THE INFLUENCE OF LATERAL BOUNDARY CONDITIONS ON THE ASYMPTOTICS IN THIN ELASTIC PLATES

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Abstract. Here we investigate the limits and the boundary layers of the three-dimensional displacement in thin elastic plates as the thickness tends to zero, in each of the eight main types of lateral boundary conditions on their edges: hard and soft clamped, hard and soft simple support, friction conditions, sliding edge and free plates. Relying on construction algorithms [8, 9], we establish an asymptotics of the displacement combining inner and outer expansions. We describe the two first terms in the outer expansion: these are Kirchhoff-Love displacements satisfying prescribed boundary conditions that we exhibit. We also study the first boundary layer term: when the transverse component is clamped, it has generically non-zero transverse and normal components, whereas when the transverse component is free, the first boundary layer term is of bending type and has only its in-plane tangential component non-zero.

Key words. Thin Plates, Linear Elasticity, Singular Perturbation, Boundary Layer, Asymptotic Expansion

AMS subject classifications. 73K10, 73C35, 35J25, 35B25

Introduction. The problem of thin elastic plate bending in linearized elastostatics has been addressed for more than 150 years (the first correct model was presented in a paper by Kirchhoff [18] published in 1850). But, due to the singular perturbation nature of the problem as the thickness of the plate tends to zero, it is not straightforward to perform a rigorous mathematical analysis of characteristic fields and tensors, solutions of the three-dimensional equations. However the knowledge of accurate asymptotics allows first an evaluation of the validity of mechanical models and second the construction of simplified and performing numerical models.

In the case when the plate is clamped along its lateral boundary, the situation is now well-known, at least theoretically: The comparison between 3D and 2D models was first performed by the construction of infinite *formal* asymptotic expansions, see Friedrichs & Dressler [15], Gol'denveizer [16], Gregory & Wan [17]. Shortly before, Morgenstern [21] was indeed the first to prove that the Kirchhoff model [18] is the correct asymptotic limit of the 3D model when the thickness approaches zero in the hard clamped, hard simply supported and free plate situations by using the Prager-Synge hypercircle theorem [29]. Next, rigorous error estimates between the 3D solution and its limit were proved by Shoikhet [31] and by Ciarlet and Destuynder [5, 13, 3]. Further terms were exhibited by Nazarov & Zorin [24], and the whole asymptotic expansion was constructed in [8, 9].

Different types of lateral boundary conditions are of interest: let us quote the soft clamped plate where the tangential in-plane component of the displacement is free, the hard simply supported plate where its normal component is free, the soft simply supported plate where both above components of the displacement are free, and also

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the totally free plate. These cases were investigated by ARNOLD & FALK [1] where an asymptotics for the Reissner-Mindlin plate was constructed, and by CHEN [2] where error bounds between the 3D solution and its limit were evaluated.

In this paper, we prove the validity of an infinite asymptotic expansion of the displacement with optimal error estimates in H^1, L^2 and energy norms. Such an expansion can be differentiated and provides then corresponding results for the stress and the strain tensors, see [7] for the clamped case. Like in [24] and [8, 9], this asymptotics includes

- An outer part containing displacements only depending on the in-plane variables x_* and on the scaled transverse variable x_3,
- An inner part containing exponentially decaying profiles (boundary layer terms), depending on two scaled variables (x_3 and $t = r/\varepsilon$ where r is the distance to the lateral boundary).

As material law, we choose to remain in the framework of homogeneous, isotropic materials, which allows to uncouple the boundary layer terms $\boldsymbol{\varphi}$ into two parts:

- The horizontal tangential component φ_s governed by a Laplace equation,
- The two other components φ_t and φ_3 governed by the bi-dimensional Lamé equations, whose solutions can themselves be uncoupled in membrane and bending modes, i.e. possessing parity properties with respect to the transverse variable: the former having an even φ_t and an odd φ_3 and the latter having converse properties.

Thus, conditions ensuring the exponential decay at infinity of solutions of the above problems can be made explicit, resulting into simple coupling formulas between the inner and outer parts of the expansion. These coupling formulas lead to the determination of boundary conditions for the limit membrane and bending problems.

The first boundary layer terms bring the quantitative limitation of accuracy of bi-dimensional models. In the clamped and simple support cases, we find a strong boundary layer term with generically non-zero membrane and bending parts, whereas in the frictional and free cases, we find a first boundary layer term which has the bending type and only the in-plane tangential component non-zero, and moreover, the sub-principal term in the outer part of the expansion is a Kirchhoff-Love displacement as usual, but with zero membrane part. Thus if the right hand side has the membrane type, the solution of the 3D Lamé equations for the free plate converges to the usual limit Kirchhoff-Love displacement with improved accuracy.

This paper contains twelve sections: in section 1 we introduce the elasticity problems and in section 2 we present our results in the form of several tables. In section 3 we give as an algorithm the construction rules for the outer part of the Ansatz, while in section 4 we formulate the boundary value problems on a half-strip governing the boundary layer profiles $\boldsymbol{\varphi}$ and give in section 5 the conditions on the data ensuring the existence of exponentially decreasing solutions to these problems. The five next sections are devoted to each of the eight types of lateral boundary conditions with more emphasis on five of them: hard and soft clamped, hard simple support, sliding edge and free plates. In section 11, we prove error estimates between the 3D solution and any truncated series from the asymptotic expansion, and analyze the regularity of the different terms in the asymptotics: whereas the outer expansion terms are smooth if the data are so, the profiles have singularities along the edges of the plate. We conclude in section 12 by considerations about relative errors between the 3D solution and a limit 2D solution, which has to be carefully chosen according to what we wish to approximate (the displacement in H^1 norm, or the strain in L^2 norm).

1. Lateral boundary conditions. We aim to study the behavior of the displacement field $\boldsymbol{u}^\varepsilon$ in a family of thin elastic three-dimensional plates Ω^ε as the thickness parameter ε tends to zero. The plates Ω^ε are constituted of a homogeneous, isotropic material with Lamé constants λ and μ and are defined as follows:

$$\Omega^\varepsilon = \omega \times (-\varepsilon, +\varepsilon) \quad \text{with } \ \omega \subset \mathbb{R}^2 \ \text{ a regular domain and } \varepsilon > 0\,.$$

Let $\Gamma^\varepsilon_\pm$ be their upper and lower faces $\omega \times \{\pm\varepsilon\}$ and Γ^ε_0 be their lateral faces $\partial\omega \times (-\varepsilon, +\varepsilon)$.

1.1. Cartesian, scaled and local coordinates. Let $\tilde{x} = (x_1,\, x_2,\, \tilde{x}_3)$ be the cartesian coordinates in the plates Ω^ε. We will often denote by x_* the in-plane coordinates $(x_1,\, x_2) \in \omega$ and by α or β the indices in $\{1, 2\}$ corresponding to the in-plane variables. The dilatation along the vertical axis ($x_3 = \varepsilon^{-1}\,\tilde{x}_3$) transforms Ω^ε into the fixed reference configuration $\Omega = \omega \times (-1, +1)$:

$$\tilde{x} = (x_*,\, \tilde{x}_3) \in \Omega^\varepsilon = \omega \times (-\varepsilon, +\varepsilon) \longmapsto x = (x_*,\, x_3) \in \Omega = \omega \times (-1, +1). \tag{1.1}$$

We also have to introduce in-plane local coordinates (r, s) in a neighborhood of the boundary $\partial\omega$. Let $\boldsymbol{n}$ be the inner unit normal to $\partial\omega$ and $\boldsymbol{\tau}$ be the tangent unit vector field to $\partial\omega$ such that the basis $(\boldsymbol{\tau}, \boldsymbol{n})$ is direct in each point of $\partial\omega$. Denote by s a curvilinear abscissa (arc length) along $\partial\omega$ oriented according to $\boldsymbol{\tau}$. Let $\mathbb{S} \sim \partial\omega$ be the set of the values of s:

$$\mathbb{S} \ni s \longmapsto \boldsymbol{\gamma}(s) \in \partial\omega.$$

For a point $x_* \in \mathbb{R}^2$, let $r = r(x_*)$ be its signed distance to $\partial\omega$ oriented along $\boldsymbol{n}$, i.e. r is this distance if $x_* \in \omega$ and r is minus this distance if $x_* \notin \omega$. If $|r|$ is small enough, there exists a unique point $x^0_* \in \partial\omega$ such that $|r| = \operatorname{dist}(x_*, x^0_*)$ and we define $s = s(x_*)$ as the curvilinear abscissa of x^0_*. Thus, we have a tubular neighborhood of $\partial\omega$ which is diffeomorphic to $(-r^0, r^0) \times \mathbb{S}$ via the change of variables $x_* \mapsto (r, s)$. And, in this tubular neighborhood, the partial derivatives ∂_r and ∂_s are well defined (and, of course, commute with each other).

We extend the vector fields $\boldsymbol{n}$ and $\boldsymbol{\tau}$ from $\mathbb{S}$ to $(-r^0, r^0) \times \mathbb{S}$ by

$$\forall r \in (-r^0, r^0), \quad \forall s \in \mathbb{S}, \qquad \boldsymbol{n}(r, s) = \boldsymbol{n}(s) \quad \text{and} \quad \boldsymbol{\tau}(r, s) = \boldsymbol{\tau}(s).$$

We have

$$\boldsymbol{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\tau} = \begin{pmatrix} n_2 \\ -n_1 \end{pmatrix}.$$

Moreover, with $R = R(s)$ the curvature radius of $\partial\omega$ at s from inside ω and $\kappa = \frac{1}{R}$ the curvature, there holds (the last identities are Frenet's relations)

$$\partial_r \boldsymbol{n} = 0, \quad \partial_r \boldsymbol{\tau} = 0 \quad \text{and} \quad \partial_s \boldsymbol{n} = -\,\kappa\,\boldsymbol{\tau}, \quad \partial_s \boldsymbol{\tau} = \,\kappa\,\boldsymbol{n}.$$

Thus, relying on the relation $x_* = \boldsymbol{\gamma}(s) + r\,\boldsymbol{n}(s)$, we obtain

$$\partial_r = n_1\partial_1 + n_2\partial_2 \quad \text{and} \quad \partial_s = (1 - \kappa\, r)(n_2\partial_1 - n_1\partial_2). \tag{1.2}$$

Of course $\partial_n = \partial_r$.

