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THE INFINITE INJURY PRIORITY METHOD[1]

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One of the most important and distinctive tools in recursion theory has been the priority method whereby a recursively enumerable (r.e.) set A is constructed by stages to satisfy a sequence of conditions $\{R_n\}_{n\in\omega}$ called *requirements*. If $n < m$, requirement R_n is given *priority* over R_m and action taken for R_m at some stage s may at a later stage $t > s$ be undone for the sake of R_n thereby *injuring* R_m at stage t. The first priority method was invented by Friedberg [2] and Muchnik [11] to solve Post's problem and is characterized by the fact that each requirement is injured at most finitely often.

Shoenfield [20, Lemma 1], and then independently Sacks [17] and Yates [25] invented a much more powerful method in which a requirement may be injured infinitely often, and the method was applied and refined by Sacks [15], [16], [17], [18], [19] and Yates [25], [26] to obtain many deep results on r.e. sets and their degrees. In spite of numerous simplifications and variations this infinite injury method has never been as well understood as the finite injury method because of its apparently greater complexity.

The purpose of this paper is to reduce the Sacks method to two easily understood lemmas whose proofs are very similar to the finite injury case. Using these lemmas we can derive all the results of Sacks on r.e. degrees, and some by Yates and Robinson as well, in a manner accessible to the nonspecialist. The heart of the method is an ingenious observation of Lachlan [7] which is combined with a further simplification of our own.

The reader need have no prior knowledge of priority arguments for in §1 we review the finite injury method using a version invented by Sacks for his Splitting Theorem [15]. In §2 we discuss the two principal obstacles in extending the strategy to the infinite injury case. We show how the obvious and well-known solution to the first obstacle has automatically solved the second and more fundamental one. We then prove the two main lemmas upon which all of the theorems depend, and from these we prove the Thickness Lemma of Shoenfield [21, p. 83].

In §3 we apply the method to derive the Yates Index Set Theorem, and results of

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[1] This research was supported by NSF grant 19958A#3. There have been several illuminating explanations of the infinite injury method including Sacks [16], Yates [24], Martin [8], Shoenfield [21] and Lachlan [5], [6], and [7]. We gratefully acknowledge our debt to all of these sources, particuarly the last. In addition this paper reflects conversations with C. G. Jockusch, Jr., A. H. Lachlan, M. Lerman, A. Manaster, D. A. Martin and R. A. Shore. This paper was presented on April 12, 1975 at a Special Session for Recursion Theory during the American Mathematical Society meeting in St. Louis.

Sacks on interpolation, upper bounds for ascending sequences of r.e. degrees, incomplete high degrees, and incomplete maximal sets. We give the Sacks Jump Theorem in §4, and the Sacks Density Theorem and related results in §§5, 6. There are several other types of "infinite-injury" constructions which we do not discuss here, such as the minimal pair constructions of Yates [24] and Lachlan [3], the Yates Theorem [1] on r.e. degrees which cannot be "joined up" to $\mathbf{0}'$, and the method of generating automorphisms of the lattice of recursively enumerable sets [22]. These and other topics will be treated in our forthcoming monograph [23].

When discussing a complex theorem informally a recursion theorist never presents the complete proof at once but explains the basic strategy for each component separately and then shows how the various strategies may be combined. As an aid to the nonspecialist we have followed this style even at the cost of some repetition.

We assume familiarity with r.e. sets and relative recursiveness [21, Chapters 4 and 5] or [14, Chapters 5 and 9]. Let $\Phi_{e,s}(X; y)$ be the result, if any, after performing s steps in the eth Turing reduction with oracle X and input y. Let $\Phi_e = \bigcup\{\Phi_{e,s}: s \in \omega\}$. We identify sets $A \subseteq \omega$ with their characteristic functions and let $A[n]$ denote the restriction of A to arguments $\leq n$. Let $A \subseteq^* B$ denote that $A - B$ is finite and $A =^* B$ denote that $A \subseteq^* B$ and $B \subseteq^* A$. Let $\deg(A)$ be the (Turing) degree of A. Further unexplained terminology or notation can be found in Rogers [14].

§1. The finite injury priority method.

1.1. *The Sacks strategy of preserving agreements.* We illustrate the finite injury priority method by proving the Friedberg-Muchnik theorem using a variation of Sacks which is just as easy as the standard method and is much more powerful. In the constructions which we shall consider the requirements $\{R_e\}_{e\in\omega}$ can be divided into the *negative* requirements $N_e = R_{2e}$ which attempt to keep elements *out* of the r.e. set A being constructed, and *positive* requirements $P_e = R_{2e+1}$ which attempt to put elements *into* A. The negative requirement N_e will always be of the form $C \neq \Phi_e(A)$, where $C \leq_T \varnothing'$ is a fixed set (usually r.e.), so that the negative requirements together assert $C \not\leq_T A$. Sacks ingeniously observed that the requirement N_e can be met by attempting to preserve *agreement* between $C_s(x)$ and $\Phi_{e,s}(A_s; x)$ contrary to intuition. Sacks originally invented the technique to prove his splitting theorem [15, Theorem 1]. This surprisingly powerful strategy can be used [10, Theorem 2.2] to obtain such unexpected corollaries as Lachlan's remarkable characterization of *hh*-simple sets [4, Theorem 3]. An r.e. set A is *simple* if $\bar{A}$ is infinite but contains no infinite r.e. set.

THEOREM 1.1 (FRIEDBERG-MUCHNIK). *For every nonrecursive r.e. set C there is a simple set A such that $C \not\leq_T A$.*

PROOF. It clearly suffices to construct A to be coinfinite and to satisfy, for all e, the requirements:

$$N_e: C \neq \Phi_e(A), \qquad P_e: W_e \text{ infinite} \Rightarrow W_e \cap A \neq \varnothing.$$

Let $\{C_s\}_{s\in\omega}$ be a recursive enumeration of C. Define $A_0 = \varnothing$. Given A_s define the following three recursive functions whose roles are obvious from their names:

(*use function*) $u(e, x, s) = \min\{z: \Phi_{e,s}(A_s[z]; x) \text{ defined}\}$ if z exists,
$= 0$ otherwise.

(*length function*) $l(e, s) = \max\{x: (\forall y < x)[C_s(y) = \Phi_{e,s}(A_s; y)]\}$.

(*restraint function*) $r(e, s) = \max\{u(e, x, s): x \leq l(e, s)\}$.

For each $e \leq s$, if $W_{e,s} \cap A_s = \phi$ and

$$(1.1) \qquad (\exists x)[x \in W_{e,s} \ \&\ x > 2e \ \&\ (\forall i \leq e)[x > r(i, s)]],$$

then enumerate the least such x in A_{s+1}. Define $A = \bigcup_s A_s$.

(Intuitively, $u(e, x, s)$ is the maximum element *used* in the above computation, and the elements $x \leq r(e, s)$ are *restrained from* A_{s+1} by requirement N_e in order to preserve the length of agreement measured by $l(e, s)$.) The negative requirement N_e is *injured at stage* $s + 1$ *by element* x if $x \leq r(e, s)$ and $x \in A_{s+1} - A_s$. These elements form an r.e. set:

(*injury set*) $I_e = \{x: (\exists s)[x \in A_{s+1} - A_s \ \&\ x \leq r(e, s)]\}$.

Note that each I_e is finite because N_e is injured at most once for each P_i, $i < e$, whereupon P_i is satisfied thereafter. (Positive requirements, of course, are never injured.)

LEMMA 1.1. $(\forall e)[C \neq \Phi_e(A)]$.

PROOF. Assume for a contradiction that $C = \Phi_e(A)$. Then $\lim_s l(e, s) = \infty$. Choose s' such that N_e is never injured after stage s'. We shall recursively compute $C(x)$ contrary to hypothesis. To compute $C(p)$ for $p \in \omega$ find some $s > s'$ such that $l(e, s) > p$. It follows by induction on $t \geq s$ that

$$(1.2) \qquad (\forall t \geq s)[l(e, t) > p \ \&\ r(e, t) \geq \max\{u(e, x, s): x \leq p\}],$$

and hence that $\Phi_{e,s}(A_s; p) = \Phi_e(A_s; p) = \Phi_e(A; p) = C(p)$. Since $s > s'$, (1.2) clearly holds unless $C_t(x) \neq C_s(x)$ for some $t \geq s$ and $x \leq p$; but if x and t are minimal then our use of "$\leq l(e, t)$" rather than "$< l(e, t)$" in the definition of $r(e, t)$ insures that the *disagreement* $C_t(x) \neq \Phi_{e,t}(A_t; x)$ is preserved forever, contrary to the hypotheses that $C = \Phi_e(A)$. Note that even though the Sacks strategy is always described as one which preserves agreements, it is crucial that we preserve at least one *disagreement* as well.

LEMMA 1.2. $(\forall e)[\lim_s r(e, s)$ *exists and is finite*$]$.

PROOF. By Lemma 1.1 choose $p = \mu x[C(x) \neq \Phi_e(A; x)]$. Choose s' sufficiently large such that, for all $s \geq s'$,

$$(\forall x < p)[\Phi_{e,s}(A_s; x) = \Phi_e(A; x)], \qquad (\forall x \leq p)[C_s(x) = C(x)], \quad \text{and}$$
$$N_e \text{ is not injured at stage } s.$$

Case 1. $(\forall s \geq s')[\Phi_{e,s}(A_s; p)$ undefined$]$. Then $r(e, s) = r(e, s')$ for all $s \geq s'$.

