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The influence of gravitational wave momentum losses on the centre of mass motion of a Newtonian binary system

M. J. Fitchett *Institute of Astronomy, Madingley Road, Cambridge CB3 0HA*

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Summary. Following Bekenstein's work on recoiling black holes, we calculate the gravitational wave linear momentum flux from a binary system of two point masses in Keplerian orbit. The quasi-Newtonian approach is adopted and the resulting motion of the centre of mass is calculated. As such a system decays via gravitational wave energy losses, the size of the orbit decreases until the components merge or become tidally disrupted. Thereafter, the centre of mass moves with the linear momentum necessary to balance that carried off by the gravitational waves. In the case of a binary black hole system, the velocity of the centre of mass could be of astrophysical significance, although numerical studies would be necessary to check this claim.

1 Introduction

In classical electrodynamics it is possible for a material system to recoil due to electromagnetic wave emission. The cause is interference between the electric dipole and electric quadrupole or magnetic dipole radiation fields. The analogous case in general relativity was considered by Peres (1962). He found that gravitational radiation can also give rise to the recoil of the emitting system, and that the effect is due to interference between the mass quadrupole and mass octupole or flow quadrupole radiation fields. Bekenstein (1973) obtained a similar result by perturbing the field equations and going beyond the usual quadrupole order. He applied his analysis to the astrophysically important case of a star collapsing to a black hole end state and was able to place an upper limit of $300\,\mathrm{km\,s^{-1}}$ on the recoil velocity that such a collapse could cause. Moncrief (1979), using the perturbation techniques of Cunningham, Moncrief & Price (1978), found that an Oppenheimer–Snyder model of a collapsing star would give rise to a typical recoil velocity of $25\,\mathrm{km\,s^{-1}}$ for small non-spherical perturbations. He also indicated that the velocities close to the Bekenstein limit could be attained for rapidly rotating collapse models.

The obvious extension of this work is to consider binary star systems. Peters & Mathews (1963) calculated the energy and angular momentum loss rates for a binary system of two point masses. They showed that gravitational wave emission circularizes the orbit and also that the orbit decays with a consequent change in the period. This effect has been observed in the binary pulsar PSR 1913 +16 and may be used to verify general relativity (Taylor 1982). Here we extend their Newtonian calculation to calculate the linear momentum flux from a binary system of two point masses in Keplerian orbit. Bekenstein's formalism is used. It is found that for typical stars the flux is very small and is only significant when the separation of the binary becomes comparable to the Schwarzschild radius of the system. Hence we would only expect a noticeable effect for a close compact object binary system. Since the calculation is performed in the Newtonian regime, it is obviously an unwarranted extrapolation to say what will happen in this context. However, our analysis at least gives an order of magnitude result. Extrapolating our results to close compact object binary systems indicates that the speed of the centre of mass can become of the order of tens of km s^{-1} for a neutron star binary system, and hundreds of km s^{-1} for a binary black hole system. Hopefully these results will stimulate numerical studies similar to those of Smarr's on colliding black holes.

In Section 2 Bekenstein's formula is introduced and the momentum flux due to gravitational wave emission calculated in three cases: a circular binary orbit, an elliptical orbit and direct radial infall. In Section 3 the motion of the centre of mass is calculated and our calculations further extended to include parabolic and hyperbolic encounters. In the case of a circular relative orbit it is found that the centre of mass moves in a circle, with the motion becoming faster as the components of the binary get closer. In Section 4 we discuss the validity of our analysis and estimate the post-Newtonian corrections to our results. In Section 5 we apply these results to various types of binary systems and conclude that it is only black hole binary systems that will give rise to astrophysically significant recoil velocities. It should of course be borne in mind that the Newtonian analysis only allows us to speculate at this point.

2 The linear momentum flux

In all that follows Greek indices lie in the range 0 to 3 and Latin indices in the range 1 to 3. Einstein's summation convention will be assumed. Bekenstein assumed the matter to be described by a symmetric stress-energy tensor $T_{\alpha\beta}$. He perturbed the field equations about a flat Minkowski background $\eta_{\alpha\beta}$ but went further than the usual quadrupole order to include octupole and angular momentum terms. He used the Landau–Lifschitz pseudotensor to evaluate the energy flux $d^2E/dtd\Omega\,(\theta, \phi)$ into solid angle $d\Omega$ in the (θ, ϕ) direction and then found the momentum flux by integration over a two-sphere at infinite, S_∞, centred on the coordinate origin

$$l^i = \int_{S_\infty} \frac{d^2E}{dtd\Omega}(\theta, \phi)\,\frac{n^i}{c}\,d\Omega \tag{2.1}$$

where n^i is a unit radial vector on S_∞. For the case of quadrupole radiation the above integral is zero; hence the need to go to higher order radiation fields. The final momentum flux he derived was

$$l^i = G\,(945\,c^6)^{-1}\,[22\,Q^{ik}\,B^{jki} - 12\,Q^{jk}\,B^{jik} - 12\,Q^{ji}\,B^{jkk}]. \tag{2.2}$$

The quantities Q^{ij} and B^{ijk} are related to the matter distribution as follows:

$$Q^{ij} = \frac{\partial^3}{\partial t^3}\int T^{oo}\,c^{-2}(3\,x^i x^j - r^2\,\delta^{ij})\,dV \tag{2.3}$$

$$B^{ijk} = c^{-1}\left(\frac{O^{ijk}}{5} - 2A^{ijk}\right) \tag{2.4}$$

where

$$O^{ijk}=\frac{\partial^4}{\partial t^4}\int T^{oo}c^{-2}\left(5x^ix^j-\frac{5}{3}r^2\delta^{ij}\right)x^k\,dV \tag{2.5}$$

$$A^{ijk}=\frac{\partial^3}{\partial t^3}\int c^{-1}\left(M^{okj}x^i+M^{oki}x^j-\frac{2}{3}M^{okl}x^l\delta^{ij}\right)dV \tag{2.6}$$

and $M^{\alpha\beta\gamma}=T^{\alpha\beta}x^\gamma-T^{\alpha\gamma}x^\beta$ (the auxiliary angular momentum tensor). Q^{ij} is what is sometimes denoted as $\dddot{Q}^{ij}$, the third time derivative of the matter's quadrupole moment. Similarly O^{ijk} is the fourth time derivative of the matter's octupole moment, and A^{ijk} represents the angular momentum contributions.

