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THE INEQUALITIES OF COMMUTATORS ON WEAK HERZ SPACES

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ABSTRACT. In this paper, the boundedness of some commutators related to linear operators on weak Herz spaces are obtained.

1. Introduction

Let $b \in BMO(\mathbb{R}^n)$ and T be a standard Calderon-Zygmund operator. The commutator $[b, T]$ generated by b and T is defined by

$$[b, T]f(x) = b(x)Tf(x) - T(bf)(x).$$

A classical result of Coifman, Rochberg and Weiss [2] states that commutator $[b, T]$ is bounded on $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$. Chanillo [1] considered the similar question when Calderon-Zygmund operator is replaced by the fractional integral operator. In recent years, the theory of Herz type spaces has been developed (see [3], [6]). Lu and Yang [7] generalized these results to the case of Herz spaces (also see [5]), in fact, they have proved that if $[b, T]$ is bounded on L^q for some $q \in (1, \infty)$, then $[b, T]$ is bounded on Herz space $K_q^{\alpha,p}(\mathbb{R}^n)$ for any $\alpha \in (-n/q, n(1 - 1/q))$ and $p \in (0, \infty]$ only under certain very weak local conditions on the size of T. The main purpose of this paper is to consider the boundedness of commutators on weak Herz spaces $W\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ when $\alpha = n(1 - 1/q)$. It was observed that commutator $[b, T]$ is not be of weak type (1.1). In fact, Perez proved that $[b, T]$ satisfy $L(\log L)$ type inequalities (see [9]). We also show that commutator $[b, T]$ satisfy $L(\log L)$ type estimates in Herz spaces when $\alpha = n(1 - 1/q)$, in addition, we get the weak boundedness of commutators in Herz spaces when b satisfies certain condition. Let us first introduce some notations (see [3], [6]).

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Let $B_k = \{x \in \mathbb{R}^n : |x| \le 2^k\}$, $A_k = B_k \setminus B_{k-1}$, $k \in \mathbb{Z}$. Let $\chi_k = \chi_{A_k}$ for $k \in \mathbb{Z}$, where χ_E is the characteristic function of the set E.

DEFINITION 1. Let $0 < p,\ q < \infty$, $\alpha \in \mathbb{R}$.

(1) The homogeneous Herz space is defined by

$$\dot{K}_q^{\alpha,p}(\mathbb{R}^n) = \{f \in L_{loc}^q(\mathbb{R}^n \setminus \{0\}) : ||f||_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} < \infty\},$$

where

$$||f||_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} = \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}||f\chi_k||_{L^q}^p\right]^{1/p}.$$

(2) The nonhomogeneous Herz space is defined by

$$K_q^{\alpha,p}(\mathbb{R}^n) = \{f \in L_{loc}^q(\mathbb{R}^n) : ||f||_{K_q^{\alpha,p}(\mathbb{R}^n)} < \infty\},$$

where

$$||f||_{K_q^{\alpha,p}(\mathbb{R}^n)} = \left[||f\chi_{B_0}||_{L^q}^p + \sum_{k=1}^{\infty} 2^{k\alpha p}||f\chi_k||_{L^q}^p\right]^{1/p}.$$

DEFINITION 2. Let $0 < p, q < \infty$, $\alpha \in \mathbb{R}$. For $k \in \mathbb{Z}$ and measurable function $f(x)$ on $\mathbb{R}^n$, let $m_k(\lambda, f) = |\{x \in A_k : |f(x)| > \lambda\}|$; for $k \in \mathbb{N}$, let $\tilde{m}_k(\lambda, f) = m_k(\lambda, f)$ and $\tilde{m}_0(\lambda, f) = |\{x \in B_0 : |f(x)| > \lambda\}|$.

(1) The homogeneous weak Herz space is defined by

$$W\dot{K}_q^{\alpha,p}(\mathbb{R}^n) = \{f : ||f||_{W\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} < \infty\},$$

where

$$||f||_{W\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} = \sup_{\lambda>0} \lambda \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} m_k(\lambda, f)^{p/q}\right]^{1/p}.$$

(2) The nonhomogeneous weak Herz space is defined by

$$WK_q^{\alpha,p}(\mathbb{R}^n) = \{f : ||f||_{WK_q^{\alpha,p}(\mathbb{R}^n)} < \infty\},$$

where

$$||f||_{WK_q^{\alpha,p}(\mathbb{R}^n)} = \sup_{\lambda>0} \lambda \left[\sum_{k=0}^{\infty} 2^{k\alpha p} \tilde{m}_k(\lambda, f)^{p/q}\right]^{1/p}.$$

DEFINITION 3. Let $b \in BMO(\mathbb{R}^n)$. The commutators of the maximal operator and the fractional maximal operator are defined, respectively, by

$$M_b f(x) = \sup_{r>0} |B(x,r)|^{-1} \int_{B(x,r)} |b(x) - b(y)||f(y)|dy$$

and

$$M_b^{\lambda} f(x) = \sup_{r>0} |B(x,r)|^{-1/\lambda'} \int_{B(x,r)} |b(x) - b(y)|\,|f(y)|dy,$$

where $1 \le \lambda \le \infty$ and $1/\lambda + 1/\lambda' = 1$.