1.2. Cartesian, scaled and local tensors. The displacement and traction tensors in Ω^ε are denoted $\boldsymbol{u}^\varepsilon$ and $\boldsymbol{T}^\varepsilon$ and their cartesian components are $(u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon)$ and $(T_1^\varepsilon, T_2^\varepsilon, T_3^\varepsilon)$. As $\boldsymbol{u}$ is covariant, it is naturally transformed by the scaling (1.1) into $\boldsymbol{u}(\varepsilon)$ according to

$$u_\alpha(\varepsilon)(x) = u_\alpha^\varepsilon(\tilde{x}),\ \alpha = 1, 2, \qquad u_3(\varepsilon)(x) = \varepsilon\, u_3^\varepsilon(\tilde{x}), \tag{1.3}$$

whereas $\boldsymbol{T}$ which is contravariant is transformed according the same laws as the volume force field $\boldsymbol{f}^\varepsilon$: by the scaling (1.1) $\boldsymbol{f}^\varepsilon$ is transformed into $\boldsymbol{f}(\varepsilon)$

$$f_\alpha(\varepsilon)(x) = f_\alpha^\varepsilon(\tilde{x}),\ \alpha = 1, 2, \qquad f_3(\varepsilon)(x) = \varepsilon^{-1} f_3^\varepsilon(\tilde{x}). \tag{1.4}$$

In the tubular neighborhood $(-r^0, r^0) \times \mathbb{S}$, in view of (1.2) we can introduce the in-plane normal and tangential components of $\boldsymbol{u}$ and $\boldsymbol{T}$ by

$$u_n = n_1 u_1 + n_2 u_2 \quad \text{and} \quad u_s = (1 - \kappa\, r)(n_2 u_1 - n_1 u_2), \tag{1.5a}$$

$$T_n = n_1 T_1 + n_2 T_2 \quad \text{and} \quad T_s = (1 - \kappa\, r)^{-1}(n_2 T_1 - n_1 T_2). \tag{1.5b}$$

1.3. The equations of elasticity. As standard, let $e(\boldsymbol{u})$ denote the linearized strain tensor $e_{ij}(\boldsymbol{u}) = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$ associated with the displacement $\boldsymbol{u}$. Then the stress tensor $\sigma(\boldsymbol{u})$ is given by Hooke's law $\sigma(\boldsymbol{u}) = A\, e(\boldsymbol{u})$ where the rigidity matrix $A = (A_{ijkl})$ of the material is given by $A_{ijkl} = \lambda\, \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$. The inward traction field at a point on the boundary is $\boldsymbol{T}$ defined as $\sigma(\boldsymbol{u})\, \boldsymbol{n}$ where $\boldsymbol{n}$ is the unit interior normal to the boundary.

We make the assumption that the boundary conditions on the upper and lower faces $\Gamma_\pm^\varepsilon$ of the plate are of traction type. On the lateral face Γ_0^ε we are going to consider the eight 'canonical' choices of boundary conditions which will be denoted by (i) where $\mathbf{i} = 1, \cdots, 8$. Indeed, on the lateral boundary Γ_0^ε we can distinguish three natural components in the displacements or the tractions: normal, horizontal tangential, vertical, and we obtain 8 'canonical' lateral boundary conditions, according to how we choose to prescribe the displacement or the traction for each component.

TABLE 1.1
Lateral boundary conditions.

(i)	Type	Dirichlet	Neumann	$A_{(i)}$	$B_{(i)}$
(1)	hard clamped	$\boldsymbol{u} = 0,$		$\{n, s, 3\}$	
(2)	soft clamped	$u_n,\ u_3 = 0,$	$T_s = 0$	$\{n, 3\}$	$\{s\}$
(3)	hard simply supported	$u_s,\ u_3 = 0,$	$T_n = 0$	$\{s, 3\}$	$\{n\}$
(4)	soft simply supported	$u_3 = 0,$	$T_n,\ T_s = 0$	$\{3\}$	$\{n, s\}$
(5)	frictional I	$u_n,\ u_s = 0,$	$T_3 = 0$	$\{n, s\}$	$\{3\}$
(6)	sliding edge	$u_n = 0,$	$T_s,\ T_3 = 0$	$\{n\}$	$\{s, 3\}$
(7)	frictional II	$u_s = 0,$	$T_n,\ T_3 = 0$	$\{s\}$	$\{n, 3\}$
(8)	free		$\boldsymbol{T} = 0$		$\{n, s, 3\}$

On Γ_0^ε, we recall that the normal component of $\boldsymbol{u}$ is $u_n = u_1 n_1 + u_2 n_2$, its horizontal tangential component is $u_s = u_1 n_2 - u_2 n_1$ and its vertical component is u_3. Similar notations apply to $\boldsymbol{T}$. To each boundary condition (i) corresponds two complementary sets of indices $A_{(i)}$ and $B_{(i)}$ where $A_{(i)}$ is attached to the Dirichlet conditions of $\textcircled{i}$, i.e. $u_a = 0$ for each index $a \in A_{\textcircled{i}}$: these are the *stable* conditions. The Neumann conditions are $T_b = 0$ for each index $b \in B_{\textcircled{i}}$ and appear as *natural* conditions.

To each boundary condition $\textcircled{i}$ is associated the space of displacements $V_{\textcircled{i}}(\Omega^\varepsilon)$ of the $\boldsymbol{v} \in H^1(\Omega^\varepsilon)^3$ such that $v_a = 0$ for all $a \in A_{\textcircled{i}}$, and the space $\mathcal{R}_{\textcircled{i}}$ of the rigid motions satisfying the Dirichlet conditions of $V_{\textcircled{i}}$. Then, the variational formulation of the problem consists in finding

$$(1.6)\quad \begin{cases} \boldsymbol{u}^\varepsilon \in V_{\textcircled{i}}(\Omega^\varepsilon) \\ \forall\, \boldsymbol{v} \in V_{\textcircled{i}}(\Omega^\varepsilon), \quad \displaystyle\int_{\Omega^\varepsilon} A\, e(\boldsymbol{u}^\varepsilon) : e(\boldsymbol{v}) = \int_{\Omega^\varepsilon} \boldsymbol{f}^\varepsilon \cdot \boldsymbol{v} \ + \int_{\Gamma_+^\varepsilon} \boldsymbol{g}^{\varepsilon,+} \cdot \boldsymbol{v} \ - \int_{\Gamma_-^\varepsilon} \boldsymbol{g}^{\varepsilon,-} \cdot \boldsymbol{v}, \end{cases}$$

where $\boldsymbol{f}^\varepsilon$ represents the volume force and $\boldsymbol{g}^{\varepsilon,\pm}$ the prescribed horizontal tractions. If the right hand side satisfies the correct compatibility condition (orthogonality to all $\boldsymbol{v} \in \mathcal{R}_{\textcircled{i}}(\Omega^\varepsilon)$), then there exists a unique solution to (1.6) satisfying the orthogonality conditions $\int_{\Omega^\varepsilon} \boldsymbol{u}^\varepsilon \cdot \boldsymbol{v} = 0$ for all $\boldsymbol{v} \in \mathcal{R}_{\textcircled{i}}(\Omega^\varepsilon)$.

After the scaling (1.3), an asymptotic expansion of $\boldsymbol{u}(\varepsilon)$ makes sense if the scaled data have comparable behaviors as ε is varying. To this aim, we make the assumption on the right hand sides that they are given by profiles in x_3, namely

$$(1.7a)\qquad f_\alpha^\varepsilon(\tilde{x}) = f_\alpha(x),\ \alpha = 1,2, \qquad \varepsilon^{-1} f_3^\varepsilon(\tilde{x}) = f_3(x),$$

$$(1.7b)\qquad \varepsilon^{-1} g_\alpha^{\varepsilon,\pm}(\tilde{x}) = \overset{\pm}{g}_\alpha(x_*),\ \alpha = 1,2, \qquad \varepsilon^{-2} g_3^{\varepsilon,\pm}(\tilde{x}) = \overset{\pm}{g}_3(x_*),$$

compare with (1.4) for the homogeneities. To simplify, we assume that the profiles $\boldsymbol{f}$ and $\boldsymbol{g}^\pm$ are regular up to the boundary, i.e. $\boldsymbol{f} \in \mathscr{C}^\infty(\overline{\Omega})^3$ and $\boldsymbol{g}^\pm \in \mathscr{C}^\infty(\overline{\omega})^3$.

After scaling (1.3) and assumption (1.7), problem (1.6) is transformed into a new boundary value problem on Ω, where now the operators depend on the small parameter ε: The variational formulation of the problem for the scaled displacement $\boldsymbol{u}(\varepsilon)$ consists in finding

$$(1.8)\quad \begin{cases} \boldsymbol{u}(\varepsilon) \in V_{\textcircled{i}}(\Omega) \\ \forall\, \boldsymbol{v} \in V_{\textcircled{i}}(\Omega), \quad \displaystyle\int_{\Omega} A\, \theta(\varepsilon)(\boldsymbol{u}(\varepsilon)) : \theta(\varepsilon)(\boldsymbol{v}) = \int_{\Omega} \boldsymbol{f} \cdot \boldsymbol{v} \ + \int_{\Gamma_+} \boldsymbol{g}^{+} \cdot \boldsymbol{v} \ - \int_{\Gamma_-} \boldsymbol{g}^{-} \cdot \boldsymbol{v}, \end{cases}$$

where $V_{\textcircled{i}}(\Omega)$ is the space of the geometrically admissible displacements $\boldsymbol{v} \in H^1(\Omega)^3$ associated with the problem with lateral boundary conditions $\textcircled{i}$, and $\theta(\varepsilon)(\boldsymbol{v})$ denotes the scaled linearized strain tensor defined by

$$(1.9)\quad \theta_{\alpha\beta}(\varepsilon)(\boldsymbol{v}) := e_{\alpha\beta}(\boldsymbol{v})\,, \quad \theta_{\alpha 3}(\varepsilon)(\boldsymbol{v}) := \varepsilon^{-1}\, e_{\alpha 3}(\boldsymbol{v})\,, \quad \theta_{33}(\varepsilon)(\boldsymbol{v}) := \varepsilon^{-2}\, e_{33}(\boldsymbol{v})\,,$$

for $\alpha, \beta = 1,2$; note that there holds $\theta(\varepsilon)(\boldsymbol{u}(\varepsilon)) = e(\boldsymbol{u}^\varepsilon)$.

