Tohoku Math. J.
61 (2009), 253–265

THE INDEX OF ELLIPTIC UNITS IN Z_p-EXTENSIONS, II

HASSAN OUKHABA

(Received September 25, 2007, revised December 15, 2008)

Abstract. In this paper we continue to explore the index of elliptic units. In a previous article we determined the asymptotic behavior in $\boldsymbol{Z}_p$-extensions of the p-part of this index divided by the p-part of the ideal class number. We proved the existence of an invariant μ_∞ which governs this behavior, and gave sufficient conditions for the vanishing of μ_∞. Here we give examples with nonzero μ_∞, especially in the case of anticyclotomic $\boldsymbol{Z}_p$-extensions.

1. Introduction. Let k be an imaginary quadratic field and let H be the Hilbert class field of k . Let F be a finite abelian extension of k and let $\mathcal{O}_F$ (resp. $\mathcal{O}_F^\times$) be the ring of integers (resp. the group of units) of F. Let us also denote by $\mathcal{C}_F$ the group of elliptic units of F defined by Rubin in [12]. Let h_F be the ideal class number of F. In [7] we studied the behavior of the quotient $[\mathcal{O}_K^\times : \mathcal{C}_K]/h_K$, where K runs through all the finite extensions of k containing F and contained in a given $\boldsymbol{Z}_p$-extension of F abelian over k. We proved the following result. Let p be a prime number and let F_∞ be a $\boldsymbol{Z}_p$-extension of F abelian over k. If n is a nonnegative integer, then we let F_n be the unique subextension of F_∞/F of degree p^n over F. If A is a positive integer, then we denote by A_p the p-part of A, that is, the exact power of p that divides A. If $H \subset F$, then there exist $\mu_\infty \in \boldsymbol{N}$ and $\nu_\infty \in \boldsymbol{Z}$ such that

$$[\mathcal{O}_{F_n}^\times : \mathcal{C}_{F_n}]_p = p^{\mu_\infty p^n + \nu_\infty}(h_{F_n})_p\,,$$

for all sufficiently large n. Further, a sufficient condition to have $\mu_\infty = 0$ may be stated as follows. Let $S_{F_\infty,F}$ be the set of prime ideals of $\mathcal{O}_k$ that ramify in F/k but not in F_∞/F. If the decomposition group of $\mathfrak{q}$ in F_∞/k is infinite for all $\mathfrak{q} \in S_{F_\infty,F}$, then we have $\mu_\infty = 0$. Moreover there exists $c_{F_\infty} \in \boldsymbol{Q}^\times$ such that

$$[\mathcal{O}_{F_n}^\times : \mathcal{C}_{F_n}] = c_{F_\infty} h_{F_n}\,,$$

for all sufficiently large n. Let us observe that, in all the cases where the classical Iwasawa main conjecture for imaginary quadratic fields is well formulated and proved, we have $\mu_\infty = 0$. Rubin has proved this conjecture in the semi-simple case, cf. [12], and Bley proved it in a more general context, cf. [1]. One may ask if our μ_∞ has any interpretation in terms of the μ-invariant of the objects of Iwasawa main conjecture.

The aim of this paper is to give examples for which μ_∞ is not zero. Indeed, under some additional hypotheses, we are able to compute μ_∞ when $S_{F_\infty,F}$ contains exactly three prime ideals, all of them split completely in F_∞/F. We conclude by giving numerical examples in the case of the anticyclotomic $\boldsymbol{Z}_p$-extensions.

2000 *Mathematics Subject Classification.* Primary 11G16; Secondary 11R23.

1.1. Notation. All our number fields are considered as subfields of $\boldsymbol{C}$, the field of complex numbers. If F is a finite abelian extension of k, then we denote by μ_F the group of roots of unity in F and by w_F its order. Let $\mathfrak{a}$ be a fractional ideal of k. If $\mathfrak{a}$ is integral then we let $\hat{\mathfrak{a}}$ be the product of the nonzero prime ideals of $\mathcal{O}_k$ that divides $\mathfrak{a}$. If $\mathfrak{a}$ is prime to the conductor of F/k, then we denote by $(\mathfrak{a}, F/k)$ the automorphism of F/k associated to $\mathfrak{a}$ by the Artin map. Usually we shall use G_F to denote the group $\mathrm{Gal}(F/k)$. The inertia group in F/k of a non zero prime ideal $\mathfrak{p}$ of $\mathcal{O}_k$ will be denoted $T_{\mathfrak{p}}(F)$. Let $\mathfrak{m}$ be a nonzero ideal of $\mathcal{O}_k$. Then we denote by $k_{\mathfrak{m}}$ the ray class field of k modulo $\mathfrak{m}$. We denote by $N(\mathfrak{m})$ (resp. $e_{\mathfrak{m}}$) the cardinal number of the ring $\mathcal{O}_k/\mathfrak{m}$ (resp. $\mathbf{Z}/\mathbf{Z} \cap \mathfrak{m}$). The number of roots of unity in k that are equivalent to 1 modulo $\mathfrak{m}$ will be denoted by $r_{\mathfrak{m}}$. The cardinal number of a finite set X will be denoted $\#X$ or $|X|$ as well.

2. Elliptic units. Let F be a finite abelian extension of k. In this section we recall the definition of $\mathcal{C}_F$ and give the index formula (12). As we shall see below, this index formula uses an intermediate group of elliptic units which we denote Ω_F.

2.1. The group Ω_F. The "discriminant-quotients" are one of the ingredients used to construct elliptic units. Recall that the discriminant of a lattice L of $\boldsymbol{C}$ is

$$\Delta(L) = g_2(L)^3 - 27g_3(L)^2 ,$$

i.e., the discriminant of the equation

$$\wp'(z, L)^2 = 4\wp(z, L)^3 - g_2(L)\wp(z, L) - g_3(L) ,$$

satisfied by the Weierstrass $\wp$-function $\wp(z, L)$ and its derivative $\wp'(z, L)$. It is well known that the theory of modular functions and the Shimura reciprocity law have the following important consequence. For all fractional ideal $\mathfrak{a}$ of k, the quotient

$$\frac{\Delta(\mathcal{O}_k)}{\Delta(\mathfrak{a})} \tag{1}$$

is in H. Moreover, if $\tau \in \mathrm{Gal}(H/k)$ then

$$\left(\frac{\Delta(\mathcal{O}_k)}{\Delta(\mathfrak{a})}\right)^{\tau} = \frac{\Delta(\mathfrak{b})}{\Delta(\mathfrak{a}\mathfrak{b})} ,$$

where $\mathfrak{b}$ is any fractional ideal of k satisfiying $(\mathfrak{b}, H/k) = \tau^{-1}$. See for instance [5, chap.11 Corollary and chap.12 Theorem 5]. Let us denote by Q the subgroup of $H^{\times}$ generated by all the quotients

$$\frac{\Delta(\mathfrak{a})}{\Delta(\mathfrak{b})} ,$$

where $\mathfrak{a}$ and $\mathfrak{b}$ run through the set of fractional ideals of k. Let σ be in $\mathrm{Gal}(H/k)$, $\mathfrak{a}$ a fractional ideal of k and $x \in k$ be chosen so that $(\mathfrak{a}, H/k) = \sigma^{-1}$ and $\mathfrak{a}^h = x\mathcal{O}_k$ ($h = h_k = [H : k]$). Then the number

$$\varphi_{(1)}(\sigma) = x^{12}\Delta(\mathfrak{a})^h$$

depends only upon σ. This invariant will appear in the Kronecker limit formula (6).

