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THE INERTIAL ASPECTS OF STEIN'S CONDITION $H - C^*HC \gg 0$

BY

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ABSTRACT. To each bounded operator C on the complex Hilbert space $\mathcal{H}$ we associate the vector space $\mathcal{K}_C$ consisting of those $x \in \mathcal{H}$ for which $C^n x \to 0$ as $n \to \infty$. We let $\alpha(C)$ denote the dimension of the closure of $\mathcal{K}_C$ and we set $\beta(C) = \dim(\mathcal{K}_C^{\perp})$. Our main theorem states that if H is Hermitian and if $H - C^*HC$ is positive and invertible then $\alpha(C) \le \pi(H)$, $\beta(C) = \nu(H)$, and $\beta(C) \ge \delta(H)$ where $(\pi(H), \nu(H), \delta(H))$ is the inertia of H. (That is, $\pi(H) = \dim(\text{Range } E[(0, \infty)])$ where E is the spectral measure of H; $\nu(H) = \pi(-H)$; and $\delta(H) = \dim(\text{Ker } H)$.) We also show (1) that in general no stronger conclusion is possible, (2) that, unlike previous inertia theorems, our theorem allows 1 to lie in $\sigma(C)$, the spectrum of C, and (3) that the main inertial results associated with the hypothesis that $\text{Re}(HA)$ is positive and invertible can be derived from our theorem. Our theorems (1) characterize C in the extreme cases that either $\pi(H) = 0$ or $\nu(H) = 0$, and (2) prove that $\alpha(C) = \pi(H)$, $\beta(C) = \nu(H)$, $\delta(H) = 0$ if either $1 \notin \sigma(C)$ or $\beta(C) < \infty$.

1. Introduction. Let $B(\mathcal{H})$ denote the algebra of bounded linear operators on the complex Hilbert space $\mathcal{H}$. We shall assume that $\mathcal{H}$ is separable. (This causes no loss of generality because at each stage we deal with only finitely many operators, and so every higher dimensional space is an orthogonal direct sum of separable summands which reduce all the operators we are discussing. Thus it can be shown that the validity of our theorems in the separable case implies their validity in the general case.)

Let π_+ denote the open right half-plane $\text{Re } z > 0$, let $\pi_- = -\pi_+$, and let π_0 denote the imaginary axis. Suppose that $A \in B(\mathcal{H})$ and that there exists a spectral measure E (cf. [2, p. 888]) such that for $\eta = +, -, 0$:

(1) The domain of E contains π_η,

(2) $E_\eta A = AE_\eta$ where $E_\eta = E(\pi_\eta)$, and

(3) $\sigma(A_\eta) \subseteq \pi_\eta$ where A_η denotes A restricted to $\mathcal{H}_\eta$, the range of E_η.

Let $\mathfrak{M}_A$ denote the set of all such spectral measures, and letting E range over $\mathfrak{M}_A$ set

$$\pi(A) = \min_E \dim \text{range } E_+, \quad \nu(A) = \min_E \dim \text{range } E_-, \quad \delta(A) = \sup_E \dim \text{range } E_0.$$

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Then $\mathrm{In}_\pi(A) = (\pi(A), \nu(A), \delta(A))$ is the *inertia* of A. Sometimes there exists an $E \in \mathfrak{M}_A$ such that $\mathrm{In}_\pi(A) = (\dim \text{range } E_+, \dim \text{range } E_-, \dim \text{range } E_0)$. Two important examples of this are discussed in §7. One is the normal operators. Here the special spectral measure is the one constructed in the spectral theorem. The other is the operators A such that $\sigma(A)$, the spectrum of A, misses $\{0 < |\mathrm{Re}\, z| < \epsilon\}$ for some positive ϵ (depending on A). Here the special E_η is determined by integrating the resolvent of A around an appropriate Jordan curve which encircles $\pi_\eta \cap \sigma(A)$.

The prototype for our results is the following Main Inertia Theorem. Its claim to this title comes from the fact (expounded in [1] and [4]) that very general forms of two "classical" inertia theorems, viz. Lyapunov's theorem and Sylvester's theorem, can be derived from it. The finite dimensional case of the following theorem appears in Ostrowski and Schneider [4]. Taussky [6] and Williams [8] also contain parts of it. We shall write $T \gg 0$ to mean that $T \in B(\mathcal{H})$ is positive and invertible.

Theorem A (Main Inertia Theorem [1]). *Suppose* $A \in B(\mathcal{H})$.

(a) *There exists a Hermitian* H *such that* $\mathrm{Re}(HA) \gg 0$ *if and only if* $\sigma(A) \cap \pi_0 = \emptyset$.

(b) *If* $\mathrm{Re}(HA) \gg 0$ *then* $\mathrm{In}_\pi(A) = \mathrm{In}_\pi(H)$.

In [7] Taussky showed that questions about Lyapunov's condition $\mathrm{Re}(HA) \gg 0$ can often be translated into questions about Stein's condition $H - C^*HC \gg 0$ by means of the linear fractional transformation $w = \phi(z) = (z-1)/(z+1)$ and its inverse $z = \psi(w) = (1+w)/(1-w)$. A direct computation shows:

(1) If $\mathrm{Re}(HA) \gg 0$ and if $C = \phi(A)$ exists (i.e. $-1 \notin \sigma(A)$) then $H - C^*HC \gg 0$.

(2) If $H - C^*HC \gg 0$ and if $A = \psi(C)$ exists (i.e. $1 \notin \sigma(C)$) then $\mathrm{Re}(HA) \gg 0$.

If we let $\Delta_+, \Delta_-, \Delta_0$ denote the sets of complex numbers z defined by the respective conditions $|z| < 1$, $|z| > 1$, $|z| = 1$, then $\phi(\pi_+) = \Delta_+$, $\phi(\pi_-) = \Delta_- \cup \{\infty\}$, and $\phi(\pi_0) = \Delta_0 \setminus \{1\}$. (Also $\psi(\Delta_+) = \pi_+$, $\psi(\Delta_-) = \pi_- \setminus \{-1\}$, and $\psi(\Delta_0) = \pi_0 \cup \{\infty\}$.) Thus in passing from theorems of Lyapunov "type" to ones of Stein "type", In_π, which we defined with respect to the sets π_η, is replaced by an inertia In_Δ, which is defined by simply replacing π_η by Δ_η in our definition of In_π.

When translated in this way Theorem A becomes

Theorem B. *Suppose that* $C \in B(\mathcal{H})$ *and* $1 \notin \sigma(C)$.