Denoting by $\mathcal{R}_{\textcircled{i}}(\Omega)$ the space of rigid motions satisfying the Dirichlet conditions of $V_{\textcircled{i}}(\Omega)$, the compatibility condition becomes

$$(1.10)\qquad \forall\, \boldsymbol{v} \in \mathcal{R}_{\textcircled{i}}(\Omega), \quad \int_{\Omega} \boldsymbol{f} \cdot \boldsymbol{v} \ + \int_{\Gamma_+} \boldsymbol{g}^{+} \cdot \boldsymbol{v} \ - \int_{\Gamma_-} \boldsymbol{g}^{-} \cdot \boldsymbol{v} = 0,$$

and $\boldsymbol{u}(\varepsilon)$ satisfies the orthogonality condition

$$(1.11)\qquad \forall\, \boldsymbol{v} \in \mathcal{R}_{\textcircled{i}}(\Omega), \quad \int_{\Omega} \boldsymbol{u}(\varepsilon) \cdot \boldsymbol{v} = 0\,.$$

Problem (1.8) can be written in the boundary value problem form (1.12)-(1.14) on Ω as follows. To formulate it, we use the repeated index convention. Moreover $\boldsymbol{u}_*$ is a condensed notation for (u_1, u_2), $\operatorname{div}_* \boldsymbol{u}_*$ denotes $\partial_1 u_1 + \partial_2 u_2$ and Δ_* denotes the horizontal Laplacian $\partial_{11} + \partial_{22}$. The in-plane components, indexed by $\alpha = 1, 2$, and the vertical component of the interior equations in Ω are:

$$2\mu\,\partial_3 e_{\alpha 3}(\boldsymbol{u}) + \lambda\,\partial_{\alpha 3} u_3 + \varepsilon^2\big((\lambda+\mu)\partial_\alpha \operatorname{div}_* \boldsymbol{u}_* + \mu\,\Delta_* u_\alpha\big) = -\varepsilon^2 f_\alpha, \tag{1.12a}$$

$$(\lambda+2\mu)\partial_{33} u_3 + \varepsilon^2\big(\lambda\,\partial_3 \operatorname{div}_* \boldsymbol{u}_* + 2\mu\,\partial_\beta e_{\beta 3}(\boldsymbol{u})\big) = -\varepsilon^4 f_3. \tag{1.12b}$$

The boundary conditions on the horizontal sides $\Gamma_\pm := \{x_3 = \pm 1\} \cap \partial\Omega$ are

$$2\mu\, e_{\alpha 3}(\boldsymbol{u}) = \varepsilon^2 g_\alpha^\pm, \qquad \alpha = 1, 2, \tag{1.13a}$$

$$(\lambda+2\mu)\partial_3 u_3 + \varepsilon^2\,\lambda \operatorname{div}_* \boldsymbol{u}_* = \varepsilon^4 g_3^\pm\,. \tag{1.13b}$$

The boundary conditions on the lateral side $\Gamma_0 = \partial\omega \times (-1, 1)$ can be written as

$$u_a = 0, \quad \forall a \in A_{ⓘ} \qquad \text{and} \qquad T_b = 0, \quad \forall b \in B_{ⓘ}. \tag{1.14}$$

The normal, tangential horizontal and vertical components of the traction $\boldsymbol{T} = \boldsymbol{T}(\varepsilon)$ on Γ_0 are given by respectively

$$T_n(\varepsilon) = \lambda\,\partial_3 u_3(\varepsilon) + \varepsilon^2\big(\lambda \operatorname{div}_* \boldsymbol{u}_*(\varepsilon) + 2\mu\,\partial_n u_n(\varepsilon)\big), \tag{1.15a}$$

$$T_s(\varepsilon) = \varepsilon^2 \mu\big(\partial_s u_n(\varepsilon) + \partial_n u_s(\varepsilon) + 2\kappa\, u_s(\varepsilon)\big), \tag{1.15b}$$

$$T_3(\varepsilon) = \mu\big(\partial_n u_3(\varepsilon) + \partial_3 u_n(\varepsilon)\big). \tag{1.15c}$$

2. Description of results. We first state the common features of the asymptotics of the scaled displacement $\boldsymbol{u}(\varepsilon)$, next deduce the asymptotics of the displacement $\boldsymbol{u}^\varepsilon$ in the thin plates. Then we describe the first terms of the asymptotics in each of the eight lateral boundary conditions.

2.1. Common features. Just as in the well-known situation of the clamped plate, the scaled displacement $\boldsymbol{u}(\varepsilon)$ tends in Ω to a Kirchhoff-Love displacement. Let us recall:

Definition 2.1. *A displacement $\boldsymbol{u}$ in Ω is called a* Kirchhoff-Love displacement *if there exist a displacement $\boldsymbol{\zeta}_* = (\zeta_1, \zeta_2)$ in the mean surface ω and a function ζ_3 on ω such that*

$$\boldsymbol{u} = (\zeta_1 - x_3\partial_1\zeta_3,\ \zeta_2 - x_3\partial_2\zeta_3,\ \zeta_3).$$

The function $\boldsymbol{\zeta} := (\boldsymbol{\zeta}_*, \zeta_3)$ is called the generator *of $\boldsymbol{u}$, and the de-scaled displacement associated with $\boldsymbol{u}$ in Ω^ε has exactly the same form with x_3 replaced by $\tilde{x}_3$. Then*

$$\boldsymbol{u}_{\mathrm{KL,b}} = (-x_3\partial_1\zeta_3, -x_3\partial_2\zeta_3,\ \zeta_3) \quad \textit{and} \quad \boldsymbol{u}_{\mathrm{KL,m}} = (\zeta_1, \zeta_2, 0) \tag{2.1}$$

are respectively the bending and membrane parts of $\boldsymbol{u}$.

The asymptotics of $\boldsymbol{u}(\varepsilon)$ contains three types of terms for $k \geq 0$:

- $\boldsymbol{u}^k_{\mathrm{KL}}$: Kirchhoff-Love displacements with 'generating functions' $\boldsymbol{\zeta}^k = (\boldsymbol{\zeta}^k_*, \zeta^k_3)$, i.e. $\boldsymbol{u}^k_{\mathrm{KL}}(x) = \big(\boldsymbol{\zeta}^k_*(x_*) - x_3\nabla_*\zeta^k_3(x_*), \zeta^k_3(x_*)\big)$,
- $\boldsymbol{v}^k$: displacements with zero mean value: $\forall x_* \in \overline{\omega},\ \int_{-1}^{+1} \boldsymbol{v}^k(x_*, x_3)\, dx_3 = 0$,
- $\boldsymbol{w}^k$: exponentially decreasing profiles as $t \to +\infty$

and can be written as:

$$\boldsymbol{u}(\varepsilon)(x) \simeq \boldsymbol{u}^0_{\rm KL} + \varepsilon \boldsymbol{u}^1(x, \frac{r}{\varepsilon}) + \cdots + \varepsilon^k \boldsymbol{u}^k(x, \frac{r}{\varepsilon}) + \cdots \tag{2.2}$$

where

$$\begin{array}{rclll} \boldsymbol{u}^1(x,t) & = & \boldsymbol{u}^1_{\rm KL} \qquad\quad + \chi(r)\, \boldsymbol{w}^1(t,s,x_3) & \text{with} & w^1_3 = 0, \\ \boldsymbol{u}^k(x,t) & = & \boldsymbol{u}^k_{\rm KL} + \boldsymbol{v}^k + \chi(r)\, \boldsymbol{w}^k(t,s,x_3) & \text{for} & k \geq 2\,, \end{array} \tag{2.3}$$

with χ a cut-off function equal to 1 in a neighborhood of $\partial\omega$.

Theorem 2.2. *Let $\boldsymbol{u}(\varepsilon)$ be the unique solution of problem* (1.8) *satisfying the mean value conditions* (1.11). *Then there exist Kirchhoff-Love generators $\boldsymbol{\zeta}^k$ for $k \geq 0$, displacements with zero mean value $\boldsymbol{v}^k$ for $k \geq 2$ and profiles $\boldsymbol{w}^k$ for $k \geq 1$ such that there holds $\forall N \geq 0$*

$$\|\boldsymbol{u}(\varepsilon)(x) - \boldsymbol{u}^0_{\rm KL}(x) - \sum_{k=1}^{N} \varepsilon^k \boldsymbol{u}^k(x, \frac{r}{\varepsilon})\|_{H^1(\Omega)^3} \leq C\, \varepsilon^{N+1/2} \tag{2.4}$$

with $\boldsymbol{u}^k(x, \frac{r}{\varepsilon})$ given in (2.3).

