The index of Dirac operators on manifolds with fibered boundaries

Eric Leichtnam Rafe Mazzeo Paolo Piazza

Abstract

Let X be a compact manifold with boundary ∂X, and suppose that ∂X is the total space of a fibration

$$Z \to \partial X \to Y\,.$$

Let D_Φ be a generalized Dirac operator associated to a Φ-metric g_Φ on X. Under the assumption that D_Φ is fully elliptic we prove an index formula for D_Φ. The proof is in two steps: first, using results of Melrose and Rochon, we show that the index is unchanged if we pass to a certain b-metric $g_b(\epsilon)$. Next we write the $b-$ (i.e. the APS) index formula for $g_b(\epsilon)$; the Φ-index formula follows by analyzing the limiting behaviour as $\epsilon \searrow 0$ of the two terms in the formula. The interior term is studied directly whereas the adiabatic limit formula for the eta invariant follows from work of Bismut and Cheeger.

1 Introduction

Let X be an even dimensional, compact, oriented spin manifold with boundary such that ∂X is the total space of a fibration $Z^\ell \to \partial X \xrightarrow{\phi} Y^k$. (Thus $\dim X = \ell + k + 1 = 2m$.) There are many interesting index formulæ for twisted Dirac operators D on X corresponding to various different classes of complete metrics g on the interior of X. Under certain hypotheses which ensure that D is either Fredholm, or at least has finite L^2 index, and that the usual Atiyah-Singer density has finite integral, the goal is to identify the index defect, i.e. the difference between $\mathrm{Ind}(D)$ and the Atiyah-Singer integrated characteristic form. Most prominent, of course, is the Atiyah-Patodi-Singer theorem when g has asymptotically cylindrical

1991 *Mathematics Subject Classification :* 58J20, 58J28.
Key words and phrases : Dirac operators, index theory, adiabatic limit, eta invariant.

ends, in which case the index defect is (minus one half) the eta invariant of the induced twisted Dirac operator on ∂X [9]. This does not take advantage of the fibred boundary structure. Two interesting classes of metrics which do take this into account are the fibred boundary and fibred cusp metrics, also called $\Phi-$ and $d-$ metrics, respectively. These appear naturally in many interesting geometric settings, cf. [4]: for example, complete Ricci flat metrics are often $\Phi-$metrics, while locally symmetric metrics with $\mathbb{Q}-$rank one cusps are $d-$metrics.

To define these, introduce the following notation. Fix a splitting $T(\partial X) = T_V(\partial X) \oplus T_H(\partial X)$ into vertical and horizontal subspaces, where $T_V(\partial X) = T(\partial X/Y)$ is the fibre tangent bundle, and $T_H(\partial X)$ is identified with $\phi^*(TY)$. We consider metrics $\tilde{g}$ on ∂X and h on Y so that ϕ is a Riemannian submersion. This means that the restriction of $\tilde{g}$ to $T_H(\partial X)$ is identified with $\phi^* h$, and the subbundles $T_H(\partial X)$ and $T(\partial X/Y)$ are orthogonal. We write $\tilde{g} = \phi^* h + \kappa$, where κ is a symmetric two-tensor on ∂X which is positive definite on $T(\partial X/Y)$.

Let x be a defining function for ∂X in some neighbourhood of the boundary. Suppose also that h and κ are allowed to depend smoothly on x, all the way to $x = 0$. Then an exact b-metric and an exact cusp (c-) metric on X are ones which have the form

$$\frac{dx^2}{x^2} + \tilde{g}, \qquad \frac{dx^4}{x^4} + \tilde{g}$$

in this neighbourhood, respectively; likewise, exact $\Phi-$ metrics and exact $d-$metrics have the forms

$$\frac{dx^2}{x^4} + \frac{\phi^* h}{x^2} + \kappa, \qquad \text{and} \qquad \frac{dx^2}{x^2} + \phi^* h + x^2 \kappa,$$

respectively, in this neighbourhood. (The term ‘exact’ in each of these refers to the fact that there are no cross-terms, at least to principal order; this is a natural, but not a serious assumption, and there are generalizations of the ideas and formulæ we discuss here to the various ‘nonexact’ settings). For simplicity in all of the discussion below, we usually label a metric as g_b, g_c, g_Φ and g_d to indicate that it is one of these four types. Note also that when discussing $b-$ and $c-$metrics, it is not important that $\tilde{g}$ respect the fibration structure (nor, of course, even that ∂X have such a structure).

