, one should read the principal values of the corresponding Cauchy integrals.

Introducing the complex function $G_\ell(z)$ by

$$(3.12)\qquad G_\ell(z) = \frac{1}{2\pi i}\int_{-1}^{1}\frac{P_\ell(\xi)+iT_\ell(\xi)}{\xi-z}\,d\xi, \qquad z = x+iy,$$

which is analytic in the complex plane with the exclusion of a straight cut along [-1,1], we arrive at a Hilbert problem for $G_\ell(z)$, the solution of which may be derived by application of standard techniques (MUSKHELISHVILI [5],[8]). We find

$$(3.13)\qquad \begin{aligned} G_\ell(z) &= \frac{1}{4\pi}\,[Q_\ell^{(1)}(z) - iQ_\ell^{(2)}(z)] + \\ &+ \frac{i(1-\nu)}{\pi\sqrt{\kappa}}\,H(z)\sum_{n=0}^{\ell} q_{\ell,n}(iz)^n + \frac{iPH(z)}{2\pi}\,\delta_{\ell 0}, \end{aligned}$$

where

$$(3.14)\qquad \begin{aligned} \kappa &= 3 - 4\nu, \\ \beta &= \frac{1}{2\pi}\log(\kappa), \\ H(z) &= (z+1)^{-\frac{1}{2}+i\beta}(z-1)^{-\frac{1}{2}-i\beta} \end{aligned}$$

and the real coefficients $q_{\ell,n}$ are determined from the expansion

$$(3.15)\qquad \frac{i\sqrt{\kappa}}{\kappa+1}\,\frac{Q_\ell^{(1)}(z)-iQ_\ell^{(2)}(z)}{H(z)} = \sum_{n=0}^{\ell} q_{\ell,n}(iz)^n + O(|z|^{-1}), \quad (|z| \to \infty).$$

Writing

$$\left.\begin{matrix} c\ell(x;\beta) \\ s\ell(x;\beta) \end{matrix}\right\} = \frac{1}{\sqrt{1-x^2}}\begin{Bmatrix} \cos \\ \sin \end{Bmatrix}\left(\beta\log\left(\frac{1+x}{1-x}\right)\right),$$

and by application of one of Plemelj's formulas we finally arrive at the following expressions for the stresses $P_\ell(x)$ and $T_\ell(x)$

$$P_\ell(x) = \frac{4(1-\nu)^2}{\pi\kappa}\left[c\ell(x;\beta)\sum_{n=0}^{[\ell/2]} q_{\ell,2n}(-1)^n x^{2n} - s\ell(x;\beta)\sum_{n=0}^{[(\ell-1)/2]} q_{\ell,2n+1}(-1)^n x^{2n+1}\right] +$$

$$+\frac{2P(1-\nu)}{\pi\sqrt{\kappa}}\, c\ell(x;\beta)\,\delta_{\ell 0}, \tag{3.17}$$

$$T_\ell(x) = \frac{4(1-\nu)^2}{\pi\kappa}\left[c\ell(x;\beta)\sum_{n=0}^{[(\ell-1)/2]} q_{\ell,2n+1}(-1)^n x^{2n+1} + s\ell(x;\beta)\sum_{n=0}^{[\ell/2]} q_{\ell,2n}(-1)^n x^{2n}\right] +$$

$$+\frac{2P(1-\nu)}{\pi\sqrt{\kappa}}\, s\ell(x;\beta)\,\delta_{\ell 0}. \tag{3.18}$$