(a) *There exists a Hermitian* H *such that* $H - C^*HC \gg 0$ *if and only if* $\sigma(C) \cap \Delta_0 = \emptyset$.

(b) *If* $H - C^*HC \gg 0$ *then* $\mathrm{In}_\Delta(C) = \mathrm{In}_\pi(H)$.

This theorem is not strictly analogous to Theorem A since we have assumed throughout it that $1 \notin \sigma(C)$. We shall provide several alternative assumptions which permit the same conclusions, but without some extra hypothesis part (a) can be false (we give examples) and $\mathrm{In}_\Delta(C)$, which appears in part (b), may not be defined. In fact, assuming that $\mathrm{In}_\Delta(C)$ exists is one adequate alternative to assuming that $1 \notin \sigma(C)$. But the listing of substitute hypotheses is incidental to the aim of this paper. Our aim and our central result is a general inertia theorem which applies to every pair C, H satisfying $H - C^*HC \gg 0$, whose proof does not depend on those of Theorems A and B, and from which those theorems and their variants may be derived. We shall now formulate this theorem.

For each $C \in B(\mathcal{H})$ let $\mathcal{K}_C$ denote the vector space consisting of those $x \in \mathcal{H}$ such that $C^n x \to 0$ as $n \to \infty$. Let $\alpha(C)$ denote the dimension of the closure of $\mathcal{K}_C$ and set $\beta(C) = \dim(\mathcal{K}_C^\perp)$.

Theorem C. *Suppose that* $H - C^*HC \gg 0$ *where* $C, H \in B(\mathcal{H})$ *and* H *is Hermitian. Then*:

(a) $\mathcal{K}_C$ *is a closed invariant subspace for* C.

(b) $\alpha(C) \leq \pi(H)$, $\beta(C) = \nu(H)$, *and* $\beta(C) \geq \delta(H)$.

(c) *No numerical relations hold, in general, among* $\alpha(C)$, $\beta(C)$, $\pi(H)$, $\nu(H)$, $\delta(H)$, *except those listed in* (b) *and their consequences.*

(d) *If* $\beta(C) < \infty$ *then* $\Delta_0 \cap \sigma(C) = \emptyset$.

(e) $\mathrm{In}_\Delta(C)$ *exists if and only if* $\Delta_0 \cap \sigma(C) = \emptyset$.

(f) *If* $\mathrm{In}_\Delta(C)$ *exists then* $\mathrm{In}_\Delta(C) = (\alpha(C), \beta(C), 0) = \mathrm{In}_\pi(H)$.

2. **Properties of the solutions to $H - C^*HC \gg 0$.** In this section we establish some of Theorem C. Part (a) is Theorem 2.4, and Theorems 2.6 and 2.8 give part (b). We assume throughout that C, H satisfy $H - C^*HC \gg 0$. We defer to §§4 and 5 the question of whether such pairs C, H exist. Let $BH(\mathcal{H})$ denote the set of Hermitian elements of $B(\mathcal{H})$ with the norm topology it inherits. We denote by $\mathcal{S}$, or sometimes $\mathcal{S}(\mathcal{H})$, the subset of $B(\mathcal{H}) \times BH(\mathcal{H})$ consisting of the pairs (C, H) for which $H - C^*HC \gg 0$.

Proposition 2.1. $\mathcal{S}$ *is an open subset of* $B(\mathcal{H}) \times BH(\mathcal{H})$.

Proof. The set $\{D \gg 0\}$ is an open subset of $BH(\mathcal{H})$ and its inverse image under the continuous map $(C, H) \to H - C^*HC$ is just $\mathcal{S}$. □

Proposition 2.2. *If* $(C, H) \in \mathcal{S}$ *then* $H, C^*HC, (C^*)^2HC^2, \cdots$ *is a decreasing sequence.*

Proof. $(C^*)^n HC^n - (C^*)^{n+1}HC^{n+1} = (C^*)^n(H - C^*HC)C^n \geq 0$. □

Theorem 2.3. *If* $(C, H) \in \mathcal{S}$ *and* $x \in \mathcal{H}$ *then*:

(a) *The sequence* $C^n x$ *either converges to* 0 *or diverges to* ∞.

(b) $\lim_{n\to\infty} C^n x = 0$ *if and only if the sequence* $(HC^n x, C^n x)$ *is bounded below.*

Proof. Set $D = H - C^*HC$ and pick $m > 0$ so that $D \geq mI$. By Proposition 2.2 the sequence (HC^nx, C^nx) is decreasing for each x. If for some x it is bounded below, it will converge, and then

$$0 \leq m\|C^nx\|^2 \leq (DC^nx, C^nx) = (HC^nx, C^nx) - (HC^{n+1}x, C^{n+1}x) \to 0 \quad \text{as } n \to \infty.$$

Hence $C^nx \to 0$.

If for some x the sequence (HC^nx, C^nx) is unbounded, then the inequality $\|H\| \cdot \|C^nx\|^2 \geq |(HC^nx, C^nx)|$ shows that $\|C^nx\| \to \infty$ as $n \to \infty$. □

For each $C \in B(\mathcal{H})$ both $\mathcal{K}_C$ and its closure $\overline{\mathcal{K}}_C$ are easily seen to be invariant vector spaces for C.

Theorem 2.4. *If $(C, H) \in \mathcal{S}$ then $\mathcal{K}_C$ is closed.*

Proof. We shall omit the subscript C. Proposition 2.2 shows that for each $x \in \mathcal{H}$

$$(Hx, x) \geq (HCx, Cx) \geq \cdots \geq (HC^nx, C^nx).$$

Since $C^nx \to 0$ if $x \in \mathcal{K}$, it follows that $(Hx, x) \geq 0$ for every $x \in \mathcal{K}$ and hence for every $x \in \overline{\mathcal{K}}$.

To see that $\overline{\mathcal{K}} \subseteq \mathcal{K}$ suppose that $x \in \overline{\mathcal{K}}$. Since $\overline{\mathcal{K}}$ is invariant under C we know that $C^nx \in \overline{\mathcal{K}}$ and, from the preceding paragraph, that $(HC^nx, C^nx) \geq 0$. Now Theorem 2.3(b) implies that $C^nx \to 0$, i.e. that $x \in \mathcal{K}$. □

Corollary 2.5. *If $m > 0$ and $H - C^*HC \geq mI$ then $(Hx, x) \geq m\|x\|^2$ for every $x \in \mathcal{K}_C$.*

Proof. The proof of Theorem 2.4 shows that $(Hy, y) \geq 0$ for $y \in \mathcal{K}_C$ and that $Cx \in \mathcal{K}_C$ if $x \in \mathcal{K}_C$. Thus if $x \in \mathcal{K}_C$ and if $y = Cx$ we have $(Hx, x) \geq ([mI + C^*HC]x, x) = m\|x\|^2 + (Hy, y) \geq m\|x\|^2$. □

If H is Hermitian and if $\mathcal{H}_\eta$ for $\eta = +, -, 0$ are the closed orthogonal subspaces used in defining $\mathrm{In}_\pi(H)$ then for $\eta = +, -, 0$ we shall denote by P_η the orthogonal projection onto $\mathcal{H}_\eta$.