We adopt a coordinate system with origin at the centre of mass of the binary, and with the x-axis aligned with the radius vector when the two components of the binary are at periastron. Then, if the separation of the components is d when the angle between the radius vector and the x-axis is θ, we have by standard Newtonian orbit theory

$$d=\frac{a(1-e^2)}{1+e\cos\theta}\qquad \dot{\theta}=\frac{[G(m_1+m_2)a(1-e^2)]^{1/2}}{d^2} \tag{2.7}$$

where e is the eccentricity of the relative orbit, a its semi-major axis and m_1, m_2 are the masses of the components. Since the motion of the matter is known we can adopt a quasi-Newtonian approach and use the motion to evaluate the quantities required by Bekenstein's formula. Considering the bodies as point masses, $T^{\alpha\beta}$ is given by

$$T^{\alpha\beta}=\sum_{n=1}^{2}m_n\delta_n u_n^\alpha u_n^\beta \tag{2.8}$$

where $\delta_n=\delta\,[x-x_n(t)]$ and u_n^α is the nth particle's four-velocity. In the Newtonian regime, where $u^i/c \ll 1$, we have

$$T^{oo}\simeq\sum_{n=1}^{2}m_n\delta_n c^2$$

$$T^{oi}\simeq\sum_{n=1}^{2}m_n\delta_n c\dot{x}_n^i \tag{2.9}$$

$$T^{ij}\simeq\sum_{n=1}^{2}m_n\delta_n\dot{x}_n^i\dot{x}_n^j$$

where the dot denotes d/dt.

The point mass assumption greatly simplifies the integrals. For example, Q^{ij} becomes

$$Q^{ij}=\frac{\partial^3}{\partial t^3}\sum_{n=1}^{2}m_n(3x_n^ix_n^j-d_n^2\delta^{ij}) \tag{2.10}$$

where $d_1=dm_2(m_1+m_2)^{-1}$ and $d_2=dm_1(m_1+m_2)^{-1}$. Similarly we find

$$O^{ijk}=\frac{\partial^4}{\partial t^4}\sum_{n=1}^{2}m_n\left(5x_n^ix_n^j-\frac{5}{3}dn^2\delta^{ij}\right)x_n^k \tag{2.11}$$

$$A^{ijk}=\frac{\partial^3}{\partial t^3}\sum_{n=1}^{2}m_n\left(2x_n^ix_n^j\dot{x}_n^k-x_n^i\dot{x}_n^jx_n^k-\dot{x}_n^ix_n^jx_n^k-\frac{2}{3}d_n^2\delta^{ij}\dot{x}_n^k+\frac{2}{3}\delta^{ij}\dot{x}_n^lx_n^lx_n^k\right). \tag{2.12}$$

The procedure now is to substitute for the known Newtonian values of x_n^j in the above expressions and calculate the Q^{ij}'s and B^{ijk}'s, and hence the linear momentum flux l^i.

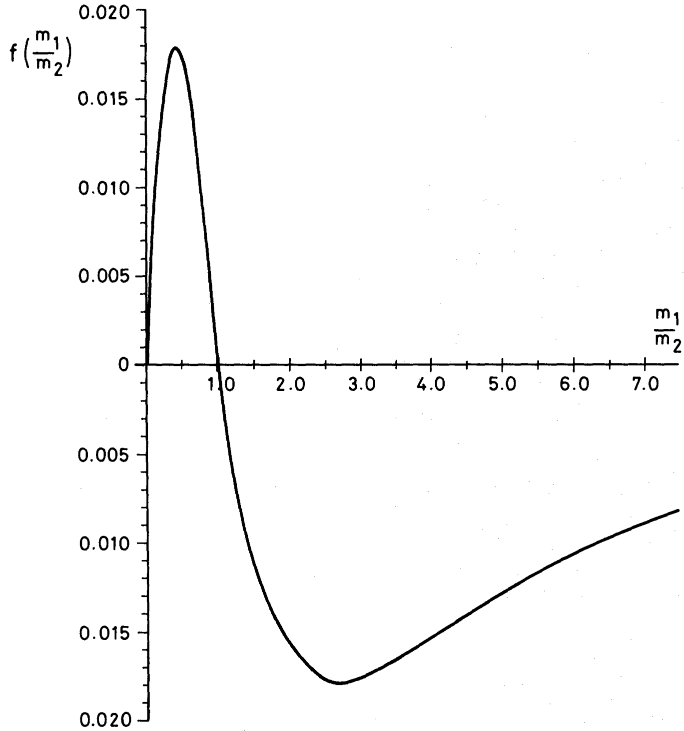

Figure 1. This shows the form of the function $f(m_1/m_2)$. The maximum and minimum of the function are attained at $m_1/m_2 = 0.38$ and 2.6 respectively.

We are therefore neglecting the effect of the waves on the orbit. In the case of a circular relative orbit, the symmetry of the problem greatly simplifies the calculation; the other two cases are algebraically tedious.

For a circular relative orbit symmetry considerations imply that the magnitude of the momentum flux vector is constant (if the orbit is assumed not to decay by gravitational wave energy losses) and that the angle it makes with the radius vector is fixed. It is therefore only necessary to evaluate **l** for the case $\theta = 0$ in order to know its value for all θ. For a circular orbit we also have the simplifying feature that $\dot{\theta}$ = constant = Ω_c, with

$$\Omega_c = [G(m_1 + m_2)]^{1/2} a^{-3/2}. \tag{2.13}$$

Assuming that $\theta = 0$ at time $t = 0$, we have $\theta(t) = \Omega_c t$. From the equations (2.10), (2.11) and (2.12) we see that the fourth and lower order time derivatives of the x_n^j are required at $\theta = t = 0$. Assuming that at $t = 0$ the particles lie on the x-axis with the particle of mass m_1 at positive x, we have

$$\frac{d^p}{dt^p} x_n^1 \bigg|_{\theta=0} = (-1)^{p/2} \left(\frac{1}{2}\right) [1 + (-1)^p] (-1)^{n+1} d_n \Omega_c^p \tag{2.14}$$

$$\frac{d^p}{dt^p} x_n^2 \bigg|_{\theta=0} = (-1)^{(p-1)/2} \left(\frac{1}{2}\right) [1 + (-1)^{p+1}] (-1)^{n+1} d_n \Omega_c^p \tag{2.15}$$

where we have assumed that the orbit is not changing due to gravitational wave emission. Performing the differentiations in the Q^{ij} expression and substituting for the differentiated terms, we find

$$Q^{12}\,|_{\theta=0} = Q^{21}\,|_{\theta=0} = -12\,\Omega_c^3 (m_1 d_1^2 + m_2 d_2^2) \tag{2.16}$$

and all other $Q^{ij}\,|_{\theta=0}$ are zero. Since only two components of Q^{ij} are non-zero the range of