Assume that X and Y are spin, and fix spin structures on each of these manifolds; there is an induced spin structure on the fibres $\phi^{-1}(Y) := Z_y \subset \partial X$. The ($\mathbb{Z}_2-$graded) spin bundles on X, ∂X and Z_y are denoted S, S^∂ and S^{Z_y}, respectively. Let $E \to X$ be an hermitian complex vector bundle endowed with a unitary connection. Fixing also a metric g of any of the types above, we obtain a twisted Dirac operator

$$D_g^+ : C^\infty(X, E \otimes S^+) \to C^\infty(X, E \otimes S^-).$$

If g is of one of the preceding types, then we also write D_b, D_c, D_Φ, D_d for the corresponding Dirac operator to indicate its asymptotic type. The associated boundary operator D^∂ induces a family of Dirac operators $\{D_y^\partial\}_{y \in Y}$, where each D_y^∂ acts on $C^\infty(\phi^{-1}(y), E \otimes S_y^\partial)$.

For $b-$ and $c-$ metrics, the simplest form of the APS occurs when D^∂ is invertible, although the general result if this is not satisfied is not much more difficult. In the other two settings, however, the analogous hypothesis is the

1.1 Assumption. *For some $\delta > 0$,*

$$\operatorname{spec}(D_y^\partial) \cap (-\delta, \delta) = \emptyset, \quad \forall\, y \in Y \tag{1.2}$$

The index formula for D_g is known when g is a metric of type b, c or d; as already noted, the first of these is just the APS theorem, while the second in fact reduces to this theorem in a rather simple way. (This is proved below.) The index formula for $d-$metrics is due in the special case of locally symmetric metrics to Müller [11], and in this general geometric setting was accomplished by Vaillant [14]. The index defect in this case is the integral over Y of the Bismut-Cheeger eta form. (Actually, Vaillant's result holds under the weaker hypothesis that $\ker D_y^\partial$ has constant rank, in which case the index formula has an additional boundary contribution.)

Assuming (1.1), D_Φ^+ is a fully elliptic operator in the (pseudo)differential $\Phi-$calculus developed in [7] and [14], and the parametrix construction there shows that D_Φ^+ is Fredholm acting between the appropriate (Φ-) Sobolev spaces. Answering a question raised in [7], we prove here that

1.3 Theorem. *Assuming (1.1), and using the notation above, we have*

$$\operatorname{Ind}(D_\Phi^+) = \int_X \widehat{A}(X, g_\Phi) \wedge \operatorname{Ch} E - \frac{1}{2}\int_Y \widehat{A}(Y, h) \wedge \widetilde{\eta}, \tag{1.4}$$

where $\widetilde{\eta} \in \Omega^*(Y)$ is the Bismut-Cheeger eta form [3] for the boundary family $(D_y^\partial)_{y\in Y}$.

While it is likely that the index formula for this operator can be obtained by methods similar to those employed in [14] for d-metrics, that proof is very long and difficult, and it is a reasonable goal to obtain this formula as a consequence either of that theorem or of the APS theorem.

The equality of the Φ-index and the d-index (when the boundary family is invertible) has been recently proved by Sergiu Moroianu [10] by reducing it directly to Vaillant's theorem [14]: if $g_d = x^2 g_\Phi$, and both are exact, then $\operatorname{Ind}(D_\phi) = \operatorname{Ind}(D_d)$. By Vaillant [14],

$$\operatorname{Ind}(D_d) = \int_X \widehat{A}(X, g_d) \wedge \operatorname{Ch} E - \frac{1}{2}\int_Y \widehat{A}(Y, h) \wedge \widetilde{\eta},$$

so it suffices to show that the first integral on the right is the same as the corresponding one for g_Φ, i.e. that

1.5 Lemma. $\int_X \widehat{A}(X, g_\Phi) \wedge \operatorname{Ch} E = \int_X \widehat{A}(X, g_d) \wedge \operatorname{Ch} E.$

Notice that the two integrals are well defined: this is discussed in [14, Section 1]. Lemma 1.5 follows simply because $\widehat{A}(X, g_\Phi) = \widehat{A}(X, g_d)$ pointwise, by conformal invariance. Note too that by a standard transgression argument, the integrals are equal even when g_d and $x^2 g_\Phi$ coincide only in a neighbourhood of ∂X.