Let us point out that the 'physical' displacement $\boldsymbol{u}^\varepsilon$ expands like $\boldsymbol{u}(\varepsilon)$ in the following way in the sense of asymptotic expansions

$$\begin{aligned} \boldsymbol{u}^\varepsilon \simeq \frac{1}{\varepsilon}\tilde{\boldsymbol{u}}^0_{\rm KL,b} + \tilde{\boldsymbol{u}}^0_{\rm KL,m} + \tilde{\boldsymbol{u}}^1_{\rm KL,b} + \varepsilon(\tilde{\boldsymbol{u}}^1_{\rm KL,m} + \tilde{\boldsymbol{u}}^2_{\rm KL,b} + \tilde{\boldsymbol{v}}^1 + \boldsymbol{\varphi}^1) + \dots \\ \dots + \varepsilon^k(\tilde{\boldsymbol{u}}^k_{\rm KL,m} + \tilde{\boldsymbol{u}}^{k+1}_{\rm KL,b} + \tilde{\boldsymbol{v}}^k + \boldsymbol{\varphi}^k) + \cdots \end{aligned} \tag{2.5}$$

where

- $\tilde{\boldsymbol{u}}^k_{\rm KL,b}$ and $\tilde{\boldsymbol{u}}^k_{\rm KL,m}$ are the bending and membrane parts on Ω^ε of the Kirchhoff-Love displacement with generator $\boldsymbol{\zeta}^k$;
- $\tilde{\boldsymbol{v}}^k = \tilde{\boldsymbol{v}}^k(x_*, \frac{\tilde{x}_3}{\varepsilon})$, i.e. does not depend on ε in the scaled domain Ω;
- $\boldsymbol{\varphi}^k = \boldsymbol{\varphi}^k(\frac{r}{\varepsilon}, s, \frac{\tilde{x}_3}{\varepsilon})$ is a boundary layer profile.

The links with expansion (2.2) on the thin plates are simply provided by the following relations

$$\left\{ \begin{array}{l} \tilde{\boldsymbol{u}}^k_{\rm KL,b}(\tilde{x}) = \varepsilon\, \boldsymbol{u}^k_{\rm KL,b}(x), \quad \tilde{\boldsymbol{u}}^k_{\rm KL,m}(\tilde{x}) = \boldsymbol{u}^k_{\rm KL,m}(x), \\ \tilde{\boldsymbol{v}}^k = (\boldsymbol{v}^k_*, v^{k+1}_3) \quad \text{and} \quad \boldsymbol{\varphi}^k = (\boldsymbol{w}^k_*, w^{k+1}_3)\,. \end{array} \right. \tag{2.6}$$

In Table 3.1, we will give the formulas linking the displacements $\boldsymbol{v}$ to the Kirchhoff-Love generators. These formulas do not depend on the nature of the lateral boundary conditions. In particular, the first non-Kirchhoff displacement $\tilde{\boldsymbol{v}}^1 = (\boldsymbol{0}, v^2_3)$ is completely determined by $\boldsymbol{\zeta}^0$, *cf* Destuynder [14] for a similar formula:

$$\tilde{\boldsymbol{v}}^1(x_*, x_3) = \frac{\lambda}{6(\lambda+2\mu)}\Big(0,\ 0,\ -6x_3\, {\rm div}_*\, \boldsymbol{\zeta}^0_* + (3x_3^2 - 1)\, \Delta_* \zeta^0_3\Big). \tag{2.7}$$

2.2. Specific features: The Kirchhoff-Love generators. The generators $\boldsymbol{\zeta}^k_*$ and ζ^k_3 of the above Kirchhoff displacements are solutions of membrane and bending equations respectively, with boundary conditions on $\partial\omega$. Let us first write down the Dirichlet and Neumann conditions associated with the membrane and bending operators. Then we describe the boundary operators and data associated with the generators.

2.2.1. Membrane. The bilinear form associated with the membrane operator L^{m} (plane stress model)

$$L^{\mathrm{m}}\boldsymbol{\zeta}_* = \mu\,\boldsymbol{\Delta}_*\boldsymbol{\zeta}_* + (\tilde\lambda+\mu)\nabla_*\,\mathrm{div}_*\,\boldsymbol{\zeta}_* \tag{2.8}$$

is $\int_\omega \tilde\lambda\, e_{\alpha\alpha}(\boldsymbol{\zeta}_*)\, e_{\beta\beta}(\boldsymbol{\eta}_*) + 2\mu\, e_{\alpha\beta}(\boldsymbol{\zeta}_*)\, e_{\alpha\beta}(\boldsymbol{\eta}_*)$ with the homogenized Lamé coefficient

$$\tilde\lambda = \frac{2\lambda\mu}{\lambda+2\mu}\,. \tag{2.9}$$

In normal and tangential components, *cf* (1.5)

$$\zeta_n = n_1\zeta_1 + n_2\zeta_2 \quad\text{and}\quad \zeta_s = (1-\kappa\, r)(n_2\zeta_1 - n_1\zeta_2),$$

the Dirichlet traces are simply (ζ_n,ζ_s) on $\partial\omega$, and the Neumann traces are

$$T^{\mathrm{m}}_n(\boldsymbol{\zeta}_*) = \tilde\lambda\,\mathrm{div}_*\,\boldsymbol{\zeta}_* + 2\mu\,\partial_n\zeta_n, \tag{2.10a}$$

$$T^{\mathrm{m}}_s(\boldsymbol{\zeta}_*) = \mu(\partial_s\zeta_n + \partial_n\zeta_s + 2\kappa\,\zeta_s), \tag{2.10b}$$

where ∂_n and ∂_s are defined in (1.2).

2.2.2. Bending. The bilinear form associated with the bending operator L^{b},

$$L^{\mathrm{b}}\zeta_3 = (\tilde\lambda+2\mu)\Delta^2_*\zeta_3 \tag{2.11}$$

is $\int_\omega \tilde\lambda\,\partial_{\alpha\alpha}\zeta_3\ \partial_{\beta\beta}\eta_3 + 2\mu\,\partial_{\alpha\beta}\zeta_3\ \partial_{\alpha\beta}\eta_3$. Its Dirichlet traces are ζ_3 and $\partial_n\zeta_3$ on $\partial\omega$, whereas the Neumann traces are

$$M_n(\zeta_3) = \tilde\lambda\,\Delta_*\zeta_3 + 2\mu\,\partial_{nn}\zeta_3, \tag{2.12a}$$

$$N_n(\zeta_3) = (\tilde\lambda+2\mu)\partial_n\Delta_*\zeta_3 + 2\mu\,\partial_s(\partial_n+\kappa)\partial_s\zeta_3. \tag{2.12b}$$

The mechanical interpretation of these boundary operators is that M_n corresponds to the 'Kirchhoff bending moment' and N_n corresponds to the 'Kirchhoff shear force' on the lateral side of the plate (up to constants only depending on λ and μ).

2.2.3. Boundary value problems for the Kirchhoff-Love generators. The $\boldsymbol{\zeta}^k_*$ and ζ^k_3 are solution of equations of the type

$$L^{\mathrm{m}}(\boldsymbol{\zeta}^k_*) = \boldsymbol{R}^k_{\mathrm{m}} \quad\text{in }\omega,\qquad \gamma^{\mathrm{m},1}(\boldsymbol{\zeta}^k_*) = \gamma^k_{\mathrm{m},1}\ \text{ and }\ \gamma^{\mathrm{m},2}(\boldsymbol{\zeta}^k_*) = \gamma^k_{\mathrm{m},2}\quad\text{on }\partial\omega, \tag{2.13a}$$

$$L^{\mathrm{b}}(\zeta^k_3) = R^k_{\mathrm{b}} \quad\text{in }\omega,\qquad \gamma^{\mathrm{b},1}(\zeta^k_3) = \gamma^k_{\mathrm{b},1}\ \text{ and }\ \gamma^{\mathrm{b},2}(\zeta^k_3) = \gamma^k_{\mathrm{b},2}\quad\text{on }\partial\omega, \tag{2.13b}$$

(see Table 3.1 for expressions of the right hand sides $\boldsymbol{R}^k_{\mathrm{m}}$ and R^k_{b}) where the boundary operators $\gamma^{\mathrm{m},j}$ and $\gamma^{\mathrm{b},j}$, $j=1,2$, depend on the nature of lateral boundary conditions according to table 2.1.

2.2.4. Boundary data for $\boldsymbol{\zeta}^0$. For conditions ① – ④, the boundary data $\gamma^0_{\mathrm{m},j}$ and $\gamma^0_{\mathrm{b},j}$, $j=1,2$, are all zero, whereas for conditions ⑤ – ⑧, only the membrane boundary data $\gamma^0_{\mathrm{m},j}$, $j=1,2$, are always zero.

In the cases ⑤ and ⑦, we assume for simplicity that ω is simply connected. Then $\gamma^0_{\mathrm{b},1}$ which is the trace of ζ^0_3 on $\partial\omega$, is a prescribed constant (so that ζ^0_3 has a zero mean value in accordance with the orthogonality condition (1.11)) which is given by the scalar product of R^0_{b} versus the solution of a typical problem of type (2.13b). The other boundary data $\gamma^0_{\mathrm{b},2}$ is zero.

Table 2.1
Boundary operators for the Kirchhoff-Love generators.

	Membrane part		**Bending part**	
	$\gamma^{\mathrm{m},1}(\boldsymbol{\zeta}_*)$	$\gamma^{\mathrm{m},2}(\boldsymbol{\zeta}_*)$	$\gamma^{\mathrm{b},1}(\zeta_3)$	$\gamma^{\mathrm{b},2}(\zeta_3)$
①	ζ_n	ζ_s	ζ_3	$\partial_n\zeta_3$
②	ζ_n	$T^{\mathrm{m}}_s(\boldsymbol{\zeta}_*)$	ζ_3	$\partial_n\zeta_3$
③	$T^{\mathrm{m}}_n(\boldsymbol{\zeta}_*)$	ζ_s	ζ_3	$M_n(\zeta_3)$
④	$T^{\mathrm{m}}_n(\boldsymbol{\zeta}_*)$	$T^{\mathrm{m}}_s(\boldsymbol{\zeta}_*)$	ζ_3	$M_n(\zeta_3)$
⑤	ζ_n	ζ_s	ζ_3	$\partial_n\zeta_3$
⑥	ζ_n	$T^{\mathrm{m}}_s(\boldsymbol{\zeta}_*)$	$\partial_n\zeta_3$	$N_n(\zeta_3)$
⑦	$T^{\mathrm{m}}_n(\boldsymbol{\zeta}_*)$	ζ_s	ζ_3	$M_n(\zeta_3)$
⑧	$T^{\mathrm{m}}_n(\boldsymbol{\zeta}_*)$	$T^{\mathrm{m}}_s(\boldsymbol{\zeta}_*)$	$M_n(\zeta_3)$	$N_n(\zeta_3)$

In the cases ⑥ and ⑧ the boundary condition related to $\gamma^{\mathrm{b},2} = N_n$ is given by

$$N_n(\zeta_3^0) = \frac{3}{2}\left(\int_{-1}^{+1} x_3\, f_n\, dx_3 + g_n^+ + g_n^-\right)\Bigg|_{\partial\omega} . \tag{2.14}$$

The mechanical interpretation of the right hand side in this relation reads that this expression has the dimension of a moment and can be understood as a prescribed moment on the lateral side of the plate, generated by f_n, g_n^+ and g_n^-. Obviously, this right hand side is zero, if the supports of the data f_n and $g_n^\pm$ avoid $\overline{\Gamma}_0$ and $\partial\omega$, respectively. The other boundary data $\gamma^0_{\mathrm{b},1}$ is zero.

