is more important in affecting the slope than is the α dependence of $k/(\alpha C)$. Figures 8 and 9 also indicate that the reaction order with respect to oxygen partial pressure is much higher (about *unity* from the results of Figure 8, with its approximations) than with respect to H^+ ion activity.

Figure 6 shows that V decays to potentials positive of V_{ss}, this amount increasing with the amount of elongation of the electrode. The present model attributes this phenomenon to an accelerated oxygen reduction on the undisturbed oxide sites during the decay, induced by the greater negative charge on the metal side of the double layer than existed before the electrode was deformed. This same reaction on the unbroken oxide areas is the reason for V_{max} and V_h being dependent on oxygen partial pressure (cf. Fig. 2*a*) whereas Hagyard and Williams[8] found these potentials to be independent of P_{O_2}.

The authors wish to acknowledge financial support from the Atomic Energy Commission (contract AT(11-1)1144), Army Ordnance (contract DA-ARO(D)-31-124-G298), and Univ. of Utah Research Fund. Also, they wish to thank S. Hardy, J. Yates, and B. Staker for assistance in calculations and in preparation of the drawings and manuscript.

[1] Glasstone, S., K. J. Laidler, and H. Eyring, *The Theory of Rate Processes* (New York: McGraw-Hill Book Co., 1941), chap. 10.
[2] Funk, A., J. Giddings, C. Christensen, and H. Eyring, these PROCEEDINGS, **43**, 421 (1957).
[3] Funk, A., J. Giddings, C. Christensen, and H. Eyring, *J. Phys. Chem.*, **61**, 1179 (1957).
[4] Funk, A., D. Chakravarty, H. Eyring, and C. Christensen, *Z. Physik Chem.*, **15**, 64 (1958).
[5] Giddings, J., A. Funk, C. Christensen, and H. Eyring, *J. Electrochem. Soc.*, **106**, 91 (1959).
[6] Spear, C. D., M. E. Wadsworth, and M. A. Cook, Tech. Report No. IV, Petroleum Research Fund Advisory Board, Am. Chem. Soc., May 1, 1960.
[7] Miner, B. A., T. N. Andersen, and H. Eyring, Technical Report No. 47, U. S. Atomic Energy Commission, October 31, 1963. Contract no. AT(11-1)-1144.
[8] Hagyard, T., and J. B. Williams, *Trans. Far. Soc.*, **57**, 2288 (1961).
[9] Milner, P. C., *J. Electrochem. Soc.*, **107**, 343 (1960).
[10] Cabrera, N., and N. F. Mott, *Rep. Prog. Phys.*, **12**, 163 (1948).
[11] Dignam, M. J., *J. Electrochem. Soc.*, **107**, 483 (1960).
[12] Dewald, J. F., *J. Electrochem. Soc.*, **102**, 1 (1955).
[13] Plumb, R. C., *J. Phys. Chem.*, **66**, 866 (1962). Experiments have been performed by Plumb in which the aluminum was first oxidized in air for various times, *t*, and then immersed in an aqueous solution. The potential in solution was measured before any further oxidation could occur. The V versus log t (time of air oxidation) slopes were found to be 0.1–0.2 volts.

*THE INDEPENDENCE OF THE CONTINUUM HYPOTHESIS, II**

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Communicated by Kurt Gödel, November 27, 1963

This paper is a continuation of reference 1, in which we began a proof of the fact that the Continuum Hypothesis cannot be derived from the other axioms of set theory, including the Axiom of Choice. We use the same notation as employed in reference 1.

THEOREM 2. *$\mathfrak{N}$ is a model for Z-F set theory.*

The proof will require several lemmas. The first two lemmas express the principle that forcing is a notion which is formalizable in the original model $\mathfrak{M}$.