2. Main results and their proofs

We begin with the boundedness of the commutators M_b and M_b^{λ} on weak Herz spaces, which will be useful to the main results in this paper and are themselves of independent interest.

THEOREM 1. *Let $b \in BMO(\mathbb{R}^n)$ and $0 < p \le 1 < q < \infty$, $\alpha = n(1 - 1/q)$. Then for any $f \in K_q^{\alpha,p}(\mathbb{R}^n)$ and $\lambda > 0$, there exist constant $C > 0$ independent on f and λ, such that*

$$\left[\sum_{k=0}^{\infty} 2^{k\alpha p} \tilde{m}_k(\lambda, M_b f)^{p/q}\right]^{1/p} \le C\lambda^{-1} ||f||_{K_q^{\alpha,p}(\mathbb{R}^n)} \left(1 + \log^+(\lambda^{-1}||f||_{K_q^{\alpha,p}(\mathbb{R}^n)})\right).$$

Proof. Let $f \in K_q^{\alpha,p}(\mathbb{R}^n)$ and $f(x) = \sum_{j=0}^{\infty} \lambda_j a_j(x)$, where supp $a_j \subset B_j$, $||a_j||_{L^q} \le C2^{-j\alpha}$ for $j \in N \cup \{0\}$ and $||f||_{K_q^{\alpha,p}(\mathbb{R}^n)} \sim \left(\sum_{j=0}^{\infty} |\lambda_j|^p\right)^{1/p}$. We write

$$\left[\sum_{k=0}^{\infty} 2^{k\alpha p} \tilde{m}_k(\lambda, M_b f)^{p/q}\right]^{1/p} \le C \left[\sum_{k=0}^{3} 2^{k\alpha p} \tilde{m}_k(\lambda, M_b f)^{p/q}\right]^{1/p}$$

$$+ C \left[\sum_{k=4}^{\infty} 2^{k\alpha p} \tilde{m}_k(\lambda, M_b f)^{p/q}\right]^{1/p} \equiv I + II.$$

For I, by the boundedness of M_b on $L^q(\mathbb{R}^n)$ for $1 < q < \infty$ (see [10]) and $0 < p \leq 1$, we have

$$\begin{aligned}
I &\leq C\lambda^{-1}||f||_{L^q}\left(\sum_{k=0}^{3} 2^{k\alpha p}\right)^{1/p} \leq C\lambda^{-1}\sum_{j=0}^{\infty}|\lambda_j|\,||a_j||_{L^q} \\
&\leq C\lambda^{-1}\sum_{j=0}^{\infty}|\lambda_j|2^{-j\alpha} \leq C\lambda^{-1}\sum_{j=0}^{\infty}|\lambda_j| \\
&\leq C\lambda^{-1}\left(\sum_{j=0}^{\infty}|\lambda_j|^p\right)^{1/p} \leq C\lambda^{-1}||f||_{K_q^{\alpha,p}(\mathbb{R}^n)}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
II &\leq C\left[\sum_{k=4}^{\infty} 2^{k\alpha p}\tilde{m}_k\left(\lambda/2, \sum_{j=0}^{k-3}|\lambda_j|M_b a_j\right)^{p/q}\right]^{1/p} \\
&\quad + C\left[\sum_{k=4}^{\infty} 2^{k\alpha p}\tilde{m}_k\left(\lambda/2, M_b(\sum_{j=k-2}^{\infty}|\lambda_j|a_j)\right)^{p/q}\right]^{1/p} \\
&\equiv II_1 + II_2.
\end{aligned}$$

Using the boundedness of M_b on $L^q(\mathbb{R}^n)$, we have

$$\begin{aligned}
II_2 &\leq C\lambda^{-1}\left[\sum_{k=4}^{\infty} 2^{k\alpha p}||\sum_{j=k-2}^{\infty}\lambda_j a_j||_{L^q}^p\right]^{1/p} \\
&\leq C\lambda^{-1}\left[\sum_{k=4}^{\infty} 2^{k\alpha p}\sum_{j=k-2}^{\infty}|\lambda_j|^p 2^{-j\alpha p}\right]^{1/p} \\
&\leq C\lambda^{-1}\left[\sum_{j=0}^{\infty}|\lambda_j|^p\sum_{k=0}^{j+2} 2^{(k-j)\alpha p}\right]^{1/p} \\
&\leq C\lambda^{-1}\left(\sum_{j=0}^{\infty}|\lambda_j|^p\right)^{1/p} \\
&\leq C\lambda^{-1}||f||_{K_q^{\alpha,p}(\mathbb{R}^n)};
\end{aligned}$$