From these expressions it follows that for $|x| \to 1$, $P_\ell(x)$ and $T_\ell(x)$ are oscillating unbounded functions.
If $\nu = 1/2$ then according to (3.14) $\beta = 0$ and the oscillating behaviour of the expressions (3.17), (3.18) vanishes. We have evaluated the first three functions in the expansions (3.6) and we found

$$P_0(x) = \frac{2(1-\nu)P}{\pi\sqrt{\kappa}}\, c\ell(x;\beta),$$

$$P_1(x) = -\frac{A_0 P}{2\pi\sqrt{\kappa}}\,[2\beta c\ell(x;\beta) - x s\ell(x;\beta)], \tag{3.19}$$

$$P_2(x) = \frac{2(1-\nu)P}{\pi\sqrt{\kappa}}\,[(4\beta^2 B_1^+ - 2(\tfrac{1}{4}+\beta^2)B_1^- + B_1^- x^2)c\ell(x;\beta) +$$

$$- 2\beta(B_1^+ - B_1^-)x s\ell(x;\beta)],$$

together with

$$T_0(x) = \frac{2(1-\nu)P}{\pi\sqrt{\kappa}}\, s\ell(x;\beta),$$

$$T_1(x) = -\frac{A_0 P}{2\pi\sqrt{\kappa}}\,[2\beta s\ell(x;\beta) + x c\ell(x;\beta)] \tag{3.20}$$

$$T_2(x) = \frac{2(1-\nu)P}{\pi\sqrt{\kappa}}[(4\beta^2 B_1^+ - 2(\tfrac{1}{4}+\beta^2)B_1^- + B_1^- x^2)s\ell(x;\beta) +$$

$$+ 2\beta(B_1^+ - B_1^-)x c\ell(x;\beta)].$$

The expressions for $P_0(x)$ and $T_0(x)$ agree with those, obtained by MUSKHELISHVILI [5, p.466] except for a factor 2 which appears to be an error in [5] (see also [9, p.64]).

The penetration d follows by substitution of (2.10) and (3.6) into $(3.5)^2$. We find for the nondimensional case

$$d = 2P \log b + \sum_{\ell=0}^{\infty} d_\ell b^{-\ell} \tag{3.21}$$

where, for $\ell = 0,1,2$ the coefficients d_ℓ are given by

$$\begin{aligned} \frac{d_0}{P} &= -2 \log 2 - 2\gamma - B_0^- - 2\mathrm{Re}\ \psi(\tfrac{1}{2}-i\beta), \\ \frac{d_1}{P} &= \frac{2A_0\beta}{1-\nu}, \\ \frac{d_2}{P} &= 4\beta^2[B_1^- - 2B_1^+] - B_1^- - \frac{A_0^2}{4(1-\nu)^2}(\tfrac{1}{4}+\beta^2) \end{aligned} \tag{3.22}$$

where γ is Euler's constant and $\psi(z)$ denotes the logarithmic derivative of the Gamma function.