LEMMA 6. *There is an enumeration $\mathfrak{a}_\alpha$ of all limited statements by means of the ordinal numbers of $\mathfrak{M}$, such that the usual formal operations performed on statements are expressible by means of definable functions in $\mathfrak{M}$ of the indices α, for example, forming negations, conjunctions, replacing variables by particular sets, etc. Furthermore, the ordering corresponds to the definition of forcing given by transfinite induction in Definition 6.*

LEMMA 7. *Let $\mathfrak{a}(x,y)$ be a fixed unlimited statement containing two unbound variables x and y. The relation $\Phi_\mathfrak{a}(P,\alpha,\beta)$ which says that P forces $\mathfrak{a}(F_\alpha,F_\beta)$ and β is the least such ordinal, is definable in $\mathfrak{M}$.*

This follows from the fact that using Lemma 6 the relation "P forces $\mathfrak{a}_\alpha$" can be formalized in Z-F as a statement about P and α. A given unlimited statement can also be handled since, after a finite number of replacements of variables, it is reduced to a limited statement.

Definition 9: For $\mathfrak{a}(x,y)$ as above, put $\Gamma_\mathfrak{a}(\alpha) = \sup\{\beta | \exists\ P,\alpha_1 < \alpha,\ \Phi_\mathfrak{a}(P,\alpha_1,\beta)\}$.

LEMMA 8. *Let $\mathfrak{a}(x,y)$ be a fixed unlimited statement, α an ordinal. For each $\alpha' < \alpha$ either there is no F_β such that $\mathfrak{a}(F_{\alpha'},F_\beta)$ or such an F_β exists with $\beta \leq \Gamma_\mathfrak{a}(\alpha)$.*

Proof: If β is the least ordinal such that $\mathfrak{a}(F_{\alpha'},F_\beta)$, then $\mathfrak{a}(F_{\alpha'},F_\beta)$ must be forced by some P_n which clearly implies $\beta < \Gamma_\mathfrak{a}(\alpha)$.

LEMMA 9. *Let $\mathfrak{a}(x,y)$ be an unlimited statement of the form*

$$Q_1x_1Q_2x_2, \ldots, Q_nx_n\mathfrak{b}(x,y,x_1, \ldots, x_n)$$

where $\mathfrak{b}$ has no quantifiers and Q_i are either existential or universal quantifiers. In $\mathfrak{N}$, assume $\mathfrak{a}$ defines y as a single-valued function of x. Then for each α there exist ordinals $\gamma_0, \ldots, \gamma_n$ such that for $x \in T_\alpha$, there exist $y \in F_{\gamma_0}$ such that $\mathfrak{a}(x,y)$ and for $\langle x,y\rangle$ in $T_\alpha \times T_{\gamma_0}$, the statement $\mathfrak{a}(x,y)$ holds if and only if $\tilde{\mathfrak{a}}(x,y)$ holds where $\tilde{\mathfrak{a}}$ is the statement formed by restricting the quantifiers Q_i in $\mathfrak{b}$ to range over F_{γ_i}.

Proof: Lemma 8 implies the existence of γ_0 such that for $x \in T_\alpha$, there is a $y \in F_{\gamma_0}$ such that $\mathfrak{a}(x,y)$. Define γ_k by induction as follows: let $g_k(x,y,x_1, \ldots, x_{k-1},z)$ be the condition

(*i*) if Q_k is universal,

$$\sim Q_{k+1}x_{k+1}, \ldots, Q_nx_n\mathfrak{b}(x,y,x_1, \ldots, x_{k-1},z,x_{k+1} \ldots x_n) \quad \text{or}$$

(*ii*) if Q_k is existential,

$$Q_{k+1}x_{k+1}, \ldots, Q_nx_n\mathfrak{b}(x,y,x_1, \ldots, x_{k-1},z,x_{k+1} \ldots x_n).$$

Lemma 8 implies that for some γ_k, for all $\langle x,y,x_1, \ldots, x_{k-1}\rangle \in T_\alpha \times F_{\gamma_0} \times \ldots \times F_{\gamma_{k-1}}$, either no z exists such that $g_k(x,y,x_1, \ldots, x_{k-1},z)$ or there is such a $z \in F_{\gamma_k}$. This clearly implies the lemma.