The proof of (1.4) here is indirect too, but it involves only a reduction to the much simpler APS theorem. We shall use the technique of adiabatic limit, as described below. We first deform g_Φ to a b-metric $g_b(\epsilon)$. The index is unchanged through this deformation, and hence equals the index of the Dirac operator corresponding to $g_b(\epsilon)$. This follows from the analysis of Melrose and Rochon [8], specifically their construction of parametrices which are uniform in an adiabatic parameter ϵ for certain parts of this metric deformation. In the (APS) index formula for this $b-$ metric we then take the limit as $\epsilon \to 0$. The fact that the eta invariant term has the correct limiting behaviour follows from the Bismut-Cheeger theory [3] so it remains only to analyze the limiting behaviour of the interior integral, which is the new calculation here. The particular metric family $g_b(\epsilon)$ is chosen because the Atiyah-Singer integrand for it has the best behaviour in the limit.

In the initial stages of our work, the plan was to develop a more direct deformation connecting g_Φ and g_b and to use a parametrix method to analyze this adiabatic limit. However, just at this time the paper of Melrose and Rochon [8] appeared, and Lemma C.1 there (i.e. Lemma 2.7 below) allowed us to develop the particular and much shorter route presented here. By relying on their substantial and deep work, as well as that of [3], we are able to give a fairly quick proof of this index formula.

It should be possible, and would still be of genuine interest, to prove the $\Phi-$index theorem directly using heat equation methods. In particular, one would hope to obtain another derivation of the fundamental Bismut-Cheeger result in the course of this.

We conclude this discussion by noting that Lauter and Moroianu [6] prove formula (1.4) in the special case $Y = S^1$. In fact, in their earlier paper [5], they also treat the case where Y is arbitrary and establish a less precise index formula using homological methods based on ideas of Melrose-Nistor. We refer also to [12] for a related formula when $\phi : \partial X = S^1 \times S^2 \to S^2$.

We shall prove formula (1.4) assuming that E is the trivial line bundle $X \times \mathbb{C}$. This is for notational simplicity only, and the general formula may be deduced using exactly the same reasoning. In the next section we introduce the sequence of metric homotopies and prove that the index is unchanged under these deformations. In the third section we analyze the other side of the index formula, and especially its behaviour in the adiabatic limit.

Acknowledgments. We wish to thank Richard Melrose and Frédéric Rochon for explaining their work to us, and also Sergiu Moroianu for making some valuable suggestions on an early draft of this note. This work was initiated during a visit by the third author to Stanford University, and he wishes to thank that department for its hospitality. The research of Eric Leichtnam and Paolo Piazza is partially supported by a CNR-CNRS bilateral project. Rafe Mazzeo was supported by the NSF grant DMS-0505709.

2 Reduction of $\mathrm{Ind}(D_\Phi)$ to $\mathrm{Ind}(D_b)$

In order to avail ourselves of the work of Melrose and Rochon, the homotopy of metrics we consider consists of the following steps: first deform g_Φ to the cusp metric

$$g_c^1(\epsilon) := \frac{dx^2}{x^4} + \frac{\phi^* h}{(x+\epsilon)^2} + \kappa; \tag{2.1}$$

next, deform $g_c^1(\epsilon)$ to the cusp metric

$$g_c^0(\epsilon) := \frac{dx^2}{x^4} + \frac{\phi^* h}{\epsilon^2} + \kappa; \tag{2.2}$$

from here deform in succession to the following three b-metrics:

$$g_b^0(\epsilon) := \frac{dx^2}{x^2} + \frac{\phi^* h}{\epsilon^2} + \kappa \tag{2.3}$$

$$g_b^1(\epsilon) := \frac{dx^2}{x^2} + \frac{\phi^* h}{(x+\epsilon)^2} + \kappa \tag{2.4}$$

$$g_b^2(\epsilon) := \frac{(dx)^2}{x^2(x+\epsilon)^2} + \frac{\phi^* h}{(x+\epsilon)^2} + \kappa \tag{2.5}$$

Of course we have only specified the forms of these metrics in a fixed collar neighbourhood of ∂X, but we can extend these to the interior arbitrarily, and standard results show that neither their indices nor the integrals depend on these extensions.

We denote by $D_*^j(\epsilon)$, $* = c, b$ and $j = 0, 1, 2$, the Dirac operators associated to these metrics, respectively.