2.2.5. Boundary data for $\boldsymbol{\zeta}^1$. For conditions ① – ④, all the boundary data for $\boldsymbol{\zeta}^1$ are special traces of $\boldsymbol{\zeta}^0$, according to the next table (we recall that κ is the curvature of $\partial\omega$)

Table 2.2
Boundary data for $\boldsymbol{\zeta}^1$.

	Membrane part		**Bending part**	
	$\gamma^1_{\mathrm{m},1}$	$\gamma^1_{\mathrm{m},2}$	$\gamma^1_{\mathrm{b},1}$	$\gamma^1_{\mathrm{b},2}$
①	$c_1^{①}\,\mathrm{div}_*\,\boldsymbol{\zeta}^0_*$	0	0	$c_4^{①}\,\Delta_*\zeta^0_3$
②	$c_1^{②}\,\mathrm{div}_*\,\boldsymbol{\zeta}^0_*$	$c_2^{②}\,\partial_s\,\mathrm{div}_*\,\boldsymbol{\zeta}^0_*$	0	$c_4^{②}\,\Delta_*\zeta^0_3$
③	$c_1^{③}\,\kappa^2\,\zeta^0_n$	0	0	$c_4^{③}\,\kappa^2\,\partial_n\zeta^0_3$
④	$c_1^{④}\,\kappa\,\mathrm{div}_*\,\boldsymbol{\zeta}^0_*$	$c_2^{④}\,\partial_s\,\mathrm{div}_*\,\boldsymbol{\zeta}^0_*$	0	$(c_4^{④}\,\kappa^2 + c_5^{④}\,\partial_{ss})\,\partial_n\zeta^0_3$

Here, the constants $c_j^{ⓘ}$ depend only on λ and μ and come from typical boundary layer profiles.

In contrast to the four 'clamped' lateral conditions, for the four 'free' lateral conditions ⑤ – ⑧ the boundary conditions related to the membrane part $\boldsymbol{\zeta}^1_*$ are all zero, which combined with the fact that the interior right hand side $\boldsymbol{R}^1_{\mathrm{m}}$ is zero yields that $\boldsymbol{\zeta}^1_*$ is itself zero.

The traces of ζ_3^1 are generically not zero: in cases ⑤ and ⑦ (and if ω is simply connected) all traces can be expressed with the help of the function

$$L(s) = \left[-\frac{2}{3}(\tilde{\lambda}+2\mu)\partial_n\Delta_*\,\zeta_3^0 + \int_{-1}^{+1} x_3\, f_n\, dx_3 + g_n^+ + g_n^- \right]\Bigg|_{\partial\omega} . \tag{2.15}$$

In cases ⑥ and ⑧ the prescribed values of the traces involve more complicated operators. We write the boundary data for ζ_3^1 in a condensed form in the next table.

TABLE 2.3
Boundary data for ζ_3^1.

	$\gamma^1_{\mathrm{b},1}$	$\gamma^1_{\mathrm{b},2}$
⑤	$\Lambda^{⑤}$	0
⑥	0	$P^{⑥}(\zeta_3^0) + \kappa K^{⑥}(f_n, \overset{\pm}{g_n})$
⑦	$\Lambda^{⑦}$	$c_4^{⑦}\, L$
⑧	$c_3^{⑧}\,\partial_s(\partial_n+\kappa)\partial_s\zeta_3^0$	$P^{⑧}(\zeta_3^0) + \kappa K^{⑧}(f_n, \overset{\pm}{g_n})$

Here $\Lambda^{⑤}$ and $\Lambda^{⑦}$ are special double primitives of L on $\partial\omega$. $P^{⑥}$ is a linear combination of $\partial_s\kappa^2\partial_s$, $(\kappa\partial_s)^2$ and $\kappa\partial_n\Delta_*$, and $P^{⑧}$ of $\kappa\partial_n\Delta_*$, $\partial_s(\kappa(\partial_n+\kappa))\partial_s$ and $\kappa\partial_s(\partial_n+\kappa)\partial_s$. Finally, $K^{⑥}$ and $K^{⑧}$ are operators preserving the support with respect to the in-plane variables.

2.3. Specific features: The first boundary layer profile. For conditions ① – ④, the first boundary layer profile $\boldsymbol{\varphi}^1$ can be described as a sum of three terms in tensor product form in the variables s and (t, x_3) with $t = \frac{r}{\varepsilon}$:

$$\boldsymbol{\varphi}^1 = \ell^{\mathrm{m}}(s)\,\bar{\boldsymbol{\varphi}}^{\mathrm{m}}(t,x_3) + \ell^{\mathrm{b}}(s)\,\bar{\boldsymbol{\varphi}}^{\mathrm{b}}(t,x_3) + \ell^{\mathrm{s}}(s)\,\bar{\boldsymbol{\varphi}}^{\mathrm{s}}(t,x_3) . \tag{2.16}$$

Here $\bar{\boldsymbol{\varphi}}^{\mathrm{m}}$, $\bar{\boldsymbol{\varphi}}^{\mathrm{b}}$ and $\bar{\boldsymbol{\varphi}}^{\mathrm{s}}$ are typical profiles only depending on the Lamé constants and whose components have special parities with respect to x_3: $\bar{\boldsymbol{\varphi}}^{\mathrm{m}}$ is a membrane displacement whereas $\bar{\boldsymbol{\varphi}}^{\mathrm{b}}$ and $\bar{\boldsymbol{\varphi}}^{\mathrm{s}}$ are bending displacements, moreover some of their components are zero, which is summarized in the next table.

TABLE 2.4
Typical boundary layer profiles.

Components	$\bar{\boldsymbol{\varphi}}^{\mathrm{m}}$	$\bar{\boldsymbol{\varphi}}^{\mathrm{b}}$	$\bar{\boldsymbol{\varphi}}^{\mathrm{s}}$
Normal	even	odd	0
Horizontal tangential	0	0	odd
Vertical	odd	even	0

The functions ℓ are given as traces of $\boldsymbol{\zeta}^0$ along the boundary $\partial\omega$ according to table 2.5.

Again in contrast to the four 'clamped' lateral conditions, the normal and transverse components of the first boundary layer profile $\boldsymbol{\varphi}^1$ are always zero in the cases ⑤ – ⑧. Only the in-plane tangential component φ_s^1 is generically non-zero, and it is odd with respect to x_3. This means that $\boldsymbol{\varphi}^1$ is a bending displacement.

Table 2.5
Lateral traces coming up in the first boundary layer profile.

Case	ℓ^{m}	ℓ^{b}	ℓ^{s}
① and ②	$\operatorname{div}_* \boldsymbol{\zeta}^0_*$	$\Delta_* \zeta^0_3$	0
③	$\kappa\, \zeta^0_n$	$\kappa\, \partial_n \zeta^0_3$	0
④	$\operatorname{div}_* \boldsymbol{\zeta}^0_*$	$\kappa\, \partial_n \zeta^0_3$	$\partial_s(\partial_n \zeta^0_3)$

Table 2.6
The first boundary layer profile.

Case	ℓ^{s}	$\bar\varphi^{\mathrm{s}}$
⑤	$\partial_s \zeta^0_3$	$\bar\varphi^{\mathrm{s}}_{\mathrm{Dir}}$
⑥	$\kappa \partial_s \zeta^0_3$	$\bar\varphi^{\mathrm{s}}_{\mathrm{Neu}}$
⑦	$\partial_s \zeta^0_3$	$\bar\varphi^{\mathrm{s}}_{\mathrm{Dir}}$
⑧	$(\partial_n + \kappa)\partial_s \zeta^0_3$	$\bar\varphi^{\mathrm{s}}_{\mathrm{Neu}}$

The component φ^1_s can be written in tensor product form $\ell^{\mathrm{s}}(s)\,\bar\varphi^{\mathrm{s}}(t, x_3)$ according to table 2.6. Here $\bar\varphi^{\mathrm{s}}_{\mathrm{Dir}}$ and $\bar\varphi^{\mathrm{s}}_{\mathrm{Neu}}$ are solutions on the half strip $\mathbb{R}^+ \times (-1, 1)$ of special boundary problems for the Laplace operator, see Lemmas 5.5 and 5.7.

Note the presence of κ in front of the traces for the hard simple support case ③ and for the sliding edge case ⑥ (compare also with [2] and [27] respectively): due to the possibility of reflecting the solution across any flat part of the boundary, the existence of boundary layer terms is linked to non-zero curvature.

3. Inner – Outer expansion Ansatz.

3.1. The Ansatz. The determination of the asymptotics (2.2) can be split into two steps. The first one consists in finding *all* suitable power series

$$\underline{\boldsymbol{u}}(\varepsilon)(x) \simeq \underline{\boldsymbol{u}}^0(x) + \varepsilon \underline{\boldsymbol{u}}^1(x) + \cdots + \varepsilon^k \underline{\boldsymbol{u}}^k(x) + \cdots \tag{3.1}$$

which solve in the sense of asymptotic expansions the interior equations (1.12) in Ω and conditions (1.13) of traction on the horizontal sides $\Gamma_\pm$. We refer to Maz'ya, Nazarov & Plamenevskii [19, Ch. 15] for general developments relating to the structure of expansion (3.1).