B^{ijk}'s required is limited and we can write Bekenstein's expression as

$$l^1 = G(945c^6)^{-1} Q^{12}[20B^{211} - 12(B^{112} + B^{222} + B^{233})]$$
$$l^2 = G(945c^6)^{-1} Q^{12}[20B^{122} - 12(B^{111} + B^{221} + B^{133})] \quad (2.17)$$
$$l^3 = 0$$

where all quantities are evaluated at $\theta = 0$. The necessary B^{ijk}'s can be evaluated in the same manner as the Q^{ij}'s and the following results are obtained

$$B^{111} = 22\Omega_c^4 c^{-1}\Delta$$
$$B^{212} = -18\Omega_c^4 c^{-1}\Delta \quad (2.18)$$
$$B^{221} = -23\Omega_c^4 c^{-1}\Delta$$

where $\Delta = m_1 d_1^3 - m_2 d_2^3$ and all other B^{ijk}'s required in (2.17) are zero. Substituting for Q^{12} and the B^{ijk}'s in (2.17) we find

$$l^2|_{\theta=0} = \frac{464}{105} G\Omega_c^7 c^{-7}\Delta\sigma$$
$$l^1|_{\theta=0} = l^3|_{\theta=0} = 0 \quad (2.19)$$

where $\sigma = m_1 d_1^2 + m_1 d_2^2$. We conclude that the momentum flux for a circular orbit is at right angles to the radius vector and in the plane of the orbit. We can therefore deduce that for arbitrary θ

$$\mathbf{l}(\theta) = l^2|_{\theta=0}(-\sin\theta, \cos\theta, 0). \quad (2.20)$$

Substituting for Ω_c, d_1 and d_2 in this expression we find that

$$\mathbf{l}(\theta) = \frac{29}{210\sqrt{2}} G^{-1}c^4 \left[\frac{ac^2}{2G(m_1+m_2)}\right]^{-11/2} \left[\left(1+\frac{m_2}{m_1}\right)^{-2}\left(1+\frac{m_1}{m_2}\right)^{-3}\left(1-\frac{m_1}{m_2}\right)\right]$$
$$\times(-\sin\theta, \cos\theta, 0). \quad (2.21)$$

We can see from this expression that the linear momentum flux is zero for equal mass components and that it is largest when a, the separation of the components is as small as possible, i.e. of the order of the Schwarzschild radius of the system. Henceforth we set

$$R_s = 2G(m_1+m_2)c^{-2}$$
$$f(m_1/m_2) = (1+m_2/m_1)^{-2}(1+m_1/m_2)^{-3}(1-m_1/m_2). \quad (2.22)$$

Generalizing to an elliptical orbit is more difficult because we no longer have the simplifications imposed by circular symmetry. We have used an algebraic computer language (CAMAL) to perform the differentiation of the x_n^j's and to evaluate the Q^{ij}'s and B^{ijk}'s. These are listed in the appendix. Here we simply state the final result

$$l^1(\theta) = -\Gamma(\theta)\sin\theta\,[58 + 175\,e\cos\theta + e^2(12 + 160\cos^2\theta) + e^3(20\cos\theta + 90\cos^3\theta)]$$
$$l^2(\theta) = +\Gamma(\theta)\,[58\cos\theta + e(175\cos^2\theta - 9) + e^2(160\cos^3\theta - 2\cos\theta)$$
$$+ e^3(2 + 3\cos^2\theta + 45\cos^4\theta)] \quad (2.23)$$
$$l^3(\theta) = 0$$

where

$$\Gamma(\theta) = \frac{8G}{105}(1+e\cos\theta)^4\Omega_c^7 c^{-7}\sigma\Delta(1-e^2)^{-1/2}. \quad (2.24)$$

Setting $e=0$ in the above expressions we recover the expression for the case of a circular relative orbit as would be expected. Substituting for Ω_c, σ and Δ we find

$$\Gamma(\theta)=\frac{1}{420\sqrt{2}}c^4G^{-1}\left(\frac{a(1-e^2)}{R_s}\right)^{-11/2}f\left(\frac{m_1}{m_2}\right)(1+e\cos\theta)^4. \tag{2.25}$$

Finally we consider the case of two point masses falling radially towards one another. If the point masses m_1, m_2 are at positions $\mathbf{r}_1$, $\mathbf{r}_2$ respectively, we have the Newtonian equations of motion

$$\ddot{\mathbf{r}}_1=-Gm_2\left(1+\frac{m_1}{m_2}\right)^{-2}\mathbf{r}_1|\mathbf{r}_1|^{-3}=-\beta_1\mathbf{r}_1|\mathbf{r}_1|^{-3} \tag{2.26}$$

and

$$\ddot{\mathbf{r}}_2=-Gm_1\left(1+\frac{m_2}{m_1}\right)^{-2}\mathbf{r}_2|\mathbf{r}_2|^{-3}=-\beta_2\mathbf{r}_2|\mathbf{r}_2|^{-3}. \tag{2.27}$$