Case 2. $\Phi_{e,t}(A_t; p)$ is defined for some $t \geq s'$. Then $\Phi_{e,s}(A_s; p) = \Phi_{e,t}(A_t; p)$ for all $s \geq t$ because $l(e, s) \geq p$, and so, by the definition of $r(e, s)$, the computation $\Phi_{e,t}(A_t; p)$ is preserved and N_e is not injured after stage s'. Thus $\Phi_e(A; p) = \Phi_{e,s}(A_s; p)$. But $C(p) \neq \Phi_e(A; p)$. Thus

$$(\forall s \geq t)[C_s(p) \neq \Phi_{e,s}(A_s; p) \ \&\ l(e, s) = p \ \&\ r(e, s) = r(e, t)].$$

Hence, $r(e, t) = \lim_s r(e, s)$.

LEMMA 1.3. $(\forall e)[W_e$ *infinite* $\Rightarrow W_e \cap A \neq \varnothing]$.

PROOF. By Lemma 1.2, let $r(e) = \lim_s r(e, s)$ and $R(e) = \max\{r(i): i \leq e\}$. Now if $(\exists x)[x \in W_e \;\&\; x > R(e) \;\&\; x > 2e]$ then $W_e \cap A \neq \varnothing$. Note that $\overline{A}$ is infinite by the clause "$x > 2e$" in (1.1), and hence A is simple. $\square$

1.2. *The Sacks splitting theorem.* Sacks invented the above preservation method (which plays a crucial role in the later infinite injury arguments) to prove the following theorem.

THEOREM 1.2 (SACKS SPLITTING THEOREM). *Let B and C be r.e. sets such that C is nonrecursive. Then there exist r.e. sets A_0 and A_1 such that*

(*a*) $A_0 \cup A_1 = B$ *and* $A_0 \cap A_1 = \varnothing$, *and*

(*b*) $C \not\leq_T A_i$, *for* $i = 0, 1$.

PROOF. Let $\{B_s\}_{s\in\omega}$ and $\{C_s\}_{s\in\omega}$ be recursive enumerations of B and C such that $B_0 = \varnothing$ and $|B_{s+1} - B_s| = 1$ for all s. It suffices to give recursive enumerations $\{A_{i,s}\}_{s\in\omega}$, $i = 0, 1$, satisfying the single positive requirement

$$P: x \in B_{s+1} - B_s \Rightarrow [x \in A_{0,s+1} \text{ or } x \in A_{1,s+1}],$$

and the negative requirements for $i = 0$, 1 and all e,

$$N_e^i: C \neq \Phi_e(A_i).$$

Define $A_{i,0} = \varnothing$. Given $A_{i,s}$ define the recursive functions $l^i(e, s)$ and $r^i(e, s)$ as in § 1.1 but with $A_{i,s}$ in place of A_s. Let $x \in B_{s+1} - B_s$. Choose $\langle e', i' \rangle$ to be the least $\langle e, i \rangle$ such that $x \leq r^i(e, s)$, and enumerate $x \in A_{i \dot{+} 1,s+1}$. If $\langle e', i' \rangle$ fails to exist, enumerate $x \in A_{0,s+1}$. This defines A_i, $i = 0, 1$.

To see that the construction succeeds define the injury set I_e^i as in § 1.1 but with A_i in place of A. It follows by induction on $\langle e, i \rangle$ that, for $i = 0$, 1 and all e,

(1) $C \neq \Phi_e(A_i)$,

(2) $\lim_s r^i(e, s)$ exists and is finite, and

(3) I_e^i is finite. $\square$

The r.e. sets A_i *automatically* satisfy the further property $A_i' \equiv_T \varnothing'$ as we shall see in Remark 4.5.

1.3. *Remarks and extensions.* Theorems 1.1 and 1.2 hold under the weaker hypothesis $C \leq_T \varnothing'$ in place of C r.e. by the same proof. If $C \leq_T \varnothing'$ then by the Limit Lemma of Shoenfield [21, p. 29] there is recursive sequence of recursive functions $\{C_s(x): s \in \omega\}$ such that $C(x) = \lim_s C_s(x)$ for all x. Use C_s in the definition of $l(e, s)$ as above.

Finite injury arguments are characterized by the fact that the injury set I_e is finite for each e. In §2 we shall consider cases where I_e is infinite although usually recursive. Note that Lemma 1.1 holds by virtually the same proof as above if we assume "I_e recursive" in place of "I_e finite". This will be crucial for the infinite injury method.

§2. The infinite injury priority method.

2.1. *The objective.* In many finite injury constructions of an r.e. set A (such as Theorem 1.1) we can specify a recursive array of r.e. sets $\{W_{p(e)}\}_{e\in\omega}$ and can define the positive requirements $\{P_e\}_{e\in\omega}$ by $P_e: W_{p(e)} \cap A \neq \varnothing$. We now consider constructions in which the positive requirements are of the form

$$P_e: W_{p(e)} \subseteq^* A, \tag{2.1}$$

(where $X \subseteq^* Y$ denotes that $X - Y$ is finite) so that a single positive requirement may contribute infinitely many elements to A. In the simplest cases the r.e. sets $\{W_{p(e)}\}_{e \in \omega}$ will be specified prior to the construction of A and will even be recursive (but not uniformly in e). As in §1 the negative requirement N_e will assert $C \neq \Phi_e(A)$ for some fixed set C, $\phi <_T C \leq_T \phi'$. For each N_e we would like a restraint function $\hat{r}(e, s)$ so that exactly as in §1 we can enumerate x in A at stage $s + 1$ for the sake of P_e just if $x \in W_{p(e),s+1}$ and $x > \hat{r}(i, s)$, for all $i \leq e$. The negative requirement can now be injured infinitely often by those positive requirements P_i, $i < e$, but reasonable hypotheses on the sets $W_{p(i)}$, $i < e$, will enable us to meet N_e as in Lemma 1.1.

The main difficulty will be that some P_e remains unsatisfied because of the restraint functions $\hat{r}(i, s)$, $i \leq e$, which may now be unbounded in s (i.e. $\limsup_s \hat{r}(i, s) = \infty$). To satisfy P_e it clearly suffices to define $\hat{r}(e, s)$ such that

$$\liminf_s \hat{R}(e, s) < \infty, \tag{2.2}$$

where $\hat{R}(e, s) = \max\{\hat{r}(i, s) : i \leq e\}$, because then P_e has a "window" through the negative restraints at least infinitely often.

2.2. *The two obstacles.* The *first* obstacle to achieving (2.2) is that if we let $\hat{r}(e, s)$ be $r(e, s)$ as defined in §1 then we may have $\lim_s r(e, s) = \infty$ for some e, even though requirement N_e is satisfied. Requirement N_0 having highest priority is never injured and hence $\lim_s r(0, x) < \infty$. Thus P_0 is satisfied but P_0 may contribute infinitely many elements to A and injure N_1 infinitely often. Even though N_1 may be satisfied, we may have $\lim_s r(1, s) = \infty$ and hence P_1 not satisfied. (Perhaps $\Phi_1(A; 0)$ is undefined but $C(0) = \Phi_{1,s}(A_s; 0)$ for almost every s, and $\lim_s u(1, 0, s) = \infty$. For example, suppose $\Phi_{1,s}(X, 0) = C(0)$ just if $n \notin X$ for some even $n < s$, but P_0 eventually forces every even number into A so that $\Phi_1(A; 0)$ is undefined.)

This difficulty arises only if there are infinitely many stages s such that $A_s[u] \neq A_{s+1}[u]$ where $u = u(1, x, s)$. Thus, we can easily remove the first obstacle by replacing $\Phi_{e,s}$ everywhere by $\hat{\Phi}_{e,s}$, defined in §2.3 below, and letting $\hat{r}(e, s)$ denote the resulting restraint function. If $C \neq \Phi_e(A)$ then we shall have $\liminf_s \hat{r}(e, s) < \infty$. (A similar device was used by Sacks, Yates and others.)

The *second* obstacle to (2.2) is that $\lim_s \hat{R}(e, s) = \infty$ even though $\liminf_s \hat{r}(i, s) < \infty$ for each $i \leq e$. (For example, N_1 and N_2 may together permanently restrain all elements because their restraint functions do not drop back simultaneously.) This is a more serious obstacle which required complex solutions and made a natural definition of $\hat{r}(e, s)$ seem unlikely. However, by a very ingenious observation[2]

[2] The heart of this method is Lachlan's observation [7] that using a device like $\hat{\Phi}_{e,s}$ the restraint functions $\hat{r}(e, s)$ will all drop back simultaneously at each true stage. However, at each stage s Lachlan defined the restraint functions $\hat{r}(e, s)$ by a series of substages e before which the positive requirements P_i, $i < e$, had already acted. By defining $\hat{r}(e, s)$ only in terms of $\{A_t : t \leq s\}$ we can avoid mentioning the positive requirements and substages, and can isolate and prove the two crucial lemmas once and for all. Furthermore, Lachlan specified a recursive sequence of recursive sets $\{D_i\}_{i \in \omega}$ prior to the construction (roughly equivalent to $\{\omega^{(i)}\}_{i \in \omega}$ in the notation of Lemma 2.3) and required in the definition of his counterpart to $\hat{\Phi}_{e,s}(A_s; x)$ that $z \in \bigcup\{D_i : i < e\}$. This is unnecessary in our version and would hamper applications to theorems like the incomplete maximal set (Theorem 3.8) where we cannot specify $W_{p(e)}$ for (2.1) before the construction.

of Lachlan we shall see that the naive solution to the first obstacle has already removed the second.