For II_1, denoting $b_j = |B_j|^{-1}\int_{B_j} b(y)dy$, by the properties of $BMO(\mathbb{R}^n)$ (see [12]), we have, for $x \in A_k$ with $j \le k-3$,

$$\begin{aligned} &M_b a_j(x) \\ &\le C2^{-kn}\int_{B_j}|b(x)-b(y)|\,|a_j(y)|dy \\ &\le C2^{-kn}\left(|b(x)-b_j|\,||a_j||_{L^q}|B_j|^{1-1/q} + ||b||_{BMO}||a_j||_{L^q}|B_j|^{1-1/q}\right) \\ &\le C2^{-kn}(|b(x)-b_k| + k||b||_{BMO}). \end{aligned}$$

Therefore,

$$\begin{aligned} II_1 &\le C\left[\sum_{k=4}^{\infty} 2^{k\alpha p}\tilde{m}_k\left(\lambda/4, C2^{-kn}|b(x)-b_k|\sum_{j=0}^{\infty}|\lambda_j|\right)^{p/q}\right]^{1/p} \\ &\quad + C\left[\sum_{k=4}^{\infty} 2^{k\alpha p}\tilde{m}_k\left(\lambda/4, Ck2^{-kn}||b||_{BMO}\sum_{j=0}^{\infty}|\lambda_j|\right)^{p/q}\right]^{1/p} \\ &\equiv II_1^{(1)} + II_1^{(2)}. \end{aligned}$$

Using John-Nirenberg inequality, we deduce

$$\begin{aligned} II_1^{(1)} &\le C\left[\sum_{k=4}^{\infty} 2^{k\alpha p}\left(\exp\left(-\frac{c2^{kn}\lambda}{||b||_{BMO}\sum_{j=0}^{\infty}|\lambda_j|}\right)2^{kn}\right)^{p/q}\right]^{1/p} \\ &\le C\left[\sum_{k=0}^{\infty} 2^{k\alpha p+knp/q}\exp\left(-\frac{c\lambda 2^{kn}}{||b||_{BMO}\sum_{j=0}^{\infty}|\lambda_j|}\right)\right]^{1/p} \\ &\le C\left[\int_0^{\infty} x^{p-1}\exp\left(-\frac{c\lambda x}{||b||_{BMO}\sum_{j=0}^{\infty}|\lambda_j|}\right)dx\right]^{1/p} \\ &= C\lambda^{-1}||b||_{BMO}\sum_{j=0}^{\infty}|\lambda_j|\left(\int_0^{\infty} t^{p-1}e^{-t}dt\right)^{1/p} \\ &\le C\lambda^{-1}\left(\sum_{j=0}^{\infty}|\lambda_j|^p\right)^{1/p} \\ &\le C\lambda^{-1}||f||_{K_q^{\alpha,p}(\mathbb{R}^n)}. \end{aligned}$$

For $II_1^{(2)}$, by using the fact: if there exist $y > 1$ such that $2^x/x < y$ holds for $x > 3$, then $2^x \leq cy \log_2 y$, we see that, for $k > 3$, if $|\{x \in A_k : C2^{-kn}k||b||_{BMO} \sum_{j=0}^{\infty} |\lambda_j| > \lambda/4\}| \neq 0$,

$$1 < 2^{kn}/kn < C\lambda^{-1}||b||_{BMO} \sum_{j=0}^{\infty} |\lambda_j|,$$

and thus

$$2^{kn} \leq C\lambda^{-1} \left(\sum_{j=0}^{\infty} |\lambda_j| \right) \left[1 + \log^+ \left(\lambda^{-1} \sum_{j=0}^{\infty} |\lambda_j| \right) \right].$$

