Prog. Theor. Phys. Vol. 79, No. 6, June 1988, Progress Letters

The Inertial Subrange and Non-Positive Lyapunov Exponents in Fully-Developed Turbulence

Michio YAMADA and Koji OHKITANI*

Disaster Prevention Research Institute, Kyoto University, Uji 611
**Department of Physics, Kyoto University, Kyoto 606*

(Received February 10, 1988)

The Lyapunov vectors in scalar models of two- and three-dimensional fully-developed turbulence are investigated. It is found that the inertial subrange is characterized as a support of Lyapunov vectors which correspond to *non-positive* Lyapunov exponents in the interior of the attractor.

One of the most fundamental properties of fully-developed Navier-Stokes turbulence is that it has the inertial range, a wavenumber range where each statistical quantity has a definite scaling law. In three-dimensional (3D) turbulence, the relevant parameters in that range are just the energy dissipation rate ε and the viscosity ν, and in particular, the energy spectrum has the celebrated scaling law of Kolmogorov,[1] $E(k)=\varepsilon^{1/4}\nu^{5/4}E_{3e}(k/k_d)$, $k_d=\varepsilon^{1/4}\nu^{-3/4}$. In two-dimensional (2D) turbulence the enstrophy dissipation rate η takes the place of ε, and the energy spectrum takes the BKL scaling form,[2] $E(k)=\eta^{1/6}\nu^{3/2}E_{2e}(k/k_d)$, $k_d=\eta^{1/6}\nu^{-1/2}$. Here both E_{3e} and E_{2e} are non-dimensional functions.

Recently attempts have been made to understand these scaling properties of fully-developed turbulence from a viewpoint of the theory of chaotic dynamical systems.[3),4)] But the Navier-Stokes equation still remains beyond the reach of the methods devised in such a theory owing to the insufficient capacity of present computers. At the present stage, therefore, it may be fruitful to introduce some tractable models (so-called scalar models, for example) which exhibit a chaotic behaviour and embody the desired scaling law.

In this direction Grappin et al. treated a scalar model of 3D-MHD turbulence,[5)] and obtained the $k^{-5/3}$-law of the time-averaged energy spectrum of a chaotic solution, and the Kaplan-Yorke dimension of the attractor consistent with Kolmogorov's scaling law. Scalar models of fully-developed turbulence of ordinary fluid have also been proposed both in 2D and 3D cases.[6)~8)] These models exhibit chaotic behaviour with BKL and Kolmogorov's scaling laws, respectively. Moreover, a definite scaling property is found with respect to the Lyapunov exponents λ_j for j less than the Kaplan-Yorke dimension D, when they are ordered as $\lambda_j \geq \lambda_{j+1}$.

In this paper we investigate how the inertial range is characterized in terms of the Lyapunov exponents in the framework of the previously reported scalar models of 2D and 3D fully-developed turbulence.[7),8)] These models are defined in the discrete wavenumber (Fourier) space where the wavenumbers are taken as $k_n=k_0 2^n (n=1 \sim N)$. In 2D case we take Gledzer's model,[6)] in which a real dependent variable u_n is associated with each wavenumber k_n, and the evolution equation of u_n is given by

$$(d/dt+\nu k_n^2+\nu' k_n^{-1}\theta_n)u_n=c_n^{(1)}u_{n+1}u_{n+2}+c_n^{(2)}u_{n-1}u_{n+1}+c_n^{(3)}u_{n-1}u_{n-2}$$
$$+f^{(1)}\delta_{n,10}+f^{(2)}\delta_{n,11}\,,$$
$$c_n^{(1)}=k_n,\ c_n^{(2)}=-5/4\cdot k_{n-1},\ c_n^{(3)}=1/4\cdot k_{n-2}\,,$$
$$c_1^{(2)}=c_1^{(3)}=c_2^{(3)}=c_{N-1}^{(1)}=c_N^{(1)}=c_N^{(2)}=0\,,$$

where $\theta_n=1(n\leqq 10)$, $0(n\geqq 11)$ and δ Kronecker's delta, $f^{(1)}=f^{(2)}=0.002$, $\nu'=9\times 10^{-7}$. In 3D case we take u_n as a complex variable and the evolution equation is given by

$$(d/dt+\nu k_n^2)u_n=i[c_n^{(1)}u_{n+1}{}^*u_{n+2}{}^*+c_n^{(2)}u_{n-1}{}^*u_{n+1}{}^*+c_n^{(3)}u_{n-1}{}^*u_{n-2}{}^*]+f\delta_{n,4}\,,$$
$$c_n^{(1)}=k_n,\ c_n^{(2)}=-k_{n-1}/2,$$
$$c_n^{(3)}=-k_{n-2}/2\,,$$
$$c_1^{(2)}=c_1^{(3)}=c_2^{(3)}=c_{N-1}^{(1)}$$
$$=c_N^{(1)}=c_N^{(2)}=0\,,$$

where $i=\sqrt{-1}$, $f=(1+i)\times 5\times 10^{-3}$ and * denotes the complex conjugate. Note that in 2D case the model is constructed in N-dimensional space, while in 3D case it is in (real) $2N$-dimensional space because a complex variable u_n includes two real variables; the real part $u_n^{(R)}$ and the imaginary part $u_n^{(I)}$.