4. A POLYNOMIAL SHAPED STAMP

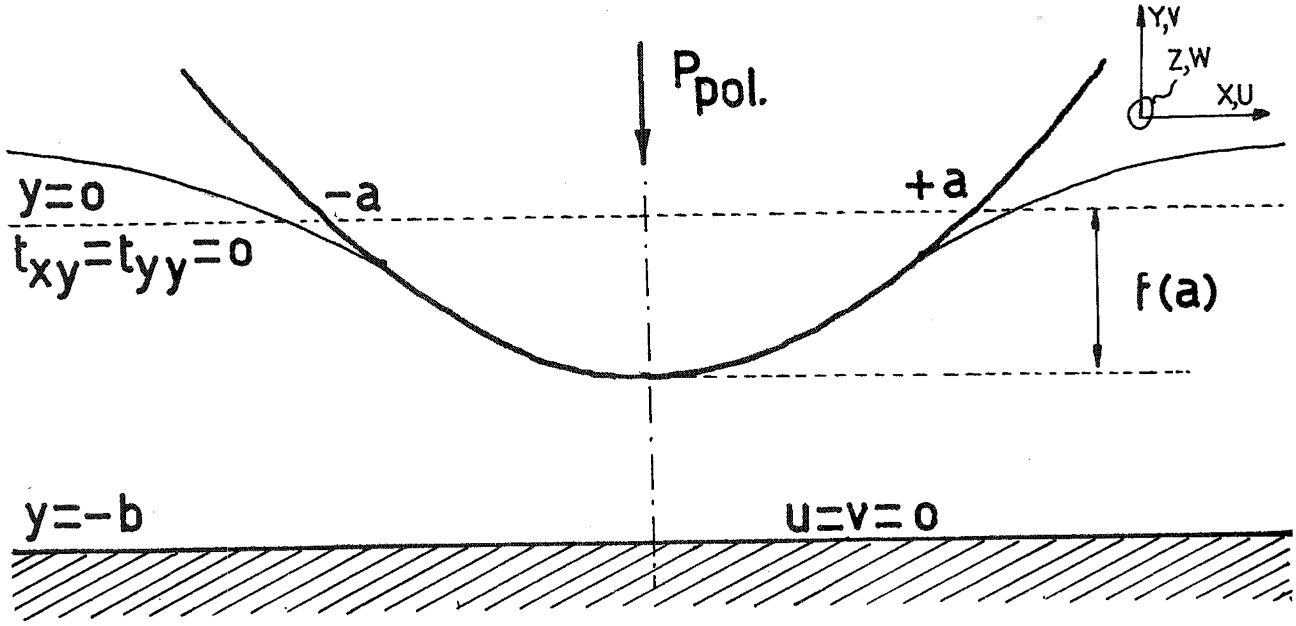
In this Section the problem is treated of the elastic layer with thickness b indented by a rigid stamp whose shape is given by the function

$$y = \Phi(x), \quad \Phi(0) = 0. \tag{4.1}$$

We shall assume that, for $x \geq 0$, $\Phi(x)$ is a polynomial having a degree M and that it is an even function in x. Moreover it is assumed that the friction between the layer and the indentor is sufficient to prevent any slip. Therefore, once a point of the upper surface of the layer has been brought into contact with the stamp, its displacement u in the x-direction cannot change any further. This can be expressed by the condition

$$\frac{\partial u}{\partial a}(x,0;a) = 0, \quad |x| < a. \tag{4.2}$$

In the following, the dependence of the stresses and the displacements on the parameter a will be brought into account by adding it to the list of arguments.



- fig. 2 -

Let f(a) be the normal displacement of the center of the stamp (i.e. the point (0,0)) due to a normal load per unit of length $P_{pol.}$. Then the boundary conditions at the upper surface of the layer are

$$t_{yy}(x,0;a) = t_{xy}(x,0;a) = 0, \qquad |x| > a,$$

$$(4.3) \qquad \left.\begin{aligned} u(x,0;a) &= F(x), \\ v(x,0;a) &= -f(a)+\Phi(x), \end{aligned}\right\} \quad |x| < a,$$

where F(x) does not depend on a.

MOSSAKOWSKI [3] argued that these kind of problems may be attacked incrementally which means that the parameter a will be treated as a time-like variable with the assumption of static equilibrium during the indentation. Let d(a) be the penetration of a flat stamp with width 2a subject to a total load per unit of length, aG. The total load of the corresponding non-dimensional problem will then be equal to one. We introduce new functions u_0, v_0, t_{yy0}, t_{xy0} by

$$u_0(x,y;a) = \frac{\partial}{\partial a} u(x,y;a) \frac{d(a)}{f'(a)}, \quad v_0(x,y;a) = \frac{\partial}{\partial a} v(x,y;a) \frac{d(a)}{f'(a)},$$

(4.4)

$$t_{yy0}(x,y;a) = \frac{\partial}{\partial a} t_{yy}(x,y;a) \frac{d(a)}{f'(a)}, \quad t_{xy0}(x,y;a) = \frac{\partial}{\partial a} t_{xy}(x,y;a) \frac{d(a)}{f'(a)}.$$

Differentiation of (4.3) with respect to a and substitution of (4.4) into the results yields the relations

$$\begin{aligned} &t_{yy0}(x,0;a) = t_{xy0}(x,0;a) = 0, \qquad |x| > a, \\ &\left.\begin{aligned} u_0(x,0;a) &= 0 \\ v_0(x,0;a) &= -d(a) \end{aligned}\right\}, \qquad |x| < a, \end{aligned} \tag{4.5}$$

which agree with the conditions (3.1).