Robert-Ramachandra invariants are also important ingredients in the construction of elliptic units. To define them, it is necessary to introduce first the Weierstrass σ-function $\sigma(z, L)$ defined for a lattice L of $\boldsymbol{C}$ by the infinite product

$$\sigma(z, L) = z \prod_{\omega \in L} \left(1 - \frac{z}{\omega}\right) e^{z/\omega + (z/\omega)^2/2} .$$

This is a holomorphic function on all $\boldsymbol{C}$ with simple zeros at the points of L and no other zeros. The logarithmic derivative $\zeta(z, L)$ of $\sigma(z, L)$ is called the Weierstrass ζ-function. It is equal to the infinite sum

$$\zeta(z, L) = \frac{d \log(\sigma(z, L))}{dz} = \frac{1}{z} + \sum_{\omega \in L} \left(\frac{1}{z - \omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right),$$

which converges absolutely and uniformly on every compact subset of $\boldsymbol{C} - L$. Thus, $\zeta(z, L)$ is a meromorphic function on $\boldsymbol{C}$ with simple poles at the points of L and no other poles. We have the identity

$$\frac{d\zeta(z, L)}{dz} = -\wp(z, L) .$$

Since $\wp(z, L)$ is elliptic with respect to L, there exists a group homomorphism $\eta(\cdot, L) : L \to \boldsymbol{C}$ such that

$$\eta(\omega, L) = \zeta(z + \omega, L) - \zeta(z, L)$$

for all $\omega \in L$ and all $z \in \boldsymbol{C} - L$. Let us extend $\eta(\cdot, L)$ to the field $\boldsymbol{C}$ to obtain an $\boldsymbol{R}$-linear map from $\boldsymbol{C}$ into itself, still denoted $\eta(\cdot, L)$. Then Robert-Ramachandra invariants are defined by using the function

$$\varphi(z, L) = (e^{-\eta(z,L)z/2} \sigma(z, L))^{12} \Delta(L) . \tag{2}$$

Indeed, let $\mathfrak{m} \neq (1)$ be a nonzero proper ideal of $\mathcal{O}_k$. Let σ be an element of $\mathrm{Gal}(k_{\mathfrak{m}}/k)$ and $\mathfrak{a}$ be a nonzero ideal of $\mathcal{O}_k$ prime to $\mathfrak{m}$ such that $\sigma = (\mathfrak{a}, k_{\mathfrak{m}}/k)$, then

$$\varphi_{\mathfrak{m}}(\sigma) = \varphi(1, \mathfrak{a}^{-1}\mathfrak{m})^{e_{\mathfrak{m}}} \tag{3}$$

is nonzero and depends only on σ. We call $\varphi_{\mathfrak{m}}(\sigma)$ the Robert-Ramachandra invariant associated to σ. Here also the theory of modular functions and Shimura reciprocity law give

$$\varphi_{\mathfrak{m}}(\sigma) \in \mathcal{O}_{k_{\mathfrak{m}}} \quad \text{and} \quad \varphi_{\mathfrak{m}}(\sigma)^{\sigma'} = \varphi_{\mathfrak{m}}(\sigma\sigma') , \tag{4}$$

for all $\sigma, \sigma' \in \mathrm{Gal}(k_{\mathfrak{m}}/k)$. See [5, Chap. 19, p. 263, Theorem 2] to get integrality and [ibid. p. 265, Theorem 3] to get the galois action. In [8], G. Robert shows that the functions $\varphi(z, L)$ satisfy very remarkable distribution relations from which he deduces, thanks to (4), the following norm formulas. Let $\mathfrak{q}$ be a non zero prime ideal of $\mathcal{O}_k$. Then we have

$$N_{k_{\mathfrak{mq}}/k_{\mathfrak{m}}}(\varphi_{\mathfrak{mq}}(1))^{r_{\mathfrak{m}}/r_{\mathfrak{mq}}} = \begin{cases} \varphi_{\mathfrak{m}}(1)^{e_{\mathfrak{mq}}/e_{\mathfrak{m}}} , & \text{if } \mathfrak{q} | \mathfrak{m} \\ [\varphi_{\mathfrak{m}}(1)]^{(e_{\mathfrak{mq}}/e_{\mathfrak{m}})(1 - (\mathfrak{q}, k_{\mathfrak{m}}/k)^{-1})} , & \text{if } \mathfrak{q} \nmid \mathfrak{m} \text{ and } \mathfrak{m} \neq (1) \\ \left(\dfrac{\Delta(\mathcal{O}_k)}{\Delta(\mathfrak{q})} \right)^{e_{\mathfrak{q}}} , & \text{if } \mathfrak{m} = (1) . \end{cases}$$

Since Robert-Ramachandra invariants are algebraic integers, we deduce from the above that $\varphi_{\mathfrak{m}}(1)$ is a unit of $k_{\mathfrak{m}}$ if $\mathfrak{m}$ is divisible by at least two prime ideals. If $\mathfrak{m} = \mathfrak{q}^e$, where $\mathfrak{q}$ is a prime ideal of $\mathcal{O}_k$, then

$$\varphi_{\mathfrak{m}}(1)\mathcal{O}_{k_{\mathfrak{m}}} = \mathfrak{q}_{\mathfrak{m}}^{u}\,, \tag{5}$$

where $\mathfrak{q}_{\mathfrak{m}}$ is the product of the prime ideals of $k_{\mathfrak{m}}$ that divide $\mathfrak{q}$ and $u = 12r_{\mathfrak{m}}e_{\mathfrak{m}}/w_k$.

The last property we would like to recall are the Kronecker limit formulas which may be stated as follows. Let us set $h_{\mathfrak{m}} = h$ if $\mathfrak{m} = (1)$ and $h_{\mathfrak{m}} = 1$ otherwise. Let χ be a nontrivial complex character of $\mathrm{Gal}(k_{\mathfrak{m}}/k)$. Then we have

$$L'(0,\chi) = -\frac{1}{12r_{\mathfrak{m}}e_{\mathfrak{m}}h_{\mathfrak{m}}}\sum_{\sigma\in G_{\mathfrak{m}}}\chi(\sigma)\log(|\varphi_{\mathfrak{m}}(\sigma)|^2)\,, \tag{6}$$

where $G_{\mathfrak{m}} = \mathrm{Gal}(k_{\mathfrak{m}}/k)$ and $s \mapsto L(s,\chi)$ is the L-function associated to χ defined, for the complex numbers s such that $\mathrm{Re}(s) > 1$, by the Euler product

$$L(s,\chi) = \prod_{\mathfrak{l}\nmid\mathfrak{m}}(1-\chi(\mathfrak{l})N(\mathfrak{l})^{-s})^{-1}\,,$$

where $\mathfrak{l}$ runs through all the non zero prime ideals of $\mathcal{O}_k$ not dividing $\mathfrak{m}$ (cf. [2]).

