# THE INERTIA OF THE WATER SURROUNDING A VIBRATING SHIP

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1. The important elements which determine the natural frequency of a vibrating ship are its mass and the distribution of this mass, and the rigidity of the hull structure. In very general terms,

$$\text{The natural frequency varies as } \sqrt{\frac{\text{rigidity}}{\text{mass}}}.$$

It has long been recognized that the water surrounding a vibrating ship produces an effect equivalent to a very considerable increase in the mass of the ship. In the present paper we discuss this water inertia effect. The discussion will be limited to vertical flexural vibration of two and three nodes.

2. At the present time the natural frequency of a ship can be calculated, if there is any attempt at accuracy, only by more or less empirical methods; that is, by comparison with similar ships of known frequency. The unknown elements which have so far prevented a purely rational calculation are the water inertia and the rigidity of the hull in flexure and shear.

We believe that the first item can be evaluated with fair accuracy by the methods explained in this paper. Accurate knowledge regarding the second is still lacking. It is recognized that the flexural rigidity of a hull is somewhat less than that based on the nominal I of its cross-section, probably due to the tendency of compression members to shirk their load. This point can be cleared up only by careful experiment. The classic experiments on the *Wolf* give practically the only results at present available.† These were made some twenty-five years ago on a hull of very light scantlings. Until further experiments on hull rigidity have been made, the frequency calculations must rest upon an empirical basis.

3. While this matter of the calculation of hull frequency is of great interest, we take this opportunity to point out again that the road to the ideal vibrationless ship does not lie in that direction. Such a ship can be produced only by the elimination of all periodic exciting forces exerted on the hull from engines or propellers. So long as such periodic forces exist, some vibration will exist in the hull at all speeds, although its amplitude may be very small. If no periodic forces act upon the hull, there will be no vibration in it at any speed, regardless of its natural frequency. Therefore, every effort should be made to avoid such periodic forces, and if they can be avoided the natural frequency of the hull is unimportant.

4. The evaluation of the water inertia by experimental model methods offers very

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†J. H. Biles, "The Strength of Ships with Special Reference to Experiments and Calculations made upon H. M. S. *Wolf*. Trans. Inst. of N. A., 1905.

considerable difficulties. Experiments have been made by Nicholls, which we discuss later, using a series of wooden blocks fastened to a steel strip, but the results are not applicable to ship forms.

5. We attempt to evaluate the water inertia by purely mathematical methods. No exact solution is possible, but with certain approximations and assumptions a result can be obtained which we believe to be sufficiently accurate for any practical purpose. We consider these approximations in order.

6. In Fig. 1, Plate 1, the full lines represent the underbody of a ship and the dotted lines its "image," obtained by inverting the underbody about the water plane. It is assumed (A) that for vertical vibration the inertia of the water surrounding the actual ship with a free water surface is one-half of that of the ship and its image together, vibrating totally immersed in a fluid extending to infinity in all directions. With such an assumption the median plane is no longer a constant pressure surface, but the error involved is of the order of the square of the velocity and, because of the small amplitude of vibration of a ship, will be of very small magnitude. For transverse vibration this assumption will not hold, and we make no attempt in this paper to evaluate the water inertia for this case.

7. It is assumed (B) that the frequency of a ship, and consequently the water inertia, is the same whether it is moving or stationary. Experiments have been made on this point, the frequency being determined with the ship under way and when at rest with the propeller disconnected, and no change in the frequency could be detected.

8. It is assumed (C) that no eddies or other discontinuities in the fluid motion, due to the vibration, are present. This is justified by the extremely small magnitude of the fluid velocities, so that perfect stream-line motion should obtain even with abrupt changes of shape.

9. Since the length of a ship is several times its beam, it is evident that the motion of the water will, for the greater part, be parallel to transverse planes.

Consider a ship vibrating vertically. The kinetic energy of the surrounding water at the instant of maximum velocity has a certain value. Imagine the water to be confined by a series of thin transverse partitions extending to infinity so that it could move only in transverse planes. Then the kinetic energy of the fluid would be greater than if it were not so confined. We designate the ratio of these two kinetic energies as the longitudinal inertia coefficient $J$.

$$J = \frac{\text{Actual K. E. of surrounding fluid}}{\begin{array}{c}\text{K. E. of surrounding fluid if motion is confined}\\ \text{to transverse planes.}\end{array}} \tag{1}$$

$J$ obviously depends on the proportions and shape of the underwater body, and will approach unity as the ratio of length to beam becomes greater and greater.

10. Among the limited number of three dimensional problems in fluid motion for which an exact solution has been obtained is that of the ellipsoid of revolution. In Appendix I we give the solution for the flow around an ellipsoid of revolution vibrating perpendicularly to its axis in two and three nodes. We assume (D) that the longitudinal inertia coefficient $J$ for a vibrating ship is the same as for such a vibrating ellipsoid of revolution having the same ratio of length to beam.

A closer approximation would probably be given if we took an ellipsoid of three unequal axes, making these the length, beam, and twice the draft, but we have been unable to obtain a mathematical solution for this case.

A graph of $J$ for two and three noded vibration is given in Plate 4. With assumption (D) the problem is reduced to one of two dimensional flow.

11. While the problem of finding the two dimensional stream-line flow past a cylinder of specified shape has never been solved, the powerful methods of conformal representation yield solutions for a great variety of shapes, some of which may approximate more or less closely to actual ship sections. Among these shapes around which the flow can be determined mathematically, we find that those given by the parametric equations

$$X = [(1 + a) \cos \theta + b \cos 3\theta] \div (1 + a + b)$$
$$Y = [(1 - a) \sin \theta - b \sin 3\theta] \div (1 + a + b)$$

give an approximation to actual ship sections. The approximation is especially good for the sections with flat bottoms and rounded bilges amidships where the water inertia effect is most important.

In Plates 2 and 3 are drawn the sections having the foregoing parametric equations for various values of Half Beam/Draft and constant $b$. The quarter section of the complete cylinder (ship and image) is drawn in each case.

12. For a cylinder of elliptical section moving in a fluid the kinetic energy of the surrounding fluid per unit length is given by*

$$2T = \frac{\pi B^2 w v^2}{g}$$

where

$T$ = kinetic energy.
$B$ = half breadth perpendicular to the motion.
$w$ = weight per unit volume of fluid.
$g$ = gravity constant.
$v$ = velocity.

In this analysis we also take

$L$ = half length or semi-major axis of ellipsoid.
$B$ = half beam or semi-minor axis of ellipsoid.
$D$ = draft.

For any other shape of section of the same axial breadth we write the energy as

$$2T = \frac{C \pi B^2 w v^2}{g}$$

and call $C$ the inertia coefficient for that section.

The surrounding fluid has the same effect as the addition to the cylinder of a mass per unit length

$$M = C \pi B^2 w$$

$\frac{w}{g}$ ?

For our ship sections we will write the additional inertia mass to be added per unit length as

$$M = \tfrac{1}{2}\, C J \pi B^2 w \qquad (2)$$

*Lamb, Hydrodynamics, Par. 70 (11) or see Formula (9), Par. 29 of this paper with $b = 0$.

where $J$ is the appropriate longitudinal inertia coefficient for the length beam ratio of the ship and the 2 is introduced since we are interested in the ship alone and not in ship and image together, as explained in Par. 6.

13. In Plates 2 and 3 the number noted on each section is the inertia coefficient $C$ for that section. These sections are drawn for ratios of Half Beam/Draft = of .2, .4, .6, .8, 1.0, 1.2, 1.5, 2.0, as marked on each. The sections marked 1.0 are ellipses.

There is also shown on each section the inertia coefficient for the rectangle and the inertia coefficient for the rhombus of the same axial dimensions. These are obtained as explained in Appendixes III and IV. Plate 4 gives $C$ for the rectangle and rhombus for a range of values of $B/D$. We assume (E) that the inertia coefficient for actual ship sections can be determined by comparison with the figures of Plates 2 and 3. Unit

14. The calculation for a ship is carried out as follows:

The body plan is drawn for the usual ten divisions. It is necessary to estimate $C$ for each of the stations. This may be done by visual comparison of the body plan with the sections of Plates 2 and 3, or more accurately as follows: For each station the value of Half Beam/Draft = $H$ is determined. Using a pair of proportional dividers, each section is transferred to tracing paper, its dimension being expanded or contracted so that it will have the same half beam and draft as the section in Plates 2 or 3 having the nearest larger $H$ value. Another section is drawn for the nearest smaller $H$ value.

Laying the sections on tracing paper over the appropriate figures in Plates 2 and 3, the $C$ value of each may be estimated. The $C$ value of the actual ship section is obtained by interpolation between the two values of $C$ for adjacent $H$ values.

For each of the stations the water inertia mass per foot of length will be given by (2), where $B$ is the half breadth on the water line at that station.

The curve of water inertia over the length of the ship is thus constructed and added to the ordinary weight curve. The combined weight curve is used to calculate the natural frequency by any of the methods which have been devised.

15. As an illustration we give the calculation for the ship whose weight curve was given by Nicholls.* The characteristics of the ship are as given on Plate 5. While the lines there shown are not the actual ones belonging to the weight curve given, they have the same overall dimensions and coefficients and will serve for purposes of illustration. The values of $C$ at each station were determined as per Par. 14 and are noted on each station. The $J$ value from Plate 4 is .86, corresponding to $L/B = 10.2$. Calculating the water inertia mass at each station by (2), the curve of water mass is drawn to the same scale as the weight curve. The total water inertia mass is 1,615 tons. The actual weight of the ship is 1,378 tons. The distribution of this mass is equally important with its magnitude, however. Calculating by the tabular method given by the writer,† we find the curves for the neutral axis of the bending, when the water inertia is and is not taken into account, to be as shown on Plate 5. With the water inertia allowed for, the nodes are nearer the center and the ratio, amplitude at ends over amplitude at center, is considerably greater, than when no such allowance is made. In the example cited this ratio has been raised from 2.6 to 3.6. The frequency has been lowered in the ratio $1 \div \sqrt{1.68}$, or 23 per cent. Thus, while the mass of the ship has been increased 117 per cent, this would be equivalent in lowering the frequency to an increase of only 68 per cent, provided the 68 per cent increase had the same distribution as the ordinary weight curve. The difference

*"The Vibration of Ships," H. W. Nicholls. Trans. Inst. of Naval Architects, 1924.

†"Vibration and Engine Balance in Diesel Ships." Soc. Naval Arch. and Marine Engineers, 1927.

between these two percentages, the actual increase of mass and the virtual increase, shows that the distribution of this mass is a vital factor and must be taken into account.