Solving the $\ddot{\mathbf{r}}_1$ equation leads to the solution

$$\begin{aligned}|\mathbf{r}_1|&=\alpha_1(1+\cos\eta)\\ t&=\delta_1(\eta+\sin\eta)\end{aligned} \tag{2.28}$$

where α_1 is specified by the initial separation, a, of m_1 and m_2 and the equation of motion requires that δ_1 be given by $\alpha_1^3=Gm_2(1+m_1/m_2)^{-2}\delta_1^2$. Similarly we can solve for $\mathbf{r}_2$ and hence evaluate the Q^{ij}'s and B^{ijk}'s. This gives

$$Q^{22}=Q^{33}=-\frac{1}{2}Q^{11}=-2\sin\eta\left(\frac{m_1\alpha_1^2\beta_1}{\delta_1r_1^3}+\frac{m_2\alpha_2^2\beta_2}{\delta_2r_2^3}\right) \tag{2.29}$$

all other Q^{11}'s being zero. As in the circular case this restricts the B^{ijk}'s required. We find

$$l^1\propto Q^{11}[2B^{111}+11(B^{221}+B^{331})]$$

$$l^2=l^3=0$$

with

$$B^{221}=B^{331}=-\frac{1}{2}B^{111}=-\frac{1}{c}\left[\frac{6-\cos\eta}{(1+\cos\eta)^3}\right]\left(\frac{m_2\beta_2}{\delta_2^2}-\frac{m_1\beta_1}{\delta_1^2}\right). \tag{2.30}$$

Assuming that the particles have initial separation a we can calculate α_i, β_i, δ_i and hence arrive at the momentum flux

$$\mathbf{l}(\eta)=-\frac{8}{105}c^4G^{-1}\left(\frac{a}{R_s}\right)^{-11/2}f\left(\frac{m_1}{m_2}\right)H(\eta)(1,0,0) \tag{2.31}$$

where

$$H(\eta)=\sin\eta(6-\cos\eta)(1+\cos\eta)^{-6}. \tag{2.32}$$

3 Motion of the centre of mass

The well-known 'quadrupole formula', originally derived for linearized gravity, has been found to be valid for nearly Newtonian gravitationally bound systems (Breuer & Rudolph 1981) even when the internal gravity of the components is strong (Kates 1980). Although it has not been proven, we believe that Bekenstein's formula, derived for linearized gravity, will also be applicable under these circumstances. Under this assumption we can write

$$\frac{d}{dt}\mathbf{P}_{\mathrm{PN}}=-\mathbf{l} \tag{3.1}$$

where **I** is the momentum flux we have calculated and $\mathbf{P}_{\rm PN}$ is the third-post-Newtonian three-momentum of the system with respect to a frame, F_0, in which the centre of mass is initially at rest. In the Newtonian limit this can be replaced by the Newtonian three-momentum $\mathbf{P}_{\rm N}$. This equation is not strictly true in the sense that $\mathbf{P}_{\rm PN}$ is evaluated with respect to F_0 whereas the frame with respect to which **I** is calculated is moving with the velocity of the centre of mass. However, as will be seen, the velocity of the centre of mass is not relativistic and so to a first approximation (3.1) holds. The errors introduced by these assumptions will be further discussed in Section 4. If **R** is the position vector of the centre of mass with respect to F_0, then (3.1) becomes

$$[(m_1+m_2)\,\mathbf{R}]^{\cdot\cdot} = -\mathbf{I}(t). \tag{3.2}$$

For the circular case we have

$$(m_1+m_2)\,\ddot{\mathbf{R}} = -\frac{29}{210\sqrt{2}}\,c^4 G^{-1}\left(\frac{a}{R_{\rm s}}\right)^{-11/2} f\left(\frac{m_1}{m_2}\right)(-\sin\Omega_{\rm c}t, \cos\Omega_{\rm c}t, 0) \tag{3.3}$$

which can be integrated to give

$$\dot{\mathbf{R}} = -c\left[\frac{29}{105}\left(\frac{a}{R_{\rm s}}\right)^{-4} f\left(\frac{m_1}{m_2}\right)\right](\cos\Omega_{\rm c}t, \sin\Omega_{\rm c}t, 0) \tag{3.4}$$

and

$$\mathbf{R} = a\left[\frac{29\sqrt{2}}{105}\left(\frac{a}{R_{\rm s}}\right)^{-7/2} f\left(\frac{m_1}{m_2}\right)\right](-\sin\Omega_{\rm c}t, \cos\Omega_{\rm c}t, 0). \tag{3.5}$$

The centre of mass therefore moves with speed

$$c\left(\frac{29}{105}\right)\left(\frac{a}{R_{\rm s}}\right)^{-4} f\left(\frac{m_1}{m_2}\right)$$

in a circle of radius

$$a\left(\frac{29\sqrt{2}}{105}\right)\left(\frac{a}{R_{\rm s}}\right)^{-7/2} f\left(\frac{m_1}{m_2}\right).$$

Since gravitational radiation energy losses tend to circularize the orbit (Peters & Mathews 1963; Peters 1964), the circular orbit will in fact be of relevance in many astrophysical contexts. Highly eccentric orbits might, however, arise through capture processes or the fission of rotating collapsing stars since, if the components are sufficiently close, there may not be sufficient time for the orbits to circularize.

For the elliptical case, the velocity of the centre of mass can be shown to be given by

$$\dot{\mathbf{R}}(\theta) = -c\left\{\frac{1}{210}\left[\frac{a(1-e^2)}{R_{\rm s}}\right]^{-4} f\left(\frac{m_1}{m_2}\right)\right\}\,[\alpha(\theta), \beta(\theta), 0]. \tag{3.6}$$

Here

$$\alpha(\theta) = \sum_{j=0}^{5} \frac{\alpha_j(\cos\theta)^{j+1}}{(j+1)}$$

with the α_j's given by

$$\begin{aligned} &\alpha_0 = 2(29+6e^2) && \alpha_3 = 10e^3(54+e^2) \\ &\alpha_1 = e(291+34e^2) && \alpha_4 = 250e^4 \\ &\alpha_2 = 8e^2(71+4e^2) && \alpha_5 = 45e^5 \end{aligned} \tag{3.7}$$

and

$$\beta(\theta) = \lambda_0\theta + \lambda_1 \sin\theta + \lambda_2 \left[\frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right] + \lambda_3 \left[\sin\theta - \frac{1}{3}\sin^3\theta\right]$$
$$+ \lambda_4 \left[\frac{3}{8}\theta + \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta\right] + \lambda_5 \left[\frac{5}{8}\sin\theta + \frac{5}{48}\sin 3\theta + \frac{1}{80}\sin 5\theta\right]$$
$$+ \lambda_6 \left[\frac{5\theta}{16} + \frac{15}{64}\sin 2\theta + \frac{3}{64}\sin 4\theta + \frac{1}{192}\sin 6\theta\right]$$

with the λ_j's given by

$$\begin{aligned} \lambda_0 &= e(2e^2-9) & \lambda_4 &= 3e^3(180+e^2) \\ \lambda_1 &= 2(2e^4-10e^2+29) & \lambda_5 &= 250e^4 \\ \lambda_2 &= e(2e^4-10e^2+291) & \lambda_6 &= 45e^5. \\ \lambda_3 &= 4e^2(142+e^2) \end{aligned} \tag{3.8}$$