LEMMA 10. *The Axiom of Replacement holds in $\mathfrak{N}$.*

Proof: If $\mathfrak{a}(x,y)$ defines y as a single-valued function of x in $\mathfrak{N}$, then for any α if $D = \{x | \exists z,\ z \in F_\alpha\ \&\ \mathfrak{a}(z,x)\}$ then by Lemma 9, D is defined by a condition in which all variables are restricted to lie in fixed sets F_{γ_i}, which by the definition of the sets F_α implies that D is a set in $\mathfrak{N}$.

The only other axiom to verify which is nontrivial, is the Axiom of the Power Set. The proof we give follows closely the method in reference 2 used to prove that $V = L$ implies the Continuum Hypothesis.

LEMMA 11. *Let W be a set in $\mathfrak{M}$, consisting of conditions P, such that if P_1 and P_2 belong to W, then $P_1 \cup P_2$ is not an admissible condition (i.e., contains a contradiction). Then W is a countable set (in $\mathfrak{M}$).*

Proof: Define sequences n_k and P_j as follows. Put $n_1 = 1$ and P_1 the first P in W. (We assume the P are well-ordered.) If n_k and P_j for $j \leq n_k$ are defined, put R_k equal to the set of all conditions $n \epsilon a_\delta$ or $\sim n \epsilon a_\delta$ such that they or their negations are contained in some $P_j, j \leq n_k$. Let P_j, $n_k < j \leq n_{k+1}$, be finitely many P in W, such that for all P in W, $\exists j$, $n_k < j \leq n_{k+1}$ and P and P_j have precisely the same intersection with R_k. This is possible since R_k is a finite set. We claim that W consists only of the P_j. For if $P \epsilon W$, then since P is a finite set of conditions, and $R_k \subseteq R_{k+1}$, there exists a k such that $P \cap R_k = P \cap R_{k+1}$. Let $n_k < j \leq n_{k+1}$, such that $P_j \cap R_k = P \cap R_k$. Then if P is not equal to P_j, since $P \cap R_{k+1} \subseteq P_j$ and $P_j \subseteq R_{k+1}$, $P \cup P_j$ is an admissible condition, which contradicts the hypothesis.

Definition 10: Put $C(P,\alpha) = \beta$ if P forces $F_\beta \epsilon F_\alpha$ and for $P' \supset P$, $\gamma < \beta$, P' does not force $F_\gamma \epsilon F_\alpha$. If no such β exists, put $C(P,\alpha) = 0$.

The function C is definable in $\mathfrak{M}$, by virtue of the general principle contained in Lemma 6.

LEMMA 12. *For any α, there are only countably many (in $\mathfrak{M}$) β such that for some P, $C(P,\alpha) = \beta$.*

Proof: For each such β, pick one P such that $C(P,\alpha) = \beta$. Then the set of all such P must be countable by Lemma 10.

LEMMA 13. *Let S be an infinite set of ordinals in $\mathfrak{M}$. There exists a set S' of ordinals, $S' \supset S$, $\bar{\bar{S}}' = \bar{\bar{S}}$ such that S' is closed under $J(i,\alpha,\beta,\gamma)$, $K_i(\alpha)$, $C(P,\alpha)$, $I(\alpha)$, for all P and α, β, $\gamma \epsilon S$. Also $\alpha \epsilon S'$ implies $\alpha + 1 \epsilon S'$.*

The statement $\bar{\bar{S}}' = \bar{\bar{S}}$, above, means that with respect to $\mathfrak{M}$, the sets S and S' are of the same cardinality.

LEMMA 14. *Let S be a set of ordinals closed under the operations in Lemma 13, and such that if $\alpha \leq 3\aleph_\tau$, $\alpha \epsilon S$. Then there is a map g mapping S 1-1 onto an initial segment of ordinals which preserves J, K_i, I, N, and such that if $N(\alpha) = 0$ (or 9), $g(\alpha) = \beta$ is the first ordinal such that $N(\beta) = 0$ (or 9) and β is greater than $g(\alpha')$ for $\alpha' < \alpha$. Also, g is the identity for $\alpha \leq 3\aleph_\tau$.*