Thus, the functions u_0, v_0, t_{yy0}, t_{xy0} are known from the results of the preceding Section where $P = aG$. Hence, if the function $\phi(a)$, defined by

$$\phi(a) = \frac{f'(a)}{d(a)}, \tag{4.6}$$

is known, then the problem in this Section is solved by integration of the expressions (4.4). Integration of $(4.4)^2$ with respect to a, assuming that $v(x,y;a)$ vanishes for $a = 0$, taking $y = 0$ and using the last condition of (4.3) leads to the Volterra integral equation

$$\Phi(x) = \int_0^{|x|} [d(\xi)+v_0(x,0;\xi)]\phi(\xi)d\xi, \qquad |x| < a. \tag{4.7}$$

Since t_{xy}, v are odd functions and t_{yy}, u are even functions in x, it is sufficient to consider the case $x > 0$ only. By means of the results of Section 3 it can easily be shown that the kernel of the above integral equation is of the form

$$d(\xi) + v_0(x,0;\xi) = \xi \sum_{\ell=0}^{\infty} \gamma_\ell\left(\frac{x}{\xi}\right)\left(\frac{\xi}{b}\right)^\ell, \tag{4.8}$$

where $\gamma_\ell\left(\frac{x}{\xi}\right)$ follows from

$$\gamma_\ell(s) = \frac{1-\nu}{\pi} \int_{-1}^{1} P_\ell(\eta) \log\left|\frac{s-\eta}{\eta}\right| d\eta + \sum_{k=0}^{\ell} \gamma_{\ell,k} s^k, \qquad |s| \le 1. \tag{4.9}$$

For $\ell = 0,1,2$ and $0 \leq k \leq \ell$ the coefficients $\gamma_{\ell,k}$ are given by

$$
\begin{aligned}
\gamma_{0,0} &= - \frac{(1-2\nu)(1-\nu)}{2\pi\sqrt{\kappa}} [\mathrm{Im}\,\psi(\tfrac{3}{4}-\tfrac{1}{2}i\beta) - \mathrm{Im}\,\psi(\tfrac{1}{4}-\tfrac{1}{2}i\beta)], \\
\gamma_{1,0} &= \frac{(1-2\nu)A_0}{4\pi\sqrt{\kappa}}, \\
\gamma_{2,0} &= - \frac{(1-2\nu)(1-\nu)}{\pi\sqrt{\kappa}} [2B_1^+ - B_1^-], \\
\gamma_{2,2} &= \frac{(1-\nu)B_1^-}{2\pi}, \\
\gamma_{1,1} &= \gamma_{2,1} = 0.
\end{aligned} \tag{4.10}
$$

The equation (4.7) can be solved by expanding $\phi(\xi)$ in the following series

$$
\phi(\xi) = \xi^{-1} \sum_{m=0}^{\infty} \phi_m(\xi) \left(\frac{\xi}{b}\right)^m. \tag{4.11}
$$

Substituting (4.8) and (4.11) into (4.7), equating parts with equal powers of b^{-1} and differentiating with respect to x, we arrive at a system of integral equations

$$
\begin{aligned}
&- \frac{(1-\nu)}{\pi} \sum_{k=0}^{\ell} \int_0^1 \phi_{\ell-k}(\xi x)\xi^{\ell-1}\left[\int_{-1}^{1} \frac{P_k(\eta)}{\eta - \frac{1}{\xi}} d\eta \right] d\xi \;+ \\
&+ \sum_{k=0}^{\ell} \sum_{m=1}^{k} m\gamma_{k,m} \left[\int_0^1 \phi_{\ell-k}(\xi x)\xi^{\ell-m} d\xi \right] = \Phi'(x)\delta_{\ell 0}, \qquad \ell = 0,1,2,\ldots
\end{aligned} \tag{4.12}
$$

the solution of which is found by putting

$$
\phi(\xi) = \xi^{-1} \sum_{m=0}^{\infty} \sum_{k=0}^{M} \phi_{m,k}\, \xi^k \left(\frac{\xi}{b}\right)^m \tag{4.13}
$$

where the coefficients $\phi_{m,k}$ follow from (4.12) by equating parts with equal powers of x.