Theorem 2.6. *If $(C, H) \in \mathcal{S}$ then* (1) $\alpha(C) \leq \pi(H)$, *and* (2) $\nu(H) + \delta(H) \leq \beta(C)$.

Proof. By Corollary 2.5 there exists an $m > 0$ such that $m\|x\|^2 \leq (Hx, x)$ for every $x \in \mathcal{K}_C$. Since $(Hx, x) \leq 0$ for every $x \in (\mathcal{H}_- \oplus \mathcal{H}_0)$ it follows that $\mathcal{K}_C \cap (\mathcal{H}_- \oplus \mathcal{H}_0) = \{0\}$. We draw two conclusions. First that since $\mathrm{Ker}(P_+) = \mathcal{H}_- \oplus \mathcal{H}_0$, the projection P_+ is 1-1 when it is restricted to $\mathcal{K}_C$. So $\alpha(C) \leq \pi(H)$. And second that the orthogonal projection of $\mathcal{H}$ onto $\mathcal{K}_C^\perp$ is 1-1 when restricted to $\mathcal{H}_- \oplus \mathcal{H}_0$. So $\nu(H) + \delta(H) \leq \beta(C)$. □

Theorem 2.7. *Suppose that $(C, H) \in \mathcal{S}$ and let $\mathcal{L}$ be a finite dimensional subspace of $\mathcal{K}_C^\perp$. Then there exists a positive integer n for which $P_-C^n: \mathcal{L} \to \mathcal{H}_-$ is* 1-1.

Proof. Since $\dim \mathfrak{L} < \infty$, for each positive integer k the set $V_k = \{x \in \mathfrak{L} \mid \|x\| = 1 \text{ and } (HC^kx, C^kx) \geq 0\}$ is compact. Furthermore, $V_{k+1} \subseteq V_k$ for each k because $(C^*)^{k+1}HC^{k+1} \leq (C^*)^kHC^k$. If $z \in \bigcap_{k=1}^{\infty} V_k$ then $(HC^kz, C^kz) \geq 0$ for every k, and by Theorem 2.3(b) $C^kz \to 0$. Thus $z \in \mathcal{K}_C \cap \mathfrak{L} \subseteq \mathcal{K}_C \cap \mathcal{K}_C^{\perp} = \{0\}$ and yet $\|z\| = 1$. So no such z can exist and $\bigcap_{k=1}^{\infty} V_k$ must be empty. By the finite intersection property some V_k, say V_n, must be empty. Hence if $x \in \mathfrak{L}$ and $\|x\| = 1$ then $0 > (HC^nx, C^nx) \geq (HP_-C^nx, P_-C^nx)$. Thus $P_-C^nx \neq 0$ and P_-C^n: $\mathfrak{L} \to \mathcal{H}_-$ is 1-1. □

Theorem 2.8. *If* $(C, H) \in \mathcal{S}$ *then* (1) $\beta(C) = \nu(H)$, (2) $\delta(H) \leq \beta(C)$, *and* (3) $\alpha(C) = \pi(H)$ *and* $\delta(H) = 0$ *if* $\beta(C) < \infty$.

Proof. First we show that $\beta(C) \leq \nu(H)$. Theorem 2.7 shows that $\nu(H)$ is at least as large as the dimension of any finite dimensional subspace of $\mathcal{K}_C^{\perp}$. So when $\beta(C) < \infty$ we have $\beta(C) \leq \nu(H)$. But when $\beta(C) = \infty$, Theorem 2.7 shows that $\nu(H)$ cannot be finite. Thus $\beta(C) = \nu(H)$ since we assumed that $\mathcal{H}$ is separable.

Combining $\beta(C) \leq \nu(H)$ with the inequality $\nu(H) + \delta(H) \leq \beta(C)$ given in Theorem 2.6(2) leads immediately to the conclusion that $\beta(C) = \nu(H)$, $\delta(H) \leq \beta(C)$, and $\delta(H) = 0$ if $\beta(C) < \infty$. When $\nu(H) = \beta(C) < \infty$ and $\delta(H) = 0$ it follows that $\alpha(C) = \pi(H)$ because $\alpha(C) + \beta(C) = \dim(\mathcal{H}) = \pi(H) + \nu(H) + \delta(H)$. □

Theorem 2.7 shows that the maps P_-C^n can be used to embed finite dimensional subspaces of $\mathcal{K}_C^{\perp}$ into $\mathcal{H}_-$. Certain infinite dimensional spaces which are of interest in inertia theory can be similarly embedded.

Theorem 2.9. *Let* $(C, H) \in \mathcal{S}$ *and suppose that* $\mathfrak{L}$ *is a closed invariant subspace of* C *such that* $\sigma(C|_{\mathfrak{L}}) \subset \Delta_-$. *Then there exists an integer* $n \geq 0$ *such that* $P_-C^n: \mathfrak{L} \to \mathcal{H}_-$ *is* 1-1.

Proof. Let $Q = P_+ + P_0$ and set $H_\epsilon = H + \epsilon Q$ for $\epsilon > 0$. Then $H_\epsilon Q \geq \epsilon Q$, and if $x \in \operatorname{Ker}(P_-C^k)$ we have

$$\begin{aligned}(H_\epsilon C^kx, C^kx) &= (H_\epsilon P_-C^kx, C^kx) + (H_\epsilon QC^kx, C^kx)\\ &\geq \epsilon(QC^kx, C^kx) = \epsilon\|(Q + P_-)C^kx\|^2 = \epsilon\|C^kx\|^2.\end{aligned}$$

Furthermore if $\epsilon > 0$ is picked so small that $(C, H_\epsilon) \in \mathcal{S}$ we obtain

$$0 \leq (H_\epsilon x, x) - (H_\epsilon C^kx, C^kx) \leq (H_\epsilon x, x) - \epsilon\|C^kx\|^2.$$

Thus $\mathfrak{L} \cap \operatorname{Ker}(P_-C^k) \subseteq \{x \in \mathfrak{L} \mid \epsilon\|C^kx\| \leq (H_\epsilon x, x)\}$ and we shall finish the proof by showing that if k is large enough then this set is just $\{0\}$. It suffices to show that our assumptions on $\mathfrak{L}$ imply that the sequence $a_k = \inf\{\|C^kx\|: x \in \mathfrak{L} \text{ and } \|x\| = 1\}$ diverges to ∞.