We will see in the sequel that all the terms in the suitable series (3.1) are strictly determined *except* the elliptic traces of the Kirchhoff-Love generators $\boldsymbol{\zeta}^k$. The second step which we will initiate in the next section, consists in finding the profiles $\boldsymbol{w}^k$ so that $\sum_k \varepsilon^k \boldsymbol{w}^k(r\varepsilon^{-1}, s, x_3)$ solves equations (1.12) in Ω with zero volume force, conditions (1.13) of zero traction and so that the lateral boundary conditions (1.14) are satisfied by the complete Ansatz. The outcome will be that the existence of *exponentially decaying* profiles is subordinated to the determination of the remaining degrees of freedom in the series (3.1).

3.2. The algorithms of the outer expansion part. This section is devoted to the construction of the most general power series (3.1) solving (1.12)-(1.13). Let us introduce the two operators A and B which associate with a displacement $\boldsymbol{u}$ in Ω

a volume force in Ω and tractions on the horizontal sides on $\Gamma_\pm$ according to:

$$A\boldsymbol{u} = \Big(2\mu\,\partial_3 e_{\alpha 3}(\boldsymbol{u}) + \lambda\,\partial_{\alpha 3}u_3,\ (\lambda+2\mu)\partial_{33}u_3\ ;\ 2\mu\, e_{\alpha 3}(\boldsymbol{u})\big|_{\Gamma_\pm},\ (\lambda+2\mu)\partial_3 u_3\big|_{\Gamma_\pm}\Big)$$

$$B\boldsymbol{u} = \Big((\lambda+\mu)\partial_\alpha \mathrm{div}_*\boldsymbol{u}_* + \mu\,\Delta_* u_\alpha,\ \lambda\,\partial_3\,\mathrm{div}_*\boldsymbol{u}_* + 2\mu\,\partial_\beta e_{\beta 3}(\boldsymbol{u})\,;\,0\big|_{\Gamma_\pm},\ \lambda\,\mathrm{div}_*\boldsymbol{u}_*\big|_{\Gamma_\pm}\Big)$$

the first group of arguments being the in-plane volume forces, the second, the transverse volume force, and similarly for the tractions. Solving (1.12)-(1.13) by a power series (3.1) is equivalent to solve the system of equations

$$\tag{3.2}\begin{cases} A\underline{\boldsymbol{u}}^k & = \ 0 & \text{for } \ k=0,1,\\ A\underline{\boldsymbol{u}}^2 + B\underline{\boldsymbol{u}}^0 & = \ \big(-f_\alpha\ ,\ 0\ ;\ g_\alpha^\pm\big|_{\Gamma_\pm}\ ,\ 0\big|_{\Gamma_\pm}\big), & \\ A\underline{\boldsymbol{u}}^4 + B\underline{\boldsymbol{u}}^2 & = \ \big(0\ ,\ -f_3\ ;\ 0\big|_{\Gamma_\pm}\ ,\ g_3^\pm\big|_{\Gamma_\pm}\big), & \\ A\underline{\boldsymbol{u}}^k + B\underline{\boldsymbol{u}}^{k-2} & = \ 0, & \text{for } \ k=3 \ \text{ and } \ k\geq 5. \end{cases}$$

It is well known that the solutions of the problem $A\underline{\boldsymbol{u}} = 0$ are the Kirchhoff-Love displacements. Thus $\underline{\boldsymbol{u}}^0 = \boldsymbol{u}^0_{\mathrm{KL}}$ and $\underline{\boldsymbol{u}}^1 = \boldsymbol{u}^1_{\mathrm{KL}}$, with generators $\boldsymbol{\zeta}^0$ and $\boldsymbol{\zeta}^1$.

In order to solve the series of equations of *odd order* $A\underline{\boldsymbol{u}}^k + B\underline{\boldsymbol{u}}^{k-2} = 0$, let us introduce the operator V.

Definition 3.1. *The operator $V : \boldsymbol{\zeta} \mapsto V\boldsymbol{\zeta}$ is defined from $\mathscr{C}^\infty(\overline{\omega})^3$ into $\mathscr{C}^\infty(\overline{\Omega})^3$ by*

$$\tag{3.3}\begin{aligned} (V\boldsymbol{\zeta})_\alpha &= \bar{p}_2\,\partial_\alpha\,\mathrm{div}_*\,\boldsymbol{\zeta}_* + \bar{p}_3\,\partial_\alpha \Delta_*\zeta_3\\ (V\boldsymbol{\zeta})_3 &= \bar{p}_1\,\mathrm{div}_*\,\boldsymbol{\zeta}_* + \bar{p}_2\,\Delta_*\zeta_3 \end{aligned}$$

with $\bar{p}_j$ for $j=1,2,3$ the polynomials in the variable x_3 of degrees j defined as

$$\tag{3.4}\begin{aligned} &\bar{p}_1(x_3) = -\,\frac{\tilde\lambda}{2\mu}\,x_3, \qquad \bar{p}_2(x_3) = \frac{\tilde\lambda}{4\mu}\,\big(x_3^2 - \frac{1}{3}\big),\\ &\bar{p}_3(x_3) = \frac{1}{12\mu}\,\big((\tilde\lambda + 4\mu)\,x_3^3 - (5\tilde\lambda + 12\mu)\,x_3\big). \end{aligned}$$

Here $\tilde\lambda$ still denotes the 'homogenized' Lamé coefficient $2\lambda\mu(\lambda+2\mu)^{-1}$.

With L^{m} the membrane operator (2.8), direct computations yield

Lemma 3.2. *Let $\boldsymbol{\zeta}$ belong to $\mathscr{C}^\infty(\overline{\omega})^3$ and let $\boldsymbol{u}_{\mathrm{KL}}$ be the associated Kirchhoff-Love displacement. Then the field $V\boldsymbol{\zeta}$ is the unique solution with zero mean values on each fiber $x_* \times (-1,1)$ of the problem*

$$\tag{3.5} A(V\boldsymbol{\zeta}) + B(\boldsymbol{u}_{\mathrm{KL}}) = \big(L^{\mathrm{m}}\boldsymbol{\zeta}_*\ ,\ 0\ ;\ 0\big|_{\Gamma_\pm}\ ,\ 0\big|_{\Gamma_\pm}\big)\,.$$

Then, if $L^{\mathrm{m}}\boldsymbol{\zeta}^1_* = 0$, we can take $\underline{\boldsymbol{u}}^3 = \boldsymbol{u}^3_{\mathrm{KL}} + V\boldsymbol{\zeta}^1$. In order to proceed, we remark that each component of $B(V\boldsymbol{\zeta})$ can be split into two parts, both of them being the product of a polynomial in x_3 and of $\Delta_*\,\mathrm{div}_*\,\boldsymbol{\zeta}_*$ or $\Delta_*^2\zeta_3$, or a derivative of these expressions. With the bending operator (2.11) we easily obtain that if $L^{\mathrm{m}}\boldsymbol{\zeta}_*$ and $L^{\mathrm{b}}\zeta_3$ are zero, then $B(V\boldsymbol{\zeta})$ is zero, too. Thus, the odd part of the outer Ansatz is solved, since we obtain by an induction argument:

Proposition 3.3. *For any $k = 1, 3, 5, \dots$ let $\boldsymbol{\zeta}^k$ be such that $L^{\mathrm{m}}\boldsymbol{\zeta}^k_* = 0$ and $L^{\mathrm{b}}\zeta^k_3 = 0$. Then, setting for $k = 3, 5, \dots$*

$$\underline{\boldsymbol{u}}^k = \boldsymbol{u}^k_{\mathrm{KL}} + V\boldsymbol{\zeta}^{k-2}\,,$$

we obtain all the solutions of the odd order equations in system (3.2).

Let us consider now the equations of *even* order. The operator A is block triangular and its diagonal is made of ordinary Neumann problems on the interval $(-1, 1)$. So actually, in order to have solvability for these problems, compatibility conditions are required on the right-hand sides. Conversely, if the problems are solvable, the solutions are uniquely determined if we require that they have a mean value zero on each fiber $x_* \times (-1, 1)$ with $x_* \in \overline{\omega}$.

With $\underline{\boldsymbol{u}}^0 = \boldsymbol{u}^0_{\mathrm{KL}}$, we will find $\underline{\boldsymbol{u}}^2$ being of the form $\boldsymbol{u}^2_{\mathrm{KL}} + V\boldsymbol{\zeta}^0 + G(\boldsymbol{f}, \boldsymbol{g}^\pm)$, where G is another solution operator. But prior to this, we need two sorts of primitive of an integrable function u on the interval $(-1, +1)$:

Notation 3.4. *Let us introduce:*

- *The primitive of u with zero mean value on $(-1, +1)$*

$$\oint^{x_3} u\, dy_3 \;:=\; \int_{-1}^{x_3} u(y_3)\, dy_3 - \frac{1}{2}\int_{-1}^{+1}\int_{-1}^{z_3} u(y_3)\, dy_3\, dz_3,$$

- *The primitive of u which vanishes in -1 and 1 if u has a zero mean value on $(-1, +1)$ and which is even, resp. odd, if u is odd, resp. even*

$$⨍^{y_3} u\, dz_3 \;:=\; \frac{1}{2}\left(\int_{-1}^{y_3} u(z_3)\, dz_3 - \int_{y_3}^{+1} u(z_3)\, dz_3\right).$$