Let $\mathfrak{f}$ be the conductor of the extension F/k. If $\mathfrak{f} \neq (1)$ then for all ideal $\mathfrak{g} \neq (1)$ of $\mathcal{O}_k$ that divides $\mathfrak{f}$, we set

$$\varphi_{F,\mathfrak{g}} = N_{k_{\mathfrak{g}}/k_{\mathfrak{g}}\cap F}(\varphi_{\mathfrak{g}}(1))^{e(\mathfrak{f},\mathfrak{g})}\,,\quad e(\mathfrak{f},\mathfrak{g}) = \frac{w_k e_{\mathfrak{f}}}{r_{\mathfrak{g}}e_{\mathfrak{g}}}\,. \tag{7}$$

The algebraic integers $(\varphi_{F,\mathfrak{g}})^h$ are introduced for the first time in [3, p. 307]. Kubert and Lang call them Kersey invariants. One may consider them as suitable normalisations of Robert-Ramachandra invariants. Let $\mathfrak{q}$ be a prime ideal of $\mathcal{O}_k$ such that $\mathfrak{g}\mathfrak{q}|\mathfrak{f}$. Then we have

$$N_{k_{\mathfrak{g}\mathfrak{q}}\cap F/k_{\mathfrak{g}}\cap F}(\varphi_{F,\mathfrak{g}\mathfrak{q}}) = \begin{cases} \varphi_{F,\mathfrak{g}}\,, & \text{if } \mathfrak{q}|\mathfrak{g} \\ [\varphi_{F,\mathfrak{g}}]^{1-(\mathfrak{q},k_{\mathfrak{g}}\cap F/k)^{-1}}\,, & \text{if } \mathfrak{q}\nmid\mathfrak{g} \text{ and } \mathfrak{g}\neq(1) \\ N_{H/H\cap F}\left(\dfrac{\Delta(\mathcal{O}_k)}{\Delta(\mathfrak{q})}\right)^{e_{\mathfrak{f}}}\,, & \text{if } \mathfrak{g}=(1)\,. \end{cases} \tag{8}$$

Now we are ready to define the group Ω_F.

DEFINITION 2.1. Let us set $Q_F = N_{H/H\cap F}(Q)^{e_{\mathfrak{f}}}$ and let $\mathcal{P}_F$ be the galois submodule of $F^\times$ generated by μ_F, Q_F and all $\varphi_{F,\mathfrak{g}}$ with $\mathfrak{g}|\mathfrak{f}$ and $\mathfrak{g}\neq(1)$. Then we let

$$\Omega_F = \mathcal{P}_F \cap \mathcal{O}_F^\times\,.$$

2.2. The group $\mathcal{C}_F$. The most elegant definition of $\mathcal{C}_F$ uses the elliptic functions $\Psi(\cdot\,;L,L') : z \mapsto \Psi(z;L,L')$ introduced by G. Robert in [9] and [11], parametrized by the pairs of lattices (L,L') of $\boldsymbol{C}$ such that $L\subset L'$ and $[L':L]$ is prime to 6. It is interesting to compare Robert's definition of $\Psi(z;L,L')$ with that proposed by D. Kubert in [4]. We do not give here any of these definitions. Instead, we recall some significant properties of special values taken by the functions $\Psi(\cdot\,; L, L')$ when L and L' are fractional ideals of k of a certain type. See [10] and [11].

Let $\mathfrak{m} \neq (1)$ be a proper nonzero ideal of $\mathcal{O}_k$. Let $\mathfrak{a}$ be a nonzero ideal of $\mathcal{O}_k$ prime to $6\mathfrak{m}$, then $\Psi(1; \mathfrak{m}, \mathfrak{a}^{-1}\mathfrak{m})$ is in $k_{\mathfrak{m}}$. Moreover, we have

$$\Psi(1; \mathfrak{m}, \mathfrak{a}^{-1}\mathfrak{m})^{12e_{\mathfrak{m}}} = \varphi_{\mathfrak{m}}(1)^{N(\mathfrak{a})-(\mathfrak{a}, k_{\mathfrak{m}}/k)} . \tag{9}$$

If $\mathfrak{m} = \mathfrak{q}$ is a nonzero prime ideal of $\mathcal{O}_k$, then the value $\Psi(1; \mathfrak{q}, \mathfrak{a}^{-1}\mathfrak{q})$ is related to the "discriminant-quotients" by the formula

$$N_{k_{\mathfrak{q}}/H}(\Psi(1; \mathfrak{q}, \mathfrak{a}^{-1}\mathfrak{q}))^{12w_k/r_{\mathfrak{q}}} = \left(\frac{\Delta(\mathcal{O}_k)}{\Delta(\mathfrak{q})}\right)^{N(\mathfrak{a})-(\mathfrak{a}, H/k)} . \tag{10}$$

Now we have the necessary materials for the definition of $\mathcal{C}_F$. For each nonzero integral ideal $\mathfrak{m} \neq (1)$ of $\mathcal{O}_k$, we define $\mathcal{C}_{F,\mathfrak{m}}$ to be the subgroup of $\mathcal{O}_F^{\times}$ generated by μ_F and the norms

$$N_{k_{\mathfrak{m}}/k_{\mathfrak{m}}\cap F}(\Psi(1; \mathfrak{m}, \mathfrak{a}^{-1}\mathfrak{m}))^{\sigma-1} ,$$

where σ is in $\mathrm{Gal}(F/k)$ and $\mathfrak{a}$ runs through the set of all nonzero integral ideals of k prime to $6\mathfrak{m}$.

DEFINITION 2.2. We denote by $\mathcal{C}_F$ the subgroup of $\mathcal{O}_F^{\times}$ generated by all the $\mathcal{C}_{F,\mathfrak{m}}$ with $\mathfrak{m} \neq (1)$. Also we set $V_F = \mu_F \mathcal{C}_F^{12w_k e_{\mathfrak{f}}}$, where $\mathfrak{f}$ is the conductor of F.