2.3 *The main lemmas.* The lemmas in this subsection are very general and apply to any r.e. set A whether it and its recursive enumeration $\{A_s\}_{s\in\omega}$ are *given* (i.e. specified by the "opponent" as in Lachlan [5]), or whether *we* ourselves are enumerating A during the construction by recursively defining A_{s+1} in terms of $\{A_t: t \leq s\}$.

To remove the first obstacle of §2.2 we replace $\Phi_{e,s}$ everywhere by the following $\hat{\Phi}_{e,s}$. Given $\{A_t: t \leq s\}$ define

$$a_s = \begin{cases} \mu x[x \in A_s - A_{s\div 1}] & \text{if } A_s - A_{s\div 1} \neq \varnothing, \\ \max(A_s \cup \{s\}) & \text{otherwise;} \end{cases}$$

$$\hat{\Phi}_{e,s}(A_s; x) = \begin{cases} \Phi_{e,s}(A_s; x) & \text{if defined and } u(e, x, s) < a_s, \\ \text{undefined} & \text{otherwise;} \end{cases}$$

$$\hat{u}(e, x, s) = \begin{cases} u(e, s, x) & \text{if } \hat{\Phi}_{e,s}(A_s; x) \text{ is defined,} \\ 0 & \text{otherwise;} \end{cases}$$

and

$$T = \{s: A_s[a_s] = A[a_s]\}.$$

If $\{A_s\}_{s\in\omega}$ is any recursive enumeration of an r.e. set A we refer to T as the set of *true* (nondeficiency) stages of this enumeration.[3] Note that T is infinite and $T \equiv_T A$ uniformly in A [14, p. 140]. If $\Phi_e(A; x) = y$ then clearly $\lim_s \hat{\Phi}_{e,s}(A_s; x) = y$ as before. The crucial point about $\hat{\Phi}_{e,s}$ is that for any true stage t any *apparent* computation $\hat{\Phi}_{e,t}(A_t; x) = y$ is a *true* computation $\Phi_e(A; x) = y$. Namely,

$$(2.3) \quad \begin{aligned} &(\forall t \in T)[\hat{\Phi}_{e,t}(A_t; x) = y \Rightarrow \\ &\qquad (\forall s \geq t)[\hat{\Phi}_{e,s}(A_s; x) = \Phi_e(A; x) = y \;\&\; \hat{u}(e, x, s) = u(e, x, t)]], \end{aligned}$$

because if $\hat{\Phi}_{e,t}(A_t; x)$ is defined then $u(e, x, t) < a_t$ and $A_t[a_t] = A[a_t]$.

Given $\varnothing <_T C \leq_T \varnothing'$ fix a recursive sequence of recursive functions $\{C_s\}_{s\in\omega}$ such that $C(x) = \lim_s C_s(x)$. Given $\{A_v: v \leq s\}$ we define, as in §1:

(*length function*) $\hat{l}(e, s) = \max\{x: (\forall y < x)[C_s(y) = \hat{\Phi}_{e,s}(A_s; y)]\}$.

(*modified length function*)

$$\hat{m}(e, s) = \max\{x: (\exists v \leq s)[x \leq \hat{l}(e, v)] \;\&\; (\forall y \leq x)\,[A_s[u(e, y, v)] = A_v[u(e, y, v)]]\}.$$

(*restraint function*) $\hat{r}(e, s) = \max\{\hat{u}(e, x, s): x \leq \hat{m}(e, s)\}$.

(*injury set*) $\hat{I}_e = \bigcup_s \hat{I}_{e,s}$, where $\hat{I}_{e,s} = \{x: (\exists v \leq s)[x \leq \hat{r}(e, v) \;\&\; x \in A_{v+1} - A_v]\}$.

(The point of $\hat{m}$ is to record a length of agreement established at some stage v so long as the "$\Phi_e(A)$ side" is unchanged even though a change in the "C side" at some stage $s > v$ may cause $\hat{l}(e, s) < \hat{l}(e, v)$.) For many applications such as Lemma 2.3 below we could use $\hat{l}$ instead of $\hat{m}$ in the definition of $\hat{r}$ exactly as in §1.

[3] These stages were also called *nondeficiency stages* and were used by Dekker in constructing hypersimple sets [14, p. 140]. These stages are a measure of the nonrecursiveness of an r.e. set A since a recursive set has a recursive enumeration in which every stage is true.

However, the use of $\hat{m}$ immediately yields by (2.3) certain convenient properties such as Remark 2.5 below and also

$$(2.4) \qquad (\forall t \in T)(\forall s \geq t)[\hat{m}(e, t) \leq \hat{m}(e, s) \,\&\, \hat{r}(e, t) \leq \hat{r}(e, s)].$$

The infinite injury method depends upon the following two lemmas whose proofs are very similar to those of Lemma 1.1 and Lemma 1.2 respectively.

LEMMA 2.1 (INJURY LEMMA). *If $C \not\leq_T \hat{I}_e$ then $C \neq \Phi_e(A)$.*

LEMMA 2.2 (WINDOW LEMMA). *Let T be the set of true stages in the enumeration $\{A_s\}_{s \in \omega}$ of the r.e. set A. If $C \neq \Phi_e(A)$ then $\lim_{t \in T} \hat{r}(e, t) < \infty$. (Hence, if $C \neq \Phi_i(A)$, for all $i \leq e$, then $\lim_{t \in T} \hat{R}(e, t) < \infty$, where $\hat{R}(e, s) = \max\{\hat{r}(i, s): i \leq e\}$, thereby satisfying (2.2).)*

PROOF OF INJURY LEMMA. Assume for a contradiction that $C = \Phi_e(A)$. Then $\lim_s \hat{l}(e, s) = \infty$. Fixing $\hat{I}_e$ as an oracle we shall compute C contrary to hypothesis. To compute $C(p)$ for $p \in \omega$ find some s such that $\hat{l}(e, s) > p$ and

$$(\forall x \leq p)(\forall z)[z \leq u(e, x, s) \Rightarrow [z \notin \hat{I}_e \text{ or } z \in A_s]].$$

Such s exists since $C = \Phi_e(A)$. By the same remarks as in Lemma 1.1 it follows by induction on $t \geq s$ that

$$(\forall t \geq s)[\hat{l}(e, t) > p \,\&\, \hat{r}(e, t) \geq \max\{u(e, x, s): x \leq p\}],$$

and hence that $\Phi_{e,s}(A_s; p) = \Phi_e(A_s; p) = \Phi_e(A; p) = C(p)$. □

PROOF OF WINDOW LEMMA. Assume $C \neq \Phi_e(A)$. Define $p = \mu x[C(x) \neq \Phi_e(A; x)]$. Choose s' sufficiently large such that, for all $s \geq s'$,

$$(\forall x < p)[\hat{\Phi}_{e,s}(A_s; x) = \Phi_e(A; x)] \quad \text{and} \quad (\forall x \leq p)[C_s(x) = C(x)].$$

Case 1. $(\forall t \geq s')[t \in T \Rightarrow \hat{\Phi}_{e,t}(A_t; p)$ undefined$]$. Then for any $t \geq s'$, such that $t \in T$, we have $\hat{m}(e, t) = \hat{l}(e, t) = p$ and $\hat{r}(e, t) = \max\{u(e, x, s'): x < p\}$.

Case 2. $\hat{\Phi}_{e,t}(A_t; p)$ is defined for some $t \in T$, $t \geq s'$. Then $\Phi_e(A; p) = \Phi_{e,s}(A_s; p)$ for all $s \geq t$ by (2.3). But $C(p) \neq \Phi_e(A; p)$. Hence, by the definitions we have

$$(\forall s \geq t)[\hat{l}(e, s) = p \,\&\, \hat{m}(e, s) \leq \hat{m}(e, t)],$$

and thus

$$(\forall s \geq t)[\hat{m}(e, t) = \hat{m}(e, s) \,\&\, \hat{r}(e, t) = \hat{r}(e, s)] \quad \text{by (2.4).} \qquad \square$$

2.4. *Applications of the lemmas.* Our first application is to prove the Thickness Lemma which is the paradigm of the infinite injury method. It was first proved by Shoenfield [20, p. 173] in the case where $C = \varnothing'$. Then Sacks [18] and Yates [25] each independently developed stronger forms of the infinite injury method by combining different devices for handling infinite injury with preservation methods as in §1.1. The latter will handle the case $C <_T \varnothing'$ below, which is necessary for almost all applications to degrees. Later Shoenfield [21, Chapter 16] formulated a stronger version of the Thickness Lemma, and showed how it could be used to yield results of Sacks and Yates. Fix a 1:1 recursive pairing function τ from $\omega \times \omega$ onto ω [14, p. 64] and let $\langle x, y\rangle$ denote $\tau(x, y)$. For any set A and $x \in \omega$ define the "column" $A^{(x)} = \{\langle x, y\rangle: \langle x, y\rangle \in A\}$, and $A^{(<x)} = \bigcup\{A^{(z)}: z < x\}$. A subset $A \subseteq B$ is a *thick* subset of B if $A^{(x)} =^* B^{(x)}$ for all x, and B is *piecewise recursive* if $B^{(x)}$ is recursive for each x.