Definition 3.5. *The operator $G : (\boldsymbol{f}, \boldsymbol{g}^\pm) \mapsto G(\boldsymbol{f}, \boldsymbol{g}^\pm)$ is defined from $\mathscr{C}^\infty(\overline{\Omega})^3 \times \mathscr{C}^\infty(\overline{\omega})^6$ into $\mathscr{C}^\infty(\overline{\Omega})^3$ by*

$$\begin{cases} (G(\boldsymbol{f}, \boldsymbol{g}^\pm))_3 &= 0 \\ (G(\boldsymbol{f}, \boldsymbol{g}^\pm))_\alpha &= \dfrac{1}{2\mu}\displaystyle\oint^{x_3}\left[-2⨍^{y_3} f_\alpha + \big(g^+_\alpha - g^-_\alpha + \int_{-1}^{+1} f_\alpha\big)\, y_3 + g^+_\alpha + g^-_\alpha\right] dy_3\,. \end{cases}$$

The reason for the introduction of G is

Lemma 3.6. *For any $(\boldsymbol{f}, \boldsymbol{g}^\pm) \in \mathscr{C}^\infty(\overline{\Omega})^3 \times \mathscr{C}^\infty(\overline{\omega})^6$, $G(\boldsymbol{f}, \boldsymbol{g}^\pm)$ is the unique solution with zero mean values on each fiber $x_* \times (-1, 1)$ of the problem*

$$A\big(G(\boldsymbol{f}, \boldsymbol{g}^\pm)\big) = \left(-f_\alpha + \frac{1}{2}\left[\int_{-1}^{+1} f_\alpha\, dx_3 + g^+_\alpha - g^-_\alpha\right]\,,\ 0\ ;\ g^\pm_\alpha\big|_{\Gamma_\pm}\,,\ 0\big|_{\Gamma_\pm}\right).$$

Now, we can see that if we set

$$\boldsymbol{R}^0_{\mathrm{m}}(x_*) = -\frac{1}{2}\left[\int_{-1}^{+1} \boldsymbol{f}_*(x_*, x_3)\, dx_3 + \boldsymbol{g}^+_*(x_*) - \boldsymbol{g}^-_*(x_*)\right], \tag{3.6}$$

for any $\boldsymbol{\zeta}^0_*$ satisfying the membrane equation $L^{\mathrm{m}}(\boldsymbol{\zeta}^0_*) = \boldsymbol{R}^0_{\mathrm{m}}$, the displacement $\underline{\boldsymbol{u}}^2 = \boldsymbol{u}^2_{\mathrm{KL}} + V\boldsymbol{\zeta}^0 + G(\boldsymbol{f}, \boldsymbol{g}^\pm)$ solves the equation of order $k = 2$ of system (3.2). We denote by $\boldsymbol{v}^2 = V\boldsymbol{\zeta}^0 + G$ its part with zero mean values on each fiber $x_* \times (-1, 1)$.

In order to go further in solving the even part of the Ansatz, we are going to introduce a residual operator $F = (F_*, F_3)$ and a new solution operator W.

DEFINITION 3.7. (i) *The operator* $F : \boldsymbol{v} \mapsto F\boldsymbol{v} = (F_*\boldsymbol{v}, F_3\boldsymbol{v})$ *is defined from* $\mathscr{C}^\infty(\overline{\Omega})^3$ *into* $\mathscr{C}^\infty(\overline{\omega})^3$ *by*

$$\left\{\begin{array}{lcl} F_3\boldsymbol{v} & = & \mu \displaystyle\int_{-1}^{+1} \partial_\beta e_{\beta 3}(\boldsymbol{v})\, dy_3\,, \\ F_\alpha \boldsymbol{v} & = & \dfrac{\tilde\lambda}{2} \displaystyle\int_{-1}^{+1} \fint^{y_3} \partial_{\alpha\beta} e_{\beta 3}(\boldsymbol{v})\, dz_3\, dy_3\ . \end{array}\right.$$

(ii) *The operator* $W : \boldsymbol{v} \mapsto W\boldsymbol{v}$ *is defined from* $\mathscr{C}^\infty(\overline{\Omega})^3$ *into itself by*

$$\left\{\begin{array}{lcl} W_3\boldsymbol{v} & = & -\displaystyle\oint^{x_3} \Big(\frac{\tilde\lambda}{2\mu}\operatorname{div}_* \boldsymbol{v}_* + \frac{\tilde\lambda}{\lambda}\fint^{y_3}\partial_\beta e_{\beta 3}(\boldsymbol{v})\Big) dy_3 \\ W_\alpha\boldsymbol{v} & = & -\displaystyle\oint^{x_3}\Big(\partial_\alpha W_3\boldsymbol{v} + \fint^{y_3}\big(\frac{\lambda}{\mu}\,\partial_{\alpha 3}W_3\boldsymbol{v} + \frac{\lambda+\mu}{\mu}\,\partial_\alpha \operatorname{div}_*\boldsymbol{v}_* + \Delta_* v_\alpha\big)\Big) dy_3. \end{array}\right.$$

With these operators, we can prove

LEMMA 3.8. *Let* $\boldsymbol{v}$ *in* $\mathscr{C}^\infty(\overline{\Omega})^3$ *be a displacement field with zero mean values on each fiber* $x_* \times (-1,1)$, $x_* \in \overline{\omega}$. *Then* $W\boldsymbol{v}$ *has also zero mean values on each fiber* $x_* \times (-1,1)$ *and solves the problem*

$$A(W\boldsymbol{v}) + B(\boldsymbol{v}) = \big(0\ ,\ 0\ ;\ \pm F_*(\boldsymbol{v})\big|_{\Gamma_\pm}\ ,\ \mp F_3(\boldsymbol{v})\big|_{\Gamma_\pm}\big).$$

Now, it is natural to search for $\underline{\boldsymbol{u}}^4$ with the form $\boldsymbol{u}^4_{\rm KL} + V\boldsymbol{\zeta}^2 + W(V\boldsymbol{\zeta}^0 + G) + H$. In view of Lemmas 3.2 and 3.8, with such an Ansatz, H has to solve the problem

$$AH = \big(-L^{\rm m}(\boldsymbol{\zeta}^2_*)\ ,\ -f_3\ ;\ \mp F_*(V\boldsymbol{\zeta}^0 + G)\big|_{\Gamma_\pm}\ ,\ g_3^\pm \pm F_3(V\boldsymbol{\zeta}^0 + G)\big|_{\Gamma_\pm}\big). \tag{3.7}$$

Thus, it is important to have more information about $F(V\boldsymbol{\zeta}^0 + G)$. It is not difficult to check:

LEMMA 3.9. *For all* $\boldsymbol{\zeta}$ *in* $\mathscr{C}^\infty(\overline{\omega})^3$ *we have*

$$F_*(V\boldsymbol{\zeta}) = 0 \qquad \textit{and} \qquad F_3(V\boldsymbol{\zeta}) = -\tfrac{1}{3}L^{\rm b}\zeta_3\ .$$

Moreover, we have

$$F_3(G) = \frac{1}{2}\operatorname{div}_*\left[\int_{-1}^{+1} x_3\, \boldsymbol{f}_*\, dx_3 + \boldsymbol{g}^+_* + \boldsymbol{g}^-_*\right]. \tag{3.8}$$

Then there holds

LEMMA 3.10. *Let* $R^0_{\rm b}$ *be defined as*

$$R^0_{\rm b} = \frac{3}{2}\left[\int_{-1}^{+1} f_3\, dx_3 + g_3^+ - g_3^- + \operatorname{div}_*\Big(\int_{-1}^{+1} x_3\, \boldsymbol{f}_*\, dx_3 + \boldsymbol{g}^+_* + \boldsymbol{g}^-_*\Big)\right], \tag{3.9}$$

and $\boldsymbol{R}^2_{\rm m}$ *be defined as*

$$\boldsymbol{R}^2_{\rm m} = F_*(G) - \frac{\tilde\lambda}{4\mu}\nabla_*\left[\int_{-1}^{+1} x_3\, f_3\, dx_3 + g_3^+ + g_3^-\right]. \tag{3.10}$$

If there hold $L^{\mathrm{b}}(\zeta_3^0) = R_{\mathrm{b}}^0$ *and* $L^{\mathrm{m}}(\boldsymbol{\zeta}_*^2) = \boldsymbol{R}_{\mathrm{m}}^2$*, then equation* (3.7) *admits a unique solution* $H = H(\boldsymbol{f}, \boldsymbol{g}^{\pm})$ *with zero mean values on each fiber* $x_* \times (-1,1)$ *which is given by*

$$\left\{\begin{array}{rcl} H_3 & = & \dfrac{1}{2(\lambda+2\mu)} \displaystyle\oint^{x_3} \Big[\big(-2 \displaystyle\int^{y_3} f_3\big) + g_3^+ + g_3^- \Big] dy_3 \\ H_\alpha & = & -\displaystyle\oint^{x_3} \Big[\partial_\alpha H_3 + \dfrac{1}{\mu}\, y_3\, F_*(G) + \dfrac{\lambda}{\mu} \displaystyle\int^{y_3} \Big\{ \partial_{\alpha 3} H_3 - \dfrac{1}{2} \displaystyle\int_{-1}^{+1} \partial_{\alpha 3} H_3 \, dz_3 \Big\} \Big] dy_3 \, . \end{array}\right.$$

Thus, we have found $\underline{\boldsymbol{u}}^4$ as $\boldsymbol{u}_{\mathrm{KL}}^4 + \boldsymbol{v}^4$ where $\boldsymbol{v}^4$ has zero mean values on each fiber $x_* \times (-1,1)$: $\boldsymbol{v}^4$ is given by $V\boldsymbol{\zeta}^2 + W(V\boldsymbol{\zeta}^0 + G) + H = V\boldsymbol{\zeta}^2 + W\boldsymbol{v}^2 + H$.