Setting $e = 0$ in the above equations, we recover the circular orbit result. The interesting difference between the elliptical and the circular case is the additional non-periodic terms in $\beta(\theta)$ which are linear in θ. The implication is that the centre of mass will undergo periodic motion with a superposed drift in the negative y direction. The reason for this drift is that, in our coordinate system, the x-axis is an axis of symmetry while, for $m_1 \neq m_2$, the y-axis is not. Roughly speaking, the momentum flux vector lies at right angles to the radius vector. Over the half orbit $-\pi/2 < \theta < +\pi/2$ the net linear momentum lost from the system will therefore be in the positive y direction. For the half orbit $+\pi/2 < \theta < 3\pi/2$ the net linear momentum lost from the system will be in the opposite direction but of a different magnitude because the path followed is different in shape to that followed between $-\pi/2$ and $+\pi/2$. Hence there is a net linear momentum flux in the y direction. On the other hand, applying the same argument to the orbit decomposed as $0 < \theta < \pi$ and $\pi < \theta < 2\pi$ shows that there will be equal but opposite momentum losses along the x-axis in each half of the orbit. In the case $e = 0$, where symmetry about the y-axis is restored, there is no net loss of momentum over one orbit since $\lambda_0 = \lambda_2 = \lambda_4 = \lambda_6 = 0$.

As we will later show, the periodic motion of the centre of mass is a very small effect in the Newtonian regime. However, the drift effect, being a cumulative effect, could conceivably become large even in the Newtonian regime (where $a/R_s \gg 1$). We therefore consider the cumulative effects of the drift velocity in more detail. After each orbit the drift velocity in the negative y direction increases by an amount proportional to $[\beta(2\pi) - \beta(0)]$, the actual value being given by v_d where

$$v_d = c\left\{\frac{1}{210} f\left(\frac{m_1}{m_2}\right) \left[\frac{a(1-e^2)}{R_s}\right]^{-4} e\left(\frac{273}{2} + \frac{399}{2}e^2 + \frac{259}{16}e^4\right) 2\pi\right\}. \tag{3.9}$$

Since we are assuming that the orbit does not undergo precession, in principle the drift velocity in the negative y direction could become relativistic, the velocity after N orbits being given by the relativistic addition formula

$$\mathrm{v}(N) = c\,\frac{\{1 - [(1 - v_d/c)/(1 + v_d/c)]^N\}}{\{1 + [(1 - v_d/c)/(1 + v_d/c)]^N\}}. \tag{3.10}$$

However, in practice the perihelion advance of the orbit due to general relativistic effects will prevent the final drift velocity becoming relativistic. If the perihelion advances by an angle δ for each orbit then after N orbits the drift velocity will be given by

$$\mathbf{v}_N = \sum_{n=0}^{N} v_d (\sin n\delta, -\cos n\delta, 0) \tag{3.11}$$

so long as the drift velocity is non-relativistic. Performing the summation we find

$$\mathbf{v}_N = \frac{1}{2} v_d \left[\cot\frac{\delta}{2}(1-\cos N\delta)+\sin N\delta, -(1+\cos N\delta+\sin N\delta \cot\delta/2), 0\right]. \tag{3.12}$$

Considered as a function of N this is maximized for

$\tan N\delta = -\tan\delta$

and

$$N\delta \in [\pi/2, \pi] \text{ (modulo } 2\pi) \tag{3.13}$$

when

$$|\mathbf{v}_N| = \frac{v_d}{\sin(\delta/2)}$$

and

$N = (\pi/\delta) - 1.$

Since we are working in the Newtonian regime where $\delta \ll 1$, the maximum modulus of the drift velocity is, to a good approximation, $2v_d/\delta$. Since δ has the value

$$\delta = \frac{6\pi G(m_1+m_2)}{(1-e^2)ac^2} \tag{3.14}$$

the maximum modulus of the drift velocity is given by

$$|\mathbf{v}_d^{MAX}| = c\left\{\frac{2\pi}{105} f\left(\frac{m_1}{m_2}\right) e\left[\frac{273}{2}+\frac{399}{2}e^2+\frac{259}{16}e^4\right](3\pi)^{-4}\delta^3\right\}. \tag{3.15}$$

This dominates the periodic motion of the centre of mass by a factor $1/\delta$. Assuming the largest possible values of e and $f(m_1/m_2)$, $|\mathbf{v}_d^{max}| \leq 14.4\,\delta^3$ km s^{-1} and therefore the cumulative effects in the Newtonian regime, where $\delta \ll 1$, are small. For example, in the case of the binary pulsar PSR 1913 +16 this corresponds to a velocity of 1.6×10^{-12} km s^{-1}. The time-scale of variation of the cumulative effects of the drift is (a/R_s) times the orbital time-scale. When we are no longer in the Newtonian regime and (a/R_s) is of order unity, the precession rate becomes comparable to the orbital angular velocity and our simple elliptical orbit calculation is no longer valid.