The first main fact is the

2.6 Lemma. *Assuming (1.1), then each of the operators $D_*^j(\epsilon)$ is fully elliptic when $\epsilon > 0$ is sufficiently small.*

Full ellipticity in either the $b-$ or $c-$ pseudodifferential calculi is simply the assumption that not only the interior symbol but also the boundary 'indicial operator' is invertible. This follows from Theorem (4.41) in [3] when ϵ is small. Using the full ellipticity, one may construct parametrices modulo compact remainders in the appropriate pseudodifferential calculi. Hence each of the operators $D_*^j(\epsilon)$ is Fredholm on the appropriate geometric Sobolev spaces.

From now on we shall omit mention that the hypothesis (1.1) is always in force here. Furthermore, we shall always assume that $0 < \epsilon < \epsilon_0$ for some sufficiently small ϵ_0.

We deform to $g_b^2(\epsilon)$, rather than any of the 'simpler' b-metrics because this is the metric for which we can more effectively analyze the limit of the Atiyah-Singer integrand as $\epsilon \searrow 0$.

We now present a series of lemmata which state that the indices of the Dirac operators remains the same through this entire deformation.

The first step uses the work Melrose and Rochon and is the most serious one analytically. [8].

2.7 Lemma. $\mathrm{Ind}(D_\Phi) = \mathrm{Ind}(D_c^1(\epsilon))$.

Proof. In Appendix C of [8], Melrose and Rochon consider an adiabatic metric deformation connecting a Φ metric to a c-metric. Actually, they consider a slightly more general situation where ∂X is the total space of a tower of fibrations $\partial X \to \tilde{Y} \to Y$, and a corresponding transition between a $\Phi-$metric associated to the first fibration and a $\Phi-$metric associated to the second. By constructing parametrices in an adiabatic calculus, they prove in Proposition C.1 of [8] that the indices of the Dirac operators associated to the metrics in this family remain invariant in this passage to an adiabatic limit. This assumes that the 'boundary symbols, i.e. the normal operators $\mathrm{ad}(P)$ and $N(P)$ are invertible, which follows directly from our hypothesis (1.1). ∎

2.8 Lemma. *We have* $\mathrm{Ind}(D^1_c(\epsilon)) = \mathrm{Ind}(D^0_c(\epsilon))$, $\mathrm{Ind}(D^0_b(\epsilon)) = \mathrm{Ind}(D^1_b(\epsilon))$ *and* $\mathrm{Ind}(D^1_b(\epsilon)) = \mathrm{Ind}(D^2_b(\epsilon))$.

Proof. In each case we simply follow the obvious homotopy of metrics. Thus, for the cusp setting, let

$$g_c(t,\epsilon) := \frac{dx^2}{x^4} + \frac{\phi^* h}{(tx+\epsilon)^2} + \kappa\,, \qquad 0 \le t \le 1, \tag{2.9}$$

so that $g_c(0,\epsilon) = g^0_c(\epsilon)$, $g_c(1,\epsilon) = g^1_c(\epsilon)$. The indicial family of the corresponding Dirac operators $D_c(t,\epsilon)$ is independent of t, and hence each $D_c(t,\epsilon)$ is Fredholm, so the index is constant. The argument in the other two cases is the same. ■

2.10 Lemma. $\mathrm{Ind}(D^0_c(\epsilon)) = \mathrm{Ind}(D^0_b(\epsilon))$.

Proof. As in [8], Lemma (14.1), $\mathrm{Ind}(D^0_c(\epsilon)$ equals the index for the incomplete metric $du^2 + \phi^* h/\,\epsilon^2 + \kappa$ with APS boundary conditions. Since the boundary operator is invertible, this also equals $\mathrm{Ind}(D^0_b(\epsilon))$. ■

Taken together, this chain of equality gives the

2.11 Proposition. $\mathrm{Ind}(D_\Phi) = \mathrm{Ind}(D^2_b(\epsilon))$.

3 The adiabatic limit

At this point we simplify notation and simply write $g(\epsilon)$ instead of $g^2_b(\epsilon)$.

We begin with the

3.1 Proposition. *Assuming, as always, that (1.1) holds, then for ϵ sufficiently small,*

$$\mathrm{Ind}(D_\Phi) = \int_X AS(g(\epsilon)) - \frac{1}{2}\eta(D^\partial_{g(\epsilon)}). \tag{3.2}$$

Proof. Define $\xi = x/(x+\epsilon)$, so that $d\xi/\xi = \epsilon\, dx/x(x+\epsilon)$. In terms of this new boundary defining function, $g(\epsilon) := \epsilon^{-2}\,\hat{g}$, where

$$\hat{g} = \frac{d\xi^2}{\xi^2} + \epsilon^2 \left(\frac{\phi^* h}{(x+\epsilon)^2} + \kappa \right).$$

The middle term on the right has been kept expressed in terms of x simply to emphasize that $\hat{g}$ is an exact b-metric which induces ϵ^2 times the boundary metric induced by $g(\epsilon)$.