It can be stated that this large increase in mass is, in general, consistent with the elastic properties of this hull and its known frequency, but due to lack of accurate knowledge regarding these elastic properties no precise correlation of the various factors which fix the frequency can be made at the present time.

16. Certain general conclusions can be drawn.

The flat bottom amidships is of the greatest importance. Due to this the greater part of the added mass is amidships, with the result that the ratio of amplitude at ends over amplitude amidships is much greater than would otherwise obtain.

The water inertia is nearly independent of draft so that the natural frequency will vary only slowly with the displacement, and it will not vary as the square root of the displacement for different displacements of the same ship.

17. It is of interest to compare the observed frequency in the experiments of Nicholls* with those calculated by the foregoing methods. Nicholls' experiments were made with steel bars, supported at their free-free nodes, to which were fastened wooden blocks. The frequency was determined in air and at various depths of immersion. The calculated frequencies in water have been obtained from the observed air frequency on the assumption that the frequencies vary inversely as the square root of the actual mass plus the water inertia mass.

*Case 1.*—Bar of steel 30″ × 2″ × .312″ carrying wood of thickness 1¼″. Weight of steel and wood in air 102.7 ounces.

| | In air | In water | | | |
|---|---|---|---|---|---|
| Depth of immersion | .......... | 0 | ½″ | 1″ | 1¼″ |
| Observed frequency | 89.4 | 80.26 | 78.77 | 77.96 | 77.59 |
| Calculated frequency | .......... | 81.0 | 78.3 | 77.3 | 76.8 |

*Case 3.*—Bar of steel 30″ × 2″ × .119″ carrying wood of thickness 1¼″. Weight of steel and wood in air 48.54 ounces.

| | In air | In water | | | |
|---|---|---|---|---|---|
| Depth of immersion | .......... | 0 | ½″ | 1″ | 1¼″ |
| Observed frequency | 49.6 | 40.57 | 39.13 | 38.61 | 38.50 |
| Calculated frequency | .......... | 40.8 | 38.6 | 37.9 | 37.4 |

In the foregoing the $L/B$ ratio is 15, the corresponding $R$ value is .908, and for the nodal position of a free-free bar $J$ would be .93. Due to the restriction of the transverse flow by the rectangular section, it is evident that $J$ will be somewhat smaller, and we have arbitrarily assumed it at .85. The $C$ values were taken from Plate 4.

*Case 2.*—Wood triangular in section, weight of bar and wood in air 94.18 ounces. The observed frequency in air was 89.48 V. P. S. and in water at 1¼ inches depth of immersion

*T. I. N. A., 1924.

82.83. The calculated frequency is 82.25. The longitudinal inertia coefficient was taken as .85 and the section coefficient .730 from the curve of Plate 4 for the rhombus.

Since a ship form resembles the ellipsoid of revolution, for which the calculation is theoretically exact, much closer than these rectangular bars, it is to be expected that the water inertia effect will be estimated much closer for such ship forms than with these experimental bars. It is possible, also, that surface tension has a slight modifying effect in these small scale experiments.

## APPENDIX I

18. The Inertia Coefficient of an Ellipsoid of Revolution, Vibrating in Two or Three Nodes.

A system of orthoganal coordinates consisting of the confocal ellipsoids of revolution $\zeta$ = constant, the confocal hyperboloids of revolution $\mu$ = constant, and the planes $\omega$ = constant, cutting the axis of the ellipsoid, is adopted. Then a solution of Laplace's equation $\nabla^2 \phi = 0$ can be expressed in terms of the Associated Legendre Functions of these coordinates. By choosing and combining the proper functions, the flow corresponding to a transverse motion of the ellipsoid, or to a bending motion of two or more nodes, can be obtained. It will be seen by equation (22) that this "bending" motion is in reality a shear, transverse section of the ellipsoid staying in the same plane. The distinction is unimportant for our purposes, however.

It is impossible in a paper of this length to give any adequate explanation of these mathematical methods, and for a preliminary study the reader should consult Chapter V, Arts. 103 to 107, of Lamb's Treatise.* The notation here used is that adopted by Lamb with the exception that $r$ is used in place of $\bar{\omega}$ and $\dot{Q}$ denotes $\frac{\partial Q}{\partial \zeta}$.

The result of this analysis is stated beginning at paragraph 25.

19. Properties of the ellipsoidal coordinates.

Referring to Fig. 2, Plate 1:

The distance between the foci is 2 $k$.

The semi-major axis of any of the ellipsoids of revolution is

$$L = k\zeta \tag{1}$$

$$\text{The semi-minor axis } B = k\sqrt{\zeta^2 - 1} \tag{2}$$

$$\text{The volume of any ellipsoid } \tfrac{4}{3}\pi k^3 \zeta(\zeta^2 - 1) \tag{3}$$

If $\zeta$, $\mu$, $\omega$ are the ellipsoidal coordinates and $x$, $r$, $\omega$ polar coordinates, then the relations between these coordinates are given by the equations

$$x = k\mu\zeta \tag{4}$$

$$r = k(1 - \mu^2)^{1/2}(\zeta^2 - 1)^{1/2} \tag{5}$$

$$\omega = \omega \tag{6}$$

The element of arc of the ellipsoid is

$$ds_\mu = \frac{k(\zeta^2 - \mu^2)^{1/2}}{(1 - \mu^2)^{1/2}}\, d\mu \tag{7}$$

*"Hydrodynamics," H. Lamb. Cambridge University Press.

and the length of an elementary normal

$$dn = \frac{k\,(\zeta^2 - \mu^2)^{1/2}}{(\zeta^2 - 1)^{1/2}}\,d\zeta \tag{8}$$

The cosine of the angle between the tangent and the $x$ axis is given by

$$\cos\theta = \frac{\zeta(1 - \mu^2)^{1/2}}{(\zeta^2 - \mu^2)^{1/2}} \tag{9}$$

20. A solution of Laplace's equation is given by

$$\phi = [\beta_1 P_1'(\mu) Q_1'(\zeta) + \beta_3 P_3'(\mu) Q_3'(\zeta)] \cos\omega \tag{10}$$

Where $P'$ and $Q'$ are the associated Legendre functions of $\mu$ and $\zeta$ and $\beta_1$ and $\beta_3$ are arbitrary constants.

The above solution represents a motion of the ellipsoid in which the axis is deformed into the arc of a parabola. The relative values of $\beta_1$ and $\beta_3$ determine the position of the axis with respect to the parabola and consequently the position of the nodes.

In like manner the solution

$$\phi = [\beta_2 P_2'(\mu) Q_2'(\zeta) + \beta_4 P_4'(\mu) Q_4'(\zeta)] \cos\omega \tag{11}$$

will represent a 3-noded motion, in which the axis is deformed into a cubical parabola.

21. The kinetic energy of the motion is given by

$$2T = -\rho \iint \phi \frac{\partial\phi}{\partial n}\,dS \;^{*} \tag{12}$$

or

$$2T = -\rho \iint \phi \frac{\partial\phi}{\partial\zeta} \frac{\partial\zeta}{\partial n}\, r\,d\omega\,ds_\mu \tag{13}$$

and substituting the values of $\frac{\partial\zeta}{\partial n}$, $r$ and $ds_\mu$ previously given, this reduces to

$$2T = -k\rho \iint \phi \frac{\partial\phi}{\partial\zeta}(\zeta_0{}^2 - 1)\,d\mu\,d\omega \tag{14}$$

$\zeta_0$ is the particular value of $\zeta$ associated with the surface of the ellipsoid, over which the integration extends.

Substituting the value of $\phi$ for the 2-noded solution:

$$\begin{aligned}2T = -k\rho(\zeta_0{}^2 - 1)\Big\{\int_{-1}^{+1}\int_0^{2\pi} & [\beta_1 P_1'(\mu) Q_1'(\zeta) + \beta_3 P_3'(\mu) Q_3'(\zeta)] \\ & \times [\beta_1 P_1'(\mu)\dot{Q}_1'(\zeta) + \beta_3 P_3'(\mu)\dot{Q}_3'(\zeta)]\Big\} \cos^2\omega\,d\mu\,d\omega\end{aligned} \tag{15}$$

where $\dot{Q}$ represents differentiation with respect to $\zeta$

We have

$$\int_0^{2\pi} \cos^2\omega\,d\omega = \pi \tag{16}$$

$$\int_{-1}^{+1} P_1'(\mu) P_3'(\mu)\,d\mu = 0 \qquad \text{Lamb Art. 87 (4)} \tag{17}$$

*Lamb. Art. (46) 14.

$$\left.\begin{aligned}\int_{-1}^{+1}[P_1'(\mu)]^2d\mu &= \tfrac{4}{3} \quad (18)\\ \int_{-1}^{+1}[P_3'(\mu)]^2d\mu &= \tfrac{24}{7} \quad (19)\end{aligned}\right\}\quad \text{Lamb Art. 87 (6)}$$

So that equation (15) reduces to

$$2T = k\rho\pi(\zeta_0{}^2 - 1)\left\{\tfrac{4}{3}\,\beta_1{}^2Q_1'(\zeta)\dot{Q}_1'(\zeta) + \tfrac{24}{7}\,\beta_3{}^2Q_3'(\zeta)\dot{Q}_3'(\zeta)\right\} \tag{20}$$

In like manner for 3-noded vibration the energy equation is

$$2T = k\rho\pi(\zeta_0{}^2 - 1)\left\{\tfrac{4}{3}\,\beta_2{}^2Q_2'(\zeta)\dot{Q}_2'(\zeta) + \tfrac{40}{9}\,\beta_4{}^2Q_4'(\zeta)\dot{Q}_4'(\zeta)\right\} \tag{21}$$

22. The velocity of the surface at any point normal to itself will be

$$\left.\begin{aligned}&(V_1 - V_3\mu^2)\cos\theta\cos\omega \qquad \text{2-noded}\\ &(V_2\mu - V_4\mu^3)\cos\theta\cos\omega \qquad \text{3-noded}\end{aligned}\right\} \tag{22}$$

and this must equal the velocity of the fluid normal to the surface which will be

$$\frac{\partial\phi}{\partial n} = \frac{\partial\phi}{\partial\zeta}\cdot\frac{\partial\zeta}{\partial n} \tag{23}$$

combining (22) (23) and (8) (9) there is obtained

$$\begin{aligned}\frac{\partial\phi}{\partial\zeta} &= -\,(V_1 - V_3\mu^2)\,\frac{\zeta(1-\mu^2)^{1/2}}{(\zeta^2-1)^{1/2}}\,k\cos\omega\\ &= [\beta_1P_1'(\mu)\dot{Q}_1'(\zeta) + \beta_3P_3'(\mu)\dot{Q}_3'(\zeta)]\cos\omega \qquad (24)\end{aligned}$$

for $\zeta=\zeta_0$. In all the following equations it will be understood that $\zeta=\zeta_f$.