Let us denote by R_F the abelian group ring $\boldsymbol{Z}[G_F]$. As we will see below, the $\boldsymbol{Q}$-algebra $\boldsymbol{Q}[G_F]$ contains an R_F-submodule U_F which is closely related to our groups of elliptic units. The investigation of U_F is the key step not only in the proof of the existence of μ_∞ but also in its computation in some special cases. To introduce U_F, we need more notations. If D is a subgroup of G_F, then we set

$$s(D) = \sum_{\sigma \in D} \sigma \in R_F .$$

Let $\mathfrak{p}$ be a nonzero prime ideal of $\mathcal{O}_k$ and let $F_{\mathfrak{p}}$ be a Frobenius automorphism at $\mathfrak{p}$ in F/k. Then we define

$$(\mathfrak{p}, F) = F_{\mathfrak{p}}^{-1} \frac{s(T_{\mathfrak{p}}(F))}{\# T_{\mathfrak{p}}(F)} .$$

For all nonzero ideal $\mathfrak{r} \neq (1)$ of $\mathcal{O}_k$, we denote by $T_{\mathfrak{r}}(F)$ the subgroup of G_F generated by the inertia groups $T_{\mathfrak{p}}(F)$ with $\mathfrak{p}|\mathfrak{r}$. If $\mathfrak{r} = (1)$ then we set $T_{(1)}(F) = \{1\}$.

DEFINITION 2.3. Let $\mathfrak{f}$ be the conductor of F/k. Let $\mathfrak{s}$ be a divisor of $\hat{\mathfrak{f}}$. If $\mathfrak{s} \neq (1)$, then we denote by $U_{\mathfrak{s}}$ or $U_{\mathfrak{s},F}$ the R_F-submodule of $\boldsymbol{Q}[G_F]$ generated by all the elements

$$\alpha(\mathfrak{r}, \mathfrak{s}) = s(T_{\mathfrak{r}}(F)) \prod_{\mathfrak{p}|\mathfrak{s}/\mathfrak{r}} (1 - (\mathfrak{p}, F)) , \quad \mathfrak{r}|\mathfrak{s} .$$

Moreover we set $U_{(1)} = U_{(1),F} = R_F$ and $U = U_F = U_{\hat{\mathfrak{f}},F}$.

To go further, we need to recall the definition of Sinnott's generalized index. Let E be a $\boldsymbol{Q}$-vector space of finite dimension d; and let M and N be two lattices of E, that is two free $\boldsymbol{Z}$-submodules of E, of rank d. Then we define the index $(M:N)$ by

$$(M:N)=|\det(\gamma)|$$

where γ is any endomorphism of the $\boldsymbol{Q}$-vector space E such that $\gamma(M)=N$. If $N\subset M$ then $(M:N)$ coincides with the usual index $[M:N]$. We also have the following transitivity formula

$$(M:P)=(M:N)(N:P)\,.$$

This leads to the identity

$$(M:N)=\frac{[M+N:N]}{[M+N:M]}\,,$$

which one may use as a definition of $(M:N)$. We refer the reader to [13] for more details about this generalized index. Here we are concerned with the R_F-modules $U_{\mathfrak{s},F}$. One may prove, exactly as in [13, Lemma 5.1], that $U_{\mathfrak{s},F}$ is a lattice of $\boldsymbol{Q}[G_F]$. Moreover, if $\mathfrak{q}$ is a prime ideal of $\mathcal{O}_k$ that divides $\mathfrak{f}$ but does not divide $\mathfrak{s}$, then the index $(U_{\mathfrak{s},F}:U_{\mathfrak{s}\mathfrak{q},F})$, which is well-defined, is a positive integer whose set of prime divisors is contained in the set of prime divisors of $\#T_{\mathfrak{q}}(F)$. Thus, if $\mathfrak{s}_0,\dots,\mathfrak{s}_e$ are the ideals defined by the relation $\mathfrak{s}_0:=(1)$ and $\mathfrak{s}_{i+1}:=\mathfrak{s}_i\mathfrak{p}_{i+1}$, where $\mathfrak{p}_1,\dots,\mathfrak{p}_e$ are the prime divisors of $\mathfrak{f}$, then the decomposition

$$(R:U)=\prod_{i=0}^{e-1}(U_{\mathfrak{s}_i}:U_{\mathfrak{s}_{i+1}})$$

of $(R:U)$ as the product of the indices $(U_{\mathfrak{s}_i}:U_{\mathfrak{s}_{i+1}})$ shows that $(R:U)$ is in $\boldsymbol{N}$. Moreover, if l is a prime number such that $l|(R:U)$, then l divides $\#\mathrm{Gal}(F/F\cap H)$.

In [6], we succeeded in computing the index $[\mathcal{O}_F^\times:\Omega_F]$. We obtained the following formula when $H\subset F$.

$$[\mathcal{O}_F^\times:\Omega_F]=h_F\frac{(12w_ke_{\mathfrak{f}})^{[F:k]-1}}{w_F/w_k}\frac{\prod_{\mathfrak{p}}[F_{\mathfrak{p}^\infty}:H]}{[F:H]}(R_F:U_F)\,, \tag{11}$$

where, for every nonzero prime ideal $\mathfrak{p}$ of $\mathcal{O}_k$, $F_{\mathfrak{p}^\infty}$ is the maximal extension of k in F unramified ouside $\mathfrak{p}$. In view of (7) and (10), one may easily check the inclusion $V_F\subset\Omega_F$ and hence the index formula

$$[\mathcal{O}_F^\times:\mathcal{C}_F]=h_F\frac{\prod_{\mathfrak{p}}[F_{\mathfrak{p}^\infty}:H]}{[F:H]}(R_F:U_F)\frac{[\Omega_F:V_F]}{w_F/w_k}\,. \tag{12}$$

3. The index $(R_F:U_{\mathfrak{s},F})$ in a very special case. In this section, we compute the index $(R_F:U_{\mathfrak{s},F})$ for the ideals $\mathfrak{s}=\mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3$, where $\mathfrak{q}_1,\mathfrak{q}_2$ and $\mathfrak{q}_3$ are prime ideals of $\mathcal{O}_k$ satisfying the following four conditions, which we shall denote by $\mathrm{cond}_p(F,d)$.

1. There exists a positive integer d such that the inertia groups $T_{\mathfrak{q}_i}(F)$, $i=1,2$ or 3 are cyclic p-groups of order p^d.
2. We have $T_{\mathfrak{q}_i}(F)\cap T_{\mathfrak{q}_j}(F)=\{1\}$ for all $i\neq j$.
3. The decomposition group $D_{\mathfrak{q}_3}(F)$ of $\mathfrak{q}_3$ in F/k is equal to $T_{\mathfrak{q}_3}(F)$.