Let $B = C|_{\mathfrak{L}} \in B(\mathfrak{L})$. Since $\limsup \|B^{-n}\|^{1/n}$ is the spectral radius of B^{-1}

there exists an integer $m > 0$ such that $\gamma = \|B^{-m}\|^{1/m} < 1$. Then for every integer $k > 0$ we have $\|B^{-k}\| \le M\gamma^k$ where $M = \max\{\gamma^{-j}\|B^{-j}\|: j = 1, \cdots, m-1\}$. (To see this write $k = qm + r$ with $0 \le r < m$ and note that $\|B^{-k}\| \le \|B^{-m}\|^q\|B^{-r}\| = (\gamma^{-r}\|B^{-r}\|)\gamma^{qm+r}$.) So

$$M^{-1}\gamma^{-k} \le \|B^{-k}\|^{-1} = \inf\{\|B^k x\| \,|\, x \in \mathfrak{L}, \|x\| = 1\}.$$

Thus $a_k \ge M^{-1}\gamma^{-k}$ which diverges to ∞ with k. □

3. Sharper inertia theorems using extra hypotheses. We begin this section by examining those $(C, H) \in \mathcal{S}$ with $\Delta_0 \cap \sigma(C) \ne \emptyset$. In this case $\sigma_r(C)$, the residual spectrum of C, must include an open annulus $a(C)$ containing Δ_0. Let $a(C)$ or $a(C, H)$ be the largest annulus of the form $\{z: 1/r < |z| < r$ and $(zC, H) \in \mathcal{S}\}$. (Since $\mathcal{S}$ is open such a set will be an annulus if $r > 1$ is small enough.) Let $\Lambda(C)$ denote the approximate point spectrum of C.

Lemma 3.1. *If* $(C, H) \in \mathcal{S}$ *then* $a(C) \cap \Lambda(C) = \emptyset$.

Proof. Let $\lambda \in a(C)$ and set $B = \lambda^{-1}C$. Since $\lambda^{-1} \in a(C)$ we can pick $m > 0$ such that $H - BHB^* \ge mI$. If $\|x\| = 1$ we have $0 < m \le (Hx, x) - (HBx, Bx)$. Furthermore, if $1 \in \Lambda(B)$ we can select x so that (HBx, Bx) is arbitrarily close to (Hx, x) and that contradicts the positivity of m. Thus $1 \notin \Lambda(B) = \lambda^{-1}\Lambda(C)$; that is $\lambda \notin \Lambda(C)$. □

Theorem 3.2. *If* $(C, H) \in \mathcal{S}$ *and* $a(C) \cap \sigma(C) \ne \emptyset$ *then* $a(C) \subset \sigma_r(C)$.

Proof. If $\sigma(C)$ does not contain $a(C)$ there must exist a $\lambda \in \partial\sigma(C) \cap a(C)$. But $\partial\sigma(C) \subset \Lambda(C)$ so by the lemma no such λ exists. Thus $a(C) \subset \sigma(C)$ and using the lemma again shows that $a(C) \subset \sigma(C)\backslash\Lambda(C)$. Since $\sigma(C)\backslash\Lambda(C)$ is contained in $\sigma_r(C)$ we are done. □

Corollary 3.3. *Suppose* $(C, H) \in \mathcal{S}$. *Then* $a(C)$ *and* $\sigma(C)$ *will be disjoint if any one of the following holds*:

(a) $a(C)\backslash\sigma(C) \ne \emptyset$.

(b) *There exists a Hermitian* K *with* $(C^*, K) \in \mathcal{S}$.

(c) $\sigma(C)$ *is totally disconnected* (*e.g.* C *is compact.*)

(d) C *is normal.*

(e) $\mathrm{In}_\Delta(C)$ *exists.*

Proof. (a) If $a(C)\backslash\sigma(C) \ne \emptyset$ then $a(C)\backslash\sigma_r(C) \ne \emptyset$ and the contrapositive of Theorem 3.2 applies.

(b) Since $\sigma_r(C)$ lies in the point spectrum of C^*, $\sigma_r(C) \subseteq \Lambda(C^*)$ and since Lemma 3.1 shows that $\Lambda(C^*)$ misses $a(C^*)$, we know that $\sigma_r(C) \cap a(C^*) = \emptyset$. The contrapositive of Theorem 3.2 applies.

(c) If $\sigma(C)$ is totally disconnected and $\sigma(C) \cap a(C) \ne \emptyset$ then $\partial\sigma(C)$, which lies in $\Lambda(C)$, also shares points with $a(C)$. Since this contradicts Lemma 3.1, $\sigma(C) \cap a(C) = \emptyset$.

(d) Since $\Lambda(C) = \sigma(C)$ when C is normal, this part follows from Lemma 3.1.

(e) If $\mathrm{In}_\Delta(C)$ exists then C and $\mathcal{H}$ can be decomposed $C = C_+ \oplus C_- \oplus C_0$, $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \oplus \mathcal{H}_0$ with $C_\eta \in B(\mathcal{H}_\eta)$ and $\sigma(C_\eta)$ contained in $\bar{\Delta}_\eta$ (the closure of Δ_η) for $\eta = +, -, 0$. If $\sigma(C) \cap \Delta_0 \neq \emptyset$ then for some η

$$\emptyset \neq \sigma(C_\eta) \cap \Delta_0 \subset \partial\sigma(C_\eta) \subseteq \Lambda(C_\eta) \subseteq \Lambda(C)$$

which contradicts Lemma 3.1. □

Part (e) of this corollary is half of Theorem C(e). The other half follows from the Riesz decomposition theorem [5, §148]. The next theorem establishes part (d) of Theorem C.

Theorem 3.4. *If* $(C, H) \in \mathcal{S}$ *and if* $a(C) \cap \sigma(C) \neq \emptyset$ *then* $\beta(C) = \infty$.