Applying the usual APS formula to $\hat{g}$ gives

$$\mathrm{Ind}(D_{\hat{g}}) = \int_X \mathrm{AS}(\hat{g}) - \frac{1}{2}\eta(D^\partial_{\hat{g}}). \tag{3.3}$$

However, clearly $D_{g(\epsilon)}$ has the same index as $D_{\hat{g}}$, which is then the same as $\mathrm{Ind}(D_\Phi)$. Furthermore, using the fact that $\hat{g}$ and $g(\epsilon)$ differ by a constant, we get both

$$\int_X \mathrm{AS}(\hat{g}) = \int_X \mathrm{AS}(g(\epsilon)), \qquad \text{and} \qquad \eta(D^\partial_{\hat{g}}) = \eta(D^\partial_{g(\epsilon)}).$$

Replacing each term in (3.3) with the corresponding quantity for $g(\epsilon)$ gives (3.2). ■

The final steps of the proof of the main theorem consist in analyzing the limiting behaviour as $\epsilon \searrow 0$ of the two terms on the right in (3.2).

3.1 Limiting behaviour of the integrand

3.4 Proposition. *The integral of the Atiyah-Singer density for $g(\epsilon)$ converges to that for g_Φ, i.e.*

$$\lim_{\epsilon \searrow 0} \int_X AS(g(\epsilon)) \; = \; \int_X AS(g_\Phi) \,. \tag{3.5}$$

Proof. This is a computation. We shall use the method of moving frames, cf. [13] for more on this formalism. Recall that if $\omega^0, \ldots, \omega^n$ is any orthonormal set of one-forms, then the connection one-forms ω_i^j are determined uniquely by the equations

$$d\omega^i = \omega^j \wedge \omega_j^i, \qquad \omega_j^i = -\omega_i^j.$$

From these we define the curvature two-forms

$$\Omega_i^j = d\omega_i^j - \omega_i^k \wedge \omega_k^j.$$

Here, and elsewhere below, summation on repeated indices is intended.

The strength of this method, of course, is that it can be adapted to the specific geometry. Thus here we shall choose the coframe for $g(\epsilon)$ as follows. Let $Y = \partial X$. Choose an orthonormal coframe $\tilde{\omega}^\alpha$, $1 \le \alpha \le k$, for (Y, h), and ω^μ, $k+1 \le \mu \le n$, for the restriction of κ to each fibre. These forms may also depend smoothly on ϵ and x (in $x \ge 0$, $\epsilon \ge 0$), and in addition, the ω^μ may also depend on $y \in Y$. In the following, we shall use the Chern convention that Roman indices $i, j, \ldots$ vary between 0 and n, while the Greek indices $\alpha, \beta, \ldots$ vary between 1 and k and $\mu, \nu, \ldots$ vary between $k+1$ and n. Now define

$$\omega^0 = \frac{dx}{x(x+\epsilon)}, \qquad \omega^\alpha = \frac{\phi^*(\tilde{\omega}^\alpha)}{x+\epsilon};$$

Then

$$\{\omega^0, \omega^1, \ldots, \omega^k, \omega^{k+1}, \ldots, \omega^n\}$$

is an orthonormal coframe for $g(\epsilon)$.

After some computation we obtain

$$\begin{aligned}
d\omega^0 &= 0 \\
&\equiv \omega^\alpha \wedge \omega_\alpha^0 + \omega^\mu \wedge \omega_\mu^0 \\
d\omega^\alpha &= -\frac{dx}{(x+\epsilon)^2} \wedge \phi^*(\tilde{\omega}^\alpha) + \frac{dx}{(x+\epsilon)} \wedge (\phi^*(\tilde{\omega}^\alpha))' + \omega^\beta \wedge \phi^*(\tilde{\omega}_\beta^\alpha) \\
&\equiv \omega^0 \wedge \omega_0^\alpha + \omega^\beta \wedge \omega_\beta^\alpha + \omega^\mu \wedge \omega_\mu^\alpha \\
d\omega^\mu &= dx \wedge (\omega^\mu)' + (x+\epsilon)\omega^\alpha \wedge E_\alpha^\mu + \omega^\nu \wedge E_\nu^\mu \\
&\equiv \omega^0 \wedge \omega_0^\mu + \omega^\alpha \wedge \omega_\alpha^\mu + \omega^\nu \wedge \omega_\nu^\mu.
\end{aligned}$$