The elliptical orbit results easily generalize to include parabolic and hyperbolic encounters. If the distance of closest approach in either case is b, then the following transformations should be made to equations (3.6) to (3.8): parabolic case, $e=1$, $a(1-e^2)\to 2b$; hyperbolic case, $e>1$, $a(1-e^2)\to b(1+e)$. In each case the net change in velocity of the centre of mass as a result of the encounter has been evaluated:

$$\Delta\mathbf{v}_p = 3532\left(\frac{b}{R_s}\right)^{-4}\left(\frac{f}{f_{max}}\right)(0,-1,0) \quad \text{km s}^{-1} \tag{3.16}$$

$$\Delta\mathbf{v}_H = -25.5\left[\frac{b(1+e)}{R_s}\right]^{-4}\left(\frac{f}{f_{max}}\right)\left\{0, e\left(273+399e^2+\frac{259}{8}e^4\right)\cos^{-1}\left(\frac{-1}{e}\right) + \left[\frac{(e^2-1)^{1/2}}{24e}\right](5943e^4+10514e^2+448), 0\right\} \quad \text{km s}^{-1} \tag{3.17}$$

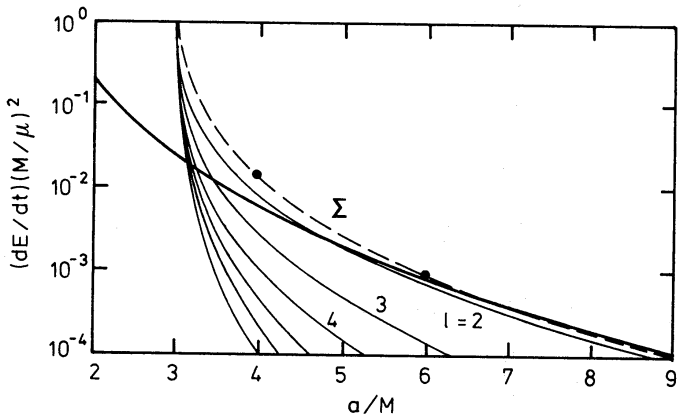

Figure 2. This shows the gravitational wave energy flux due to a test particle of mass μ moving in a circular orbit of radius a about a Schwarzschild black hole. Units in which $c = G = 1$ are used. The thin continuous lines show Detweiler's (1978) exact results for the various multipole contributions to the flux. The dotted line shows the total flux he evaluated. The thicker line shows the quasi-Newtonian prediction which is to be compared with the $l = 2$ (quadrupole) line.

in the parabolic and hyperbolic cases respectively. For large e equation (3.17) becomes

$$\Delta v_H = -25.5 \left(\frac{b}{R_s}\right)^{-4} \left(\frac{f}{f_{max}}\right) \left(0, \frac{259}{16} \pi e + 280 - \frac{259}{4} \pi, 0\right) \quad \text{km s}^{-1} \tag{3.18}$$

which is interesting because it indicates that large centre of mass velocities are possible in this case.

Finally there is the case of direct radial infall. If we assume that initially the masses are at rest, a distance a apart, and that the velocity of the centre of mass is then zero, then

$$\dot{\mathbf{R}}(\eta) = c \left[\frac{8}{105} \left(\frac{a}{R_s}\right)^{-4} f\left(\frac{m_1}{m_2}\right)\right] \left[\frac{7}{4} (1 + \cos\eta)^{-4} - \frac{1}{3} (1 + \cos\eta)^{-3} - \frac{13}{192}\right] (1, 0, 0) \tag{3.19}$$

where η is related to the time by equation (2.28).

In each of the cases discussed above the factors $f(m_1/m_2)$ and $(a/R_s)^{-4}$ are of critical importance. For example, for a circular relative orbit, the speed of the centre of mass motion is given by

$$v = c \left(\frac{29}{105}\right) \left| f\left(\frac{m_1}{m_2}\right)\right| \left(\frac{a}{R_s}\right)^{-4}. \tag{3.20}$$

The form of $f(m_1/m_2)$ is shown in Fig. 2. It has the following properties: (i) $f(m_1/m_2) = -f(m_2/m_1)$, (ii) it is extremized by $m_1/m_2 = 2.6$ or $1/2.6$ when it takes the values -0.01789 and $+0.01789$ respectively. For equal mass components, (i) shows that the centre of mass is stationary. Otherwise we can write the speed of the centre of mass in the circular case as:

$$v = 1480 \left(\frac{f}{f_{max}}\right) \left(\frac{a}{R_s}\right)^{-4} \text{ km s}^{-1}. \tag{3.21}$$

4 Validity of our approximations

The main source of error in our analysis is the neglect of post-Newtonian corrections to Newtonian quantities. For example, the dominant correction to T^{oo} is of relative magnitude v^2/c^2 (Chandrasekhar 1969), this being the first-post-Newtonian correction. In the case of a

Newtonian binary system this is of order $(a/R_S)^{-1}$. Higher order corrections also exist. For example the 2½-post-Newtonian correction corresponds to the orbital evolution due to radiation reaction, and is of relative magnitude $(a/R_S)^{-5/2}$. Another possible source of error is the application of Bekenstein's formula when the components of the binary are close. An exact analysis would result in an infinite series of products of various multipole moments of the source. We have in effect used a truncated form of the series. This will be a reasonable approximation in the Newtonian regime because the terms we have used will be the dominant ones. However, when a and R_S are comparable, so that the components of the binary move with large velocities, the remaining terms in the series may become significant.

We can only be guided by the use of the quadrupole formula under the same circumstances; Peters & Mathews (1963), using the same approximations as we have made, derived an expression for the average rate of energy loss of a binary system. They used the standard quadrupole formula and this gave

$$\left\langle \frac{dE}{dt} \right\rangle = -\frac{32}{5} \frac{G^4 (m_1 m_2)^2 (m_1 + m_2)}{(ac)^5 (1-e^2)^{7/2}} \left[1 + \frac{73}{24} e^2 + \frac{37}{96} e^4\right]. \tag{4.1}$$

If, for example, we now consider the case of a small test test particle of mass μ moving in a circular orbit around a more massive body of mass M, then this equation implies

$$\left\langle \frac{dE}{dt} \right\rangle = -\frac{32}{5} \frac{G^4 M^3 \mu^2}{(ac)^5} \tag{4.2}$$

which can be written as

$$\left| \left\langle \frac{dE}{dt} \right\rangle \left(\frac{M}{\mu}\right)^2 \right| = \frac{32}{5} \left(\frac{ac^2}{GM}\right)^{-5} \left(\frac{c^5}{G}\right). \tag{4.3}$$

Detweiler (1978) used Newman–Penrose formalism to evaluate the energy flux for the case of a small test particle moving in a circular orbit around a Kerr black hole. This analysis is exact. In Fig. 2 we compare our quasi-Newtonian result with Detweiler's exact results. It is interesting to note the similarities. For the case of a test particle in orbit around a Schwarzschild black hole, the Newtonian result slightly underestimates the energy flux when the particle is near the hole. This could be taken to indicate that the Q^{ij}'s calculated in our approximation are underestimates and that we have consequently underestimated the recoil velocity.