Let us begin with 2D case. The parameters used are $(\nu, N, k_0)=(10^{-10}, 29, 2^{-10})$. The Kaplan-Yorke dimension D is

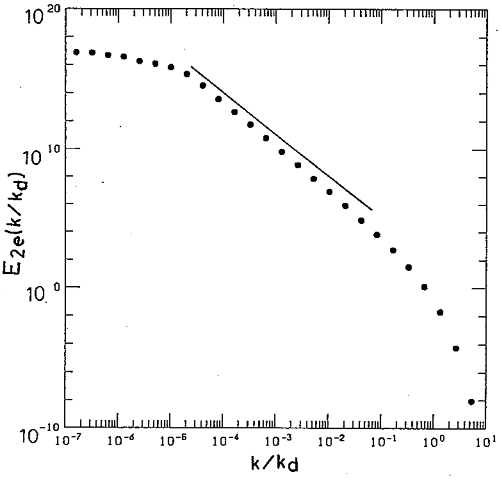

Fig. 1. The time-averaged energy spectrum for the 2D case normalized following the BKL scaling law. The straight line shows the slope of -3.

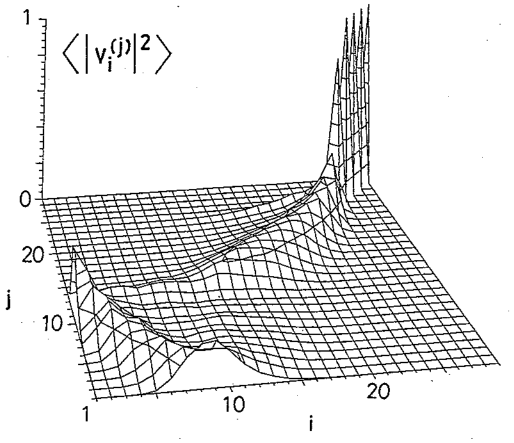

Fig. 2. The time-average of the squared components of the Lyapunov vectors $\langle|v_i^{(j)}|^2\rangle$ for the 2D case, where i and j respectively represent Fourier and Lyapunov indices. The inertial subrange extends roughly between $i=12$ and 22, while almost null and negative Lyapunov exponents in the interior of the attractor between $j=10$ and $24(\sim D)$.

23.5. The time-averaged energy spectrum $E(k_i)=\langle(u_i^2/2k_i)\rangle$ normalized following BKL scaling law is shown in Fig. 1, in which the inertial subrange with k^{-3}-spectrum is clearly seen. We calculated the Lyapunov exponents λ_j and the Lyapunov vectors $v_i^{(j)}$ accociated with them by the Gram-Schmidt orthogonalization method with the innerproduct of the vectors $\{u_i\}$ and $\{w_i\}$ being $\sum_{i=1}^N u_i w_i$. Note that $v_i^{(j)}$ is the i-th Fourier component of the j-th Lyapunov vector, and that $v_i^{(j)}$ represents an orthogonal matrix of transformation between the Fourier basis and the basis formed by the Lyapunov vectors (Lyapunov basis).

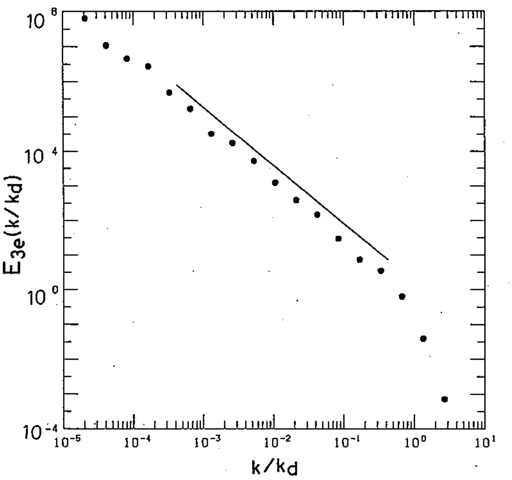

Fig. 3. The time-averaged energy spectrum for the 3D case normalized following the Kolmogorov scaling law. The straight line shows the slope $-5/3$.