LEMMA 2.3 (THICKNESS LEMMA—SHOENFIELD). *Given* $\varnothing <_T C \leq_T \varnothing'$ *and a piecewise recursive r.e. set* B *there is a thick r.e. subset* A *of* B *such that* $C \not\leq_T A$.

PROOF. Fix recursive sequences $\{B_s\}_{s\in\omega}$ and $\{C_s\}_{s\in\omega}$ such that $B = \bigcup_s B_s$ and $C = \lim_s C_s$. Let $A_0 = \varnothing$. Given $\{A_t: t \leq s\}$ define $\hat{r}(e, s)$ as above. To meet the requirements

$$P_e: B^{(e)} =^* A^{(e)} \quad \text{and} \quad N_e: C \neq \Phi_e(A),$$

we enumerate x in $A^{(e)}_{s+1}$ just if $x \in B^{(e)}_{s+1}$ and $x > \hat{r}(i, s)$ for all $i \leq e$. Let $A = \bigcup_s A_s$.

Note that $\hat{I}_e \subseteq A^{(<e)}$ because N_e is injured by P_i only if $i < e$. Thus now (and in all later theorems) we have

$$\hat{I}_e \leq_T A^{(<e)} \tag{2.5}$$

because if $x \in A^{(<e)}$, say $x \in A^{(<e)}_s$, then $x \in \hat{I}_e$ just if $x \in \hat{I}_{e,s}$.

Fix e and assume by induction that $C \neq \Phi_i(A)$ and $A^{(i)} =^* B^{(i)}$ for all $i < e$. Then $A^{(<e)} =^* B^{(<e)}$ is recursive and hence $\hat{I}_e$ is recursive. Thus $C \neq \Phi_e(A)$ by the Injury Lemma, $\lim_{t\in T} \hat{R}(e, t) < \infty$ by the Window Lemma, and $A^{(e)} =^* B^{(e)}$ by construction. $\square$

If we assume $(\forall e)[C \not\leq_T B^{(<e)}]$ instead of B piecewise recursive, then the above proof still suffices, because $\hat{I}_e \leq_T A^{(<e)} =^* B^{(<e)}$ and the Injury Lemma requires only that $C \not\leq_T \hat{I}_e$. Indeed locally we have proved that, for each e,

$$[C \not\leq_T B^{(<e)}] \Rightarrow [A^{(e)} =^* B^{(e)} \;\&\; C \neq \Phi_e(A)]$$

which suffices for the Yates index set theorems in §3 and in §6.

REMARK 2.4. Define $A^{(<e)}_s = \bigcup\{A^{(i)}_s: i < e\}$ and let T^e be the set of true stages in the enumeration $\{A^{(<e)}_s\}_{s\in\omega}$ of the r.e. set $A^{(<e)}$. Then $T^e \leq_T A^{(<e)}$ and if $\hat{I}^e \subseteq A^{(<e)}$ then

$$(\forall t \in T^e)(\forall x \leq \hat{m}(e, t))[\Phi_{e,t}(A_t; x) = y \Rightarrow \Phi_e(A; x) = y] \quad \text{and} \tag{2.6}$$

$$(\forall t \in T^e)(\forall s \geq t)[\hat{m}(e, t) \leq \hat{m}(e, s) \;\&\; \hat{r}(e, t) \leq \hat{r}(e, s)]. \tag{2.7}$$

PROOF. Since $\hat{I}_e \subseteq A^{(<e)}$, (2.6) follows as (2.3) and (2.7) follows as (2.4).

REMARK 2.5. For A and B as in Lemma 2.3 we have automatically achieved $A \leq_T B$.

PROOF. Fix e and assume that for all $i < e$ we have B-effectively computed $g(i)$ such that $A^{(i)} = \Phi_{g(i)}(B)$. Then B-effectively compute $A^{(<e)}$ and T^e. Now if $x \in B^{(e)}$, say $x \in B^{(e)}_s$, let $t' = \mu t[t \geq s \;\&\; t \in T^e]$. Hence, $x \in A^{(e)}$ just if $x \in A^{(e)}_{t'}$ by (2.7).

All these remarks can be simultaneously combined as follows. (Similar versions have been obtained using different proofs by Shoenfield [21, p. 92], Robinson [12, Theorem 1] and Yates [25].)

LEMMA 2.6 (THICKNESS LEMMA—STRONG FORM). *Given* $\varnothing <_T C \leq_T \varnothing'$ *and an r.e. set* B *there is an r.e. set* $A \subseteq B$ *such that* $A \leq_T B$ *and*

(a) $(\forall e)[C \not\leq_T B^{(<e)}] \Rightarrow [C \not\leq_T A$ *&* A *is a thick subset of* $B]$,

(b) $(\forall e)[C \not\leq_T B^{(<e)} \Rightarrow (\forall i \leq e)[C \neq \Phi_i(A) \;\&\; A^{(i)} =^* B^{(i)}]]$.

Furthermore, an index for A *can be computed uniformly in indices for* B *and* C.

§3. **Direct applications.** Many results in the literature are almost immediate corollaries of the Thickness Lemma. All the results in this section are of this form except for the incomplete maximal set (Theorem 3.8) which requires reapplying the Injury and Window Lemmas.

THEOREM 3.1 (SACKS [18, p. 108]). *Let* $\boldsymbol{d}_0 < \boldsymbol{d}_1 < \boldsymbol{d}_2 < \cdots$ *be an infinite ascending sequence of simultaneously r.e. degrees. Then there exists an r.e. degree* $\boldsymbol{a}$ *such that* $\boldsymbol{d}_0 < \boldsymbol{d}_1 < \cdots < \boldsymbol{a} < \mathbf{0}'$. (*Hence,* $\mathbf{0}'$ *is not a minimal upper bound for the sequence.*)

PROOF. Fix a recursive function h such that $\deg(W_{h(x)}) = \boldsymbol{d}_x$ for all x. Define the r.e. set B by $B^{(x)} = \{\langle x, y\rangle : y \in W_{h(x)}\}$. Let $C = \varnothing'$ and apply Lemma 2.6(a) to obtain a thick r.e. subset $A \subseteq B$ such that $C \not\leq_T A$. By thickness $W_{h(x)} \equiv_T B^{(x)} =^* A^{(x)}$ so that $\boldsymbol{d}_i < \deg(A)$ for all i. $\square$

The next corollary is a weak form of the Sacks Jump Theorem (Theorem 4.2) below and implies that there are infinitely many r.e. degrees $\boldsymbol{d}$ which are *high* ($\boldsymbol{d}' = \mathbf{0}''$).

THEOREM 3.2 (SACKS). *For any nonrecursive* $C \leq_T \varnothing'$ *there exists an r.e. set* A *such that* $A' \equiv_T \varnothing''$ *and* $C \not\leq_T A$.

PROOF. By Post's theorem [14, p. 314] any set S which is r.e. in $\varnothing'$ (such as $\varnothing''$) is in Σ_2^0 form, i.e., there is a recursive predicate $R(x, y, z)$ such that, for all x, $x \in S \Leftrightarrow (\exists y)(\forall z)R(x, y, z)$. Hence, there exists a recursive function $h(x)$ such that, for all x, $x \in S \Rightarrow W_{h(x)}$ is finite, and $x \notin S \Rightarrow W_{h(x)} = \omega$. Namely, define $W_{h(x),s} = \{v : (\forall y \leq v)(\exists z \leq s) \neg R(x, y, z)\}$. For each such S define the r.e. set B_S by $B_S^{(x)} = \{\langle x, y\rangle : y \in W_{h(x)}\}$. If A is any thick subset of B_S then $S \leq_T A'$ because

$$x \in S \Rightarrow B_S^{(x)} \text{ finite} \Rightarrow A^{(x)} \text{ finite} \Rightarrow \lim_y A(\langle x, y\rangle) = 0 \quad \text{and}$$
$$x \notin S \Rightarrow B_S^{(x)} = \omega \Rightarrow A^{(x)} =^* \omega^{(x)} \Rightarrow \lim_y A(\langle x, y\rangle) = 1.$$

(The function $F(x) = \lim_y A(\langle x, y\rangle)$ is recursive in A' by the Limit Lemma of Shoenfield [21, p. 29].)

Choose $S = \varnothing''$ and apply Lemma 2.6(a) to B_S to obtain A. $\square$

The following theorem yields a weak form of the density theorem for the special case of r.e. sets $C <_T D$ satisfying $C' <_T D'$.

THEOREM 3.3 (INTERPOLATION THEOREM—SACKS [18, p. 117]). *If* C *and* D *are r.e. sets such that* $D <_T C$ *then there exists an r.e. set* A *such that* $D \leq_T A <_T C$ *and* $A' = C'$.