The similarity between these results is encouraging, even though it is only a comparison in the test particle limit when the linear momentum flux is vanishingly small. The flux only becomes appreciable when m_1 and m_2 are comparable, in which case one might think that the test particle approach is no longer valid. However, it is interesting to note that Smarr's fully non-linear computer simulations of black hole collisions give strikingly similar results to small perturbation analyses when the test particle mass is allowed to become comparable to the mass of the hole. (See for example the articles by Smarr & Detweiler in Smarr 1978.) Work is in progress (Detweiler & Fitchett, in preparation) to compare an exact test particle momentum flux calculation with the quasi-Newtonian results presented here. Until this is complete, the results of our analysis can only be considered as indicative.

5 Astrophysical implications

Since gravitational wave energy losses cause the orbit of a binary to decay, equation (3.21) implies that the speed of the centre of mass will systematically increase, and so we must consider how small a can become. A strong lower bound on a is $(R_1 + R_2)$, the sum of the radii of the components of the binary. For a main sequence or main sequence-compact object binary, (R_1+R_2) is typically $10^5 R_s$ and so $(a/R_s)^{-4}$ is bounded above by 10^{-20}. For example, in the case of YY Eri where the components have masses of $0.76 M_\odot$ and $0.50 M_\odot$ and the orbital period is 0.321 day (Misner, Thorne & Wheeler 1973), the speed of the centre of mass is approximately $10^{-14}\,\mathrm{cm\,s^{-1}}$. Main sequence or main sequence-compact object binaries therefore produce a negligible centre of mass velocity.

In order to produce a more significant centre of mass velocity we must consider compact object binary systems in which a/R_s can decrease until one of the objects undergoes tidal disruption or, in the case of a binary black hole system, until the orbit becomes relativistically unstable and the objects coalesce. The centre of mass will then move with the velocity it had prior to the disruption or coalescence. Candidates for such a scenario and the recoil velocities associated with them are now discussed.

Clark & Eardley (1976) considered the evolution of close neutron star binaries. The orbit evolves by gravitational radiation until the less massive neutron star reaches its Roche radius whereupon it may undergo immediate tidal disruption or slow mass stripping. The stripping process always ends in tidal disruption of the less massive object. Clark & Eardley plot a graph of the minimum separation of the components (i.e. the separation at which tidal disruption occurs) against their mass. Taking the case of $m_2/m_1 = 2.6$, which is the optical case for fixed a, their graph suggests the value of 40 km as the minimum separation. For example, this occurs for $m_2 = 1.04 M_\odot$, and $m_1 = 0.40 M_\odot$. For this system R_s is 4.3 km and so, by equation (3.21), the speed of the centre of mass is approximately $200\,\mathrm{m\,s^{-1}}$. Different values of m_2/m_1 can lead to smaller values of a and hence possibly greater centre of mass speeds, but it seems that the upper bound will be of the order of a few $\mathrm{km\,s^{-1}}$. The latest values for the orbital parameters of the binary pulsar PSR 1913 +16 (Taylor 1982) imply a maximum centre of mass drift speed at present of the order of $10^{-12}\,\mathrm{km\,s^{-1}}$, which is unfortunately too small to be detectable.

Another candidate for our scenario is a neutron star–black hole binary system. This has been studied in great detail by Lattimer & Schramm (1974, 1976). In their models the neutron star eventually undergoes tidal disruption. The point at which this occurs is very sensitive to the mean neutron star density and to the mass of the black hole. It is therefore difficult to decide on an accurate figure, but it seems that for most configurations, tidal disruption occurs at $a \geqslant 4R_s$. This implies a centre of mass speed of approximately $6\,\mathrm{km\,s^{-1}}$ prior to disruption.

A cleaner and potentially more interesting system is a black hole–black hole binary system. In this case we do not need to worry about tidal disruption effects. The evolution of the orbit can continue until the last stable orbit is reached. For a test particle in orbit around a black hole, the dynamics of the orbit are well understood and the last circular stable orbits are known to lie between $a = GM/c^2$ (for a prograde orbit around a maximally rotating Kerr hole) and $9GM/c^2$ (for a retrograde orbit around a maximally rotating Kerr hole). The last stable circular orbit for a Schwarzschild hole is at $a = 6GM/c^2$. However, since these are test particle results, it is by no means clear that the same values apply to the case of a binary black hole system where the components are of similar mass. Clark & Eardley (1976) gave an approximate calculation to suggest that the closest stable circular orbit for two non-rotating black holes is $(6G/c^2)\max(m_1, m_2)$. They also found that, for the case of a Kerr black hole with another black hole corotating synchronously, this limit could be reduced to $(5G/c^2)\max(m_1, m_2)$. However, since we have justified our weak field approximation on the basis of Detweiler's work in the context of a Schwarzschild black hole, it is more reasonable to assume that:

$$a_{\min} \approx \frac{6G}{c^2}\max(m_1, m_2). \tag{5.1}$$

Then, for example, in the case of a $2.6 M_\odot$ and $1 M_\odot$ black hole binary system we have $a_{\rm min} \simeq 15.6\, GM_\odot/c^2$ and therefore $|\mathbf{v}| \sim 67\,\rm km\,s^{-1}$. We can only consider this to be an order of magnitude result because of the approximations made. The result is very sensitive to the value of a at which the system becomes unstable. It is possible that, in an exact numerical simulation, the figure of $1480\,\rm km\,s^{-1}$ (equation 3.21) could be of order the recoil velocity. In view of the size of this figure numerical calculations are of obvious importance.