4. We have $T_{\mathfrak{s}}(F) \simeq \mathbf{Z}/p^d\mathbf{Z} \times \mathbf{Z}/p^d\mathbf{Z}$.

Let us set $\mathfrak{s}_2 = \mathfrak{q}_1\mathfrak{q}_2$ and $T = T_{\mathfrak{s}_2}(F) = T_{\mathfrak{s}}(F)$. Then we have the decomposition

$$(R_F : U_{\mathfrak{s},F}) = (R_F : U_{\mathfrak{s}_2,F})(U_{\mathfrak{s}_2,F} : U_{\mathfrak{s},F}) .$$

But one may prove that $(R_F : U_{\mathfrak{s}_2,F}) = 1$ exactly as in [13, Proposition 5.2]. Let M be an R_F-submodule of $\boldsymbol{Q}[G_F]$ and let D be a subgroup of G_F. Then the kernel in M of multiplication by $1 - e_D$, where

$$e_D = \frac{s(D)}{\#D} ,$$

is equal to M^D, the maximal R_F-submodule of M on which D acts trivially. If M and N are two R_F-submodules of $\boldsymbol{Q}[G_F]$ such that $(M : N)$ is defined, then the indices $(M^D : N^D)$ and $((1-e_D)M : (1-e_D)N)$ are also defined, and we have the equality

$$(M : N) = (M^D : N^D)((1-e_D)M : (1-e_D)N) ,$$

which we now apply to $D = T_{\mathfrak{q}_3}(F)$, $M = U_{\mathfrak{s}_2,F}$ and $N = U_{\mathfrak{s},F}$. It is easy to check that $(1-e_D)U_{\mathfrak{s}_2,F} = (1-e_D)U_{\mathfrak{s},F}$. Moreover, we have

$$U_{\mathfrak{s},F}^{T_{\mathfrak{q}_3}(F)} = U_{\mathfrak{s}_2,F}(T_{\mathfrak{q}_3}(F)) + (1 - F_{\mathfrak{q}_3})[U_{\mathfrak{s}_2,F}^{T_{\mathfrak{q}_3}(F)}] ,$$

where $U_{\mathfrak{s}_2,F}(T_{\mathfrak{q}_3}(F))$ is the R_F-submodule of $\boldsymbol{Q}[G_F]$ generated by the elements

$$s(T_{\mathfrak{r}\mathfrak{q}_3}(F)) \prod_{\mathfrak{p}|\mathfrak{s}_2/\mathfrak{r}} (1 - (\mathfrak{p}, F)) , \quad \mathfrak{r}|\mathfrak{s}_2 .$$

By the condition 3, we have $(1 - F_{\mathfrak{q}_3})[U_{\mathfrak{s}_2,F}^{T_{\mathfrak{q}_3}(F)}] = 0$. Hence we have

$$(R_F : U_{\mathfrak{s},F}) = (U_{\mathfrak{s}_2,F} : U_{\mathfrak{s},F}) = [A : B] , \tag{13}$$

where A and B are the R_F-modules defined by

$$A = U_{\mathfrak{s}_2,F}^{T_{\mathfrak{q}_3}(F)}/s(T_{\mathfrak{q}_3}(F))U_{\mathfrak{s}_2,F} \quad \text{and} \quad B = U_{\mathfrak{s}_2,F}(T_{\mathfrak{q}_3}(F))/s(T_{\mathfrak{q}_3}(F))U_{\mathfrak{s}_2,F} .$$

In the following we compute the orders of A and B.

LEMMA 3.1. *The inclusion $s(T)R_F \subset U_{\mathfrak{s}_2,F}(T_{\mathfrak{q}_3}(F))$ induces a surjective homomorphim of R_F-modules*

$$h : s(T)R_F \to U_{\mathfrak{s}_2,F}(T_{\mathfrak{q}_3}(F))/s(T_{\mathfrak{q}_3}(F))U_{\mathfrak{s}_2,F} ,$$

whose kernel is equal to the R_F-submodule of $s(T)R_F$ generated by $p^d s(T)$, $\gamma = s(T)(1 - F_{\mathfrak{q}_1})$ and $\delta = s(T)(1 - F_{\mathfrak{q}_2})$.

PROOF. By its very definition and thanks to the conditions 1, 2 and 3, we see that $U_{\mathfrak{s}_2,F}(T_{\mathfrak{q}_3}(F))$ is generated as an R_F-module by the four elements

$$s(T) , \quad \gamma , \quad \delta \quad \text{and} \quad \theta = s(T_{\mathfrak{q}_3}(F))(1 - (\mathfrak{q}_1, F))(1 - (\mathfrak{q}_2, F)) .$$

Furthermore, $s(T_{\mathfrak{q}_3}(F))U_{\mathfrak{s}_2,F}$ is generated by

$$p^d s(T) , \quad \gamma , \quad \delta \quad \text{and} \quad \theta .$$

It is now clear that h is onto. Let $x \in R_F$ be an element satisfying $s(T)x$ is in $\ker h$. We have

$$s(T)x = p^d s(T)a + \gamma b + \delta c + \theta d\,,$$

for some elements $a, b, c, d \in R_F$. In particular, θd is in $\boldsymbol{Q}[G_F]^T$ since γ and δ are elements of $\boldsymbol{Q}[G_F]^T$ by their very definition. But $\theta - s(T_{\mathfrak{q}_3}(F))$ is also in $\boldsymbol{Q}[G_F]^T$ since T is generated by any two inertia groups $T_{\mathfrak{q}_i}(F)$ and $T_{\mathfrak{q}_j}(F)$ for $i \neq j$, as one may deduce from the conditions 1, 2 and 4. Hence, we deduce that $s(T_{\mathfrak{q}_3}(F))d$ is invariant under the action of T. Thus, we have $s(T_{\mathfrak{q}_3}(F))d = s(T)d'$ for some $d' \in R_F$, and then $\theta d = s(T)d'(1 - F_{\mathfrak{q}_1}^{-1})(1 - F_{\mathfrak{q}_2}^{-1})$. The lemma follows. □

LEMMA 3.2. *Let F' be the maximal extension of k in F such that $\mathfrak{q}_1$, $\mathfrak{q}_2$ and $\mathfrak{q}_3$ split completely in F'/k. Let $\overline{\mathfrak{X}} : R_F \to \boldsymbol{Q}[G_{F'}]$ be the R_F-homomorphism which sends $\sigma \in G_F$ to*

$$\overline{\mathfrak{X}}(\sigma) = \frac{1}{p^{2d}}(\sigma_{|F'})\,,$$

where $\sigma_{|F'}$ is the restriction of σ to F'. Let $\mathfrak{X}$ be the restriction of $\overline{\mathfrak{X}}$ to $s(T)R_F$. Then we have $\mathrm{Im}(\mathfrak{X}) = R_{F'}$. *Moreover, $\mathfrak{X}$ induces an isomorphism of R_F-modules*

$$s(T)R_F/\ker(h) \simeq R_{F'}/p^d R_{F'}\,.$$

PROOF. It is obvious that $\mathfrak{X}$ induces a surjective map

$$s(T)R_F/\ker(h) \to R_{F'}/p^d R_{F'}\,.$$

On the other hand, if τ is an automorphism of F'/k and σ is an extension of τ to F, then $s(T)\sigma$ is well-defined modulo $\ker(h)$. Actually, if σ_1 is some other extension of τ to F, then $\sigma_1 = \sigma\theta$ for some θ in $\mathrm{Gal}(F/F')$. But this group is generated by T, $F_{\mathfrak{q}_1}$ and $F_{\mathfrak{q}_2}$. In particular $1 - \theta$ may be written as

$$1 - \theta = (1 - F_{\mathfrak{q}_1})u + (1 - F_{\mathfrak{q}_2})v + (1 - \gamma)w\,,$$

where $u, v, w \in R_F$ and $\gamma \in T$. Thus, $s(T)(\sigma - \sigma_1)$ is in $\ker(h)$. In particular we obtain a surjective **Z**-homomorphism

$$R_{F'} \to s(T)R_F/\ker(h)\,,$$

whose kernel contains $p^d R_{F'}$. This proves the lemma. □

The structure of the R_F-module B is now entirely decided by Lemmas 3.1 and 3.2. We have

$$B = U_{\mathfrak{s}_2,F}(T_{\mathfrak{q}_3}(F))/s(T_{\mathfrak{q}_3}(F))U_{\mathfrak{s}_2,F} \simeq R_{F'}/p^d R_{F'}\,. \tag{14}$$