PROOF. Fix recursive enumerations $\{C_s\}_{s\in\omega}$ and $\{D_s\}_{s\in\omega}$ of C and D and define the recursive function

$$h(x, s) = \begin{cases} x & \text{if } \hat{\Phi}_{x,s}(C_s; x) \text{ is defined,} \\ s & \text{otherwise.} \end{cases}$$

Define $B^{(0)} = \{\langle 0, y\rangle : y \in D\}$ and $B^{(x+1)} = \{\langle x + 1, y\rangle : (\exists s)[y \leq h(x, s)]\}$. Note that, by (2.3), $B^{(x+1)}$ is finite if and only if $\Phi_x(C; x)$ is defined (if and only if $x \in C'$). Also note that $B \leq_T C$, because if $\hat{\Phi}_{x,s}(C_s; x)$ is defined and $u = \min\{z : \Phi_{x,s}(C_s[z]$ is defined$\}$, then $h(x, s) = h(x, t)$ for all $t \geq s$ just if $C[u] = C_s[u]$.

Since B is piecewise recursive in D, use Lemma 2.6(a) to choose a thick r.e. subset A of B such that $A \leq_T B$ and $C \not\leq_T A$. Now $D \leq_T A$ because $A^{(0)} =^* B^{(0)} \equiv_T D$. Next $A' \leq_T C'$ because $A \leq_T B \leq_T C$. Finally $C' \leq_T A'$ because

$$x \in C' \Rightarrow B^{(x+1)} \text{ finite} \Rightarrow \lim_y A(\langle x + 1, y\rangle) = 0 \quad \text{and}$$
$$x \notin C' \Rightarrow B^{(x+1)} = \omega^{(x+1)} \Rightarrow \lim_y A(\langle x + 1, y\rangle) = 1. \qquad \square$$

COROLLARY 3.4 (WEAK DENSITY THEOREM). *If C and D are r.e. sets such that $D <_T C$ and $D' <_T C'$ then there is an r.e. set A such that $D <_T A <_T C$.*

PROOF. By Theorem 3.3. $\square$

After proving this weak form of the density theorem, Sacks eliminated the hypothesis $D' <_T C'$ to obtain the full Density Theorem by inventing a new coding strategy to insure $A \not\leq_T D$ as we discuss in §5. Shortly thereafter Yates [25] derived the Density Theorem as a corollary of the following result on index sets. To prove his result we need two lemmas of Yates which can easily be derived without using priority methods [25, pp. 312, 314] and whose proofs we omit. Let V be an r.e. set. (As in Rogers [14, p. 304] let $S \in \Sigma_3^V$ denote that there is a predicate R^V recursive in V such that $x \in S$ just if $(\exists y)(\forall v)(\exists w)R^V(x, y, v, w)$.)

LEMMA A (YATES). $\{x: W_x \equiv_T V\} \in \Sigma_3^V$.

LEMMA B (YATES). *For any set $S \in \Sigma_3^V$ there is a recursive function $h(x)$ such that, for all x, $W_{h(x)} \leq_T V$ and*

(a) $x \in S \Rightarrow (\exists e)[W_{h(x)}^{(e)} \equiv_T V \;\&\; (\forall i < e)[W_{h(x)}^{(i)}$ *is recursive*$]]$,

(b) $x \notin S \Rightarrow (\forall e)[W_{h(x)}^{(e)}$ *recursive*$]$.

THEOREM 3.5 (INDEX SET THEOREM—YATES). *Given r.e. sets C and D such that $D <_T C$ and $S \in \Sigma_3^C$ there is a recursive function $g(x)$ such that, for all x,*

(a) $D \leq_T W_{g(x)} \leq_T C$, *and*

(b) $x \in S \Leftrightarrow W_{g(x)} \equiv_T C$.

COROLLARY 3.6 (DENSITY THEOREM—SACKS). *If D and C are r.e. and $D <_T C$ then there exists an r.e. set A such that $D <_T A <_T C$.*

PROOF (COROLLARY 3.6). Let $S = \{x: W_x \equiv_T D\}$. Then, by Lemma A, $S \in \Sigma_3^D$ and hence $S \in \Sigma_3^C$. Apply Theorem 3.5 to find $g(x)$ such that $D \leq_T W_{g(x)} \leq_T C$ and $W_x \equiv_T D$ just if $W_{g(x)} \equiv_T C$. By the recursion theorem choose x_0 such that $W_{x_0} = W_{g(x_0)}$. Then $D <_T W_{g(x_0)} <_T C$. $\square$

PROOF (THEOREM 3.5). Fix $V = C$, $S \in \Sigma_3^C$ and $h(x)$ the recursive function for S according to Lemma B. For each x define the r.e. set B_x by $B_x^{(0)} = \{\langle 0, y\rangle : y \in D\}$ and $B_x^{(e+1)} = \{\langle e + 1, y\rangle : y \in W_{h(x)}^{(e)}\}$. (Note that $B_x \leq_T C$ for all x because $W_{h(x)} \leq_T C$ and $D \leq_T C$.) For each x apply Lemma 2.6(b) to B_x and C to find $A_x \subseteq B_x$, so that $A_x \leq_T B_x \leq_T C$. Moreover, for each x, $D \leq_T A_x$ because $A_x^{(0)} =^* B_x^{(0)} \equiv_T D$ by Lemma 2.6(b) with $e = 0$. By the uniformity of Lemma 2.6 there is a recursive function $g(x)$ such that $W_{g(x)} = A_x$. Now if $x \notin S$ then $B_x^{(e)}$ is recursive for all $e > 0$ by Lemma B(b) whence $C \not\leq_T A_x$ by Lemma 2.6(b). If $x \in S$ then by Lemma B(a) and the definition of B choose e such that

$$B_x^{(e)} \equiv_T C \;\&\; (\forall i)[0 < i < e \Rightarrow B_x^{(i)} \text{ is recursive}].$$

Hence, $B_x^{(<e)} \equiv_T D <_T C$. Therefore, by Lemma 2.6(b) $A_x^{(e)} =^* B_x^{(e)} \equiv_T C$, so that $C \leq_T A_x$. Thus, $A_x \equiv_T C$ because $A_x \leq_T B_x \leq_T C$. $\square$

COROLLARY 3.7 (YATES). *If C is r.e. and $S \in \Sigma_3^C$ then $S \leq_1 \{x: W_x \equiv_T C\}$.*

PROOF. If $C \equiv_T \varnothing$ see Rogers [14, p. 327]. Otherwise apply Theorem 3.5 with $D = \varnothing$. $\square$

Even when the Thickness Lemma does not apply directly, the Injury and Window Lemmas can be applied to prove such theorems as the following by Sacks [16] which implies the existence of an incomplete maximal set. An r.e. coinfinite set A is *maximal* if $A \subseteq W_x$ implies $A =^* W_x$ or $\omega =^* W_x$ for all x.

THEOREM 3.8 (SACKS). *For any nonrecursive set $C \leq_T \varnothing'$ there exists a maximal set A such that $C \not\leq_T A$.*

PROOF (SKETCH). We assume that the reader is familiar with the Yates maximal set construction [14, p. 235] where the positive requirement P_e asserts that, for all $i \geq e$, x_i the ith member of $\bar{A}$ achieves the highest possible e-state which is infinitely resided by $\{x_j : j \geq i\}$. Given C define N_e and $\hat{r}(e, s)$ as usual. Allow P_e to contribute x to A at stage $s + 1$ only if $x > \hat{r}(i, s)$ for all $i \leq e$. Applying Lemmas 2.1 and 2.2 it is easy to show by induction on e that for all e: N_e is satisfied, P_e is satisfied, and the set of elements contributed to A by P_e is recursive (whence $\hat{I}_{e+1}$ is recursive). $\square$

Later Martin [9] proved by a different method that an r.e. degree $\boldsymbol{a}$ contains a maximal set if and only if $\boldsymbol{a}$ is *high* ($\boldsymbol{a}' = \boldsymbol{0}''$). By Theorem 3.2 such degrees may be incomplete and hence Martin's result implies Theorem 3.8.

§4. **The Jump Theorem.** Historically Sacks' first application of the infinite injury priority method was to prove the Jump Theorem [17] which asserts that if S is r.e. in $\varnothing'$, $\varnothing' \leq_T S$, and $\varnothing <_T C \leq_T \varnothing'$ then $S \equiv_T A'$ for some r.e. set A such that $C \not\leq_T A$. Of course, $S \leq_T A'$ is easy to achieve using the corresponding r.e. set B_S (abbreviated B) defined in Theorem 3.2 because any thick r.e. subset A of B satisfies $S \leq_T A'$. Keeping *down* the jump A' requires some further restraint which is best illustrated by the following special case of the Jump Theorem. This result is dual to Theorem 3.2, and requires only a finite injury proof as in Theorem 1.1.

THEOREM 4.1. *There exists a simple set A such that $A' \equiv_T \varnothing'$ (i.e., A is low).*

PROOF. Define the positive requirement $P_e: \bar{A} \neq W_e$ as in Theorem 1.1. To insure $A' \leq_T \phi'$ it suffices to meet for each e the negative requirement

$$Q_e: (\exists^\infty s)[\hat{\Phi}_{e,s}(A_s; e) \text{ defined}] \Rightarrow \Phi_e(A; e) \text{ defined},$$

where $(\exists^\infty s)$ abbreviates "there exist infinitely many s." To see this we define the recursive function $g(e, s) = 1$ if $\hat{\Phi}_{e,s}(A_s; e)$ is defined and $g(e, s) = 0$ otherwise. If requirement Q_e is satisfied for all e then $G(e) = \lim_s g(e, s)$ exists, is the characteristic function of A', and $G \leq_T \phi'$ by the Limit Lemma [21, p. 29].