Since $f'(a) = \phi(a)d(a)$ we find by substitution of the obtained expressions for $\phi(a)$ and $d(a)$ into this relation and by integration with respect to a under the assumption of $f(0) = 0$

$$\frac{2\pi}{1-\nu} f(a) = 2\log\left(\frac{b}{a}\right) \sum_{m=0}^{\infty} \left(\frac{a}{b}\right)^m \sum_{k=1}^{M} \frac{\phi_{m,k-1}}{m+k} a^k + 2 \sum_{m=0}^{\infty} \left(\frac{a}{b}\right)^m \sum_{k=1}^{M} \frac{\phi_{m,k-1}}{(m+k)^2} a^k + \tag{4.14}$$
$$+ \sum_{m=0}^{\infty} \left(\frac{a}{b}\right)^m \sum_{k=1}^{M} \frac{a^k}{m+k} \sum_{\ell=0}^{m} d_{m-\ell}\phi_{\ell,k-1}.$$

Integrating $(4.4)^{3,4}$ with respect to a and substituting $y = 0$, the following expressions for the stresses at the upper boundary of the layer can be easily derived

$$\left.\begin{matrix} t_{xy}(x,0;a) \\ -t_{yy}(x,0;a) \end{matrix}\right\} = G \sum_{m=0}^{\infty} \sum_{k=0}^{M-1} \sum_{\ell=0}^{m} b^{-m} x^{m+k} \phi_{\ell,k} \int_{x/a}^{1} \eta^{-m-k-1} \left\{\begin{matrix} T_{m-\ell}(\eta) \\ P_{m-\ell}(\eta) \end{matrix}\right\} d\eta, \; x > 0, \tag{4.15}$$

and it will be shown in the Appendix that the integrals in (4.15) may be written in terms of Hypergeometric Functions. Suppose $\Phi(x)$ is of the form

$$\Phi(x) = Ax^M. \tag{4.16}$$

We note that A is not dimensionless but of the dimension of $(\text{length})^{1-M}$. By means of the known expansions for t_{xy0} and t_{yy0} the displacement $u_0(x,0;a)$ follows from the representation formula $(2.9)^1$. Expanding to powers of b^{-1}, inserting the result into $(4.4)^1$ together with (4.13) and integrating this with respect to a we finally obtain that $F(x)$ is of the form

$$F(x) = Ax^M \sum_{m=0}^{\infty} \alpha_m \left(\frac{x}{b}\right)^m \tag{4.17}$$

where the constants α_m are dimensionless. Integration of the expression for $t_{yy}(x,0;a)$ which is given by $(4.15)^2$ over $(-a,a)$ yields the total load P_{pol}

$$P_{pol} = G \sum_{m=0}^{\infty} \sum_{k=0}^{M-1} \frac{b^{-m} a^{m+k+1}}{m+k+1} \phi_{m,k}. \tag{4.18}$$

5. TWO APPLICATIONS

We first take

$$\Phi(x) = A|x|, \quad A > 0, \tag{5.1}$$

(a wedge) in which case $\phi_m(\xi) \equiv \phi_{m,0}$. For several values of Poison's ratio ν and m, the values of $c_m = \phi_{m,0}/A$ are given in Table 3 ($c_0 = 4/\sqrt{\kappa}$).