Let us now investigate the R_F-module A. Since $T_{\mathfrak{q}_3}(F)$ is assumed to be cyclic, the R_F-module A is equal to the second Tate cohomology group of $T_{\mathfrak{q}_3}(F)$ with coefficients in $U_{\mathfrak{s}_2,F}$, i.e.,

$$A = \hat{H}^2(T_{\mathfrak{q}_3}(F), U_{\mathfrak{s}_2,F})\,. \tag{15}$$

As we shall see in a moment, A is related to the Tate cohomology groups

$$A^n = \hat{H}^n(T_{\mathfrak{q}_3}(F), [U_{\mathfrak{q}_1,F}]^{T_{\mathfrak{q}_2}(F)})\,, \quad n \in \boldsymbol{N}\,.$$

We are able to decide the structure of A^n. But let us first point out the equality

$$[U_{\mathfrak{q}_1,F}]^{T_{\mathfrak{q}_2}(F)} = s(T_{\mathfrak{q}_2}(F))U_{\mathfrak{q}_1,F}\,,$$

and more generally the identity

$$\hat{H}^n(T_{\mathfrak{q}_j}(F), U_{\mathfrak{q}_i,F}) = 0\,, \quad \text{for all } i \neq j \tag{16}$$

which one may prove as follows. Consider the idempotent

$$e_i = \frac{s(T_{\mathfrak{q}_i}(F))}{\#T_{\mathfrak{q}_i}(F)}\,.$$

Multiplication by $(1-e_i)$ gives us the exact sequences

$$0 \to [U_{\mathfrak{q}_i,F}]^{T_{\mathfrak{q}_i}(F)} \to U_{\mathfrak{q}_i,F} \to (1-e_i)U_{\mathfrak{q}_i,F} \to 0\,,$$

$$0 \to R_F^{T_{\mathfrak{q}_i}(F)} \to R_F \to (1-e_i)R_F \to 0\,.$$

But it is easily seen that

$$(1-e_i)U_{\mathfrak{q}_i,F} = (1-e_i)R_F \quad \text{and} \quad [U_{\mathfrak{q}_i,F}]^{T_{\mathfrak{q}_i}(F)} = R_F^{T_{\mathfrak{q}_i}(F)}\,.$$

On the other hand R_F and $R_F^{T_{\mathfrak{q}_i}(F)}$ are cohomologically trivial as $T_{\mathfrak{q}_j}(F)$-modules. This implies formula (16).

LEMMA 3.3. *Let us denote $s(T)R_F$ by W, then we have*

$$A^n = \hat{H}^n(T_{\mathfrak{q}_3}(F), [U_{\mathfrak{q}_1,F}]^{T_{\mathfrak{q}_2}(F)}) \simeq \begin{cases} W/p^dW + (1-F_{\mathfrak{q}_1})W & \text{if } n \text{ is even} \\ \\ (W/p^dW)^{F_{\mathfrak{q}_1}} & \text{if } n \text{ is odd}\,. \end{cases}$$

PROOF. Let us consider an element α of $[U_{\mathfrak{q}_1,F}]^{T_{\mathfrak{q}_2}(F)}$. There exist $x, y \in R_F$ such that

$$\alpha = s(T_{\mathfrak{q}_2}(F))[s(T_{\mathfrak{q}_1}(F))x + (1-(\mathfrak{q}_1, F))y] \tag{17}$$

in view of (16). If α is invariant under the action of $T_{\mathfrak{q}_3}(F)$, then the element $s(T_{\mathfrak{q}_2}(F))y$ is in R_F^T. Indeed, since T is generated by any two inertia groups $T_{\mathfrak{q}_i}(F)$ and $T_{\mathfrak{q}_j}(F)$ for $i \neq j$, we have

$$s(T_{\mathfrak{q}_2}(F))s(T_{\mathfrak{q}_1}(F)) = s(T) \quad \text{and} \quad p^d s(T_{\mathfrak{q}_2}(F))(\mathfrak{q}_1, F) = s(T)F_{\mathfrak{q}_1}^{-1}\,.$$

Therefore we may write $s(T_{\mathfrak{q}_2}(F))y = s(T)y'$ for some $y' \in R_F$. This clearly shows that

$$[U_{\mathfrak{q}_1,F}]^{\langle T_{\mathfrak{q}_2}(F), T_{\mathfrak{q}_3}(F)\rangle} = [U_{\mathfrak{q}_1,F}]^T = s(T)R_F\,.$$

But it is also obvious that

$$s(T_{\mathfrak{q}_3}(F))[U_{\mathfrak{q}_1,F}]^{T_{\mathfrak{q}_2}(F)} = s(T)U_{\mathfrak{q}_1,F} = p^d s(T)R_F + (1-F_{\mathfrak{q}_1})s(T)R_F\,.$$

This proves the isomorphism

$$\hat{H}^2(T_{\mathfrak{q}_3}(F), [U_{\mathfrak{q}_1,F}]^{T_{\mathfrak{q}_2}(F)}) \simeq W/p^dW + (1-F_{\mathfrak{q}_1})W\,.$$

Furthermore, we may define an R_F-homomorphism $\Theta : [U_{\mathfrak{q}_1,F}]^{T_{\mathfrak{q}_2}(F)} \to W/p^d W$ by the formula

$$\Theta(\alpha) = s(T)y \bmod p^d W .$$

The map Θ is well-defined because if x, y are replaced by some other elements $x', y' \in R_F$ in the formula (17), then

$$s(T_{\mathfrak{q}_2}(F))y - s(T_{\mathfrak{q}_2}(F))y' \in R_F^T = s(T)R_F .$$

Moreover, if $s(T_{\mathfrak{q}_3}(F))\alpha = 0$ then $\Theta(\alpha)$ is invariant under the action of $F_{\mathfrak{q}_1}$. Let us suppose that $s(T)y = p^d s(T)y'$ for some $y' \in R_F$. Since $s(T_{\mathfrak{q}_2}(F))R_F$ is cohomologically trivial as a $T_{\mathfrak{q}_3}(F)$-module, we may find $z \in R_F$ such that

$$s(T_{\mathfrak{q}_2}(F))y = p^d s(T_{\mathfrak{q}_2}(F))y' + (1-\sigma)s(T_{\mathfrak{q}_2}(F))z ,$$

where σ is a generator of $T_{\mathfrak{q}_3}(F)$. If, in addition, we have $s(T_{\mathfrak{q}_3}(F))\alpha = 0$, then $s(T)x = -s(T)(1 - F_{\mathfrak{q}_1}^{-1})y'$ and

$$\alpha = [s(T_{\mathfrak{q}_2}(F))y'(p^d - s(T_{\mathfrak{q}_3}(F))) + (1-\sigma)s(T_{\mathfrak{q}_2}(F))z](1 - (\mathfrak{q}_1, F)) .$$