$$\left.\begin{aligned}P_1'(\mu) &= (1-\mu^2)^{1/2}\dot{P}_1(\mu) = (1-\mu^2)^{1/2}\\ P_3'(\mu) &= (1-\mu^2)^{1/2}\dot{P}_3(\mu) = \tfrac{3}{2}(5\mu^2-1)(1-\mu^2)^{1/2}\end{aligned}\right\} \tag{25}$$

Lamb Art (85)

Substituting the above in (24) and equating the coefficients of like functions of $\mu$ there is obtained

$$\left.\begin{aligned}\beta_3 &= \frac{2V_3k\zeta}{15\,(\zeta^2-1)^{1/2}\,\dot{Q}_3'(\zeta)}\\ \beta_1 &= -\,\frac{(V_1 - V_3/5)k\zeta}{(\zeta^2-1)^{1/2}\,\dot{Q}_1'(\zeta)}\end{aligned}\right\} \tag{26}$$

and substituting in the energy equation (20) there is obtained for 2-noded vibration

$$2T = -\,\pi\rho k^3\zeta^2\left[\tfrac{4}{3}\,(V_1 - V_3/5)^2\,\frac{Q_1'(\zeta)}{\dot{Q}_1'(\zeta)} + \frac{32}{525}\,V_3{}^2\,\frac{Q_3'(\zeta)}{\dot{Q}_3'(\zeta)}\right] \tag{27}$$

In like manner for 3-noded vibration

$$2T = -\,\pi\rho k^3\zeta^2\left[\tfrac{4}{15}\,(V_2 - 3\,V_4/7)^2\,\frac{Q_2'(\zeta)}{\dot{Q}_2'(\zeta)} + \frac{160}{9\cdot35^2}\,V_4{}^2\,\frac{Q_4'(\zeta)}{\dot{Q}_4'(\zeta)}\right] \tag{28}$$

23. The Legendre $Q$ functions are defined by the equations

$$\left.\begin{aligned}
Q_n'(\zeta) &= (\zeta^2 - 1)^{1/2}\, \dot{Q}_n(\zeta) \\
Q_1(\zeta) &= \tfrac{1}{2}\zeta \log \frac{\zeta + 1}{\zeta - 1} \\
Q_2(\zeta) &= \tfrac{1}{4}\,(3\,\zeta^2 - 1) \log \frac{\zeta + 1}{\zeta - 1} - \tfrac{3}{2}\,\zeta \\
Q_3(\zeta) &= \tfrac{1}{4}\,(5\,\zeta^3 - 3\zeta) \log \frac{\zeta + 1}{\zeta - 1} - \tfrac{5}{2}\,\zeta^2 + \tfrac{2}{3} \\
Q_4(\zeta) &= \tfrac{1}{16}\,(35\,\zeta^4 - 30\,\zeta^2 + 3) \log \frac{\zeta + 1}{\zeta - 1} - \tfrac{35}{8}\,\zeta^3 + \tfrac{55}{24}\,\zeta
\end{aligned}\right\} \tag{29}$$

Performing the necessary differentiations upon the equation (29), substituting in equations (27) and (28) and simplifying there is finally obtained for 2-noded vibration

$$2T = M\left[(V_1 - V_3/5)^2 R_1 + \tfrac{8}{175}\, V_3{}^2 R_3\right] \tag{30}$$

and for 3-noded

$$2T = M\left[\frac{(V_2 - \frac{3}{7}\, V_4)^2}{5} R_2 + \tfrac{8}{735}\, V_4{}^2 R_4\right] \tag{31}$$

where $M$ is the mass of an ellipsoid of density $\rho$ and $R_1$ $R_2$ $R_3$ $R_4$ are functions of $\zeta$ given by the equations.

$$\left.\begin{aligned}
R_1 &= -\frac{\frac{3}{2}\,\zeta \log \frac{\zeta + 1}{\zeta - 1} - \frac{3\,\zeta^2}{\zeta^2 - 1}}{\frac{3}{2}\,\zeta \log \frac{\zeta + 1}{\zeta - 1} - \frac{3\,\zeta^2 - 6}{\zeta^2 - 1}} \\
R_2 &= -\frac{\frac{3}{2}\,\zeta^2 \log \frac{\zeta + 1}{\zeta - 1} - \frac{3\,\zeta^3 - 2\,\zeta}{\zeta^2 - 1}}{\frac{3}{2}\,(2\,\zeta^2 - 1) \log \frac{\zeta + 1}{\zeta - 1} - \frac{6\,\zeta^3 - 7\,\zeta}{\zeta^2 - 1}} \\
R_3 &= -\frac{\frac{3}{2}\,(5\,\zeta^3 - \zeta) \log \frac{\zeta + 1}{\zeta - 1} - \frac{15\,\zeta^4 - 13\,\zeta^2}{\zeta^2 - 1}}{\frac{3}{2}\,(15\,\zeta^3 - 11\,\zeta) \log \frac{\zeta + 1}{\zeta - 1} - \frac{45\,\zeta^4 - 63\,\zeta^2 + 16}{\zeta^2 - 1}} \\
R_4 &= -\frac{\frac{3}{2}\,(35\,\zeta^4 - 15\,\zeta^2) \log \frac{\zeta + 1}{\zeta - 1} - \frac{105\,\zeta^5 - 115\,\zeta^3 + 16\,\zeta}{\zeta^2 - 1}}{\frac{3}{2}\,(140\,\zeta^4 - 135\,\zeta^2 + 15) \log \frac{\zeta + 1}{\zeta - 1} - \frac{420\,\zeta^5 - 685\,\zeta^3 + 259\,\zeta}{\zeta^2 - 1}}
\end{aligned}\right\} \tag{32}$$

24. We wish to obtain the ratio of the actual kinetic energy to that which would exist if the motion of the fluid were confined to transverse planes. The results for a circular cylinder show that the energy in the latter case would be equal to the energy of the ellipsoid itself if of density $\rho$.

The area of any transverse section of the ellipsoid will be $\pi r^2 = \pi k^2 (1-\mu^2)(\zeta^2-1)$ and the velocity at any section for 2-noded vibration

$$v = (V_1 - V_3 \mu^2)$$

then

$$2T' = \rho \pi k^3 \zeta (\zeta^2 - 1) \int_{-1}^{+1} (1 - \mu^2)(V_1 - V_3\mu^2)^2 \, d\mu$$

$$2T' = M\left[(V_1 - V_3/5)^2 + \tfrac{8}{175} V_3^2\right] \tag{33}$$

and likewise for 3-noded vibration

$$2T' = M\left[\frac{(V_2 - \frac{3}{7} V_4)^2}{5} + \tfrac{8}{735} V_4^2\right] \tag{34}$$

25. Summary of results and numerical evaluation.

The coefficient $J$, defined as the ratio of the kinetic energy of the flow around the vibrating ellipsoid, to that which would obtain if the flow were confined to transverse planes, is given by the equations

$$J_{2\ \text{node}} = \frac{(V_1 - V_3/5)^2 R_1 + \frac{8}{175} V_3^2 R_3}{(V_1 - V_3/5)^2 + \frac{8}{175} V_3^2}$$

$$J_{3\ \text{node}} = \frac{\frac{1}{5}(V_2 - \frac{3}{7} V_4)^2 R_2 + \frac{8}{735} V_4^2 R_4}{\frac{1}{5}(V_2 - \frac{3}{7} V_4)^2 + \frac{8}{735} V_4^2}$$

The $R$'s are functions of $\zeta$ as given by equation (32). $\zeta$ determines the proportions of the ellipsoid, the ratio semi-major axis over semi-minor axis being given by $L/B = \zeta/(\zeta^2 - 1)^{1/2}$

We have evaluated the $R$ functions for certain values of $\zeta$ and they are given in the table below with corresponding values of $L/B$. Graphs of the $R$ functions over $L/B$ as a base are also given in Plate 4.

TABLE 1

| $\zeta$ | $L/B$ | $R_1$ | $R_2$ | $R_3$ | $R_4$ | |
|---|---|---|---|---|---|---|
| 1.000 | $\infty$ | 1.000 | 1.0 | 1.0 | 1.0 | cylinder |
| 1.001 | 22.4 | .9888 | .9728 | .9539 | .9319 | |
| 1.002 | 15.82 | .9806 | .9530 | .9220 | .8902 | |
| 1.003 | 12.95 | .9731 | .9368 | .8972 | .8586 | |
| 1.005 | 10.02 | .9604 | .9103 | .8588 | .8084 | |
| 1.007 | 8.48 | .9495 | .8890 | .8291 | .7704 | |
| 1.010 | 7.11 | .9348 | .8608 | .7894 | .7248 | |
| 1.015 | 5.83 | .9141 | .8234 | .7414 | .6702 | |
| 1.020 | 5.07 | .8963 | .7932 | .7037 | .6286 | |
| 1.030 | 4.174 | .8672 | .7468 | .6476 | .5694 | |
| 1.050 | 3.275 | .8226 | .6806 | .5746 | .4944 | |
| $\infty$ | 1.0 | .5000 | .3333 | .2500 | .2000 | sphere |

26. The shape of the axis is given by the equations

$$V_1 - V_3\,\mu^2 = y \qquad \text{2-noded}$$

$$V_2\,\mu - V_4\,\mu^3 = y \qquad \text{3-noded}$$

where $\mu$ varies from 0 at the origin to + or − 1 at the ends of the major axis, so that by taking $V_1/V_3$ and $V_2/V_4$ of the proper value the nodes of the motion can be placed at any desired point.

If we take $V_3 = 5V_1$ for 2-noded vibration, the nodes will be at $\mu = \pm \sqrt{.2} = \pm .447$, or they are at .447 the length of the semi-major axis from the center. This is very nearly at the point found in a vibrating ship. Likewise, with $V = 7/3V_2$ for 3-noded vibration the nodes will be at $\mu = 0, \pm .65$, a position corresponding fairly close to the nodes of actual ships. With these ratios the equations (35) (36) reduce to

$$\left.\begin{aligned} J_2 \text{ node} &= R_3 \\ J_3 \text{ node} &= R_4 \end{aligned}\right\} \tag{37}$$

For slight variations in the nodal position the variation in $J$ would not be very great, and considering the approximate nature of the step from the ellipsoid to the ship it is considered that the equation (37) give the result with sufficient accuracy. It may be noted also that these positions of the nodes are the ones corresponding to the minimum kinetic energy of the flow.