Proof: S and g in the lemma refer to sets in the model $\mathfrak{M}$. We define g by transfinite induction. For $\alpha \leq 3\aleph_\tau$, let g be the identity. If g is defined for all β in S less than α, if $I(\alpha) = \alpha^3$ (i.e., $N(\alpha) = 0$), put $g(\alpha) = \sup\{g(\beta)|\beta < \alpha \text{ and } \beta \epsilon S\}$. If $I(\alpha) = \beta < \alpha$, then if $N(\alpha) = 9$ (i.e., $\alpha = \beta + 1$), put $g(\alpha) = g(\beta) + 1$. If $i = N(\alpha)$, $1 \leq i \leq 8$, put $g(\alpha) = J(i,g(K_1(\alpha)), g(K_2(\alpha)), g(\beta))$. One can now show by induction that if $\alpha \epsilon S$, $N(\alpha) = 0$, g maps the set of all $\beta < \alpha$ onto an initial segment. The lemma then easily follows.

LEMMA 15. *If we put $G(F_\alpha) = F_{g(\alpha)}$ for α in S, then G is an isomorphism with respect to ϵ of $A_1 = \{F_\alpha | \alpha \epsilon S\}$ onto $A_2 = \{F_{g(\alpha)} | \alpha \epsilon S\}$.*

Proof: This follows by induction on α, in the same way as in 12.6 of reference 2. Observe that in examining the operations $\mathfrak{F}_4$ and $\mathfrak{F}_5$ we need the fact that if $F_\alpha \epsilon A_1$ and is not empty, then it has a member in A_1 preceding it. This is true since S is closed under $C(P,\alpha)$, and $C(P,\alpha)$ for some P is the smallest β for which $F_\beta \epsilon F_\alpha$, if $F_\alpha \neq \phi$.

LEMMA 16. *If $F_\beta \subseteq F_\alpha$, then for some γ, $F_\beta = F_\gamma$, where $\bar{\bar{\gamma}} \leq \bar{\bar{\alpha}} + \aleph_\tau$ in $\mathfrak{M}$.*

Proof: Let S contain all $\delta \leq \alpha$, all $\delta \leq 3\aleph_\tau$ and β, and be closed under the operations in Lemma 13. Let g be the corresponding isomorphism. Then clearly $g(\delta) = \delta$ if $\delta \leq \alpha$. Thus, by Lemma 15, if we put $\gamma = g(\beta)$, since $F_\beta \subseteq F_\alpha$, $F_\gamma = F_\beta$. Since g maps S onto an initial segment, $\bar{\bar{\gamma}} \leq \bar{\bar{S}}$ and so the lemma is proved.

LEMMA 17. *The Axiom of the Power Set holds in* $\mathfrak{N}$.

Proof: Since every subset of F_α is contained in F_β, where β is the first ordinal such that $N(\beta) = 0$ and $\bar{\bar{\beta}} > \bar{\bar{\alpha}} + \aleph_\tau$, it is clear that the power set of F_α occurs in $\mathfrak{N}$.

This completes the proof that $\mathfrak{N}$ is a model, the other axioms being trivially verified. Since rank $F_\alpha \leq \alpha$, $\mathfrak{N}$ contains no new ordinals.

LEMMA 18. *If* $N(\alpha) = N(\beta) = 9$, *and* $\bar{\bar{F}}_\alpha > \bar{\bar{F}}_\beta$ *in* $\mathfrak{M}$, *then* $\bar{\bar{F}}_\alpha > \bar{\bar{F}}_\beta$ *in* $\mathfrak{N}$.

Proof: The point of this lemma is that ordinals do not change their relative cardinality in the model $\mathfrak{N}$. The added complications in the definition of forcing due to $N(\alpha) = 9$ are compensated for in the proof of this lemma, in that as α runs through the ordinals with $N(\alpha) = 9$, F_α runs through the ordinals of $\mathfrak{M}$ in a manner independent of the sequence P_n. More exactly, the map $\alpha \rightarrow F_\alpha$ is an order-preserving map of the ordinals α, $N(\alpha) = 9$, onto all the ordinals of $\mathfrak{M}$.