But it is straightforward that $(1-\sigma)[U_{\mathfrak{q}_1,F}]^{T_{\mathfrak{q}_2}(F)} = (1-\sigma)s(T_{\mathfrak{q}_2}(F))R_F$. Hence we have proved that Θ induces a monomorphism

$$\hat{H}^1(T_{\mathfrak{q}_3}(F),\ [U_{\mathfrak{q}_1,F}]^{T_{\mathfrak{q}_2}(F)}) \to (W/p^d W)^{F_{\mathfrak{q}_1}} ,$$

which is onto as one may easily check. The proof is now complete. □

LEMMA 3.4. *There exists an exact sequence of R_F-modules*

$$0 \to A^2/(1 - F_{\mathfrak{q}_2})A^2 \xrightarrow{\alpha} A \xrightarrow{\beta} (A^1)^{F_{\mathfrak{q}_2}} \to 0 .$$

PROOF. The map α is induced by the inclusion

$$[U_{\mathfrak{q}_1,F}]^{T_{\mathfrak{q}_2}(F)} = s(T_{\mathfrak{q}_2}(F))U_{\mathfrak{q}_1,F} \subset U_{\mathfrak{s}_2,F} ,$$

which extends the inclusion used to define the map h of Lemma 3.1. On the other hand, if we let σ be a generator of $T_{\mathfrak{q}_3}(F)$, then β is induced by the well-defined map

$$[U_{\mathfrak{s}_2,F}]^{T_{\mathfrak{q}_3}(F)} \to (A^1)^{F_{\mathfrak{q}_2}} ,$$

which associates to $\gamma = s(T_{\mathfrak{q}_2}(F))\mu + (1 - (\mathfrak{q}_2, F))\nu$, where μ and ν are elements of $U_{\mathfrak{q}_1,F}$, the class of $(\sigma - 1)\nu$ in A^1. Actually, the identity $0 = (\sigma - 1)\gamma$ implies the relation

$$(\sigma - 1)\nu = (\mathfrak{q}_2, F)(\sigma - 1)\nu - s(T_{\mathfrak{q}_2}(F))(\sigma - 1)\mu ,$$

from which we deduce that $(\sigma-1)\nu$ is invariant under the action of $T_{\mathfrak{q}_2}(F)$. Therefore we may rewrite it as follows: $(1-F_{\mathfrak{q}_2}^{-1})(\sigma-1)\nu = -(\sigma-1)s(T_{\mathfrak{q}_2}(F))\mu$. Hence the image of $(\sigma-1)\nu$ in A^1 is in fact in $(A^1)^{F_{\mathfrak{q}_2}}$. To prove that we have an exact sequence is straightforward. We leave the details to the interested reader. □

Let us remark that both $(A^1)^{F_{\mathfrak{q}_2}}$ and $A^2/(1-F_{\mathfrak{q}_2})A^2$ are isomorphic as R_F-modules to $R_{F'}/p^d R_{F'}$ by Lemma 3.3. The field F' was introduced in Lemma 3.2. Even though we are not sure that the exact sequence of Lemma 3.4 splits, we may use it to deduce the order of A. We have

$$\#A = p^{2d[F':k]}\,. \tag{18}$$

COROLLARY 3.5. *Suppose that there exist three prime ideals $\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3$ in $\mathcal{O}_k$ satisfiying* $\mathrm{cond}_p(F,d)$. *Then we have*

$$(R_F : U_{\mathfrak{s},F}) = p^{d[F':k]}\,, \tag{19}$$

where $\mathfrak{s} = \mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3$ and F' is the maximal extension of k in F such that $\mathfrak{q}_1$, $\mathfrak{q}_2$ and $\mathfrak{q}_3$ split completely in F'/k.

PROOF. This is a straightforward consequence of the three formulas (13), (14) and (18). □

4. $\boldsymbol{Z}_p$-extensions. Let F be a finite abelian extension of k such that $H \subset F$. Then we define

$$\mathfrak{A}_F = \frac{[F:H]}{\prod_{\mathfrak{p}}[F_{\mathfrak{p}^\infty} : H]}\,.$$

Let F_∞ be a $\boldsymbol{Z}_p$-extension of F abelian over k. In [7, Théorème 4.1], we proved the existence of a positive constant $c_\infty \in \boldsymbol{Q}^\times$ such that

$$[\Omega_{F_n} : V_{F_n}] = c_\infty \frac{w_{F_n}}{w_k}\mathfrak{A}_{F_n} \tag{20}$$

for all sufficiently large n. Let $\tilde{S}_{F_\infty,F}$ be the set of prime ideals $\mathfrak{q} \in S_{F_\infty,F}$ such that the decomposition group of $\mathfrak{q}$ in F_∞/k is finite. Let $\mathfrak{f}_0$ be the product of the prime ideals $\mathfrak{q} \in \tilde{S}_{F_\infty,F}$. Then, one may find $\nu \in \boldsymbol{N}$ such that

$$(R_{F_n} : U_{F_n})_p = p^\nu (R_{F_n} : U_{\mathfrak{f}_0,F_n})_p \tag{21}$$

for all sufficiently large n. This is a consequence of [ibid., Proposition 3.2 and Corollaire 3.4].

Let $\mathfrak{q}_1, \mathfrak{q}_2$ and $\mathfrak{q}_3$ be three prime ideals of $\mathcal{O}_k$ satisfying $\mathrm{cond}_p(F,d)$ and such that $\{\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3\} \subset \tilde{S}_{F_\infty,F}$. Then, for all $n \in \boldsymbol{N}$, the condition $\mathrm{cond}_p(F_n,d)$ is satisfied by these three prime ideals. Moreover, we have $F' = F'_n \cap F$ by their very definition. The degree $[F_n : F'_n]$ does not depend on n and $F_n = F'_n F$. In particular, we have $[F'_n : k] = p^n[F' : k]$. Therefore, by Corollary 3.5, we have

$$(R_{F_n} : U_{\mathfrak{s},F_n}) = p^{d[F':k]p^n}\,, \tag{22}$$

for all $n \in \boldsymbol{N}$, where $\mathfrak{s} = \mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3$. Now, from the index formula (12) and the identities (20) and (22), we derive the following theorem.