To meet Q_e define a second restraint function

$$\hat{q}(e, s) = \hat{u}(e, e, s). \tag{4.1}$$

The construction of A and definition of injury are the same as in Theorem 1.1 with $\hat{q}(e, s)$ in place of $r(e, s)$. (As in Theorem 1.1, Q_e is injured at most finitely often, say never after stage s'. In place of Lemma 1.1 note that Q_e is satisfied because if $\hat{\Phi}_{e,s}(A_s; e)$ is defined for some $s > s'$ then $\hat{\Phi}_{e,t}(A_t; e) = \Phi_e(A; e)$ for all $t \geq s$.) $\square$

For the full Jump Theorem below we have for each e the positive requirement $P_e: A^{(e)} =^* B^{(e)}$ and negative requirement $N_e: C \neq \Phi_e(A)$ as in §2. In addition we have the negative "pseudo-requirement" Q_e above which we cannot hope to meet for each e because we want $A' \equiv_T S$ and perhaps $\varnothing' <_T S$. Nevertheless we *attempt* to meet Q_e in exactly the same way as before by allowing P_i to injure Q_e only if $i < e$. We then prove that the injuries to Q_e (although possibly infinite in number) are sufficiently well-behaved so that we can verify $A' \leq_T S$ using the hypothesis $\varnothing' \leq_T S$. This is accomplished by defining an S-recursive function g such that, for every e, $\varphi_{g(e)}$ is the characteristic function of $A^{(e)}$.

THEOREM 4.2 (JUMP THEOREM—SACKS). *For any sets S and C such that S is r.e. in $\varnothing'$, $\varnothing' \leq_T S$, and $\varnothing <_T C \leq_T \varnothing'$ there exists an r.e. set A such that $A' \equiv_T S$ and $C \nleq_T A$.*

PROOF. Given S define the r.e. set B as in the proof of Theorem 3.2 such that, for all x,

$$x \in S \Rightarrow B^{(x)} \text{ is finite} \quad \text{and} \quad x \notin S \Rightarrow B^{(x)} = \omega^{(x)}. \tag{4.2}$$

Fix recursive sequences $\{B_s\}_{s\in\omega}$ and $\{C_s\}_{s\in\omega}$ such that $B = \bigcup_s B_s$ and $C = \lim_s C_s$. Given $\{A_t: t \leq s\}$ define $\hat{r}(e, s)$ as in §2 and $\hat{q}(e, s)$ as above. Enumerate x in $A^{(e)}_{s+1}$ just if $x \in B^{(e)}_{s+1}$ and $x > \max\{\hat{r}(i, s), \hat{q}(i, s)\}$ for all $i \leq e$. Let $A = \bigcup_s A_s$.

LEMMA 1. *For all e, $A^{(e)} =^* B^{(e)}$ and $C \neq \Phi_e(A)$. (Hence, $C \nleq_T A$ and $S \leq_T A'$.)*

PROOF. Let T be the set of true stages in the enumeration $\{A_s\}_{s\in\omega}$. Fix e and assume that P_i and N_i hold for all $i < e$. Then $A^{(<e)} =^* B^{(<e)}$ is recursive (because B is piecewise recursive); N_e is satisfied by the Injury Lemma, and $\lim_{t\in T} \hat{r}(e, t) < \infty$ by the Window Lemma. Furthermore, $\lim_{t\in T} \hat{q}(e, t) < \infty$ by (2.3). Hence, $A^{(e)} =^* B^{(e)}$ by construction.

LEMMA 2. $A' \leq_T S$.

PROOF. Define T^e to be the set of true stages of the enumeration $\{A^{(<e)}_s\}_{s\in\omega}$ (as in Remark 2.4 whose proof is very similar). Now requirement N_e or Q_e can be injured by element x only if $x \in A^{(i)}$ for some $i < e$. Thus, as in (2.6) and (2.7),

$$(\forall t \in T^e)[\hat{\Phi}_{e,t}(A_t; e) \text{ defined} \Rightarrow \Phi_e(A; e) \text{ defined}] \quad \text{and} \tag{4.3}$$

$$(\forall t \in T^e)(\forall s \geq t)[\hat{q}(e, t) \leq \hat{q}(e, s) \;\&\; \hat{r}(e, t) \leq \hat{r}(e, s)]. \tag{4.4}$$

Fix e and assume that we have S-recursively computed, for all $i < e$,

(1) whether $i \in A'$; and

(2) an index $g(i)$ such that $A^{(i)}$ has characteristic function $\varphi_{g(i)}$.

From (2) we can S-recursively compute indices for the (recursive) characteristic functions of $A^{(<e)}$ and T^e. Since $\varnothing' \leq_T S$ we now decide whether $e \in A'$ because, by (4.3),

$$e \in A' \Leftrightarrow \Phi_e(A; e) \text{ defined} \Leftrightarrow (\exists t)[t \in T^e \;\&\; \hat{\Phi}_{e,t}(A_t; e) \text{ defined}], \tag{4.5}$$

and the latter is in Σ^0_1 form and hence recursive in $\varnothing'$.

To compute $g(e)$ we first S-recursively compute an index for the recursive characteristic function of $B^{(e)}$ using (4.2) and the definition of B. Now if $x \in B^{(e)}$, say $x \in B^{(e)}_s$, define $t' = (\mu t > s)[t \in T^e]$. Then $x \in A^{(e)}$ just if $x \in A_{t'}$ by (4.4). □

For future reference note that using $\varnothing' \leq_T S$ and (4.4) we have

$$\hat{r}(e) = \liminf_s \hat{r}(e, s) \quad \text{is an } S\text{-recursive function.} \tag{4.6}$$

REMARK 4.3 (SACKS). Given S and C as in Theorem 4.2 if D is any r.e. set such that $D' \leq_T S$ and $C \nleq_T D$ then we can add to the conclusion that $D \leq_T A$.

PROOF. Replace B above by the r.e. set $\tilde{B}$ where $\tilde{B}^{(0)} = \{\langle 0, y\rangle : y \in D\}$ and $\langle x + 1, y\rangle \in \tilde{B}$ just if $\langle x, y\rangle \in B$. The construction and proofs are now the same as above except that A and $\tilde{B}$ are now "piecewise recursive in D" instead of "piecewise recursive" because $A^{(0)} =^* \tilde{B}^{(0)} \equiv_T D$. The Injury Lemma still applies in Lemma 1 because $\hat{l}_e \leq_T A^{(<e)} \leq_T D$ and $C \nleq_T D$. Now $T^e \leq_T D$ so that in (4.5) we must use a D'-oracle instead of a $\varnothing'$-oracle. □

REMARK 4.4. Given S and C as in Theorem 4.2 we can obtain A by directly applying the Thickness Lemma to C and B without introducing the second restraint function $\hat{q}(e, s)$.

PROOF. Given $C(0)$ define the partial recursive functional,

$$\Phi_{g(e)}(X; y) = \begin{cases} C(0) & \text{if } y = 0 \text{ and } \Phi_e(X; e) \text{ is defined,} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

$\therefore\ \Phi_e(A; e)$ defined $\Leftrightarrow \Phi_{g(e)}(A; 0) = C(0) \Leftrightarrow \lim_s \hat{l}(g(e), s) > 0$.

In the proof of Theorem 4.2 replace all instances of $\hat{q}(e, s)$ by $\hat{r}(g(e), s)$. $\square$

REMARK 4.5. The r.e. sets A_i of the Sacks Splitting Theorem (Theorem 1.2) automatically satisfy $A_i' \equiv_T \varnothing'$ for $i = 0, 1$.

PROOF. By the proofs of Remark 4.4 and Theorem 4.1. $\square$

§5. The Density Theorem and the coding strategy. Sacks originally proved the Density Theorem not by index sets as in Corollary 3.6 but by inventing a new coding strategy [19] for the positive requirements. This coding strategy is the major new idea in that proof and has numerous other applications.

THEOREM 5.1 (DENSITY THEOREM—SACKS). *Given r.e. sets $D <_T C$ there exists an r.e. set A such that $D <_T A <_T C$.*

PROOF.[4] Fix recursive enumerations $\{C_s\}_{s\in\omega}$, $\{D_s\}_{s\in\omega}$ of C and D. Define $A_s^{(0)} = \{\langle 0, y\rangle : y \in D_s\}$ so that $D \leq_T A$. We shall arrange $A \leq_T C$ as in Remark 2.5 by finding a C-recursive function g such that $A^{(e)} = \Phi_{g(e)}(C)$ for all e. To make both inequalities strict it suffices to meet, for all $e > 0$, the requirements

$$N_e\colon C \neq \Phi_e(A) \quad \text{and} \quad P_e\colon A \neq \Phi_e(D).$$

To meet P_e we attempt to code C into $A^{(e)}$ so that if $A = \Phi_e(D)$ then $C \leq_T D$ contrary to hypothesis.

Let $A_0 = \varnothing$. Given $\{A_t : t \leq s\}$ define $\hat{r}(e, s)$ as in §2, and define

$$\hat{l}^D(e, s) = \max\{x : (\forall y < x)[A_s(y) = \hat{\Phi}_{e,s}(D_s, y)]\}.$$

Let $\langle e, x, s\rangle$ denote $\langle e, \langle x, s\rangle\rangle$. For $t \leq s$, and $e > 1$ enumerate $\langle e, x, t\rangle$ in A_{s+1} just if $\langle e, x, t\rangle > \hat{r}(i, s)$ for all $i \leq e$, $x \in C_{s+1}$ and $x < \hat{l}^D(e, v)$ for all v, $t \leq v \leq s$. Let $A = \bigcup_s A_s$.