## APPENDIX II

### THE INERTIA COEFFICIENT FOR CYLINDERS OF APPROXIMATE SHIP SECTION

27. We show how the shapes of Plates 2 and 3 and the energy of the flow around each are obtained.

Let $x$, $y$ be the coordinates of any point of a two dimensional flow and $\phi$, $\psi$ the values of the velocity potential and stream-line functions at the same point. Then the equation

$$z = (x + iy) = f\,(\phi + i\psi)$$

where $f$ designates any arbitrary function represents a two dimensional stream-line flow.*

The flow in the $x$, $y$ plane can be further transformed to any other plane, say $X$, $Y$, by the relation

$$Z = X + iY = f_1\,(x + iy)$$

since $(X + iY)$ is still a function of $\phi + i\psi$.

We take the flow past the circle

$$x^2 + y^2 = 1 \qquad \text{or} \qquad r = 1$$

and transform it by means of the relation

$$Z = X + iY = z + \frac{a}{z} + \frac{b}{z^3} = z + az^{-1} + bz^{-3} \tag{1}$$

but

$$z = x + iy = re^{i\theta} = e^{i\theta}$$

so that

$$X + iY = e^{i\theta} + ae^{-i\theta} + be^{-3i\theta} \tag{2}$$

and since

$$e^{i\theta} = \cos\theta + i\sin\theta$$

*Lamb, Chapter IV.

we have, by equating the real and imaginary parts

$$\left.\begin{aligned} X &= (1 + a)\cos\theta + b\cos 3\,\theta \\ Y &= (1 - a)\sin\theta - b\sin 3\,\theta \end{aligned}\right\} \tag{3}$$

These are the parametric equations of the transformed circle.

The semi-axis parallel to the flow will be of length

$$X_1 = (1 + a + b)$$

and perpendicular to the flow

$$Y_1 = (1 - a + b)$$

so that the ratio Half Beam over Draft will be

$$H = B/D = \frac{1 + a + b}{1 - a + b} \tag{4}$$

28. For the flow past the circular cylinder $\phi$ and $\psi$ are given by*

$$\phi = U\left(r + \frac{1}{r}\right)\cos\theta \qquad \psi = U\left(r - \frac{1}{r}\right)\sin\theta \tag{5}$$

and for the boundary $r = 1$

$$\phi = 2\,U\cos\theta \qquad \psi = 0$$

This represents the flow past a stationary cylinder. We wish to obtain the energy of the flow of a moving cylinder in a fluid stationary at infinity; we must accordingly add to the values of $\phi$ and $\psi$ in ~~(6)~~ 5 the terms

$$\phi = -\,UX \qquad \psi = -\,UY \tag{6}$$

giving

$$\left.\begin{aligned} \phi &= U\,[(1 - a)\cos\theta - b\cos 3\,\theta] \\ \psi &= -\,U\,[(1 - a)\sin\theta - b\sin 3\,\theta] \end{aligned}\right\} \tag{7}$$

for the moving cylinder.

29. The kinetic energy per unit length of cylinder is given by

$$2T = \rho\int_0^{\theta = 2\pi} \phi\, d\psi = \pi U^2\rho\,[(1 - a)^2 + 3\,b^2] \tag{8}$$

on substituting the above values of $\phi$ and $\psi$ and integrating.

If the semi-axis of the transformed circle is to be $B$, instead of $(1 - a + b)$, the above energy must be multiplied by the ratio

$$B^2/(1 - a + b)^2$$

giving

$$2T = \pi U^2\rho B^2\left[\frac{(1 - a)^2 + 3\,b^2}{(1 - a + b)^2}\right] \tag{9}$$

The kinetic energy of a circular cylinder of radius $B$ is given by

$$2T = \pi U^2\rho B^2$$

*Lamb. Art. (68).

Therefore the inertia coefficient of the transformed circle is

$$C = \frac{(1-a)^2 + 3b^2}{(1-a+b)^2} \tag{10}$$

30. Solving equations (4) and (10) simultaneously for $a$ and $b$ leads to the equations

$$b = H\left[\frac{-(2CH - H + 1) + (H+1)\sqrt{4C-3}}{2(CH^2 - H^2 - H - 1)}\right] \tag{11}$$

$$a = \frac{(1-H)(1+b)}{(1+H)} \tag{12}$$

and the parametric equations of the transformed circles when of draft $D$ will be

$$\left.\begin{aligned} X &= D\left[\frac{(1+a)\cos\theta + b\cos 3\theta}{(1+a+b)}\right] \\ Y &= D\left[\frac{(1-a)\sin\theta - b\sin 3\theta}{(1+a+b)}\right] \end{aligned}\right\} \tag{13}$$

$X$ being the dimension in the direction of motion and the line $\theta = 0$ coinciding with the $X$ axis.

31. Each shape of Plates 2 and 3 corresponds to definite values of $C$ and $H$ marked upon it. For each of these shapes "b" was calculated by (11) and "a" by (12), then the parametric equations (13) were evaluated and a simple graphical construction used to draw the shapes. The inner curve in each case represents the limiting cusped form. The outermost curve has been taken in each case so as to obtain the best approximation to a rectangle with rounded corners. The stream-lines could be constructed for each of these shapes, but this is not essential to the present investigation.

## APPENDIX III

### THE INERTIA COEFFICIENT OF A RECTANGLE

32. In estimating the inertia coefficient for sections with sharp bilges a knowledge of its value for the rectangle is useful. While we would consider it probable that the equations for the flow past a rectangle have been given before, we have been unable to locate them.

A solution may be effected by the transformation of Schwarz.

Let

$$z = x + iy \quad \text{and} \quad W = \phi + i\psi$$

Then a rectilinear flow may be transformed into a flow with a polygonal boundary by the transformation

$$\frac{dz}{dW} = (W - \phi_1)^{\frac{\alpha_1}{\pi} - 1}(W - \phi_2)^{\frac{\alpha_2}{\pi} - 1}(W - \phi_3)^{\frac{\alpha_3}{\pi} - 1} \dots$$

where $\phi_1$, $\phi_2$, etc., are the particular values of $\phi$ at the corners of the polygon and $\alpha_1$, $\alpha_2$, etc., are the internal angles at corresponding corners.*

* For further explanation see Electricity and Magnetism, J. H. Jeans. Cambridge University Press, Art. (322), or Lamb Art. (73).

33. Applying this to the rectangle, let $\pm k$ be the values of $\phi$ at the corners of the rectangle and $\pm 1$ be the values where the central stream-line meets the rectangle, as shown in Fig. 3, Plate 1. Then

$$\alpha_1 = \alpha_4 = \tfrac{1}{2}\pi \qquad \alpha_2 = \alpha_3 = \tfrac{3}{2}\pi$$

and the transformation becomes

$$\frac{dz}{dw} = (W+1)^{-1/2}(W-1)^{-1/2}(W+k)^{1/2}(W-k)^{1/2}$$

or

$$z = \int \frac{(W^2-k^2)^{1/2}}{(W^2-1)^{1/2}}\,dW \tag{1}$$

We are interested in evaluating this around the boundaries of the rectangle only. For this $\psi = 0$ and we have

$$z = x + iy = \int \frac{(\phi^2-k^2)^{1/2}}{(\phi^2-1)^{1/2}}\,d\phi \tag{2}$$

Along the $D$ side of the rectangle $\phi$ varies from 0 to $\pm k$, and we have

$$x = \int_0^{\phi} \frac{(k^2-\phi^2)^{1/2}}{(1-\phi^2)^{1/2}}\,d\phi \qquad 1 > k > \phi \tag{3}$$

Along the $B$ side, after multiplying (2) by $i$

$$y = -\int_k^{\phi} \frac{(\phi^2-k^2)^{1/2}}{(1-\phi^2)^{1/2}}\,d\phi \qquad 1 > \phi > k \tag{4}$$

For large values of $\phi$, $x = \phi$ so that the velocity of the undisturbed stream is $-1$.

In the equation for the $D$ side (3), let

$$\phi = k \sin \theta_1 \qquad \theta_1 = \sin^{-1} \phi/k$$

Then

$$x = \int_0^{\theta_1} \frac{k^2 \cos^2 \theta_1}{\sqrt{1 - k^2 \sin^2 \theta_1}}\,d\,\theta_1 \tag{5}$$

This is an elliptic integral of known form, having the solution*

$$x = [E(\theta_1, k) - (k')^2 F(\theta_1, k)] \tag{6}$$

where $F$ and $E$ are the elliptic integrals of the first and second kind, respectively, and $k$ is the complementary modulus defined by

$$(k')^2 + k^2 = 1$$

If the half width parallel to the motion ($D$ side) be $D_1$, then

$$D_1 = \left[ E\left(\frac{\pi}{2}, k\right) - (k^1)^2 F\left(\frac{\pi}{2}, k\right) \right] \tag{7}$$

More conveniently, if

$$k = \sin \alpha \qquad k' = \cos \alpha$$

$$D_1 = \left[ E\left(\frac{\pi}{2}, \sin \alpha\right) - \cos^2 \alpha F\left(\frac{\pi}{2}, \sin \alpha\right) \right] \tag{8}$$

*The Integral Calculus, Joseph Edwards, Vol. 1, Art. (391).

Taking the equation for the $B$ side (4) and making the substitution

$$\sin\theta_2 = \frac{\sqrt{1-\phi^2}}{\sqrt{1-k^2}} \qquad \theta_2 = \sin^{-1}\frac{\sqrt{1-\phi^2}}{\sqrt{1-k^2}}$$

we have

$$y = (1-k^2)\int_0^{\theta_2} \frac{\cos^2\theta_2}{\sqrt{1-(k')^2\sin^2\theta_2}}\,d\theta_2 \tag{9}$$

the solution of which is

$$y = E\,(\theta_2, k') - k^2\,F\,(\theta_2, k') \tag{10}$$

and if the half width perpendicular to the direction of motion is $B_1$

$$B_1 = E\left(\frac{\pi}{2}, \cos\alpha\right) - \sin^2\alpha\,F\left(\frac{\pi}{2}, \cos\alpha\right) \tag{11}$$

34. The kinetic energy is given by

$$2T = -\rho\int \phi'\,\frac{\partial\phi'}{\partial n}\,dS \tag{12}$$

where $\phi'$ refers to the flow around a moving rectangle in a fluid stationary at infinity ($\phi$ refers to the flow past a stationary rectangle).