Thus assume that some element in $\mathfrak{N}$ defines a relation $\varphi(x,y)$ on $F_\beta \times F_\alpha$ which is a single-valued function from F_β onto F_α. For each $\beta' < \beta$, $N(\beta') = 9$ consider the set $H_{\beta'}$ of all γ, $N(\gamma) = 9$, such that some P forces both $\varphi(F_{\beta'},F_\gamma)$ and $(x)\,[\varphi(F_{\beta'},x) \rightarrow x = F_\gamma]$. The set $H_{\beta'}$ exists in $\mathfrak{M}$ as does the map $\beta' \rightarrow H_{\beta'}$ since the notion of forcing is expressible in $\mathfrak{M}$. We shall now show that each $H_{\beta'}$ is countable in $\mathfrak{M}$. For each element in $H_{\beta'}$ choose a corresponding P which forces the above statements. By Lemma 11, it is sufficient to show that these P are mutually incompatible. If two such P corresponding to γ_1 and γ_2 were compatible, their union would force both $\varphi(F_{\beta'},F_{\gamma_1})$ and $(x)\,[\varphi(F_{\beta'},x) \rightarrow x = F_{\gamma_2}]$. Now since $\sim F_{\gamma_1} = F_{\gamma_2}$ is forced, taking into account that $\sim F_{\gamma_1} = F_{\gamma_2}$ involves only existential quantifiers, it follows that $\sim(\varphi(F_{\beta'},F_{\gamma_1}) \rightarrow F_{\gamma_1} = F_{\gamma_2})$ is forced, which is a contradiction. Thus the union of all the $H_{\beta'}$ is of cardinality $\bar{\bar{F}}_\beta$ in $\mathfrak{M}$, since we may clearly restrict ourselves to the case where F_β is infinite. If in $\mathfrak{N}$, $\varphi(F_{\beta'},F_\gamma)$ holds for some γ, $N(\gamma) = 9$ then since all true statements in $\mathfrak{N}$ are forced by some P, γ belongs to $H_{\beta'}$. Thus since φ is onto, the union of $H_{\beta'}$ must contain all $\gamma < \alpha$, $N(\gamma) = 9$ which is impossible since $\bar{\bar{F}}_\beta < \bar{\bar{F}}_\alpha$ in $\mathfrak{M}$.

LEMMA 19. *There is a statement* $\mathcal{A}(x,y)$ *built up from the logical symbols and the set* V, *which expresses in* $\mathfrak{N}$ *the condition that* x *is an ordinal and* $F_x = y$. *Thus the Axiom of Choice holds in* $\mathfrak{N}$.

Proof: This is true because our construction differs from that of reference 2, merely in the introduction of the sets a_δ. If we use the set V, we can of course describe their ordering and so define the construction. We can thus well-order $\mathfrak{N}$ by saying F_α precedes F_β if $\alpha < \beta$ and $F_\beta \neq F_\gamma$ for $\gamma < \beta$.

LEMMA 20. *In* $\mathfrak{N}$, *we have* $\aleph_\tau \leq 2^{\aleph_0} \leq \aleph_{\tau+1}$.

Proof: By Lemma 18, the sets $\aleph_\lambda$ do not change in $\mathfrak{N}$. One can easily see that no P forces any two a_δ to be equal, hence they are distinct, which implies one half of the lemma. Our proof of the Power Set Axiom shows that every subset of ω is some F_α with $\bar{\bar{\alpha}} \leq \aleph_\tau$ or $\alpha < \aleph_{\tau+1}$. Thus Lemma 19 establishes a map of $\aleph_{\tau+1}$ onto $2^{\aleph_0}$.

We have now completed the proof of part 3 of Theorem 1. We now sketch the proof of one of the finer points involved.

LEMMA 21. *If in* $\mathfrak{M}$ $\aleph_\tau$ *is not the sum of countably many smaller cardinals, then* $2^{\aleph_0} = \aleph_\tau$ *in the model* $\mathfrak{N}$. *If it is, then* $2^{\aleph_0} = \aleph_{\tau+1}$.