ν	c_0	c_1	c2
0,0	2.309	-0.614	-2.601
0,1	2.481	-0.521	-3.460
0,2	2.697	-0.426	-4.439
0,3	2.981	-0.319	-5.900
0,4	3.381	-0.194	-8.195
0,5	4.000	0	-12.282

- Table 3 -

The stress distributions in the contact region in case of a wedge, given by (4.15) with M = 1, are shown in the figures 3 and 4, for $\nu = 0.2$ and $b/a = 5$. Also the normal stress distribution for a rigid smooth wedge, acting on a half space (SNEDDON [6]) is shown in figure 4. From (4.15) (with M = 1) one may derive the following asymptotic relations

$$\frac{t_{xy}(x,0;a)}{GA} = \frac{2(1-2\nu)}{3-4\nu} + \mathcal{O}\left(\frac{x}{a}\log\left(\frac{x}{a}\right)\right), \left(\frac{x}{a} \downarrow 0\right),$$

$$\frac{t_{xy,x}(x,0;a)}{GA} = \frac{1}{2\pi b\sqrt{\kappa}}\,[A_0c_0(1+4\beta^2)-2c_1\beta(\kappa+1)]\,\log\left(\frac{x}{a}\right) + \mathcal{O}(1), \left(\frac{x}{a} \downarrow 0\right), \tag{5.2}$$

$$\frac{t_{yy}(x,0;a)}{GA} = \frac{2(1+\kappa)}{\pi\kappa}\log\left(\frac{x}{a}\right) + \mathcal{O}(1), \left(\frac{x}{a} \downarrow 0\right).$$

Differentation of the representation formulas (2.5) for the displacements yields for $x = 0$

$$t_{xy}(0,y;a) = 0,$$

$$(5.3) \qquad |t_{yy}(0,y;a)| < \infty,$$

$$t_{yy,x}(0,y;a) = 0, \quad (y < 0),$$

so, in (0,0) the stresses are discontinuous.

Secondly, we take

$$(5.4) \qquad \Phi(x) = A|x|^2, \quad A > 0,$$

(a parabolic stamp) where $\phi_m(\xi) \equiv \phi_{m,1}\xi$. For several values of ν and m, the values of $e_m = \phi_{m,1}/A$ are given in Table 4.

ν	e_0	e_1	e_2
0.0	7.832	-2.623	14.551
0.1	8.251	-2.158	14.146
0.2	8.807	-1.705	14.367
0.3	9.567	-1.248	15.508
0.4	10.694	-0.739	18.420
0.5	12.566	0	25.723

- Table 4 -

The stress distributions in the contact region in this case, given by (4.15) with $M = 2$, $\phi_{\ell,0} = 0$, are shown in the figures 5 and 6 for $\nu = 0.2$ and $b/a = 5$. In figure 6 the normal stress distribution under a smooth parabolic stamp acting on a layer (ALBLAS and KUIPERS [2]) is shown for the same values of ν and b/a.

From (4.15) the following asymptotic relations can be found

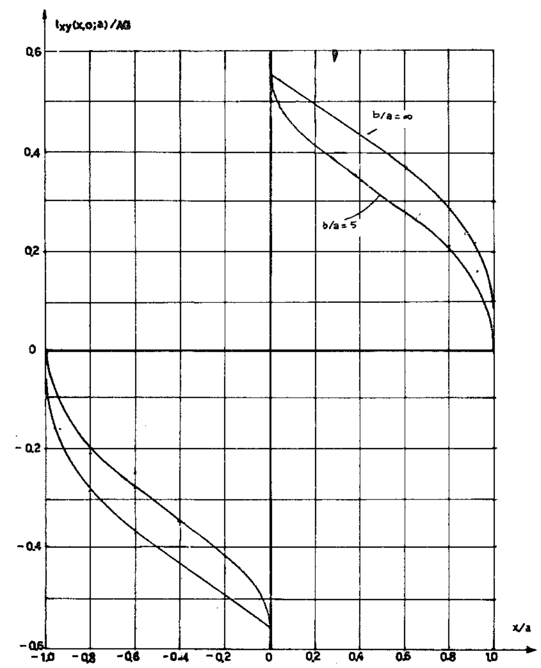
$$\frac{t_{xy}(x,0;a)}{GA} = -\frac{4e_0(1-\nu)\beta}{\pi\sqrt{\kappa}} x \log\left(\frac{x}{a}\right) + O(x), \left(\frac{x}{a} \downarrow 0\right),$$