THEOREM 4.1. *Let $\mathfrak{q}_1, \mathfrak{q}_2$ and $\mathfrak{q}_3$ be three prime ideals of $\mathcal{O}_k$ chosen as above. Then, there exists $\nu_0 \in \boldsymbol{Z}$ such that*

$$[\mathcal{O}_{F_n} : \mathcal{C}_{F_n}]_p = p^{d[F':k]p^n+\nu_0}(h_{F_n})_p(U_{\mathfrak{s},F_n} : U_{\mathfrak{f}_0,F_n})_p \tag{23}$$

for all sufficiently large n. In particular, we have

$$\mu_\infty \geq d[F' : k]\,,$$

with equality $\mu_\infty = d[F' : k]$ *if* $\tilde{S}_{F_\infty, F} = \{\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3\}$.

PROOF. The formula (23) is a direct consequence of (12), (20), (21) and (22). Since the index $(U_{\mathfrak{s},F_n} : U_{\mathfrak{f}_0,F_n})$ is a positive integer, we obtain the lower bound of μ_∞. □

4.1. The anticyclotomic $\boldsymbol{Z}_p$-extension. In this subsection, we give examples to illustrate Theorem 4.1. First we explain a method to find a finite abelian extenion F of k, and three non zero prime ideals of $\mathcal{O}_k$, say $\mathfrak{q}_1$, $\mathfrak{q}_2$ and $\mathfrak{q}_3$ that satisfy $\mathrm{cond}_p(F, d)$, where p and d are given. Further we consider the anticyclotomic $\boldsymbol{Z}_p$-extension of F denoted by F_∞^a.

Let q_1, q_2 and q_3 be prime numbers not dividing w_k. For each i we fix $\mathfrak{q}_i$ a nonzero prime ideal of $\mathcal{O}_k$ lying over q_i. Consider the ray class field $K = k_{\mathfrak{s}}$ where $\mathfrak{s} = \mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3$. Then, $T_{\mathfrak{q}_i}(K)$ is isomorphic to the multiplicative group of the finite field $\mathcal{O}_k/\mathfrak{q}_i$. In particular, $T_{\mathfrak{q}_i}(K)$ is cyclic.

Let d be a positive integer and let p be an odd prime number such that $N(\mathfrak{q}_i) \equiv 1$ modulo p^d and $N(\mathfrak{q}_i) \not\equiv 1$ modulo p^{d+1}. Let L be the maximal p-extension of H in K. Then, $T_{\mathfrak{q}_i}(L)$ is a cyclic p-group of order p^d. Moreover, $\mathrm{Gal}(L/H) = T_{\mathfrak{s}}(L)$ is the direct product of $T_{\mathfrak{q}_i}(L)$ for $i \in \{1, 2, 3\}$. Let us choose for each i a generator σ_i of $T_{\mathfrak{q}_i}(L)$. Then we let F be the subfield of L fixed by the product $\sigma_1\sigma_2\sigma_3$, i.e.,

$$F = L^{\langle \sigma_1\sigma_2\sigma_3 \rangle}\,.$$

It is easy to check that $\mathfrak{q}_1$, $\mathfrak{q}_2$ and $\mathfrak{q}_3$ satisfy the conditions 1, 2 and 4 of $\mathrm{cond}_p(F, d)$. If q_3 is inert in $k/\boldsymbol{Q}$ and satisfies the congruence $q_3 \equiv 1$ modulo q_1q_2, then the condition 3 is also satisfied. Moreover, if we suppose that q_1 and q_2 are not split in $k/\boldsymbol{Q}$, then we have $\tilde{S}_{F_\infty^a, F} = S_{F_\infty^a, F} = \{\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3\}$ and $\mu_\infty = d[F' : k]$. Here are three examples with $p = 3$, $d = 1$ and $(q_1, q_2, q_3) = (7, 13, 547)$, in which we denote μ_∞ by μ_∞^a to mean that we are considering the anticyclotomic $\boldsymbol{Z}_p$-extension of F. Let us remark that 3 is inert in the first example, split in the second and ramified in the third one.

EXAMPLE 4.2. For $k = \boldsymbol{Q}(\sqrt{-7})$, we have $\mu_\infty^a = 1$.

EXAMPLE 4.3. For $k = \boldsymbol{Q}(\sqrt{-11})$, we have $\mu_\infty^a = 1$.

EXAMPLE 4.4. For $k = \boldsymbol{Q}(\sqrt{-33})$, we have $\mu_\infty^a = [F' : k]$.

Acknowledgment. I would like to thank Professor Linsheng Yin and the department of mathematics of Tsinghua University in Beijing, where a significant part of this paper was achieved during my visit in the summer of 2006. My thanks go also to the referee and to the editorial office of Tohoku Mathematical Journal for all their suggestions to improve the paper.

REFERENCES

[1] W. BLEY, Equivariant Tamagawa number conjecture for abelian extensions of a quadratic imaginary field, Doc. Math. 11 (2006), 73–118.

[2] B. GROSS AND M. ROSEN, Fourier series and the special values of L-functions, Adv. in Math. 69 (1988), 1–31.

[3] D. S. KUBERT AND S. LANG, Modular units, Grundlehren der Mathematischen Wissenschaften 244, Springer-Verlag, New York, 1981.

[4] D. S. KUBERT, Product formulae on elliptic curves, Invent. Math. 117 (1994), 227–273.

[5] S. LANG, Elliptic functions, with an appendix by J. Tate, Addison-Wesley Publishing Co., Inc., Reading, Mass.-London-Amsterdam, 1973.

[6] H. OUKHABA, Index formulas for ramified elliptic units, Compositio Math. 137 (2003), 1–22.

[7] H. OUKHABA, Indice des unités elliptiques dans les $\boldsymbol{Z}_p$-extensions, Bull. Soc. Math. France 135 (2007), 299–322.

[8] G. ROBERT, Unités elliptiques, Bull. Soc. Math. France, Mém. No. 36, Tome 101, Société Mathématique de France, Paris, 1973.

[9] G. ROBERT, Concernant la relation de distribution satisfaite par la fonction φ associée à un réseau complexe, Invent. Math. 100 (1990), 231–257.

[10] G. ROBERT, Unités de stark et racine 12-ième canonique, Prépublication de l'institut Fourier n$^{\text{o}}$ 181, 1991.

[11] G. ROBERT, La racine 12-ième canonique $\Delta(L)^{[\underline{L}:\, L]}/\Delta(\underline{L})$, in Séminaire de Théorie des Nombres, Paris, 1989–90, Progr. Math. 102, Birkäuser Boston, Boston, MA, 1992, 209–232.

[12] K. RUBIN, The "main conjectures" of Iwasawa theory for imaginary quadratic fields, Invent. Math. 103 (1991), 25–68.

[13] W. SINNOTT, On the Stickelberger ideal and the circular units of an abelian field, Invent. Math. 62 (1980/81), 181–234.

LABORATOIRE DE MATHÉMATIQUE
16 ROUTE DE GRAY
25030 BESANÇON CEDEX
FRANCE

E-mail address: houkhaba@univ-fcomte.fr