Proof: The second part follows from Lemma 20, and the theorem of Koenig which says that the continuum is not a countable sum of smaller cardinals. Let $F_\alpha \subseteq \omega$. To prove the first part, let S be a set of indices containing all $\beta \leq \omega$, the ordinal α, and closed under J, K_i, I, $+ 1$, and C as before, and such that $\overline{\overline{S}} = \aleph_0$. The set S is definable in $\mathfrak{M}$ by virtue of the general principle of Lemma 6. For β in S define a new collection of sets G_β, defined by induction on β as follows. If $\beta \leq 3\aleph_\tau$, $G_\beta = F_\beta$. If $N(\beta) = 0$, put $G_\beta = \{G_{\beta'} | \beta' < \beta\}$, and if $N(\beta) = 9$, $G_\beta = \{G_{\beta'} | \beta' < \beta \;\&\; N(\beta') = 9\}$. If $1 \leq i = N(\beta) \leq 8$, $K_1(\beta) = \gamma_1$, $K_2(\beta) = \gamma_2$, put $G_\beta = \mathfrak{F}_i(G_{\gamma_1}, G_{\gamma_2})$. Then the correspondence $F_\beta \rightarrow G_\beta$ is an isomorphism with respect to ϵ. Clearly, $F_\alpha = G_\alpha$. Let ρ be an isomorphism with respect to ϵ of S onto a countable ordinal S'. Let $K_i' = \rho K_i \rho^{-1}$, and $N' = N\rho^{-1}$. Then our argument shows that F_α depends only upon S', K_i', N', $\rho(\alpha)$, and the set $S \cap 3\aleph_\tau$. The number of possible S' is $\aleph_1$. For each S', the number of possible K_i' and N' is $\aleph_1$, since ${\aleph_0}^{\aleph_0} = \aleph_1$ in $\mathfrak{M}$ and K_i', N' are definable in $\mathfrak{M}$. The number of countable subsets of $3\aleph_\tau$ is of cardinality $\aleph_\tau$, as follows easily from our hypothesis on $\aleph_\tau$ and the fact that the Generalized Continuum Hypothesis holds in $\mathfrak{M}$. Thus the number of possible F_α does not exceed $\aleph_\tau$ and the lemma is proved.

LEMMA 22. *If in* $\mathfrak{M}$ *the number of subsets of* $\aleph_\tau$ *of cardinality* $\aleph_1$ *is* $\aleph_\tau$, *then* $2^{\aleph_1} = \aleph_\tau$ *in* $\mathfrak{N}$.

Proof: This is very similar to Lemma 21. We merely demand that S contain all $\beta \leq \aleph_1$. The condition of the lemma may be rephrased by saying $\aleph_\tau$ is not cofinal with $\aleph_0$ or $\aleph_1$. In particular, τ may be 2.

This settles an old question of Lusin whether one can have $2^{\aleph_0} = 2^{\aleph_1}$. Other examples of this type presumably can be constructed with our method. In particular, one can construct models in which the set of constructible reals is countable, a countable union of countable sets is uncountable, etc.

We now give a short discussion of the question of how the above proof can be formalized. Let us denote by (Z-F)′ the axiom system obtained by adjoining to Z-F the axiom:

There exists a set $\mathfrak{M}$ *which is a model for Z-F.*

Observe that this axiom can be expressed as a single statement about $\mathfrak{M}$, because $\mathfrak{M}$ is a set. In the axiom system of Gödel-Bernays this would be still simpler, since only finitely many axioms are employed there. The classic argument of Gödel[2] shows that from (Z-F)′ one can deduce the existence of a set $\mathfrak{N}$ which is a model for Z-F and $V = L$. Similarly, the argument of this paper shows that (Z-F)′ implies the existence of a set $\mathfrak{N}$, which is a model for Z-F, the Axiom of Choice, and the negation of the Continuum Hypothesis. Since our additional axiom is quite readily acceptable to most mathematicians (being merely a formal expression of the Löwenheim-Skolem principle, and implied by well-known axioms such as the Axiom of an Inaccessible Cardinal), one can regard the unprovability of the Continuum Hypothesis as firmly established. However, the consistency of a formal system can also be regarded as a statement in elementary number theory, and one may ask for a proof within elementary number theory of various implications. If $(\text{Z-F})_1$ denotes Z-F with the Axiom of Choice and say $2^{\aleph_0} = \aleph_\tau$, the relevant question is, can we prove within number theory or, if need be, a system of higher type, the implication $\mathrm{Con}(\text{Z-F}) \rightarrow \mathrm{Con}(\text{Z-F})_1$. By using rather standard methods, we shall show how to prove the above implication purely within elementary number theory.