$$(5.5)\quad \begin{aligned} \frac{t_{xy,x}(x,0;a)}{GA} &= -\frac{4e_0(1-\nu)}{\pi\sqrt{\kappa}}\,\beta\,\log\left(\frac{x}{a}\right) + O(1), \left(\frac{x}{a}\downarrow 0\right), \\ \frac{t_{yy}(x,0;a)}{GA} &= -\sum_{m=0}^{\infty}\frac{b^{-m}a^{m+1}}{m+1}\sum_{\ell=0}^{m} e_\ell P_{m-\ell}(0) + O(x), \left(\frac{x}{a}\downarrow 0\right), \\ \frac{t_{yy,x}(x,0;a)}{GA} &= \frac{e_0(1-2\nu)\beta}{\sqrt{\kappa}} + O\left(\frac{x}{a}\log\left(\frac{x}{a}\right)\right), \left(\frac{x}{a}\downarrow 0\right). \end{aligned}$$

NOTE. Studying $\frac{\partial u}{\partial y}(x,0;a)$ in the last case, one may find that there exists a constant $x_0 \in (0,a)$ such that

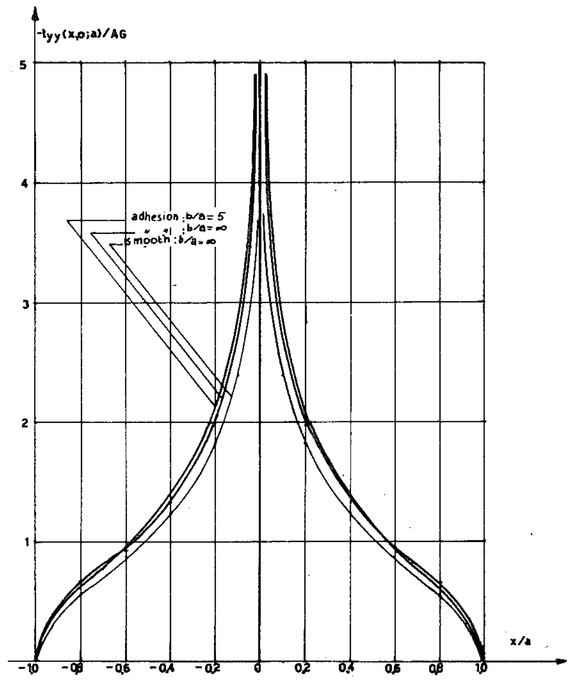
$$(5.6)\qquad \operatorname{sign}\frac{\partial u}{\partial y}(x,0;a) = \operatorname{sign}(x_0-x), \qquad (0 < x < a),$$

which means that for $x \in (0,x_0)$ the directions of the movement of a point of the upper boundary of the layer is inside while for $x \in (x_0,a)$, it is outside.