Let K_λ denote the maximal integer which satisfies this estimation. Then

$$\begin{aligned} II_1^{(2)} &\leq C \left(\sum_{k=4}^{K\lambda} 2^{k\alpha p} \cdot 2^{knp/q} \right)^{1/p} \leq C2^{K_\lambda n} \\ &\leq C\lambda^{-1} \left(\sum_{j=0}^{\infty} |\lambda_j| \right) \left[1 + \log^+ \left(\lambda^{-1} \sum_{j=0}^{\infty} |\lambda_j| \right) \right] \\ &\leq C\lambda^{-1} \left(\sum_{j=0}^{\infty} |\lambda_j|^p \right)^{1/p} \left[1 + \log^+ \left(\lambda^{-1} \left(\sum_{j=0}^{\infty} |\lambda_j|^p \right)^{1/p} \right) \right] \\ &\leq C\lambda^{-1} ||f||_{K_q^{\alpha,p}(\mathbb{R}^n)} [1 + \log^+ (\lambda^{-1} ||f||_{K_q^{\alpha,p}(\mathbb{R}^n)})]. \end{aligned}$$

Combining the estimations of $I, II_2, II_1^{(1)}$ and $II_1^{(2)}$, we gain the conclusion of the theorem. □

THEOREM 2. *Let $b \in BMO(\mathbb{R}^n)$ and b satisfy the condition L : $|b(x) - b_j| \leq C|b(x) - b_k|$ for any $k, j \in \mathbb{Z}$ and $j \leq k - 3$, $x \in A_k$. If $0 < p \leq 1 < q < \infty$, $\alpha = n(1 - 1/q)$. Then M_b is bounded from $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ (or $K_q^{\alpha,p}(\mathbb{R}^n)$) to $W\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ (or $WK_q^{\alpha,p}(\mathbb{R}^n)$).*

Proof. We only prove the homogeneous case. Let $f(x) = \sum_{j=-\infty}^{\infty} \lambda_j a_j(x) \in \dot{K}_q^{\alpha,p}(\mathbb{R}^n)$, where supp$a_j \subset B_j$, $||a_j||_{L^q} \leq C2^{-j\alpha}$ for $j \in \mathbb{Z}$ and

$||f||_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} \sim \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p\right)^{1/p}$. We write

$$\begin{aligned}
&||M_b f||_{W\dot{K}_q^{\alpha,p}} \\
&\leq C \sup_{\lambda>0} \lambda \\
&\quad \times \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left|\left\{x \in A_k : M_b\left(\sum_{j=-\infty}^{k-3} \lambda_j a_j\right)(x) > \lambda/2\right\}\right|^{p/q}\right]^{1/p} \\
&\quad + C \sup_{\lambda>0} \lambda \\
&\quad \times \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left|\left\{x \in A_k : M_b\left(\sum_{j=k-2}^{\infty} \lambda_j a_j\right)(x) > \lambda/2\right\}\right|^{p/q}\right]^{1/p} \\
&\equiv I + II.
\end{aligned}$$

For II, using the boundedness of M_b on $L^q(\mathbb{R}^n)$ for $1 < q < \infty$, and $0 < p \leq 1$, we have

$$\begin{aligned}
II &\leq C \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left\|\sum_{j=k-2}^{\infty} \lambda_j a_j\right\|_{L^q}^p\right]^{1/p} \\
&\leq C \left[\sum_{j=0}^{\infty} |\lambda_j|^p \sum_{k=-\infty}^{j+2} 2^{(k-j)\alpha p}\right]^{1/p} \\
&\leq C||f||_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}.
\end{aligned}$$

For I, using the estimation of $M_b a_j$ in the proof of Theorem 1 and condition L we see that, for $x \in A_k$ with $j \leq k-3$,

$$M_b a_j(x) \leq C 2^{-kn}(|b(x) - b_k| + ||b||_{BMO}),$$

and therefore

$$\begin{aligned}I \leq & C \sup_{\lambda>0} \lambda \\ & \times \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left| \left\{ x \in A_k : C2^{-kn} |b(x) - b_k| \sum_{j=-\infty}^{\infty} |\lambda_j| > \lambda/4 \right\} \right|^{p/q} \right]^{1/p} \\ & + C \sup_{\lambda>0} \lambda \\ & \times \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left| \left\{ x \in A_k : C2^{-kn} ||b||_{BMO} \sum_{j=-\infty}^{\infty} |\lambda_j| > \lambda/4 \right\} \right|^{p/q} \right]^{1/p} \\ \equiv & I_1 + I_2.\end{aligned}$$

Using John-Nirenberg inequality, we deduce

$$\begin{aligned}I_1 \leq & C \sup_{\lambda>0} \lambda \left[\sum_{k=-\infty}^{\infty} 2^{knp} \exp\left(- \frac{c\lambda 2^{kn}}{||b||_{BMO} \sum_{j=-\infty}^{\infty} |\lambda_j|} \right) \right]^{1/p} \\ \leq & C ||f||_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}.\end{aligned}$$