Proof. Let E denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{K}_C$ and let $F = I - E$. Since $\mathcal{K}_C$ is invariant under C we know that $C = ECE + ECF + FCF$ and that $\sigma(C) \subseteq \sigma(ECE) \cup \sigma(FCF)$. (To prove the latter let R, S, and T denote the respective results of restricting ECE to $E\mathcal{H}$, ECF to $F\mathcal{H}$, and FCF to $F\mathcal{H}$. Then $C = \left(\begin{smallmatrix} R & S \\ 0 & T\end{smallmatrix}\right)$. If $0 \notin \sigma(R) \cup \sigma(T)$ then

$$C^{-1} = \begin{pmatrix} R^{-1} & -R^{-1}ST^{-1} \\ 0 & T^{-1}\end{pmatrix}$$

exists, and so $0 \notin \sigma(C)$. When C is translated by $-\lambda I$ this argument shows that if $\lambda \notin \sigma(R) \cup \sigma(T)$ then $\lambda \notin \sigma(C)$. In other words $\sigma(C) \subset \sigma(R) \cup \sigma(T)$. Thus it suffices to prove that $\sigma(R) \subset \sigma(ECE)$ and that $\sigma(T) \subset \sigma(FCF)$. If $\lambda \notin \sigma(ECE)$ then $[R - \lambda(I|_{E\mathcal{H}})]^{-1} = (ECE - \lambda I)^{-1}|_{E\mathcal{H}}$, and so $\lambda \notin \sigma(R)$. Thus $\sigma(R) \subset \sigma(ECE)$. The proof that $\sigma(T) \subset \sigma(FCF)$ is similar.) Since $C^n x \to 0$ for every x in $\mathcal{K}_C$ it follows that $(ECE)^n x \to 0$ for every x in $\mathcal{H}$, and hence that $\sigma(ECE) \subset (\Delta_+ \cup \Delta_0)$. If the theorem is false then $\beta(C) < \infty$ and $\sigma(FCF)$ is finite. So if $\beta(C) < \infty$ then $\sigma(C) \cap \Delta_-$ is finite. However Theorem 3.2 shows that $\sigma(C) \cap \Delta_-$ is infinite. Therefore $\beta(C)$ must be infinite. □

To establish part (f) of Theorem C we assume that $\mathrm{In}_\Delta(C)$ exists and that $(C, H) \in \mathcal{S}$. Then Theorem C(e) shows that $\sigma(C) \cap a(C) = \emptyset$. Using Riesz's theorem to decompose C and $\mathcal{H}$ we obtain $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ where $\mathcal{H}_+ = \mathcal{K}_C$ and $\dim \mathcal{K}_C^\perp = \dim \mathcal{H}_-$. Thus $\mathrm{In}_\Delta(C) = (\alpha(C), \beta(C), 0)$. The remainder of part (f) is proved in the next theorem.

Theorem 3.5. *If* $(C, H) \in \mathcal{S}$ *and if* $\sigma(C) \cap \Delta_0 = \emptyset$ *then* $\mathrm{In}_\Delta(C) = \mathrm{In}_\pi(H)$.

Proof. We know that $\mathrm{In}_\Delta(C) = (\alpha(C), \beta(C), 0)$ and $\beta(C) = \nu(H)$. When $\alpha(C) = \infty$ then $\alpha(C) = \pi(H)$ because $\mathcal{H}$ is separable and $\alpha(C) \leq \pi(H)$. If $\alpha(C) < \infty$, then by changing C an arbitrarily small amount on $\mathcal{K}_C$ we can obtain an invertible operator B such that $\alpha(C) = \alpha(B)$, $\beta(C) = \beta(B)$, and $(B, H) \in \mathcal{S}$. Since B is invertible $(B^{-1}, -H) \in \mathcal{S}$, and so $\alpha(C) = \alpha(B) = \beta(B^{-1}) = \nu(-H) = \pi(H)$.

Since $1 \notin \sigma(C)$, $A = \psi(C) = (I + C)(I - C)^{-1}$ exists and $\mathrm{Re}(AH) \gg 0$. (See (2) in the Introduction.) If $\delta(H) > 0$ there exists $x \in (\mathrm{Ker}\, H)\backslash\{0\}$, and then $0 < (\mathrm{Re}(AH)x, x) = \mathrm{Re}(AHx, x) = 0$. Thus $\delta(H) = 0$. □

When $(C, H) \in \mathcal{S}$ and $\sigma(C) \cap d(C) = \varnothing$ the relation between $\sigma(H) \cap \pi_+$ and $\sigma(C) \cap \Delta_+$ can be simply summarized by $\pi(H) = \alpha(C)$. However, when $\sigma(C) \cap d(C) \neq \varnothing$ the relationship may be more complicated. For example, the next theorem shows that if $\sigma(C) \cap d(C) \neq \varnothing$ and $\pi(H) < \infty$, then $\{z: |z| < [r(C)]^{-1}\}$ (where $r(C)$ is the spectral radius of C) is contained in $\sigma(C)$. Let λ^* be the complex conjugate of λ.

Theorem 3.6. *Suppose that $(C, H) \in \mathcal{S}$ and $a(C) \cap \sigma(C) \neq \varnothing$. If there exists a $\lambda \notin \sigma(C)$ such that $1/\lambda^* \notin \sigma(C)$ (e.g. if C is invertible) then $\pi(H) = \infty$.*

Proof. Since $\Delta_0 \subset \sigma(C)$, $\lambda \notin \Delta_0$ and so either λ or $1/\lambda^*$ lies in Δ_+. There is no loss in assuming that $\lambda \in \Delta_+$. If $\theta(z) = (z - \lambda)/(1 - \lambda^* z)$ then $B = \theta(C)$ exists because $1/\lambda^* \notin \sigma(C)$. A direct computation using $C = \theta^{-1}(B)$ and $(C, H) \in \mathcal{S}$ shows that $(B, H) \in \mathcal{S}$. Since $\lambda \notin \sigma(C)$, B is invertible, and so $(B^{-1}, -H) \in \mathcal{S}$. Thus $\pi(H) = \nu(-H) = \beta(B^{-1})$. Since $\Delta_0 \subset \sigma(C)$ and since $\theta(\Delta_0) = \Delta_0$, we know that $\Delta_0 \subset \sigma(B^{-1})$. So Theorem 3.4 shows that $\beta(B^{-1}) = \infty$. □

4. The existence of solutions to $H - C^*HC \gg 0$. In this section we establish the existence of solutions H to $H - C^*HC \gg 0$ under the assumption that $\Delta_0 \cap \sigma(C) = \varnothing$. We also characterize the solutions C to $H - C^*HC \gg 0$ in the extreme cases where $H \gg 0$ and where $H \ll 0$.