Here the $'$ denotes differentiation with respect to x, and E_i^j denotes terms (involving the curvature and second fundamental form of the fibres) which are uniformly bounded (with respect to the unscaled metric $\tilde{g}$ on ∂X) along with their derivatives as $x, \epsilon \to 0$.

More specifically, in the formula for $d\omega^\alpha$, we use that d commutes with ϕ^*. The expression for $d\omega^\mu$ contains ω^α factors corresponding to the derivative of the fibre metric in the horizontal direction, and also to the variation of the horizontal subspaces in the fibre direction. We refer to [4] §5.3.1 (particularly (43)-(45)) for the precise details, but note simply that the $\omega^\alpha \wedge \omega^\nu$ components correspond to the second fundamental form in the normal direction e_α to the fibre (with respect to the scaled metric on ∂X for a given x and ϵ), and are indeed of the form $(x+\epsilon)E^\mu_\alpha$, while the $\omega^\alpha \wedge \omega^\beta$ components in $d\omega^\mu$ correspond to the curvature of the horizontal distribution, which are of the form $(x+\epsilon)^2 E^\mu_\alpha$, hence even lower order. Next, the terms E^μ_ν are precisely the connection one-forms ω^μ_ν for the metric induced by κ on the fibres; in particular, these do not involve any ω^α factors. Finally, we have included the extra terms involving the x derivative of $\tilde{\omega}^\alpha$ and ω^μ since we do allow the metric h on Y and symmetric two-tensor κ to depend smoothly on x.

Using this same E^j_i notation for all 'negligible' bounded terms, we now claim that

$$\omega^\alpha_0 = -\frac{x}{x+\epsilon}\tilde{\omega}^\alpha + xE^\alpha_0, \qquad \omega^\mu_0 = x(x+\epsilon)E^\mu_0,$$
$$\omega^\beta_\alpha = E^\beta_\alpha, \qquad \omega^\mu_\alpha = (x+\epsilon)E^\mu_\alpha, \qquad \omega^\nu_\mu = E^\nu_\mu.$$

To verify this, we simply need to show that these forms satisfy the structure equations and are skew-symmetric in their indices, for then Cartan's lemma guarantees uniqueness. The equations for all terms except the ω^μ_α (which by skew-symmetry, we *require* to be equal to $-\omega^\alpha_\mu$) are clear enough. For these terms, first note that the equation for $d\omega^\alpha$ has no vertical components, which means that $\omega^\mu \wedge \omega^\alpha_\mu$ must vanish. This means that

$$\omega^\alpha_\mu = c_{\alpha,\mu,\nu}\omega^\nu, \qquad \text{and} \qquad c_{\alpha,\mu,\nu} = c_{\alpha,\nu,\mu}.$$

(The point is that there can be no ω^β or ω^0 components.) Finally, setting this into the equation for $d\omega^\mu$, and noting that the E^ν_ν term is already accounted for by the ω^μ_ν, we must have $\omega^\mu_\alpha = (x+\epsilon)E^\mu_\alpha$, as claimed.

When computing each of the curvature two-forms Ω^j_i, we write all forms in terms of dx, $\tilde{\omega}^\alpha$ and ω^μ, which are smooth in the ordinary sense up to $\epsilon = x = 0$. We single out the particular terms which help or hurt us, and as before gather all the harmless remaining factors into terms F^j_i, which are uniformly bounded in $x, \epsilon \geq 0$. Thus, after further work, we obtain

$$\begin{aligned}
\Omega^\alpha_0 &= dx \wedge \left(-\frac{\epsilon}{(x+\epsilon)^2}\tilde{\omega}^\alpha + F^\alpha_0\right) + xF^\alpha_0, \\
\Omega^\mu_0 &= (x+\epsilon)F^\mu_0, \\
\Omega^\beta_\alpha &= F^\beta_\alpha, \\
\Omega^\mu_\alpha &= dx \wedge F^\mu_\alpha + (x+\epsilon)F^\mu_\alpha, \\
\Omega^\nu_\mu &= F^\nu_\mu.
\end{aligned}$$