Let us enumerate the axioms of Z-F, A_n. For each n, there is in Z-F a proof of the existence of a countable set $\mathfrak{M}_n$ which satisfies the axioms $A_j, j \leq n$. Furthermore, the correspondence between n and the string of symbols corresponding to such a proof is expressible in number theory.

We may also assume by reference 2 that the axiom $V = L$ is valid in $\mathfrak{M}_n$. We now assert that the proof that $\mathfrak{N}$ is a model for $A_j, j \leq p$ as well as $2^{\aleph_0} = \aleph_\tau$ can be given under the assumption that $\mathfrak{M}$ is a set satisfying A_j, for $j \leq n$ where n is a suitable number greater than p, but still an arithmetical function of p. To see this, we observe that the notion of forcing for limited statements can in Z-F be formulated for unlimited statements as well and the basic lemmas may be proved, since no special properties of $\mathfrak{M}$ are used except the transitivity of $\mathfrak{M}$. To prove that the axioms of Z-F other than the Replacement Axiom holds in $\mathfrak{N}$, as well as $2^{\aleph_0} = \aleph_\tau$ requires only finitely many axioms to hold in $\mathfrak{M}$. Each instance of the Replacement Axiom to be proved in $\mathfrak{N}$ requires that a finite number of instances of replacement used in the proof of Lemma 8 hold in $\mathfrak{M}$. Which instances are sufficient is a simple function of the number of logical symbols used in the formula $\mathfrak{a}(x,y)$ discussed. Since any contradiction in $(\text{Z-F})_1$ would involve only finitely many axioms and since we can prove the existence of a set $\mathfrak{N}$ satisfying these axioms, we would thus be led to a contradiction in Z-F itself. This mapping from contradictions in $(\text{Z-F})_1$ to contradictions in (Z-F) is expressible in an elementary number-theoretic manner which is what was to be proved. In general the statement $2^{\aleph_0} = \aleph_\tau$, for τ in $\mathfrak{M}$, may not be capable of being expressed as a statement in Z-F or may have different interpretations in different countable models $\mathfrak{M}$ or $\mathfrak{N}$. If τ is a particular natural number or $\omega^2 + 1$, etc., then it can readily be expressed in Z-F and the proof sketched goes through.

The argument given in this paper to establish the independence of the Continuum Hypothesis will certainly carry over if one adjoins to Z-F the Axiom of an Inaccessible Cardinal. It seems probable to the author that the addition of any axiom of infinity, as the term is presently understood (i.e., of axioms such as those introduced by P. Mahlo and Azriel Levy), will not alter the situation.

The author wishes to express his gratitude to Professor Kurt Gödel for his many helpful suggestions during the preparation of this manuscript, and for correcting several weak points in the previous exposition. We also would like to thank Professor Solomon Feferman for pointing out, after the author had shown $2^{\aleph_0} = \aleph_2$ in $\mathfrak{N}$, that probably $2^{\aleph_1} = \aleph_2$ would hold as well, thus resolving Lusin's problem.

* The results of this paper first appeared in April 1963 as a set of notes multilithed at Stanford University, and were presented at a lecture in Princeton at the Institute for Advanced Study on May 3, 1963.

† The author is a fellow of the Alfred P. Sloan Foundation.

[1] Cohen, P. J., "The independence of the continuum hypothesis," these PROCEEDINGS, **50**, 1143 (1963).

[2] Gödel, K., *The Consistency of the Continuum Hypothesis* (Princeton University Press, 1940).

[3] The definition of I in Lemma 1 is to be supplemented by the stipulation:

$$I(j(\alpha)) = I(j(\alpha) + 1) = j(\alpha).$$