Theorem 4.1. (See also [1, p. 102].) *Given $D \gg 0$ there exists an $H \gg 0$ such that $H - C^*HC = D$ if and only if $r(C) < 1$. When such an H exists it is unique.*

Proof. If $r(C) < 1$ the root test shows that we can define the mapping $D \to \Sigma_{k=0}^{\infty}(C^*)^k DC^k$, which is the inverse of the mapping $H \to H - C^*HC$. Then $H = \Sigma(C^*)^k DC^k \gg 0$ is unique.

If $H - C^*HC \gg 0$ for some Hermitian $H \geq 0$ there exists a $t > 1$ such that $H - (tC)^*H(tC) \gg 0$ also. By Proposition 2.2 H, $(tC^*)H(tC)$, $(tC^*)^2H(tC)^2, \cdots$ is a decreasing sequence. Since it is bounded below by 0, Theorem 2.3 shows that $(tC)^n x \to 0$ for every $x \in \mathcal{H}$. By the uniform boundedness principle for some $M \geq 0$ we have $\|(tC)^n\| \leq M$ for $n = 1, 2, 3, \cdots$. Hence $r(C) \leq t^{-1}\lim M^{1/n} < 1$. □

Theorem 4.2. (See also [1, p. 104].) *If $\sigma(C) \cap \Delta_0 = \varnothing$ then there exists a Hermitian H such that $(C, H) \in \mathcal{S}$.*

Proof. Let $C = C_+ \oplus C_-$ be the Riesz decomposition of C with respect to the spectral sets $w_\eta = \sigma(C) \cap \Delta_\eta$ for $\eta = +, -$. Then $\sigma(C)_\eta = w_\eta$ and we introduce the corresponding decomposition of $\mathcal{H}$ and the identity operator $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ and $I = I_+ \oplus I_-$. Since $\dim \mathcal{H}_- = \dim(\mathcal{H}_+^\perp)$, there exists an isometry U taking $\mathcal{H}_+^\perp$ onto $\mathcal{H}_-$. If $S = I_+ \oplus U$ then S is invertible and $SCS^{-1} = C_+ \oplus B$ where $B = U(C_-)U^{-1} \in B(\mathcal{H}_+^\perp)$. Since $\sigma(B) \subsetneq w_-$, B is invertible and $\sigma(B^{-1}) \subset \Delta_+$. By the preceding theorem there exist Hermitian operators $0 \ll K \in B(\mathcal{H}_+)$ and $0 \ll L \in B(\mathcal{H}_+^\perp)$ satisfying

$$K - C_+^*KC_+ = I_+, \quad \text{and} \quad L - (B^{-1})^*LB^{-1} = (B^{-1})^*B^{-1} \gg 0.$$

Then $B^*LB - L$ is the identity operator in $B(\mathcal{H}_+^\perp)$. If $H = K \oplus (-L) \in B(\mathcal{H})$ then $H^* = K^* \oplus (-L^*) = H$ and $(SCS^{-1})^* = C_+^* \oplus B^*$, since the direct sums are orthogonal. And so

$$H - (SCS^{-1})^*H(SCS^{-1}) = [K - C_+^*KC_+] \oplus [-(L - B^*LB)] = I \gg 0.$$

Thus $(C, S^*HS) \in \mathcal{S}$. $\square$

Lemma 4.3. *Suppose that $H \ll 0$ and let $0 \ll K = \sqrt{-H}$. Then $(C, H) \in \mathcal{S}$ if and only if $(KCK^{-1})^*(KCK^{-1}) \gg I$.*

Proof. Since $(C, H) \in \mathcal{S}$ if and only if $(KCK^{-1}, K^{-1}HK^{-1}) \in \mathcal{S}$ and since $K^{-1}HK^{-1} = -I$ we are done. $\square$

Theorem 4.4. *Let U be a partial isometry such that $U^*U = I$, and suppose that $P \gg I$ and $K \gg 0$. Then $(K^{-1}UPK, -K^2) \in \mathcal{S}$ and every $(C, H) \in \mathcal{S}$ for which $H \ll 0$ has this form.*

Proof. The first part comes directly from Lemma 4.3. For the second part we let $K = \sqrt{-H} \gg 0$ and we let UP be the polar decomposition of KCK^{-1}. By the lemma $PU^*UP \gg I$ and so the projection U^*U must be invertible. Thus $U^*U = I$ and $P \gg I$. $\square$

5. Examples showing that our results are sharp. If $(C_i, H_i) \in \mathcal{S}(\mathcal{H}_i)$ for $i = 1, 2$ then it is easy to see that (with orthogonal direct sums) $(C_1 \oplus C_2, H_1 \oplus H_2) \in \mathcal{S}(\mathcal{H}_1 \oplus \mathcal{H}_2)$. Using this fact and the following basic examples we shall construct all the examples we need to verify part (c) of Theorem C. Namely, that the only relations which hold in general between $\alpha(C)$, $\beta(C)$, $\pi(H)$, $\nu(H)$, $\delta(H)$ can be derived from $\alpha(C) \leq \pi(H)$, $\beta(C) = \nu(H)$, $\beta(C) \geq \delta(H)$. In all our examples (except (7)) $\Delta_0 \subset \sigma(C)$; for if not, then we know that $\alpha(C) = \pi(H)$, $\beta(C) = \nu(H)$, $\delta(H) = 0$ by Theorem 2.8. And examples with such behavior can be constructed trivially as in the proof of Theorem 4.2. In the examples which follow the quickest way to check that $\Delta_0 \subset \sigma(C)$ is usually to note that $\delta(H) \neq 0$ or $\alpha(C) \neq \pi(H)$. In order to have $\Delta_0 \subset \sigma(C)$ we must at least start with $\beta(C) = \nu(C) = \infty$.

Example. Let $\mathcal{H} = l^2$, the space of square summable complex sequences. Let H, D be represented by diagonal matrices $H = \operatorname{diag}[u, -1, -1, \cdots]$ and $D = \operatorname{diag}[0, 2v, 2, 2, \cdots]$. Let S denote the unilateral shift $S[x_1, x_2, \cdots] = [0, x_1, x_2, \cdots]$. Then if $C = DS$ we have $H - C^*HC = \operatorname{diag}[u + 4v, 3, 3, \cdots]$. Thus $(H, C) \in \mathcal{S}$ if $u + 4v > 0$, and $\beta(C) = \nu(H) = \infty$. We consider three such choices of (u, v):

(1) If $(u, v) = (1, 0)$ then $\alpha(C) = 1$, $\pi(H) = 1$, $\delta(H) = 0$.