Only the first of these requires more explanation. We have

$$d\omega^\alpha_0 - \omega^\beta_0 \wedge \omega^\alpha_\beta - \omega^\mu_0 \wedge \omega^\alpha_\mu$$
$$= d\left(-\frac{x}{x+\epsilon}\right) \wedge \tilde{\omega}^\alpha + dx \wedge F^\alpha_0 - \frac{x}{x+\epsilon}\left(d\tilde{\omega}^\alpha - \tilde{\omega}^\beta \wedge \tilde{\omega}^\alpha_\beta\right) + xF^\alpha_0.$$

The first terms on the right, involving dx, and the final term, correspond to the assertion above. The middle terms appear not to be of the correct form, but the particular combination in parentheses is just the structure equation for the connection one-forms and hence vanishes.

Recalling that $\dim X = n+1 = 2m$, the integral $\int_X \widehat{A}(g(\epsilon))$ is a linear combination of terms of the form:

$$\int_X \operatorname{Tr} R^{m_1}(\epsilon)\dots\operatorname{Tr} R^{m_p}(\epsilon), \qquad m_1+\dots+m_p = m.$$

To fix the ideas and simplify the notation we focus on

$$\int_X \operatorname{Tr} R^m(\epsilon),$$

since all other terms are handled the same way. In terms of the curvature two-forms,

$$\operatorname{Tr} R^m(\epsilon) \;=\; \sum \Omega_{i_1}^{i_2}\Omega_{i_2}^{i_3}\cdots\Omega_{i_m}^{i_1}. \tag{3.6}$$

Now substitute in this the expressions we have obtained for the Ω_i^j. Using the boundedness of all of the E_i^j, the only terms in any of these curvature forms which is not bounded near $x=\epsilon=0$ is Ω_0^α, and in fact only its first term $\epsilon(x+\epsilon)^{-2}dx\wedge\tilde{\omega}^\alpha$ causes difficulties. Thus we may as well suppose that this is the first term, i.e. $i_1=0$ and $i_2=\alpha$, and we can replace the entire two-form Ω_0^α by this single bad term. The final term in the entire product is either Ω_μ^0 or Ω_β^0 for some μ or β. In the former case this contains a vanishing factor $(x+\epsilon)$, while in the latter, only the part of this two-form which does not contain a dx contributes, and this has the same vanishing factor. Thus in all cases, the entire $2m$-form is bounded (though not necessarily smooth!) near $\epsilon=x=0$, and we can pass to the limit, as desired. ■

3.2 Adiabatic limit of the eta invariant

We briefly recall the context of the Bismut-Cheeger theorem [3]. Let M be an odd dimensional, compact spin manifold which is the total space of a fibration

$$Z \to M \xrightarrow{\phi} Y$$

where the base Y is also spin. We fix a connection $TM = T_H(M)\oplus T(M/Y)$, where $T_H(M)\simeq\phi^*(TY)$ and $T(M/Y)$ denotes the vertical tangent bundle. Let h be a Riemannian metric on Y and κ a symmetric two-tensor on TM which restricts to a metric on each Z_y and which annihilates the horizontal space, and introduce the Riemannian submersion metric $\tilde{g} := \phi^*h+\kappa$. Denote by ∇ and $\nabla^{M/Y}$ the Levi-Civita connection for g_M and the induced connection on $T(M/Y)$ obtained by compressing ∇ by the projections $P: TM\to T(M/Y)$. Let S be the vertical spinor bundle and $E\to M$ an additional Hermitian bundle endowed with a unitary connection. The bundle $F := S\otimes E$ is a vertical Clifford module. Finally, let $\mathbb{F} := \phi^*(\Lambda^*Y)\otimes F$. To fix the notation we assume that the fibers are even dimensional.

To this entire set of data one associates the rescaled Bismut superconnection

$$\mathbb{A}_t : C^\infty(M,\mathbb{F})\to C^\infty(M,\mathbb{F})\,,$$

cf. [2] and [1]. The operator $d\mathbb{A}_t/dt \exp(-\mathbb{A}_t^2)$ is a vertical family of smoothing operators $(\mathcal{K}_y)_{y\in Y}$ with coefficients which are differential forms on the base Y. From this family one obtains a differential form of odd degree on the base Y,

$$\mathrm{Str}\left(\frac{d\mathbb{A}_t}{dt}\exp(-\mathbb{A}_t^2)\right).$$

The value of this form at $y \in Y$ is obtained by restricting $\mathcal{K}_y$ to the diagonal $\Delta_y \subset \phi^{-1}(y) \times \phi^{-1}(y)$, taking its supertrace and then integrating over Δ_y.