(We visualize this coding strategy as follows. Fixing e, the elements $\{\langle e, x, y\rangle : x, y \in \omega\}$ are arranged in a plane. A "coding marker" is assigned to $\langle e, x, t\rangle$ at stage t if $\hat{l}^D(e, t) > x$. The marker is later removed (forever) at some stage $s > t$ if $\hat{l}^D(e, s) \leq x$. If $x \in C_{s+1}$ then all elements $\langle e, x, t\rangle$, $t \leq s$, still possessing markers at stage s and not restrained with higher priority are enumerated in A_{s+1}.)

To see that A succeeds we shall verify the requirements by induction on e and simultaneously C-recursively define $g(e)$ such that $A^{(e)} = \Phi_{g(e)}(C)$. Clearly $g(0)$ exists because $A^{(0)} \equiv_T D <_T C$. Fix $e > 0$ and assume, for all i, $0 < i < e$, that

(5.1) $C \neq \Phi_i(A)$,

(5.2) $A \neq \Phi_i(D)$,

[4] Our proof of the Density Theorem is very similar to Lachlan [7] except for our treatment of N_e as explained in §2.

(5.3) $A^{(i)}$ is recursive, and

(5.4) $A^{(i)} = \Phi_{g(i)}(C)$, where $g(i)$ has been C-recursively computed.

LEMMA 1. $C \neq \Phi_e(A)$.

PROOF. Now $A^{(<e)} \leq_T D$ because $A^{(i)}$ is recursive for $0 < i < e$. Let $\hat{I}_e$ be the injury set for N_e defined in §2. Then $\hat{I}_e \subseteq A^{(<e)}$ and hence $\hat{I}_e \leq_T A^{(<e)}$ as in (2.5). Thus, $\hat{I}_e \leq_T A^{(<e)} \leq_T D <_T C$. Hence, $C \neq \Phi_e(A)$ by the Injury Lemma.

LEMMA 2. $A \neq \Phi_e(D)$.

PROOF. By Lemma 1 and the Window Lemma $\hat{r}(e) = \liminf_s \hat{r}(e, s) < \infty$. If $A = \Phi_e(D)$ then $\lim_s \hat{l}^D(e, s) = \infty$. Since D is r.e. we can D-recursively compute the modulus function

$$M(x) = (\mu s)(\forall t \geq s)[\hat{l}^D(e, t) > x].$$

For $x > \hat{r}(e)$, $x \in C$ just if $\langle e, x, M(x)\rangle \in A$. Hence, $C \leq_T A \leq_T D$ contrary to hypothesis. (Note that M is a D-recursive function because D r.e. implies that $M(x) = (\mu s)[\hat{l}^D(e, s) > x \;\&\; D_s[u] = D[u]]$, where $u = \max\{u(e, y, s): y \leq x\}$.)

LEMMA 3. *$A^{(e)}$ is recursive.*

PROOF. By Lemma 2, let $p_e = (\mu x)[A(x) \neq \Phi_e(D; x)]$. Then $\liminf_s \hat{l}^D(e, s) = p_e$. For $x \geq p_e$ given x and t find $s \geq t$ such that $\hat{l}^D(e, s) = p_e$. Then $\langle e, x, t\rangle \in A$ just if $\langle e, x, t\rangle \in A_s$. For $x < p_e$ fix s' such that $C_{s'}[p_e] = C[p_e]$. Given t define

$$v' = (\mu v)[v \geq s' \;\&\; v \geq t \;\&\; \hat{r}(e, v) = \hat{r}(e)].$$

Then $\langle e, x, t\rangle \in A$ just if $\langle e, x, t\rangle \in A_{v'}$.

LEMMA 4. *We can C-recursively compute $g(e)$ such that $A^{(e)} = \Phi_{g(e)}(C)$.*

PROOF. Define T^e as in Remark 2.4. From $\{g(i): i < e\}$ we C-recursively compute $A^{(<e)}$ and hence T^e. Fix $\langle e, x, t\rangle$. If $x \notin C$ then $\langle e, x, t\rangle \notin A$. If $x \in C$, say $x \in C_s$, define

$$v' = (\mu v)[v \geq t \;\&\; v \geq s \;\&\; v \in T^e].$$

Then $\langle e, x, t\rangle \in A$ just if $\langle e, x, t\rangle \in A_{v'}$. □

(Notice that unlike Lemma 2 of Theorem 4.2 we do not claim here that (5.3) and (5.4) can be combined to produce a C-recursive function g such that $\varphi_{g(e)}$ is the characteristic function of $A^{(e)}$ for all $e > 0$, but merely that $A^{(e)} = \Phi_{g(e)}(C)$. The point is that even though $A^{(e)}$ is recursive for all $e > 0$, the proof of Lemma 3 above depends upon parameters p_e and $\hat{r}(e)$ which cannot be C-recursively computed uniformly in e, and the proof of Lemma 4 clearly uses a C-oracle for each x. This subtle distinction will become more apparent after comparing the functions g and h in the proof of Theorem 5.4 below, where the distinction is crucial.)

The above coding procedure has many other applications such as the following.

THEOREM 5.2 (SACKS-YATES[5]). *Given any r.e. set C such that $\varnothing <_T C <_T \varnothing'$ there exists an r.e. set A such that A is Turing incomparable with C. Furthermore, an index for A can be found uniformly from one for C.*

PROOF. It suffices to meet for all $e > 0$ the requirements $N_e: C \neq \Phi_e(A)$, and

[5] The nonuniform version of the theorem follows immediately from the Sacks Splitting Theorem. The uniform version requires an infinite injury argument and was proved by Yates [25] using index sets.

$P_e: A \neq \Phi_e(C)$. Let $K = \{e: e \in W_e\} \equiv_T \varnothing'$. In place of the hypothesis $D <_T C$ of Theorem 5.1 we use the hypothesis $C <_T K$ so that C and K play the former roles of D and C respectively. Let $A_0 = \varnothing$. Given $\{C_s\}_{s\in\omega}$, $\{K_s\}_{s\in\omega}$ and $\{A_t: t \le s\}$ as usual define

$$\hat{l}^C(e, s) = \max\{x: (\forall y < x)[A_s(y) = \hat{\Phi}_{e,s}(C_s; y)]\}.$$

For $t \le s$ enumerate $\langle e, x, t\rangle$ in A_{s+1} just if $\langle e, x, t\rangle > \hat{r}(i, s)$ for all $i \le e$, $x \in K_{s+1}$, and $x < \hat{l}^C(e, v)$, for all v, $t \le v \le s$. Let $A = \bigcup_s A_s$.

Fix e and assume by induction that, for all $i < e$, $C \neq \Phi_i(A)$, $A \neq \Phi_i(C)$, and $A^{(i)}$ is recursive. The proofs above establish (with C and K in place of D and C respectively in Lemmas 2 and 3): $C \neq \Phi_e(A)$; $A \neq \Phi_e(C)$; and $A^{(e)}$ is recursive. □

It is natural to ask to what extent the previous theorem can be combined with the jump theorem of §4 so that $A' \equiv_T S$ for some given S. Yates [26, p. 261] proved that this could be done for $S \equiv_T \varnothing''$ while still preserving the uniformity of Theorem 5.2. Later Robinson [13, Corollary 3] extended the result to any S r.e. in $\varnothing'$ such that $\varnothing' \le_T S$, but without the Yates uniformity. We prove the result in two stages as the next two theorems in order to fully expose the difficulties.

THEOREM 5.3 (YATES). *Given any r.e. set C such that $\varnothing <_T C <_T \varnothing'$ and any set S r.e. in $\varnothing'$ such that $C' \le_T S$ there exists an r.e. set A such that $A' \equiv_T S$ and A is Turing incomparable with C. Furthermore, an index for A can be found uniformly in indices for C and S.*

PROOF. The strategy for enumerating elements in $A^{(e)}$ is the same as that for P_i of Theorem 5.2 if $e = 2i$ and as that for P_i of Theorem 4.2 if $e = 2i + 1$ subject to the usual restraint function $\hat{r}(e, s)$. (By Remark 4.4 we can eliminate the second restraint function $\hat{q}(e, s)$ and still achieve $A' \le_T S$.)

For the former proofs to suffice it remains only to show that there is an S-recursive function g such that $\varphi_{g(e)}$ is the characteristic function of $A^{(e)}$ for all $e \ge 0$. Given $\{g(i): i < e\}$ compute $g(e)$ as in Theorem 4.2, Lemma 2, if e is odd, and as in Theorem 5.2, Lemma 3, if e is even. (Note that the parameters $\hat{r}(e)$ and p_e in Lemma 3 are S-recursive as functions of e using (4.6) and $C' \le_T S$.) □

THEOREM 5.4 (ROBINSON). *Given any r.e. set C such that $\varnothing <_T C <_T \varnothing'$ and set S r.e. in $\varnothing'$ such that $\varnothing' \le_T S$ there exists an r.e. set A Turing incomparable with C such that $A' \equiv_T S$.*

PROOF. By Theorem 1.2 and Remark 4.5 there exist r.e. sets D_0, D_1 such that
(1) $D_0 \cup D_1 = K$ and $D_0 \cap D_1 = \varnothing$,
(2) $C \not\le_T D_i$ for $i = 0, 1$, and
(3) $D_i' \equiv_T \phi'$ for $i = 0, 1$.