(2) If $(u, v) = (0, 1)$ then $\alpha(C) = 0$, $\pi(H) = 0$, $\delta(H) = 1$.

(3) If $(u, v) = (1, 1)$ then $\alpha(C) = 0$, $\pi(H) = 1$, $\delta(H) = 0$.

Given any $\alpha, \beta, \pi, \nu, \delta$ such that $\beta = \nu = \infty$ and $\alpha \le \pi$ we can construct a separable Hilbert space $\mathcal{H}_0$ and $(C_0, H_0) \in \mathcal{S}(\mathcal{H}_0)$ such that $\alpha = \alpha(C_0)$, $\beta = \beta(C_0)$, $\pi = \pi(H_0)$, $\nu = \nu(H_0)$, $\delta = \delta(H_0)$ as follows. Let $\mathcal{H}_0$ be the direct sum of $\alpha + (\pi - \alpha) + \delta$ copies of l^2. (If $\alpha = \pi = \infty$ take $\pi - \alpha = 0$.) Let C_0 (resp. H_0) be a direct sum of C's (resp. H's) of the types (1), (2), and (3) above. Use α copies of type (1), $\pi - \alpha$ copies of type (2), and δ copies of type (3). If $\alpha = \pi = \delta = 0$ this process fails, and in its place we use example

(4) Set $C = 2S$ and $H = -I$. Then $H - C^*HC = 3I$, $\sigma(C) = 2\Delta_+$, and $\alpha(C) = \pi(H) = \delta(H) = 0$. (This example is covered by Theorem 4.4.)

In the examples above C has a left, but no right, inverse. To find examples with C invertible and $\Delta_0 \subset \sigma(C)$ we must take $\pi(H) = \infty$ by Theorem 3.6.

Our basic example in this case is

Example. Let $\mathcal{L}$ be the Hilbert space of bilateral square summable sequences $[\cdots, x_{-1}, \langle x_0 \rangle, x_1, \cdots]$. (We always enclose the zeroth position of the sequence in angular brackets.) Let H, D be given by diagonal matrices

$$H = \operatorname{diag}[\cdots, 1, 1, \langle u \rangle, -1, -1, \cdots] \quad \text{and} \quad D = \operatorname{diag}[\cdots, \tfrac{1}{2}, \tfrac{1}{2}, \langle 1 \rangle, 2, 2, \cdots].$$

Let T denote the shift $[\cdots, x_{-1}, \langle x_0 \rangle, x_1, \cdots] \to [\cdots, x_{-2}, \langle x_{-1} \rangle, x_0, x_1, \cdots]$. If $C = DT$ then C is invertible and $H - C^*HC = \operatorname{diag}[\cdots, 3/4, 3/4, 1 - u, \langle u + 4 \rangle, 3, 3, \cdots]$ so $(H, C) \in \mathcal{S}(\mathcal{L})$ if $-4 < u < 1$.

(5) If $u = -1$ then $\alpha(C) = \delta(H) = 0$.

(6) If $u = 0$ then $\alpha(C) = 0$ and $\delta(H) = 1$.

Given α, δ we can find $(C, H) \in \mathcal{S}$ with C invertible such that $\alpha(C) = \alpha$, $\delta(H) = \delta$, and $\pi(H) = \beta(C) = \nu(H) = \infty$ as follows: When $\alpha = \delta = 0$ we use (5); when $\alpha = 0$ and $\delta > 0$ we take a direct sum of δ copies of (6). If $\alpha > 0$ and $\delta = 0$, the direct sum of (5) with α copies of the one dimensional example $(C, H) = (\tfrac{1}{2}, 1)$ works. For $\alpha > 0$ and $\delta > 0$ we use α copies of $(C, H) = (\tfrac{1}{2}, 1)$ and δ copies of (6).

Since $\mathcal{S}$ is open it follows that if $(C, H) \in \mathcal{S}$ and if B is close enough to C then $(B, H) \in \mathcal{S}$. Thus each of $\sigma(B)$ and $\sigma(C)$ is either disjoint from Δ_0 or contains it. The upper semicontinuity of σ shows that if $\sigma(C) \cap \Delta_0 = \emptyset$ and B is close enough to C then $\sigma(B) \cap \Delta_0 = \emptyset$. However, our next example shows that if $\Delta_0 \cap \sigma(C) \neq \emptyset$ C may still admit approximation by operators B with $\Delta_0 \cap \sigma(B) = \emptyset$.

Example (7). Let $\mathcal{L}$ and T be the space and operator described above. Set $C_u = DT$ where $D = \mathrm{diag}[\cdots, 2, 2, \langle u\rangle, 2, 2, \cdots]$, and let $H = \mathrm{diag}[\cdots, -1, -1, \langle 1\rangle, -1, -1, \cdots]$. Then

$$H - C_u^* H C_u = \mathrm{diag}[\cdots, 3, 3, 1 - |u|^2, \langle 3\rangle, 3, \cdots].$$

Thus $(H, C_u) \in \mathcal{S}$ if $|u| < 1$. If $u \neq 0$ then $\sigma(C_u) \subseteq 2\Delta_0$, but $\sigma(C_0) = 2(\Delta_+ \cup \Delta_0)$. To verify this either see [3, p. 210] or compute $r(C_u)$ and $r((C_u)^{-1})$ for $u \neq 0$ and note that if $|\lambda| < 1$ then $[\cdots, \lambda^2, \lambda, 1, \langle 0\rangle, 0, \cdots] \in \mathcal{L}$ is an eigenvector for the eigenvalue 2λ.

6. The main inertia theorem for $\mathrm{Re}(HA) \gg 0$. This section is devoted to deriving the Main Inertia Theorem, which is associated with Lyapunov's condition $\mathrm{Re}(HA) \gg 0$, from our inertia theorems about Stein's condition $H - C^*HC \gg 0$. Theorem B may either be viewed as a corollary of Theorems C and 4.2 or as a corollary, via the translation technique explained in the introduction, of Theorem A.

Proof of Theorem A. Let $B = cA$ where $c^{-1} = 1 + \|A\|$. Then $-1 \notin \sigma(B)$ and $\phi(B)$ exists.