For any fixed $\lambda > 0$, if $\left| \left\{ x \in A_k : C2^{-kn} ||b||_{BMO} \sum_{j=-\infty}^{\infty} |\lambda_j| > \lambda/4 \right\} \right| \neq 0$, then

$$2^{kn} \leq C\lambda^{-1} \sum_{j=-\infty}^{\infty} |\lambda_j|.$$

Let K_λ denote the maximal integer which satisfies this estimation. Then

$$\begin{aligned}I_2 \leq C \sup_{\lambda>0} \lambda \left(\sum_{k=-\infty}^{K_\lambda} 2^{k\alpha p} \cdot 2^{knp/q} \right)^{1/p} & \leq C \sup_{\lambda>0} \lambda 2^{K_\lambda n} \\ & \leq C \sum_{j=-\infty}^{\infty} |\lambda_j| \leq C ||f||_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}.\end{aligned}$$

This finishes the proof of Theorem 2. □

THEOREM 3. *Let* $b \in BMO(\mathbb{R}^n)$ *and* $1 < \lambda < \infty$, $0 < p_1 \leq p_2 \leq 1 < q_1 < \lambda$, $1/q_2 = 1/q_1 - 1/\lambda$, $\alpha = n(1 - 1/q_1)$.

(1) *For any* $s > 0$ *and* $f \in K_{q_1}^{\alpha,p_1}(\mathbb{R}^n)$, *we have*

$$\left[\sum_{k=0}^{\infty} 2^{k\alpha p_2} \tilde{m}_k(s, M_b^\lambda f)^{p_2/q_2}\right]^{1/p_2}$$
$$\leq Cs^{-1}\|f\|_{K_{q_1}^{\alpha,p_1}(\mathbb{R}^n)}\left(1+\log^+(s^{-1}\|f\|_{K_{q_1}^{\alpha,p_1}(\mathbb{R}^n)})\right).$$

(2) *Furthermore, if* b *satisfies the condition* L, *then* M_b^λ *is bounded from* $\dot{K}_{q_1}^{\alpha,p_1}(\mathbb{R}^n)$ (*or* $K_{q_1}^{\alpha,p_1}(\mathbb{R}^n)$) *to* $W\dot{K}_{q_2}^{\alpha,p_2}(\mathbb{R}^n)$ (*or* $WK_{q_2}^{\alpha,p_2}(\mathbb{R}^n)$).

Proof. Notice that if $p_2 \geq p_1$, then

$$W\dot{K}_{q_2}^{\alpha,p_1}(\mathbb{R}^n) \subset W\dot{K}_{q_2}^{\alpha,p_2}(\mathbb{R}^n) \text{ and } WK_{q_2}^{\alpha,p_1}(\mathbb{R}^n) \subset WK_{q_2}^{\alpha,p_2}(\mathbb{R}^n).$$

Thus, we only need to show the theorem in the case $p_1 = p_2$.

(1) Let $f(x) = \sum_{j=0}^{\infty}\lambda_j a_j(x) \in K_{q_1}^{\alpha,p_1}(\mathbb{R}^n)$, where every a_j are the same as in the proof of Theorem 1 and $\|f\|_{K_{q_1}^{\alpha,p_1}(\mathbb{R}^n)} \sim \left(\sum_{j=0}^{\infty}|\lambda_j|_1^p\right)^{1/p_1}$. We write

$$\left[\sum_{k=0}^{\infty} 2^{k\alpha p_2} \tilde{m}_k(s, M_b^\lambda f)^{p_2/q_2}\right]^{1/p_2} \leq C\left[\sum_{k=0}^{3} 2^{k\alpha p_2} \tilde{m}_k(s, M_b^\lambda f)^{p_2/q_2}\right]^{1/p_2}$$
$$+C\left[\sum_{k=4}^{\infty} 2^{k\alpha p_2} \tilde{m}_k(s, M_b^\lambda f)^{p_2/q_2}\right]^{1/p_2} \equiv I + II.$$

For I, using the fact that M_b^λ is of type (q_1, q_2) (see [10]) and $0 < p \leq 1$, we obtain, by using an argument similar to the proof of Theorem 1,
$I \leq Cs^{-1}\|f\|_{K_{q_1}^{\alpha,p_1}(\mathbb{R}^n)}$ and

$$II \leq C\left[\sum_{k=4}^{\infty} 2^{k\alpha p_2}\left|\left\{x \in A_k : M_b^\lambda\left(\sum_{j=0}^{k-3}\lambda_j a_j\right)(x) > s/2\right\}\right|^{p_2/q_2}\right]^{1/p_2}$$
$$+C\left[\sum_{k=4}^{\infty} 2^{k\alpha p_2}\left|\left\{x \in A_k : M_b^\lambda\left(\sum_{j=k-2}^{\infty}\lambda_j a_j\right)(x) > s/2\right\}\right|^{p_2/q_2}\right]^{1/p_2}$$
$$= II_1 + II_2.$$