From (1), (2), and $K \not\le_T C$ it follows that one of the sets, say D_0, is incomparable with C. Now using (2) and (3) apply Remark 4.3 to construct A such that $D_0 \le_T A$, $A' \equiv_T S$, and $C \not\le_T A$. But $A \not\le_T C$ because $D_0 \le_T A$ and thus $A \le_T C$ would contradict $D_0 \not\le_T C$. □

We can also use the coding strategy to simultaneously combine the results of the Jump Theorem, the Density Theorem, and the Interpolation Theorem (Theorem 3.3). Given r.e. sets $D <_T C$ suppose that $D <_T A <_T C$ and $S \equiv_T A'$. What can be said of S? Clearly it is necessary that $D' \le_T S$ and deg(S) is r.e. in C [14, Chapter 13]. The following theorem asserts that these conditions are also sufficient.

THEOREM 5.5 (JUMP INTERPOLATION THEOREM—ROBINSON). *Given r.e. sets* $D <_T C$ *and a set* S *r.e. in* C *and* $D' \leq_T S$, *then there exists an r.e. set* A *such that* $D <_T A <_T C$ *and* $A' \equiv_T S$.

PROOF. Since S is r.e. in C (and hence $S \leq_1 C'$) we can use the method of Theorem 3.3 to find an r.e. set $B \leq_T C$ such that $B^{(0)} \equiv_T D$ and, for all $x > 0$, $x \in S \Rightarrow B^{(x)}$ is finite, and $x \notin S \Rightarrow B^{(x)} = \omega^{(x)}$. The strategy for enumerating elements in $A^{(e)}$ is the same as that for P_i of Theorem 3.3 (with B as above) if $e = 2i$ and for P_i of Theorem 5.1 if $e = 2i + 1$.

Since $B \leq_T C$ we can compute (as in Lemma 4 of Theorem 5.1) a C-recursive function g such that, for all $e > 0$, $A^{(e)} = \Phi_{g(e)}(C)$. Since $D' \leq_T S$ we can also compute as in Theorem 4.2, Lemma 2, and Theorem 5.3 an S-recursive function $h(e)$ such that $\varphi_{h(e)}$ is the characteristic function of $A^{(e)}$. The rest follows as in the previous proofs. $\square$

By modifying slightly the coding procedure of Theorem 5.2 one can weaken the hypothesis there from C r.e. to $\varnothing <_T C \leq_T \varnothing'$. Further generalizations were obtained by R. W. Robinson [13, Theorem 3] using more complicated methods.

§6. **Degrees of index sets.** Let D be any r.e. set and $\boldsymbol{d}$ its degree. Following Yates [26] we define the following index sets related to $\boldsymbol{d}$.

$$G(\boldsymbol{d}) = \{e\colon W_e \equiv_T D\}, \quad G(\leq \boldsymbol{d}) = \{e\colon W_e \leq_T D\}, \quad G(\geq \boldsymbol{d}) = \{e\colon D \leq_T W_e\},$$
$$G(|\, \boldsymbol{d}) = \{e\colon W_e \text{ and } D \text{ are Turing incomparable}\}.$$

From the results of Yates in §3 (Lemma A and Corollary 3.7), $G(\boldsymbol{d})$ is Σ_3^D-complete (as defined in Rogers [14, p. 316]).

Yates then proved that if $\boldsymbol{d} < \boldsymbol{0}'$ then $G(\leq \boldsymbol{d})$ is Σ_3^D-complete. (If $\boldsymbol{d} = \boldsymbol{0}'$ there is no interest because $G(\leq \boldsymbol{d}) = \omega$.) We shall omit the proof which does not use any priority method although it is clever and delicate and requires a coding method like that used by Sacks in Theorem 5.1. Yates also proved [26, §3] by an infinite injury argument that if $\boldsymbol{0} < \boldsymbol{d} \leq \boldsymbol{0}'$ then $G(\geq \boldsymbol{d})$ is Σ_4-complete and if $\boldsymbol{0} < \boldsymbol{d} < \boldsymbol{0}'$ then $(G(|\, \boldsymbol{d})$ is Π_4-complete. We now easily derive these results from Lemma 2.6.

THEOREM 6.1 (YATES). *For any r.e. set* D, $\varnothing <_T D <_T \varnothing'$, *and any set* $S \in \Sigma_4$ *there is an r.e. sequence* $\{A_x\}_{x \in \omega}$ *of r.e. sets such that*

$$x \in S \Rightarrow A_x \equiv_T \varnothing',$$
$$x \notin S \Rightarrow A_x \textit{ is Turing incomparable with } D.$$

PROOF. By Theorem 5.2 (or Theorem 1.2) there exists an r.e. set E Turing incomparable with D. Since $S \in \Sigma_4$ and $\Sigma_4 = \Sigma_3^K$ apply Yates Lemma B of §3 above to produce an r.e array $\{W_{h(x)}\}_{x \in \omega}$ such that

$$x \in S \Rightarrow (\exists e)[W_{h(x)}^{(e)} \equiv_T K \;\&\; (\forall i < e)[W_{h(x)}^{(i)} \text{ is recursive}]],$$
$$x \notin S \Rightarrow (\forall e)[W_{h(x)}^{(e)} \text{ is recursive}].$$

For each x choose an r.e. set B_x in the obvious way such that $B_x^{(0)} \equiv_T E$ and $B_x^{(e+1)} \equiv_T W_{h(x)}^{(e)}$. Apply Lemma 2.6 with $C = D$ to produce an r.e. set $A_x \subseteq B_x$. If $x \notin S$ then, for all e, $D \not\leq_T B^{(<e)}$. Hence, (a) of Lemma 2.6 yields $D \not\leq_T A_x$ and A_x a thick subset of B_x (whence $E \leq_T A_x$ and therefore $A_x \not\leq_T D$). If $x \in S$ then $A_x \equiv_T K$ by (b) of Lemma 2.6. $\square$

COROLLARY 6.2 (YATES). *If $\boldsymbol{d}$ is r.e. and $\mathbf{0} < \boldsymbol{d} \le \mathbf{0}'$ then $G(\ge \boldsymbol{d})$ is Σ_4-complete.*

PROOF. Using $\boldsymbol{d}$ r.e. and the Tarski Kuratowski algorithm Yates easily shows [26, p. 254] that $G(\ge \boldsymbol{d}) \in \Sigma_4$. If $\boldsymbol{d} = \mathbf{0}'$ apply Corollary 3.7 and $\Sigma_3^K = \Sigma_4$. If $\boldsymbol{d} < \mathbf{0}'$ apply Theorem 6.1. □

COROLLARY 6.3 (YATES). *If $\boldsymbol{d}$ is r.e. and $\mathbf{0} < \boldsymbol{d} < \mathbf{0}'$ then $G(|\,\boldsymbol{d})$ is Π_4-complete.*

PROOF. First $G(|\boldsymbol{d}) \in \Pi_4$ because $G(\le \boldsymbol{d}) \in \Sigma_3^D \in \Sigma_4$ and $G(\ge \boldsymbol{d}) \in \Sigma_4$. Now $G(|\,d)$ is Π_4-complete by Theorem 6.1. □

Virtually the same proof yields the following stronger result of Yates [26, Theorem 2, p. 255] which generalizes Theorem 3.1.

THEOREM 6.4 (YATES). *Let $\varnothing <_T D \le_T \varnothing'$ and let $D_1 \le_T D_2 \le_T D_3 \le \cdots$ be a recursively enumerable sequence of r.e. sets such that $D \not\le_T D_i$ for all i. If $S \in \Sigma_4$ then there is a recursively enumerable sequence $\{A_x\}_{x\in\omega}$ of r.e. sets, such that for all x and i, $D_i \le_T A_x$, and*

$$x \in S \Rightarrow A_x \equiv_T \varnothing',$$
$$x \notin S \Rightarrow D \not\le_T A_x.$$

PROOF. The proof is the same as in Theorem 6.1 with the following change. For each x define an r.e. set B_x in the obvious way such that $B_x^{(2e)} \equiv_T W_{h(x)}^{(e)}$ and $B_x^{(2e+1)} \equiv_T D_e$. □

COROLLARY 6.5 (YATES). *Let $\varnothing <_T D <_T \varnothing'$ and let $D_1 \le_T D_2 \le_T D_3 \le \cdots$ be a recursively enumerable sequence of r.e. sets such that $D \not\le_T D_i$ for all i. Then there exists an r.e. set A Turing incomparable with D such that $D_i \le_T A$ for all i.*

PROOF (YATES [26, p. 260]). Let $\boldsymbol{d} = \deg(D)$. Set $S = G(\le \boldsymbol{d}) \in \Sigma_4$. Apply Theorem 6.4 to obtain $\{A_x\}_{x\in\omega}$. By the recursion theorem choose n such that $W_n = A_n$. Now $W_n \le_T D$ implies $A_n \equiv_T \varnothing'$, a contradiction. Thus, $W_n \not\le_T D$, hence $n \notin S$ and $D \not\le_T A_n$. □

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