The time-average of the squared components $\langle|v_i^{(j)}|^2\rangle$ of the Lyapunov vectors is shown in Fig. 2 as a bird's eye view. We can see that in the highest wavenumber range (the far dissipation range, roughly with $i, j>25$) there are very strong peaks which indicate a sharp correspondence between the two bases. In the same range the values of the Lyapunov exponents agree with the viscous damping rates of the corresponding wavenumber.[7),8)] We call this sharp correspondence "the strong Lyapunov-Fourier (L-F) correspondence".[9)] In the intermediate wavenumber range ($10<i, j<25$), on the other hand, there is a definite but relatively weak

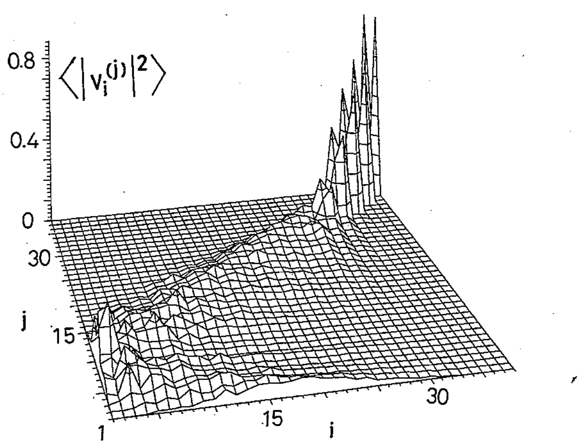

Fig. 4. The time-average of the squared components of the Lyapunov vectors $\langle|v_i^{(j)}|^2\rangle$ for the 3D case, where i and j respectively represent Fourier and Lyapunov indices. The inertial subrange extends roughly between $i=10$ and 30, while almost null and negative Lyapunov exponents in the interior of the attractor between $j=10$ and $30(\sim1.5D)$.

correspondence, which may be called "the weak L-F correspondence". We can see that this occurs at the wavenumbers in the inertial subrange, and simultaneously at the Lyapunov vectors associated with negative or almost null Lyapunov exponents whose index j is less than the Kaplan-Yorke dimension D. In other words, the inertial subrange is characterized as a support of the Lyapunov vectors which are associted with *non-positive* Lyapunov exponents in the interior*) of the attractor.

In the 3D case we set $(\nu, N, k_0)=(10^{-6}, 19, 2^{-4})$ and $D=20.3$. The energy spectrum has the Kolmogorov scaling property and the $k^{-5/3}$-form in the inertial subrange (Fig. 3), and the strong and weak correspondences similar to those in the 2D case are observed (Fig. 4). Because there are two (real) components for each wavenumber, the correspondences are not so simple as in the 2D case and the interior of the attractor extends up to $j\sim 1.5D$. It is also noted that the support of the first Lyapunov vector lies in the inertial subrange. But again we can see the sharp correspondence in the highest wavenumber range and the weak corespondence in the inertial subrange.

We note that in the Lyapunov exponents there is a clear distinction between the properties in the interior and the exterior of the attractor, while in the energy spectrum only one scaling law dominates from the inertial subrange to the far dissipation range. We have discussed by employing simple scalar models a characterization of the inertial subrange with the non-positive Lyapunov exponents, the validity of which for the Navier-Stokes turbulence is a future problem.

A detailed observation shows that the cascade process in the inertial subrange is highly temporally intermittent. In other words, the energy (enstrophy) transfer in the inertial subrange occurs like bursts, between which there are rather long quiescent periods. The weak correspondence may be attributed to these quiescent periods. The bursts, on the other hand, give rise to the instability which, especially in the 3D case, may lead to the first Lyapunov vector with its support extended in the inertial subrange. Detailed analysis on this intermittency phenomenon will be reported in the future.

The authors with to thank Professor K. Ikeda and Professor S. Kida at Kyoto University for their valuable discussions. Thanks are also due to Professor T. Tatsumi for his continuous encouragement.

1) A. N. Kolmogorov, C. R. Acad. Sci. USSR **30** (1941), 301.
2) G. K. Batchelor, Phys. Fluids Suppl. **12** (1969), II233.
R. H. Kraichnan, Phys. Fluids **10** (1967), 1417.
C. E. Leith, Phys. Fluids **11** (1968), 671.
3) J. P. Eckmann and D. Ruelle, Rev. Mod. Phys. **57** (1985), 617.
4) U. Frisch, Physica Scripta **T9** (1985), 137.
5) R. Grappin, J. Leorat and A. Pouquet, J. de Phys. **47** (1986), 1127.
C. Gloaguen, J. Leorat, A. Pouquet and R. Grappin, Physica **17D** (1985), 154.
6) E. B. Gledzer, Sov. Phys.-Dokl. **18** (1973), 216.
7) M. Yamada and K. Ohkitani, Phys. Rev. Lett. **60** (1988), 983.
8) M. Yamada and K. Ohkitani, J. Phys. Soc. Jpn. **56** (1987), 4210.
9) K. Ikeda and K. Matsumoto, J. Stat. Phys. **44** (1986), 955; Physica **29D** (1987), 223.

*) In this paper we use the terms "interior" and "exterior" for brevity simply to indicate that the index of the Lyapunov exponent, j, is respectively less or much larger than aD, where a is a constant of order unity. For the discussion of the interior and the exterior and also of the L-F correspondence of an attractor, see Ikeda and Matsumoto (1986).[9]