Along the $B$ side of the rectangle

$$\frac{\partial\phi'}{\partial n} = -1 \qquad dS = dy \qquad \phi' = (\phi - D_1)$$

and (12) becomes

$$2T = 4\,\rho\int_0^{B_1} (\phi - D_1)\,dy \tag{13}$$

Along the $D$ side (12) is zero.

From (4)

$$dy = -\,\frac{(\phi^2 - k^2)^{1/2}}{(1-\phi^2)^{1/2}}\,d\phi$$

so that

$$2T = 4\rho\left[\int_k^1 \frac{\phi\,(\phi^2-k^2)^{1/2}}{(1-\phi^2)^{1/2}}\,d\phi - B_1 D_1\right] \tag{14}$$

integrating

$$2T = 4\rho\left[\frac{\pi}{4}\cos^2\alpha - B_1 D_1\right] \tag{15}$$

Dividing by $\pi\rho\,B_1{}^2$ the section inertia coefficient $C$ is found

$$C_{\text{RECT.}} = \frac{\cos^2\alpha}{B_1{}^2} - \frac{4\,D_1}{\pi\,B_1} \tag{16}$$

35. The procedure is then to calculate $B_1$ and $D_1$ for various assumed value of $\alpha$ by (8) and (11). Then $C$ is given by (16) and $B_1/D_1 = H$ gives the proportion of the rectangle. The values of the complete elliptic integrals $E$ and $F$ or $K$ can be taken from any table of the elliptic functions.*

*Pierce's Table of Integrals, or Smithsonian Tables of Elliptic Functions.

Below are tabulated the results for certain values of $\alpha$, and these results are also plotted on Plate 4.

TABLE 2

| $\alpha$ | 0 | 15 | 20 | 25 | 30 | 35 | 40 | 43 |
|---|---|---|---|---|---|---|---|---|
| $H$ | $\infty$ | 16.8 | 8.85 | 5.23 | 3.31 | 2.18 | 1.47 | 1.165 |
| $C$ | 1.0 | 1.095 | 1.152 | 1.211 | 1.276 | 1.348 | 1.427 | 1.476 |
| | Flat Plate | | | | | | | |

| $\alpha$ | 45 | 47 | 50 | 55 | 60 | 65 | 70 | 75 |
|---|---|---|---|---|---|---|---|---|
| $H$ | 1 | .858 | .681 | .459 | .302 | .191 | .113 | .0595 |
| $C$ | 1.5131 | 1.551 | 1.611 | 1.720 | 1.846 | 1.995 | 2.177 | 2.412 |
| | Square | | | | | | | |

## APPENDIX IV

### THE INERTIA COEFFICIENT FOR A RHOMBUS MOVING AXIALLY

36. Referring to Fig. 4, Plate 1, it will be seen that $\alpha_1 = d_3 = \gamma\pi \quad \alpha_2 = \pi(3 - 2\gamma)$ and therefore the appropriate Schwarzian transformation is

$$\frac{dz}{dW} = (W^2 - 1)^{\gamma - 1} W^{2(1 - \gamma)}$$

or

$$Z = re^{i\theta} = \int \frac{W^{2(1-\gamma)}}{(W^2 - 1)^{1-\gamma}} \, dW \tag{1}$$

along the sides $\psi = 0$. Let $r$ denote the distance along these sides from the point marked $O$. Then by the elimination of $\theta$ from (1) there is obtained

$$r = \int_0^{\phi} \frac{\phi^{2(1-\gamma)}}{(1 - \phi^2)^{1-\gamma}} \, d\phi \tag{2}$$

the length of a side being

$$r_1 = \int_0^1 \frac{\phi^{2(1-\gamma)}}{(1 - \phi^2)^{1-\gamma}} \, d\phi \tag{3}$$

The definite integral can be expressed in terms of the Gamma Function giving

$$r_1 = \frac{\Gamma(\frac{3}{2} - \gamma)\,\Gamma(\gamma)}{2\Gamma(\frac{3}{2})} \tag{4}$$

37. The kinetic energy will be given by

$$2T = -\rho \int \phi' \, d\psi \tag{5}$$

where $\phi'$ and $\psi'$ refer to the flow past a moving cylinder in a stationary fluid

$$\phi' = \phi - x$$
$$d\psi' = dy$$

and

$$2T = -\rho \int (\phi - x)\, dy$$

$$2T = \rho \left[ 4 \sin \gamma\pi \int_0^1 \frac{\phi^{3-2\gamma}}{(1-\phi^2)^{1-\gamma}}\, d\phi - \begin{matrix}\text{area of}\\ \text{rhombus}\end{matrix} \right] \tag{6}$$

which will reduce to

$$2T = \rho\,[\, 2\pi\,(1-\gamma) - \text{area}\,] \tag{7}$$

The half width perpendicular to the direction of motion is

$$B_1 = r_1 \sin \gamma\pi$$

and parallel to the motion

$$D_1 = -\, r_1 \cos \gamma\pi$$

The area

$$2B_1D_1 = -\, 2r_1{}^2 \cos \gamma\pi \sin \gamma\pi \tag{8}$$

Dividing (7) by $\pi\,\rho\,B_1{}^2$ and simplifying, there is obtained the inertia coefficient $C$:

$$C = \frac{2\pi^2\,(1-\gamma)}{\Gamma^2\,(\frac{3}{2}-\gamma)\;\Gamma^2\,(\gamma)\,\sin^2\gamma\pi} + 2 \cot \gamma\pi \tag{9}$$

the proportions of the rhombus being given by

$$B_1/D_1 = H = -\,\tan \gamma\pi \tag{10}$$

The table below gives the evaluation of these equations for certain values of $\gamma$. Values of $\log \Gamma\,(X)$ can be taken from Pierce's Table of Integrals or from Brownlee's Table of $\text{Log}\,\Gamma\,(X)$.* A graph of $C$ is also shown on Plate 4. Note that for the square moving corner on the coefficient is half that of the square moving flat on, a result which can also be arrived at by considerations of superposed flows.

TABLE 3

| | | | | | | |
|---|---|---|---|---|---|---|
| $\gamma$ | .5 | .65 | .7 | .75 | .9 | .95 |
| $H$ | $\infty$ | 1.962 | 1.375 | 1.0 | .325 | .191 |
| $C$ | 1.0 | .8430 | .7986 | .7566 | .6389 | .600 |
| | Flat Plate | | | Square | | |

LIST OF ILLUSTRATIONS



*Tracts for Computers. Cambridge Univ. Press.

## DISCUSSION

The President: We are ready for the discussion on this paper.

Mr. John L. Bogert, *Member*: No one seems to want to discuss this mathematical paper, but there is one aspect of it which is very interesting to all of the members. Professor Lewis says in one part of his paper that, regardless of any theoretical treatment of this subject, the one thing which appeals to every practical man is, that vibration of a hull is something that won't occur, provided the cause of vibration in the interior of the hull is largely eliminated. He therefore stressed in his paper the importance of doing away with all those factors which go to produce vibration.

There isn't any question at all but what that is a highly desirable thing, and of course, naturally, to our minds occurs the fact that the turbine-driven ship and the electrically driven ship fall largely in that category, whereas the reciprocating driven ship is much more subject to those actions.

There is one other way in which this thing can be reached, and that is by the avoidance of synchronism of impulse; and, just as we recognize the fact that we are obliged to have a rolling ship as long as the ocean rolls, so, of course, the solution naturally has been that of destroying the synchronism of the rolling impulse by means of the Frahm anti-rolling tank and things of that kind.

I think that point hardly has been brought out here to those who haven't read the paper, and I think it is one of the strong points that Professor Lewis makes in his paper.

Mr. Thomas C. Rathbone, *Visitor*: The accuracy of prediction of the natural frequencies of complex systems, such as are represented by hulls and complicated machine structures, depends on the accuracy of information as to the mass moments and elastic scales involved, and on the closeness with which the assumed schematic system approaches the actual conditions encountered. As the author states, the more reliable predictions have been arrived at through interpolations or extrapolations based on existing experimental data. It may be pointed out that these experimental data include the water inertia effect.

The author has accomplished a valuable work in affording a quantitative evaluation of the water inertia factor. It will be extremely interesting to apply this information to the numerous cases wherein the calculated frequency did not agree with the actual frequency found.

There has been little information available as to the *damping* effect of the water. Experience on large steel land structures which are free to vibrate, and possess no damping other than internal friction of the metal itself, indicates that enormous inhibitive forces must be supplied by the water surrounding a ship, preventing the hull from responding violently and in a great variety of modes to small periodic forces. In so far as amplitude of vibration is concerned, the damping effect of the water skin is probably much more potent than the inertia effect on the natural frequency of the hull and its relation to the disturbing frequency.

Mr. J. Lockwood Taylor, D. Sc., *Visitor*: I have read with considerable interest Professor Lewis's mathematical results, having recently been engaged in some similar calculations. As Professor Lewis suspects, the case of the rectangular section has been worked out (Riabouchinski, International Congress of Mathematicians, Strasbourg, 1920), and it so happens that he anticipates the solution for a rhombus in a paper of my own which is at present in the press. I also worked out the case of the ellipsoid with two nodes, but arrived at somewhat different results from the author, who remarks that his solution really represents a shearing motion rather than actual flexure. The distinction is, however, of some importance, and if the actual boundary condition for flexure, allowing for the longitudinal movement of the surface, which is of the form

$$\frac{\partial \phi}{\partial G} = (a^2 - x^2)\frac{\partial y}{\partial G} + 2xy\frac{\partial x}{\partial G}$$

is fulfilled, instead of neglecting, as Professor Lewis virtually does, the second term, the resulting coefficients are appreciably modified. I find for $J_2$ the value

$$\left\{R_3 \times \frac{2(2 - G^2)}{(7 - 14G^2 + 9G^4)}\right\}$$

in place of $R_3$, the reduction being about 8 per cent for $G = 1.01$, while the corresponding reduction for the three-node coefficient is even greater. To take an extreme case, the coefficient replacing $R_2$, which corresponds to the case of "one-node" vibration (rotation), will be zero for the sphere instead of one-third. It is capable of mathematical proof that the inertia coefficient is independent of motion ahead, for the ellipsoid, and there is no reason to doubt that this is a perfectly general result.