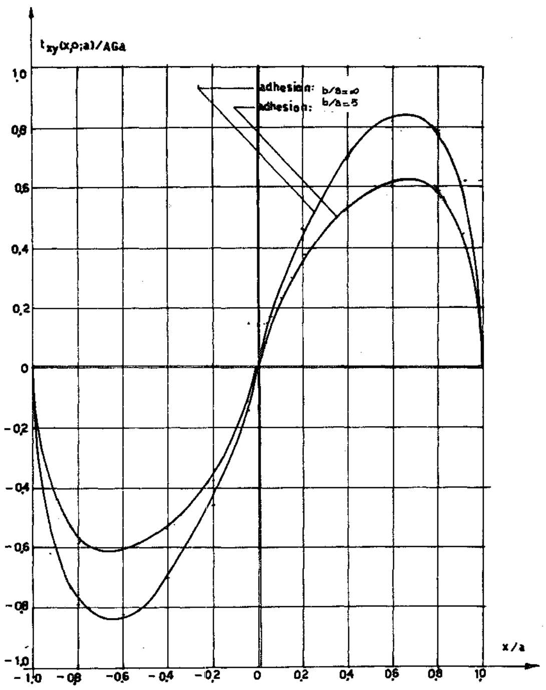
- fig. 3 -

Shear stress distributions in case of a wedge, acting on a fixed layer with thickness b, for ν = 0.2.



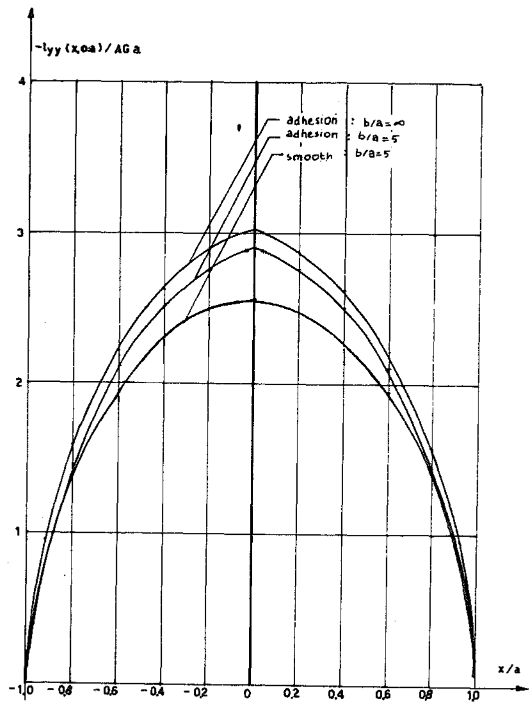
- fig. 4 -

Normal stress distributions in case of a wedge, acting on a fixed layer with thickness b, for ν = 0.2.



- fig. 5 -

Shear stress distributions for a parabolic stamp, acting on a fixed layer with thickness b, for ν = 0.2.



- fig. 6 -

Normal stress distributions for a parabolic stamp, acting on a fixed layer with thickness b, for ν = 0.2.

APPENDIX

Substituting $\eta = x/\xi$ into the integrals in (4.15) we see that they can be expressed in terms of the functions $K(n,0;x)$ where

$$K(n,m;x) = \int_x^a \xi^n(\xi-x)^{-\frac{1}{2}+m-i\beta}(\xi+x)^{-\frac{1}{2}+m+i\beta}d\xi, \quad x > 0, \quad n,m, = 0,1,2,... \tag{A1}$$

Integration by parts yields for $n \geq 1$

$$(1+2m)K(n,m;x) = 2i\beta xK(n-1,m;x)-(n-1)K(n-2,m+1;x) + $$

$$+ a^{n-1}(a-x)^{\frac{1}{2}+m-i\beta}(a+x)^{\frac{1}{2}+m+i\beta}. \tag{A2}$$

So $K(n,m;x)$ can be determined, once $K(0,m;x)$ is known. Repeated integration by parts yields

$$K(0,m;x) = \frac{(a+x)^{-1/2+m+i\beta}(a-x)^{1/2+m-i\beta}}{1/2+m-i\beta} \sum_{n=0}^{\infty} \frac{(1/2-i\beta-m)_n}{(3/2-i\beta+m)_n}\left(\frac{a-x}{a+x}\right)^n$$

$$= \frac{(a+x)^{-1/2+m+i\beta}(a-x)^{1/2+m-i\beta}}{1/2+m-i\beta} F\left(\frac{1}{2}-i\beta-m,\ 1;\frac{3}{2}-i\beta-m;\ \frac{a-x}{a+x}\right). \tag{A3}$$

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