Since M_b^λ is of type (q_1, q_2), we deduce

$$II_2 \le Cs^{-1}||f||_{K_{q_1}^{\alpha,p_1}(\mathbb{R}^n)}.$$

For II_1, using an argument similar to the proof of Theorem 1, we have, if $x \in A_k$ and $0 \le j \le k-3$,

$$M_b^\lambda a_j(x) \le C2^{-kn/\lambda'}|b(x) - b_k| + Ck2^{-kn/\lambda'}||b||_{BMO}.$$

Thus, using the argument same as the proof of Theorem 1, we deduce

$$II_1 \le Cs^{-1}||f||_{K_{q_1}^{\alpha,p_1}(\mathbb{R}^n)}\left(1 + \log^+(s^{-1}||f||_{K_{q_1}^{\alpha,p_1}(\mathbb{R}^n)})\right).$$

The proof of (2) is similar to the proof of Theorem 2, we omit the details.
This finishes the proof of Theorem 3. □

THEOREM 4. *Let* $b \in BMO(\mathbb{R}^n)$ *and* $1 < \lambda < \infty$, $0 < p_1 \le p_2 \le 1 < q_1 < \lambda$, $1/q_2 = 1/q_1(1 - p_1/\lambda)$, $\alpha_1 = n(1 - 1/q_1)$, $\alpha_2 = \alpha_1 + n(p_1/q_1 - 1)/\lambda$. *Then*

(1) *For any* $s > 0$ *and* $f \in K_{q_1}^{\alpha_1,p_1}(\mathbb{R}^n)$, *we have*

$$\left[\sum_{k=0}^{\infty} 2^{k\alpha_2 p_2}\tilde{m}_k(s, M_b^\lambda f)^{p_2/q_2}\right]^{1/p_2}$$
$$\le Cs^{-1}||f||_{K_{q_1}^{\alpha_1,p_1}(\mathbb{R}^n)}\left(1 + \log^+(s^{-1}||f||_{K_{q_1}^{\alpha_1,p_1}(\mathbb{R}^n)})\right).$$

(2) *Furthermore, if* b *satisfies the condition* L, *then* M_b^λ *is bounded from* $\dot{K}_{q_1}^{\alpha_1,p_1}(\mathbb{R}^n)$ (*or* $K_{q_1}^{\alpha_1,p_1}(\mathbb{R}^n)$) *to* $W\dot{K}_{q_2}^{\alpha_2,p_2}(\mathbb{R}^n)$ (*or* $WK_{q_2}^{\alpha_2,p_2}(\mathbb{R}^n)$).

The proof of the theorem is similar to the proof of Theorem 3, we omit the details.

Now let us state one of our main theorems.

THEOREM 5. *Let* $b \in BMO(\mathbb{R}^n)$ *and* T *be a linear operator. Suppose that the commutator* $[b, T]$ *is of weak type* (q, q) *for some* $q \in (1, +\infty)$ *and that* T *satisfies the local size condition*

$$|Tf(x)| \le C|x|^{-n}\int |f(y)|dy$$

for $f \in L^1_{loc}(\mathbb{R}^n)$, $suppf \subset A_k$ *and* $|x| \geq 2^{k+1}$ *with* $k \in \mathbb{Z}$. *Let* $0 < p \leq 1 < q < \infty$, $\alpha = n(1 - 1/q)$. *Then*

(1) *For any* $\lambda > 0$ *and* $f \in K_q^{\alpha,p}(\mathbb{R}^n)$, *we have*

$$\left[\sum_{k=0}^{\infty} 2^{k\alpha p} \tilde{m}_k(\lambda, [b,T]f)^{p/q}\right]^{1/p} \leq C\lambda^{-1}||f||_{K_q^{\alpha,p}(\mathbb{R}^n)}(1 + \log^+(\lambda^{-1}||f||_{K_q^{\alpha,p}(\mathbb{R}^n)})).$$

(2) *Furthermore, if* b *satisfies the condition* L, *then* $[b,T]$ *is bounded from* $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ *(or* $K_q^{\alpha,p}(\mathbb{R}^n)$*) to* $W\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ *(or* $WK_q^{\alpha,p}(\mathbb{R}^n)$ *).*