With regard to the author's general remarks, it is not easy to endorse the statement that hull rigidity is a comparatively unknown factor, since the recognition of the importance of shear deflection—and in exceptional cases of buckling under compression—has, by bringing into line the calculated and observed deflections in practically all the cases for which experimental data are available, removed the uncertainty in this direction. The principal element of doubt in the frequency calculation has been the inertia of the water, and, if the results of such mathematical investigations as Professor Lewis has presented can be correlated with actual ship data, it should be possible for this doubt to be removed; and the fact that I was, I believe, the first to suggest the application of "perfect fluid" hydrodynamics to this problem* gives me especial pleasure in congratulating him on his achievements in this direction.

Professor Lewis: I am in agreement with Mr. Rathbone that the damping effect of the water in suppressing the numerous possible modes of vibration of a hull is undoubtedly very great. This damping must be due either to the surface waves, produced by the vibratory motion of the vessel, or by viscosity and eddies. Beyond this little can be stated at the present time.

In preparing a paper containing original work of this character, the question of priority is always a difficult one, and I wish to thank Mr. Taylor for calling attention to the previous paper of Riabouchinski on the flow past a rectangle. In commencing this work I recognized that the surface condition stated by Mr. Taylor would represent a bending, while that I have used represents a shear. It seemed unlikely, however, that the difference would be very great for the proportion of an ellipsoid which would apply to a ship and would certainly be smaller than the error involved in passing from the ellipsoid to the ship. Further, there is no way of determining, *a priori*, whether the bending or shear conditions would give the better results for the ship. The conditions for shear, which are very con-

*Trans. N. E. C. Inst., 1927–8.

siderably simpler, were therefore used. I am surprised that Mr. Taylor should find the difference to be as great as 8 per cent. I am unable to follow Mr. Taylor in his equation, for I note that this would make $J_2$ negative for values of $G^2$ greater than 2, a result which is physically impossible. I trust that these difficulties will be cleared up with the publication of Mr. Taylor's own results.

THE PRESIDENT: Before passing on to the next paper I wish to assure Professor Lewis of our appreciation of the work he has done in preparing this paper, and of our appreciation also of the splendid work that he has done along this line in recent years. His researches and practical work in regard to vibration and the causes of vibration and elimination of those causes have been among the greatest things contributed in recent years to the development of the science of our profession.

The next paper on our list is No. 2, on "Great Lakes Bulk-Freighters," by Mr. William Bennett, Member.

Mr. Bennett presented the paper.

*To illustrate paper on "The Inertia of the Water Surrounding a Vibrating Ship," by Frank M. Lewis, Member.*

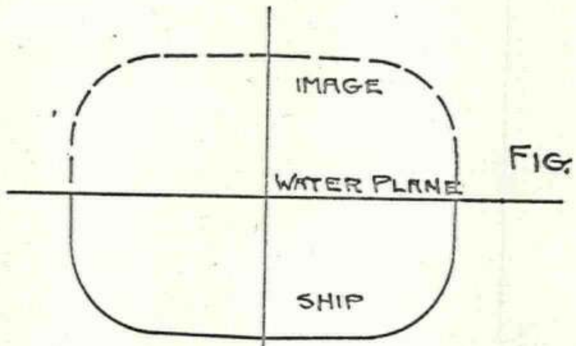



FIG. 1

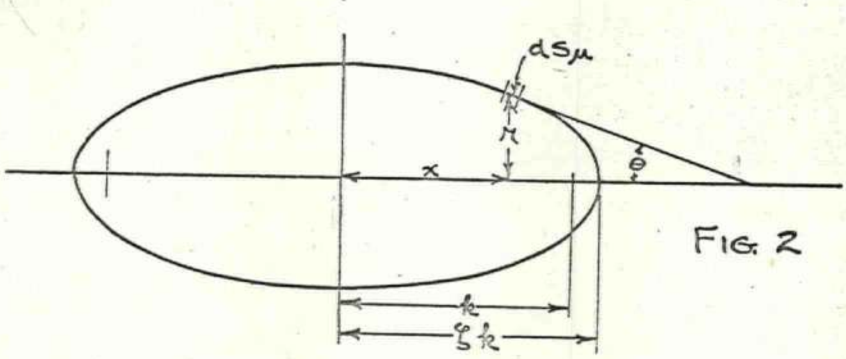



FIG. 2

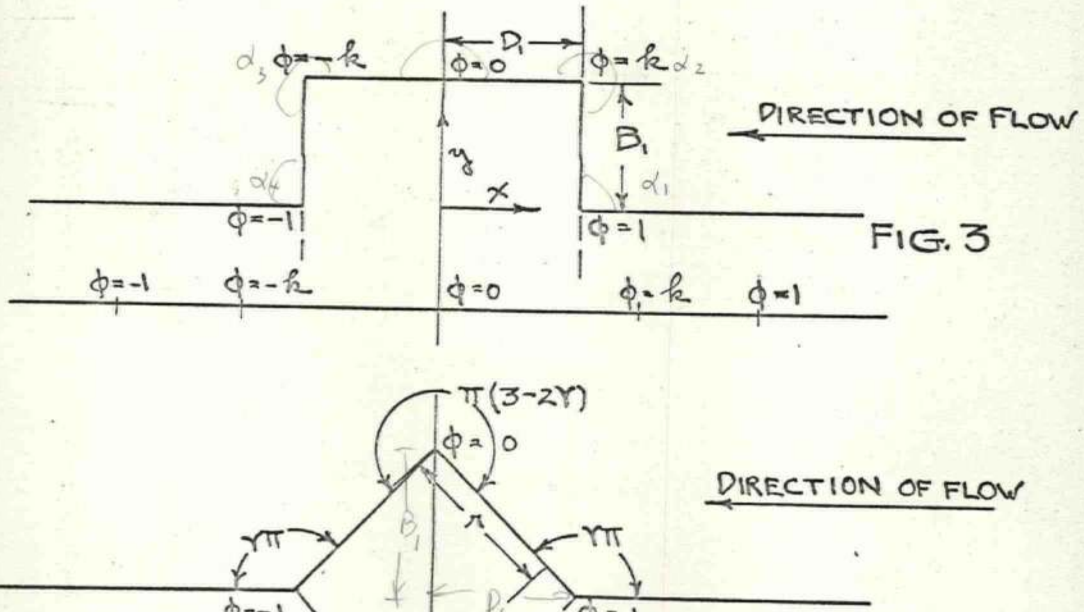



FIG. 3

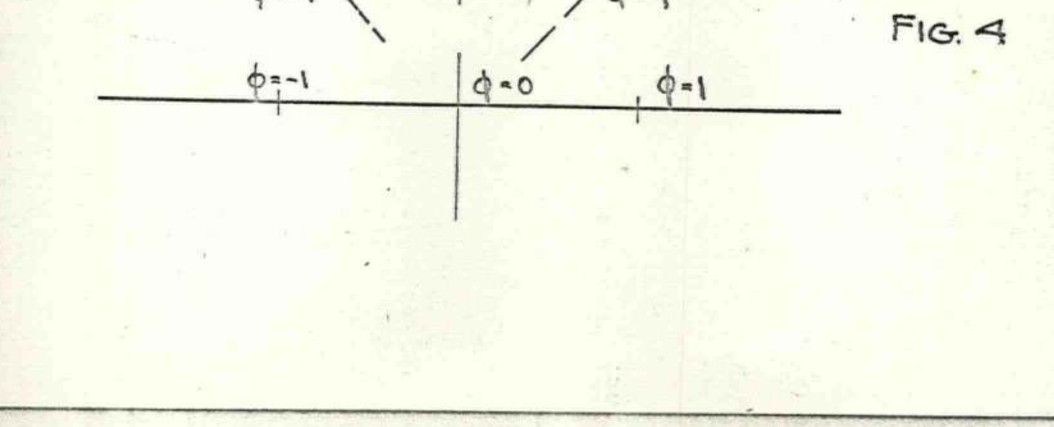
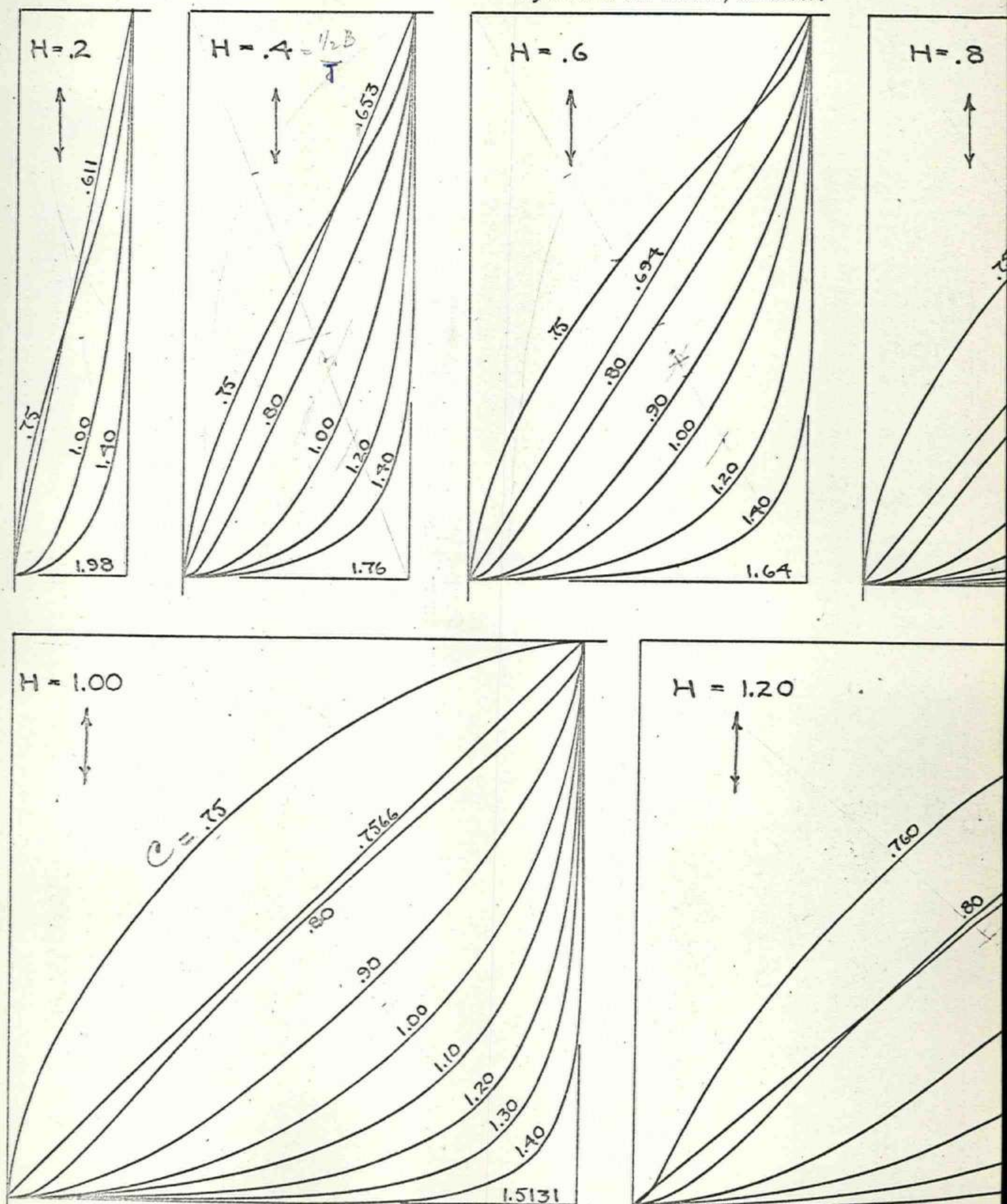