Suppose $\sigma(A) \cap \pi_0 = \emptyset$. Then $\sigma(B) \cap \pi_0 = \emptyset$ and so $\sigma(\phi(B)) \cap \Delta_0 = \emptyset$. If H is the Hermitian operator given by Theorem 4.2 such that $H - \phi(B)^* H \phi(B) \gg 0$, then $\mathrm{Re}(HA) = c^{-1}\mathrm{Re}(HB) \gg 0$, as was required.

Assume now that $\mathrm{Re}(HA) \gg 0$ for some Hermitian H. It follows that $(\phi(B), H) \in \mathcal{S}$. Since 1 is not in the range of ϕ, $1 \notin \sigma(\phi(B))$. So by Corollary 3.3(a) $\Delta_0 \cap \sigma(\phi(B)) = \emptyset$ and thus $\pi_0 = \phi^{-1}(\Delta_0 \backslash \{1\})$ is disjoint from $\sigma(B) = c\sigma(A)$. Hence $\pi_0 \cap \sigma(A) = \emptyset$. The proof of part (a) of Theorem A is finished.

To prove part (b) we assume that $\mathrm{Re}(HA) \gg 0$ and conclude, as above, that $(\phi(B), H) \in \mathcal{S}$ and $\sigma(\phi(B)) \cap \Delta_0 = \emptyset$. Then the Riesz decomposition gives $\phi(B) = C_+ \oplus C_-$ where $\sigma(C_\eta) = \Delta_\eta \cap \sigma(\phi(B))$ for $\eta = +, -$. Thus $cA = B = \phi^{-1}(C_+) \oplus \phi^{-1}(C_-)$ and by Theorem 3.5 we have

$$\pi(H) = \alpha(\phi(B)) = \pi(\phi^{-1}(C_+)) = \pi(A),$$

$$\nu(H) = \beta(\phi(B)) = \nu(\phi^{-1}(C_-)) = \nu(A),$$

and $\delta(H) = 0$. Since $\sigma(A) \cap \pi_0 = \emptyset$, $\delta(A) = 0$. Thus $\mathrm{In}_\pi(A) = \mathrm{In}_\pi(H)$. □

7. Computing $\mathrm{In}_\pi(A)$. In §1 we asserted that $\mathrm{In}_\pi(A)$ can be computed using a single spectral measure in certain cases. Now we shall prove that:

Lemma 7.1. *Suppose that $A, P, Q \in B(\mathcal{H})$ commute with each other, and that $P^2 = P$ and $Q^2 = Q$. If $\sigma(A|_{P\mathcal{H}}) \cap \sigma(A|_{Q\mathcal{H}}) = \emptyset$, then* (a) $PQ = 0$ *and* (b) $P\mathcal{H} \subset (I - Q)\mathcal{H}$.

Proof. Since $A|_{PQ\mathcal{H}}$ is a direct summand of both $A|_{P\mathcal{H}}$ and $A|_{Q\mathcal{H}}$ it follows that $\sigma(A|_{PQ\mathcal{H}}) \subset [\sigma(A|_{P\mathcal{H}}) \cap \sigma(A|_{Q\mathcal{H}})] = \emptyset$. So $PQ = 0$ and $P\mathcal{H} = P(Q + I - Q)\mathcal{H} = P(I - Q)\mathcal{H} \subset (I - Q)\mathcal{H}$. □

Theorem 7.2. *Suppose that $A \in B(\mathcal{H})$ is normal and that E is the spectral measure constructed in the spectral theorem. Then* $\mathrm{In}_\pi(A) = (\dim E_+\mathcal{H}, \dim E_-\mathcal{H}, \dim E_0\mathcal{H})$.

Proof. If $F \in \mathfrak{M}_A$ then each F_η commutes with A. Thus F_η commutes with $E(X)$ for $\eta = +, -, 0$ and for every Borel set $X \subseteq \mathbf{C}$. Lemma 7.1 shows that $E(\pi_+ + \epsilon)\mathcal{H} \subset (I - F_- - F_0)\mathcal{H} = F_+\mathcal{H}$ for every $\epsilon > 0$. Thus

$$E_+\mathcal{H} = E(\pi_+)\mathcal{H} = \text{closure}\,(\text{span} \bigcup \{E(\pi_+ + \epsilon)\mathcal{H} : \epsilon > 0\}) \subset F_+\mathcal{H}.$$

This shows that $\pi(A) = \dim E_+\mathcal{H}$, and a similar argument shows that $\nu(A) = \dim E_-\mathcal{H}$.

Lemma 7.1 shows that $F_0E(\pi_+ + \epsilon) = F_0E(\pi_- - \epsilon) = 0$ for every $\epsilon > 0$. Hence $F_0E_+ = F_0E_- = 0$ and so $F_0\mathcal{H} = F_0(E_+ + E_- + E_0)\mathcal{H} = F_0E_0\mathcal{H} \subset E_0\mathcal{H}$. Thus $\delta(A) = \dim E_0\mathcal{H}$. □

Theorem 7.3. *Suppose that $A \in B(\mathcal{H})$ and that $\sigma(A) \cap \{0 < |\mathrm{Re}\, z| < \epsilon\} = \emptyset$ for some $\epsilon > 0$. Then* $\mathrm{In}_\pi(A) = (\dim E_+\mathcal{H}, \dim E_-\mathcal{H}, \dim E_0\mathcal{H})$ *for some $E \in \mathfrak{M}_A$.*

Proof. For $\eta = +, -, 0$ let $2\pi i E_\eta$ be the integral of the resolvent of A, in the counterclockwise direction, around a rectifiable Jordan curve which misses $\sigma(A)$ but whose interior meets $\sigma(A)$ in exactly $\sigma(A) \cap \pi_\eta$. Then $E \in \mathfrak{M}_A$ and $\sigma(A|_{E_\eta\mathcal{H}}) \subset \pi_\eta$ for $\eta = +, -$. If $F \in \mathfrak{M}_A$ then each F_ξ commutes with A and must therefore commute with each E_η. So Lemma 7.1 shows that $E_\eta\mathcal{H} \subset F_\eta\mathcal{H}$ for $\eta = +, -$. Hence $\pi(A) = \dim E_+\mathcal{H}$ and $\nu(A) = \dim E_-\mathcal{H}$. Lemma 7.1 also implies that $F_0\mathcal{H} \subset E_0\mathcal{H}$, and so $\delta(A) = \dim E_0\mathcal{H}$. □

In these two theorems we have not fully exploited the method which underlies their proofs. For example certain cases in which $\sigma(A)$ has a sequence of components lying in $\pi_+ \cup \pi_-$ which "converges" to part of $\pi_0 \cap \sigma(A)$ can be handled similarly.

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