Proof. (1) Let $f(x) = \sum_{j=0}^{\infty} \lambda_j a_j\ (x) \in K_q^{\alpha,p}(\mathbb{R}^n)$, where the a_j are the same as in the proof of Theorem 1 and $||f||_{K_q^{\alpha,p}(\mathbb{R}^n)} \sim \left(\sum_{j=0}^{\infty} |\lambda_j|^p\right)^{1/p}$. We write

$$\left[\sum_{k=0}^{\infty} 2^{k\alpha p} \tilde{m}_k(\lambda, [b,T]f)^{p/q}\right]^{1/p} \leq C\left[\sum_{k=0}^{3} 2^{k\alpha p} \tilde{m}_k(\lambda, [b,T]f)^{p/q}\right]^{1/p}$$

$$+ C\left[\sum_{k=4}^{\infty} 2^{k\alpha p} m_k\left(\lambda/2, [b,T]\left(\sum_{j=0}^{k-3} \lambda_j a_j\right)\right)^{p/q}\right]^{1/p}$$

$$+ C\left[\sum_{k=4}^{\infty} 2^{k\alpha p} m_k(\lambda/2, [b,T]\left(\sum_{j=k-2}^{\infty} \lambda_j a_j\right))^{p/q}\right]^{1/p}$$

$$\equiv I_1 + I_2 + I_3.$$

Using the condition that $[b,T]$ is of weak type (q,q) and $0 < p \leq 1$, we have for $i = 1, 3$,

$$I_i \leq C\lambda^{-1}\left(\sum_{j=0}^{\infty} |\lambda_j|^p\right)^{1/p} \leq C\lambda^{-1}||f||_{K_q^{\alpha,p}(\mathbb{R}^n)}.$$

For I_2, note that $x \in A_k$ and $j \leq k-3$, by using the size condition of T, we obtain

$$\left|[b,T]\left(\sum_{j=0}^{k-3} \lambda_j a_j\right)(x)\right| \leq C|x|^{-n}\int |b(x) - b(y)|\left|\sum_{j=0}^{k-3} \lambda_j a_j(y)\right| dy$$
$$\leq CM_b f(x),$$

and thus, by Theorem 1,

$$I_2 \leq C\lambda^{-1}||f||_{K_q^{\alpha,p}(\mathbb{R}^n)} \left(1 + \log^+(\lambda^{-1}||f||_{K_q^{\alpha,p}(\mathbb{R}^n)})\right).$$

The proof of (2) may be obtained by using Theorem 2.

This finishes the proof of Theorem 5. □

THEOREM 6. *Let $b \in BMO(\mathbb{R}^n)$ and $0 < l < n$. Suppose that the linear operator T_l satisfies*

$$|T_l f(x)| \leq C|x|^{-(n-l)} \int |f(y)|dy$$

for $f \in L^1_{loc}(\mathbb{R}^n)$, supp$f \subset A_k$ and $|x| \geq 2^{k+1}$ with $k \in \mathbb{Z}$. Assume $0 < p_1 \leq p_2 \leq 1 < q_1 < n/l$, $\alpha = n(1 - 1/q_1)$, $1/q_2 = 1/q_1 - l/n$, and that $[b, T_l]$ is of weak type (q_1, q_2). Then

(1) *For any $\lambda > 0$ and $f \in K_{q_1}^{\alpha_1 p_1}(\mathbb{R}^n)$, we have*

$$\left[\sum_{k=0}^{\infty} 2^{k\alpha p_2} \tilde{m}_k(\lambda, [b, T_l]f)^{p_2/q_2}\right]^{1/p_2}$$
$$\leq C\lambda^{-1}||f||_{K_{q_1}^{\alpha,p_1}(\mathbb{R}^n)}(1 + \log^+(\lambda^{-1}||f||_{K_{q_1}^{\alpha,p_1}(\mathbb{R}^n)})).$$

(2) *Furthermore, if b satisfies the condition L, then $[b, T_l]$ is bounded from $\dot{K}_{q_1}^{\alpha,p_1}(\mathbb{R}^n)$ (or $K_{q_1}^{\alpha,p_1}(\mathbb{R}^n)$) to $W\dot{K}_{q_2}^{\alpha,p_2}(\mathbb{R}^n)$ (or $WK_{q_2}^{\alpha,p_2}(\mathbb{R}^n)$).*

The proof is similar to the proof of Theorem 3, we only notice that, if $x \in A_k$ and supp $a_j \subset B_j$ with $j \leq k - 3$,

$$\left|[b, T_l]\left(\sum_{j\leq k-3} \lambda_j a_j\right)(x)\right| \leq CM_b^{n/l} f(x),$$

then, we obtain the conclusion of Theorem 6 by using Theorem 3.