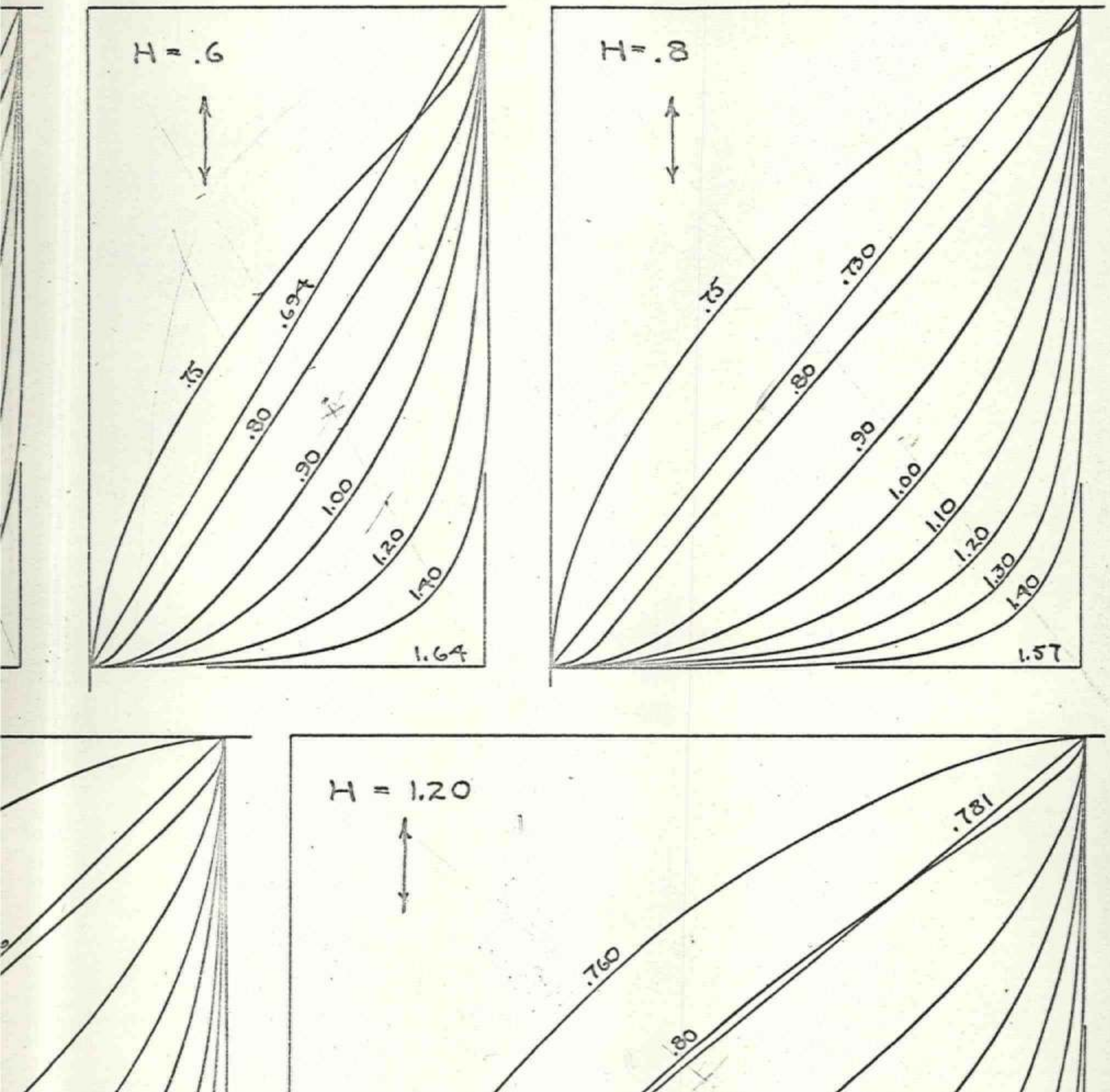

FIG. 4

*Transactions Society Naval Architects and Marine Engineers, Vol. 37, 1929.*

*To illustrate paper on "The Inertia of the Water Surrounding a Vibrating*
*by Frank M. Lewis, Member.*



paper on "The Inertia of the Water Surrounding a Vibrating Ship," by Frank M. Lewis, Member.