Assume now that the vertical family of Dirac operators $(D_y)_{y\in Y}$ associated to the data above is invertible. Then the integral

$$\int_0^\infty \frac{1}{\sqrt{\pi}}\mathrm{Str}\left(\frac{d\mathbb{A}_t}{dt}\exp(-\mathbb{A}_t^2)\right) dt$$

converges and defines the eta form $\widetilde{\eta} \in C^\infty(Y, \Lambda^* Y)$ associated to the family $(D_y)_{y\in Y}$.

The adiabatic limit formula of Bismut and Cheeger states that if $\eta(\epsilon)$ is the eta invariant for the Dirac operator associated to the metric

$$g_M(\epsilon) := \frac{\phi^* h}{\epsilon^2} + \kappa\,;$$

then

$$\lim_{\epsilon\to 0}\eta(\epsilon) = \int_Y \widehat{A}(Y,h) \wedge \widetilde{\eta}\,.$$

Applied to the boundary operator $D^{\partial}_{g(\epsilon)}$ on $M = \partial X$, we obtain the limiting behaviour of the final term in (3.2). This completes the proof of the index formula for Dirac operators associated to $\Phi-$metrics in the fully elliptic case.

References

[1] Nicole Berline, Ezra Getzler, and Michèle Vergne. *Heat kernels and Dirac operators*, volume 298 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992.

[2] Jean-Michel Bismut. The Atiyah-Singer index theorem for families of Dirac operators: two heat equation proofs. *Invent. Math.*, **83**(1):91–151, 1985.

[3] Jean-Michel Bismut and Jeff Cheeger. η-invariants and their adiabatic limits. *J. Amer. Math. Soc.*, **2**(1):33–70, 1989.

[4] Tamás Hausel, Eugenie Hunsicker, and Rafe Mazzeo. Hodge cohomology of gravitational instantons. *Duke Math. J.*, **122**(3):485–548, 2004.

[5] Robert Lauter and Sergiu Moroianu. Homology of pseudodifferential operators on manifolds with fibered cusps. *Trans. Amer. Math. Soc.*, **355**(8):3009–3046 (electronic), 2003.

[6] Robert Lauter and Sergiu Moroianu. An index formula on manifolds with fibered cusp ends. *J. Geom. Anal.*, **15**(2):261–283, 2005.

[7] Rafe Mazzeo and Richard B. Melrose. Pseudodifferential operators on manifolds with fibred boundaries. *Asian J. Math.*, **2**(4):833–866, 1998. Mikio Sato: a great Japanese mathematician of the twentieth century.

[8] Richard Melrose and Frédéric Rochon. Index in k-theory for families of fibred cusp operators. Preprint 2005.

[9] Richard B. Melrose. *The Atiyah-Patodi-Singer index theorem*, volume 4 of *Research Notes in Mathematics*. A K Peters Ltd., Wellesley, MA, 1993.

[10] Sergiu Moroianu. Fibered cusps versus d-index theory. preprint 2006.

[11] Werner Müller. *Manifolds with cusps of rank one*, volume 1244 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1987. Spectral theory and L^2-index theorem.

[12] Tom M. W. Nye and Michael A. Singer. An L^2-index theorem for Dirac operators on $S^1 \times \mathbf{R}^3$. *J. Funct. Anal.*, **177**(1):203–218, 2000.

[13] Michael Spivak. *A comprehensive introduction to differential geometry. Vol. II.* Publish or Perish Inc., Wilmington, Del., second edition, 1979.

[14] Boris Vaillant. Index- and spectral theory for manifolds with generalized fibred cusps, 2001. Dissertation, Rheinische Friedrich-Wilhelms-Universität Bonn, Bonn, 2001.

Eric Leichtnam
Institut de Jussieu et CNRS
175 Rue du Chevaleret
75013 Paris

Rafe Mazzeo
Department of Mathematics Stanford University
Stanford, CA 94305 USA
mazzeo@math.stanford.edu

Paolo Piazza
Dipartimento Di Matematica G. Castelnuovo
Universita Di Roma "La Sapienza"
P.Le Aldo Moro
00185 Rome (Italy)