Black hole binary systems may exist, either in the centres of galactic nuclei (Begelman, Blandford & Rees 1980), possibly formed as a result of galaxy mergers, or in the disc itself. The endpoint of evolution of such a system will be the spiralling coalescence of the black holes. It is possible that the combined effect of the recoil discussed here, and that associated with the formation of the new hole, could eject the newly formed hole from the galaxy. The binary recoil effect could also have implications for the case of a single rotating object which undergoes fission, the recoil effects associated with the fragments possibly resulting in ejection.

It is to be hoped that numerical simulations or test particle calculations, extrapolated beyond their usual domain of validity, will clarify the points on which we have only been able to speculate.

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References

Begelman, M. C., Blandford, R. D. & Rees, M. J., 1980. *Nature,* **287,** 307.
Bekenstein, J. D., 1973. *Astrophys. J.,* **183,** 657.
Breuer, R. A. & Rudolph, E., 1981. *Gen. Rel. Grav.,* **13,** 777.
Chandrasekhar, S., 1969. *Astrophys. J.,* **158,** 45.
Clark, J. P. A. & Eardley, D. M., 1977. *Astrophys. J.,* **215,** 311.
Cunningham, C. T., Moncrief, V. & Price, R. H., 1978. *Astrophys. J.,* **224,** 643.
Detweiler, S. L., 1978. *Astrophys. J.,* **225,** 687.
Kates, R. E., 1980. *Phys. Rev. D,* **22,** 1871.
Lattimer, J. M. & Schramm, D. N., 1974. *Astrophys. J.,* **192,** L145.
Lattimer, J. M. & Schramm, D. N., 1976. *Astrophys. J.,* **210,** 549.
Misner, C. W., Thorne, K. S. & Wheeler, J. A., 1973. *Gravitation,* Freeman, San Francisco.
Moncrief, V., 1979. *Astrophys. J.,* **238,** 333.
Peres, A., 1962. *Phys Rev.,* **128,** 2471.
Peters, P. C., 1964. *Phys Rev.,* **136,** B1224.
Peters, P. C. & Mathews, J., 1963. *Phys Rev.,* **131,** 435.
Smarr, L., (ed.) 1978. *Sources of Gravitational Radiation,* Cambridge University Press.
Taylor, J. H., 1982. *Astrophys. J.,* **253,** 908.

Appendix

The values of Q^{ij} and B^{ijk} evaluated by the computer are:

$$Q^{11} = \Omega_c^3(1-e^2)^{-9/2}\,\sigma(1+e\cos\theta)^2\,[24\cos\theta\sin\theta + 2e\sin\theta + 18e\cos^2\theta\sin\theta]$$

$$Q^{12} = Q^{21} = \Omega_c^3(1-e^2)^{-9/2}\,\sigma(1+e\cos\theta)^2\,[12 - 24\cos^2\theta + 6e\cos\theta - 18e\cos^3\theta]$$

$$Q^{22} = -\Omega_c^3(1-e^2)^{-9/2}\,\sigma(1+e\cos\theta)^2\,[24\cos\theta\sin\theta + 4e\sin\theta + 18e\cos^2\theta\sin\theta]$$

$$Q^{33} = \Omega_c^3(1-e^2)^{-9/2}\,\sigma(1+e\cos\theta)^2(2e\sin\theta).$$

All other Q^{ij} are zero.

$$B^{111} = \Omega_c^4 (1-e^2)^{-6} \Delta (1+e\cos\theta)^2 [81\cos^3\theta + 120e\cos^4\theta - 59\cos\theta - 6e - 78e\cos^2\theta - 4e^2\cos\theta]$$

$$B^{112} = \Omega_c^4 (1-e^2)^{-6} \Delta (1+e\cos\theta)^2 [81\cos^2\theta\sin\theta + 2e^2\sin\theta + 120e\cos^3\theta\sin\theta - 23\sin\theta + 45e^2\cos^4\theta\sin\theta - 24e\cos\theta\sin\theta - 9e^2\cos^2\theta\sin\theta]$$

$$B^{121} = B^{211} = \Omega_c^4 (1-e^2)^{-6} \Delta (1+e\cos\theta)^2 [81\cos^2\theta\sin\theta + 120e\cos^3\theta\sin\theta + 45e^2\cos^4\theta\sin\theta - 18\sin\theta - 12e\cos\theta\sin\theta]$$

$$B^{122} = B^{212} = \Omega_c^4 (1-e^2)^{-6} \Delta (1+e\cos\theta)^2 [63\cos\theta + 6e + 78e\cos^2\theta + 27e^2\cos^3\theta - 81\cos^3\theta - 120e\cos^4\theta - 45e^5\theta]$$

$$B^{221} = \Omega_c^4 (1-e^2)^{-6} \Delta (1+e\cos\theta)^2 [58\cos\theta + 12e + 66e\cos^2\theta + 8e^2\cos\theta + 18e^2\cos^3\theta - 81\cos^3\theta - 120e\cos^4\theta - 45e^2\cos^5\theta]$$

$$B^{222} = \Omega_c^4 (1-e^2)^{-6} \Delta (1+e\cos\theta)^2 [22\sin\theta - 81\cos^2\theta\sin\theta + 12e\cos\theta\sin\theta - 4e^2\sin\theta - 120e\cos^3\theta\sin\theta - 45e^2\cos^4\theta\sin\theta]$$

$$B^{331} = \Omega_c^4 (1-e^2)^{-6} \Delta (1+e\cos\theta)^2 [\cos\theta + 12e\cos^2\theta + 9e^2\cos^3\theta - 6e - 4e^2\cos\theta]$$

$$B^{332} = \Omega_c^4 (1-e^2)^{-6} \Delta (1+e\cos\theta)^2 [\sin\theta + 12e\cos\theta\sin\theta + 2e^2\sin\theta + 9e^2\cos^2\theta\sin\theta].$$

All other B^{ijk}'s are zero. Here σ and Δ are defined as $\sigma = a^2 m_1 (1+m_1/m_2)^{-1}$ and $\Delta = a^3(m_2 - m_1)\ (1+m_1/m_2)^{-1}(1+m_2/m_1)^{-1}$. This is the same as the earlier definition of σ and Δ where we set $d_1 = am_2(m_1+m_2)^{-1}$ and $d_2 = am_1(m_1+m_2)^{-1}$.