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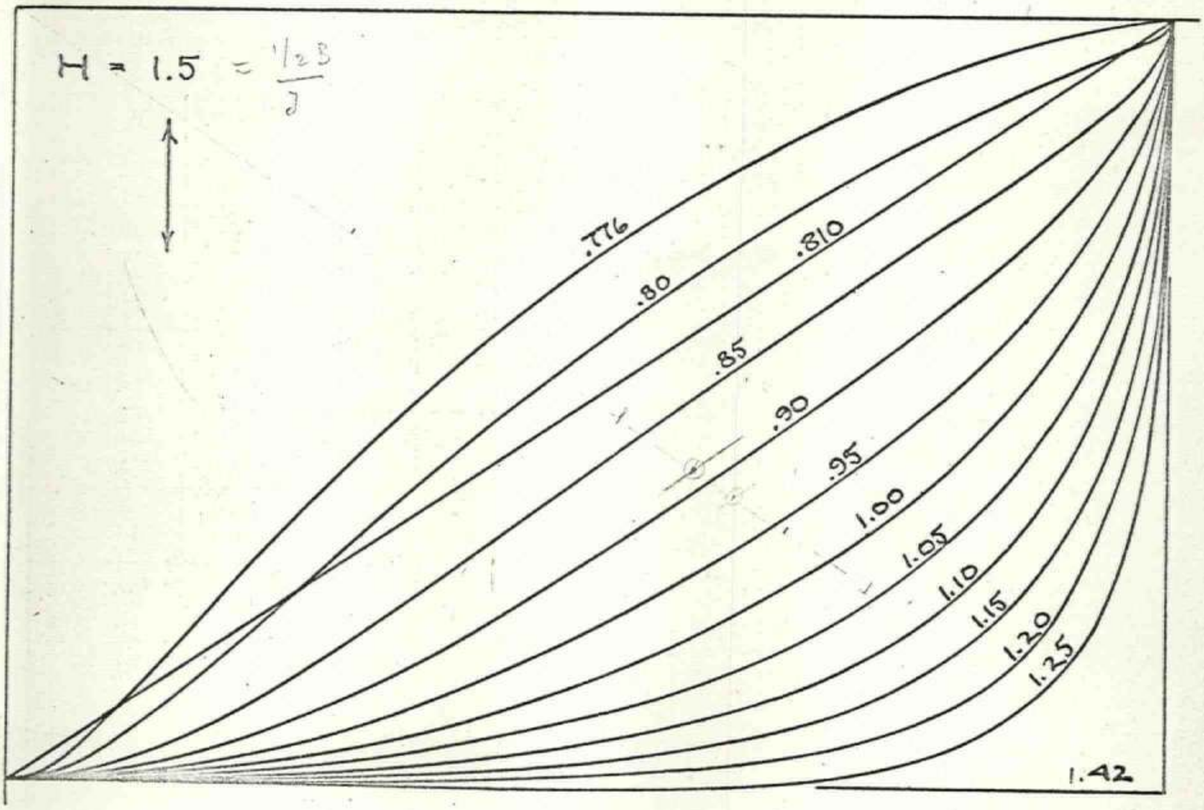
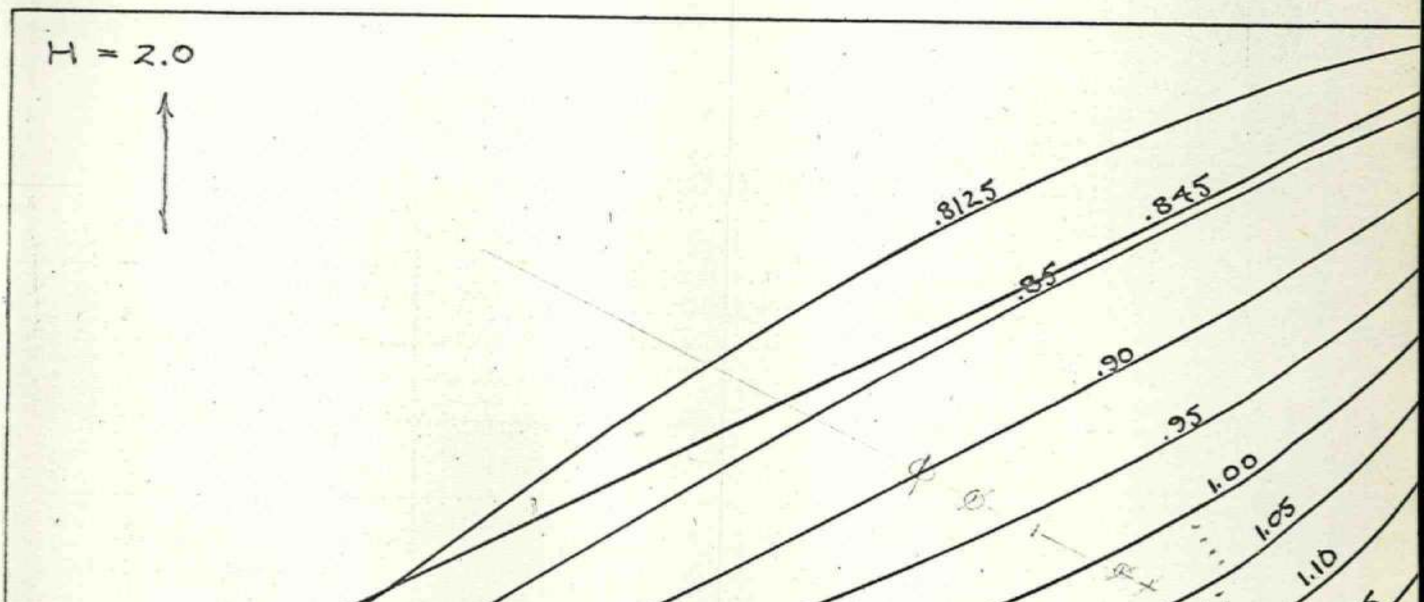
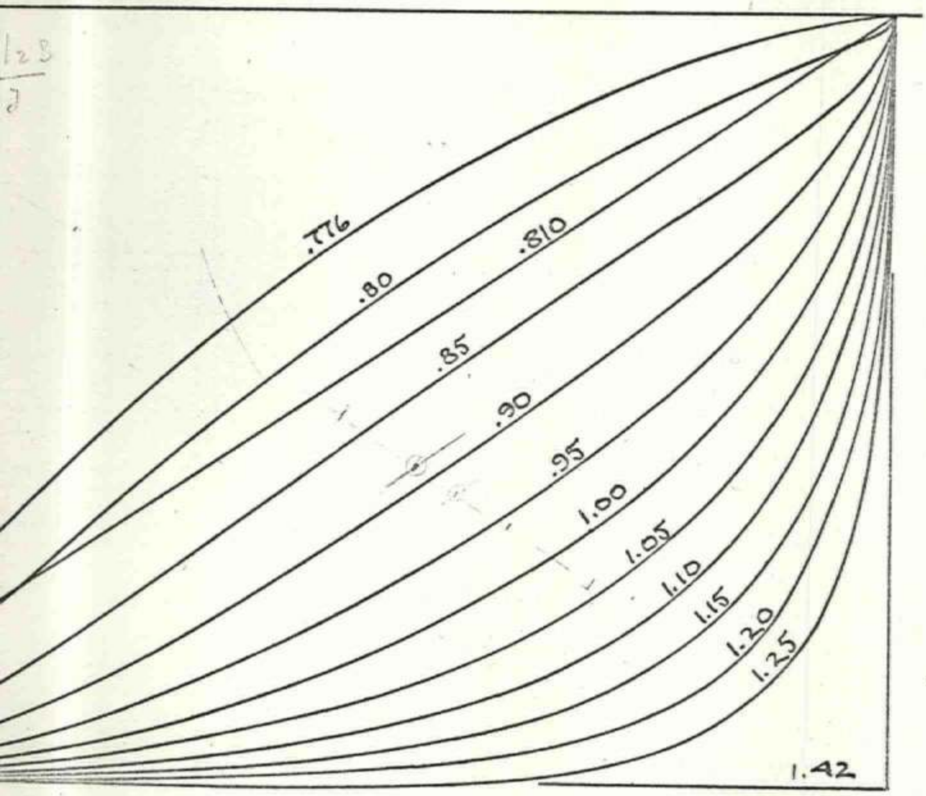
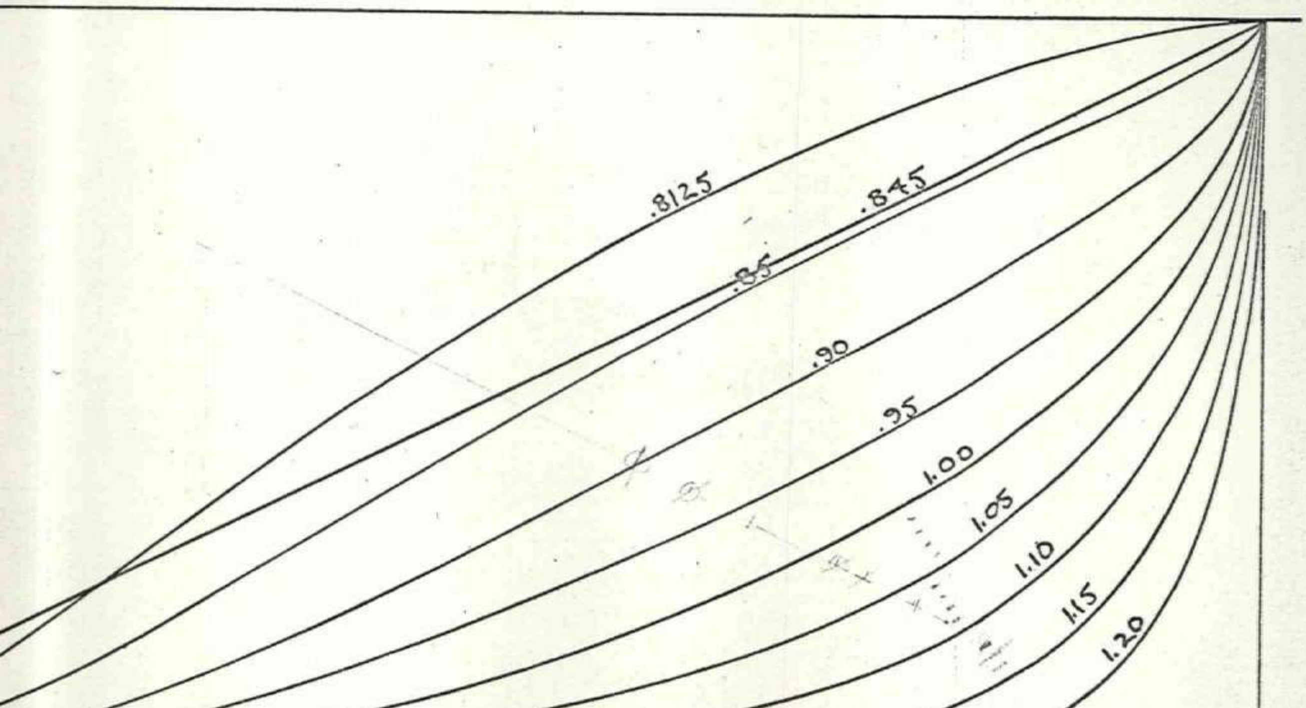


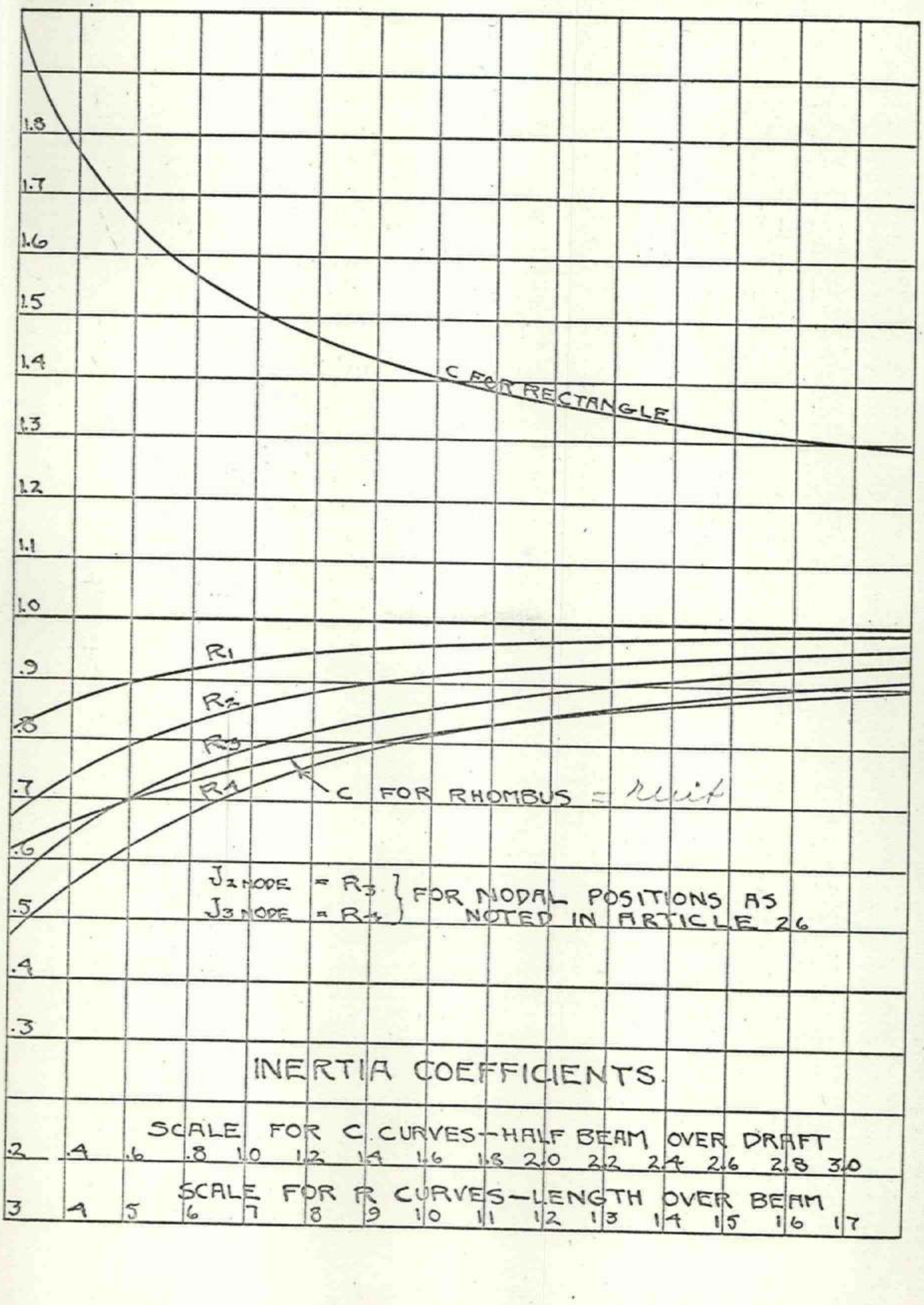



[To i]llustrate paper on "The Inertia of the Water Surrounding a Vibrating Ship," by Frank M. Lewis, Member.



To illustrate paper on "The Inertia of the Water Surrounding a Vibrating Ship," by Frank M. Lewis, Member.



*To illustrate paper on "The Inertia of the Water Surrounding a Vibrating Ship," by Frank M. Lewis, Member.*