THEOREM 7. *Let $b \in BMO(\mathbb{R}^n)$ and $0 < l < n$. Suppose that the linear operator T_l satisfies*

$$|T_l f(x)| \leq C|x|^{-(n-l)} \int |f(y)|dy$$

for $f \in L^1_{loc}(\mathbb{R}^n)$, *supp*$f \subset A_k$ *and* $|x| \geq 2^{k+1}$ *with* $k \in \mathbb{Z}$. *Assume* $0 < p_1 \leq p_2 \leq 1 < q_1 < n/l$, $\alpha_1 = n(1 - 1/q_1)$, $\alpha_2 = \alpha_1 + l(p_1/q_1 - 1)$, $1/q_2 = 1/q_1(1 - lp_1/n)$, *and that* $[b, T_l]$ *is of weak type* (q_1, q_2). *Then*

(1) *For any* $\lambda > 0$ *and* $f \in K^{\alpha_1,p_1}_{q_1}(\mathbb{R}^n)$, *we have*

$$\left[\sum_{k=0}^{\infty} 2^{k\alpha_2 p_2} \tilde{m}_k(\lambda : [b, T_l]f)^{p_2/q_2}\right]^{1/p_2} \leq C\lambda^{-1}\|f\|_{K^{\alpha_1,p_1}_{q_1}(\mathbb{R}^n)}(1 + \log^+(\lambda^{-1}\|f\|_{K^{\alpha_1,p_1}_{q_1}(\mathbb{R}^n)})).$$

(2) *Furthermore, if* b *satisfies the condition* L, *then* $[b, T_l]$ *is bounded from* $\dot{K}^{\alpha_1 p_1}_{q_1}(\mathbb{R}^n)$ *(or* $K^{\alpha_1,p_1}_{q_1}(\mathbb{R}^n)$*) to* $W\dot{K}^{\alpha_2,p_2}_{q_2}(\mathbb{R}^n)$ *(or* $WK^{\alpha_2,p_2}_{q_2}(\mathbb{R}^n)$*).*

The proof is similar. We omit the details.

COROLLARY 1. *If the size condition of* T *in Theorem 5 is replaced by*

$$|Tf(x)| \leq C \int |f(y)|\, |x - y|^{-n} dy \tag{1.1}$$

for $f \in L^1_{loc}(\mathbb{R}^n)$ *with compact support and* $x \notin$ *supp*f. *Then the conclusions of Theorem 5 also hold.*

COROLLARY 2. *If the size condition of* T *in Theorem 6 and Theorem 7 is replaced by*

$$|T_l f(x)| \leq C \int |f(y)|\, |x - y|^{-(n-l)} dy \tag{1.2}$$

for $f \in L^1_{loc}(\mathbb{R}^n)$ *with compact support and* $x \notin$ *supp*f. *Then the conclusions of Theorem 6 and Theorem 7 also hold.*

REMARK 1. The size conditions (1.1) and (1.2) are satisfied by many operators in harmonic analysis, such as Calderon-Zygmund operators, Fefferman's singular multiplier, Ricci-Stein's oscillatory singular integral, the Bochner-Riesz operators at the critical index, fractional integral operators and so on. Thus, the weak type estimates of these operators in Herz spaces are obtained.

REMARK 2. If b does not to satisfy the condition L, the weak type estimates of $[b, T]$ in the homogeneous Herz space is still an open problem.

References

[1] S. Chanillo, *A note on commutators*, Indiana Univ. Math. J. **31** (1982), 7–16.
[2] R. Coifman, R. Rochberg, and G. Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math. **103** (1976), 611–635.
[3] G. Hu, S. Lu and, D. Yang, *The weak Herz spaces*, J. of Beijing Normal University (Natural science) **31** (1997), 27–36.
[4] ———, *The applications of weak Herz spaces*, Adv. Math. (China) **26** (1997), 417–428.
[5] Z. Liu, *Boundedness of commutators of Calderon-Zygmund singular integral operator on Herz type spaces*, Adv. Math. (China) (to appear).
[6] S. Lu and D. Yang, *The decomposition of the weighted Herz spaces and its applications*, Sci. China Ser. A **38** (1995), 147–158.
[7] ———, *The continuity of commutators on Herz type spaces*, Michigan Math. J. **44** (1997), 255–281.
[8] S. Lu, L. Tang, and D. Yang, *Boundedness of commutators on homogeneous Herz spaces*, Sci. China Ser. A **41** (1998), 1023–1033.
[9] C. Perez, *Endpoint estimates for commutators of singular integral operators*, J. Func. Anal. **128** (1995), 163–185.
[10] C. Segovia and J. L. Torrea, *Vector valued commutators and applications*, Indiana Univ. Math. J. **38** (1989), 959–971.
[11] ———, *Higher order commutators for vector valued Calderon -Zygmund operators*, Trans. Amer. Math. Soc. **336** (1993), 537–556.
[12] E. M. Stein, *Harmonic Analysis:real-variable methods, orthogonality and oscillatory integrals*, Princeton Univ. Press, Princeton, NJ, 1993.

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