Next, we search for a $\underline{\boldsymbol{u}}^6$ with the form $\boldsymbol{u}_{\mathrm{KL}}^6 + V\boldsymbol{\zeta}^4 + W\boldsymbol{v}^4 + Y$. In view of Lemmas 3.2 and 3.8, with such an Ansatz, Y has to solve the problem

$$AY = \big(-L^{\mathrm{m}}(\boldsymbol{\zeta}_*^4)\,,\ 0\,;\ \mp F_*(\boldsymbol{v}^4)\big|_{\Gamma_\pm}\ ,\ \pm F_3(\boldsymbol{v}^4)\big|_{\Gamma_\pm}\big). \tag{3.11}$$

This problem is solvable if
(i) $F_3(\boldsymbol{v}^4)$ is zero, which holds true if $L^{\mathrm{b}}\zeta_3^2 = 3F_3(W\boldsymbol{v}^2 + H)$,
(ii) $L^{\mathrm{m}}(\boldsymbol{\zeta}_*^4) = F_*(\boldsymbol{v}^4)$, compare Lemma 3.10.
Then $Y = Y(\boldsymbol{\zeta}_*^4)$ solves equation (3.11), with the solution operator Y defined as

Definition 3.11. *For* $\boldsymbol{\zeta}_* \in \mathscr{C}^\infty(\overline{\omega})^2$, $Y = Y(\boldsymbol{\zeta}_*)$ *is defined as*

$$Y_3 = 0 \quad \textit{and} \quad Y_* = -2\,\tilde{\lambda}^{-1}\,\bar{p}_2\,L^{\mathrm{m}}(\boldsymbol{\zeta}_*)\,.$$

And from now on, the solution of the series of equations (3.2) is self-similar. Summarizing, we obtain by induction that every expansion (3.1) solving (1.12)-(1.13) can be described according to Table 3.1 below, where G and H are a condensed notation for $G(\boldsymbol{f}, \boldsymbol{g}^{\pm})$ and $H(\boldsymbol{f}, \boldsymbol{g}^{\pm})$ respectively and $\boldsymbol{R}_{\mathrm{m}}^k$ and R_{b}^k are the prescribed values for $L^{\mathrm{m}}(\boldsymbol{\zeta}_*^k)$ and $L^{\mathrm{b}}(\zeta_3^k)$ respectively (note that $\boldsymbol{R}_{\mathrm{m}}^0$, $\boldsymbol{R}_{\mathrm{m}}^2$ and R_{b}^0 are defined in (3.6), (3.10) and (3.9)).

Table 3.1
Algorithm formulas.

k	$\underline{\boldsymbol{u}}^k$	$\boldsymbol{v}^k$	$\boldsymbol{y}^{k-2}$	$\boldsymbol{R}_{\mathrm{m}}^k$	R_{b}^k
0	$\boldsymbol{u}_{\mathrm{KL}}^0$	—	—	$\boldsymbol{R}_{\mathrm{m}}^0$	R_{b}^0
2	$\boldsymbol{u}_{\mathrm{KL}}^2 + \boldsymbol{v}^2$	$V\boldsymbol{\zeta}^0 + \boldsymbol{y}^0$	G	$\boldsymbol{R}_{\mathrm{m}}^2$	$3F_3(W\boldsymbol{v}^2 + H)$
4	$\boldsymbol{u}_{\mathrm{KL}}^4 + \boldsymbol{v}^4$	$V\boldsymbol{\zeta}^2 + \boldsymbol{y}^2$	$W\boldsymbol{v}^2 + H$	$F_*\boldsymbol{v}^4$	$3F_3(W\boldsymbol{v}^4 + Y\boldsymbol{\zeta}_*^4)$
$2\ell{+}2$	$\boldsymbol{u}_{\mathrm{KL}}^{2\ell+2} + \boldsymbol{v}^{2\ell+2}$	$V\boldsymbol{\zeta}^{2\ell} + \boldsymbol{y}^{2\ell}$	$W\boldsymbol{v}^{2\ell} + Y\boldsymbol{\zeta}_*^{2\ell}$	$F_*\boldsymbol{v}^{2\ell+2}$	$3F_3(W\boldsymbol{v}^{2\ell+2} + Y\boldsymbol{\zeta}_*^{2\ell+2})$
1	$\boldsymbol{u}_{\mathrm{KL}}^1$	—	—	0	0
$2\ell{+}1$	$\boldsymbol{u}_{\mathrm{KL}}^{2\ell+1} + \boldsymbol{v}^{2\ell+1}$	$V\boldsymbol{\zeta}^{2\ell-1}$	—	0	0

Here, the even order terms and the odd order ones are independent from each other. We will see later on that they are connected by the lateral boundary conditions via the boundary layer terms. We emphasize that each term $\underline{\boldsymbol{u}}^k$ in the algorithm is the sum of two terms $\underline{\boldsymbol{u}}^k = \boldsymbol{u}_{\mathrm{KL}}^k + \boldsymbol{v}^k$ with $\boldsymbol{u}_{\mathrm{KL}}^k$ representing the general solution of homogeneous Neumann problems for ordinary differential equations over each fiber $x_* \times (-1,1)$ and $\boldsymbol{v}^k$ being particular solutions of inhomogeneous ordinary Neumann problems across the thickness with mean value zero.

3.3. Formulas for the determined part of the displacements. The formulas in Table 3.1 giving the $\boldsymbol{v}^k$ yield in a straightforward way that

$$
\begin{aligned}
\boldsymbol{v}^{2k+1} &= V\boldsymbol{\zeta}^{2k-1}, \\
\boldsymbol{v}^{2k+2} &= \sum_{\ell=0}^{k} W^\ell \circ V\boldsymbol{\zeta}^{2(k-\ell)} + \sum_{\ell=0}^{k-2} W^\ell \circ Y\boldsymbol{\zeta}_*^{2(k-\ell)} \\
&\quad + W^k \circ G(\boldsymbol{f},\boldsymbol{g}^\pm) + W^{k-1}\circ H(\boldsymbol{f},\boldsymbol{g}^\pm)
\end{aligned}
\tag{3.12}
$$

with the convention that $W^{-1}=0$ and $W^0 = \mathrm{Id}$.

Using the definitions of V and W, we can prove

LEMMA 3.12. *For $\ell = 0,1,\cdots$, we have the following formulas for the iterates $W^\ell \circ V$*

$$
\begin{aligned}
(W^\ell \circ V\boldsymbol{\zeta})_\alpha &= \bar r_{2\ell+2}\,\partial_\alpha \Delta_*^\ell \operatorname{div}_* \boldsymbol{\zeta}_* \quad + \quad \bar r_{2\ell+3}\,\partial_\alpha\Delta_*^{\ell+1}\zeta_3 \\
(W^\ell \circ V\boldsymbol{\zeta})_3 &= \bar q_{2\ell+1}\,\Delta_*^\ell \operatorname{div}_* \boldsymbol{\zeta}_* \quad + \quad \bar q_{2\ell+2}\,\Delta_*^{\ell+1}\zeta_3
\end{aligned}
\tag{3.13}
$$

with $\bar q_j$, $\bar r_j$ the polynomials in the variable x_3 of degrees j and of parities j defined recursively as

$$
\bar q_1 = \bar p_1, \quad \bar q_2 = \bar p_2, \quad \bar r_2 = \bar p_2, \quad \bar r_3 = \bar p_3,
$$

with $\bar p_j$ for $j = 1,2,3$ the polynomials defined in (3.4)*, and*

$$
\begin{aligned}
\bar q_j(x_3) &= -\oint^{x_3}\Big(\frac{\tilde\lambda}{2\mu}\bar r_{j-1} + \frac{\tilde\lambda}{2\lambda}\fint^{y_3}(\bar q_{j-2} + \bar r'_{j-1})\Big)dy_3, && \textit{for } j \ge 3 \\
\bar r_j(x_3) &= -\oint^{x_3}\Big(\bar q_{j-1} + \fint^{y_3}(\frac{\lambda}{\mu}\,\bar q'_{j-1} + \frac{\lambda+2\mu}{\mu}\,\bar r_{j-2})\Big)dy_3, && \textit{for } j \ge 4.
\end{aligned}
\tag{3.14}
$$

Similarly, using the definition of Y we are able to show

LEMMA 3.13. *For $\ell = 0,1,\cdots$, we have the following formulas for the iterates $W^\ell \circ Y$*

$$
\begin{aligned}
(W^\ell \circ Y\boldsymbol{\zeta}_*)_\alpha &= \bar s_{2\ell+2}\,\partial_\alpha\Delta_*^\ell \operatorname{div}_*\boldsymbol{\zeta}_* + \bar t_{2\ell+2}\,\Delta_*^{\ell+1}\zeta_\alpha \\
(W^\ell \circ Y\boldsymbol{\zeta}_*)_3 &= \bar s_{2\ell+1}\,\Delta_*^\ell \operatorname{div}_*\boldsymbol{\zeta}_*
\end{aligned}
\tag{3.15}
$$

with $\bar s_j$ and $\bar t_j$ the polynomials in the variable x_3 of degrees j and of parities j defined recursively as

$$
\bar s_1 = 0, \quad \bar s_2 = -\frac{\lambda+2\mu}{\lambda}\,\bar p_2 \quad \textit{and} \quad \bar t_2 = -\frac{3\lambda+2\mu}{\lambda}\,\bar p_2
$$

with $\bar p_2$ given in (3.4)*, and for $\ell \ge 1$:*

$$
\begin{aligned}
\bar s_{2\ell+1}(x_3) &= -\oint^{x_3}\Big(\frac{\tilde\lambda}{2\mu}(\bar s_{2\ell} + \bar t_{2\ell}) + \frac{\tilde\lambda}{2\lambda}\fint^{y_3}(\bar s_{2\ell-1} + \bar s'_{2\ell} + \bar t'_{2\ell})\Big)dy_3, \\
\bar s_{2\ell+2}(x_3) &= -\oint^{x_3}\Big(\bar s_{2\ell+1} + \fint^{y_3}(\frac{\lambda}{\mu}\,\bar s'_{2\ell+1} + \frac{\lambda+\mu}{\mu}\,(\bar s_{2\ell} + \bar t_{2\ell}) + \bar s_{2\ell})\Big)dy_3, \\
\bar t_{2\ell+2}(x_3) &= -\oint^{x_3}\Big(\fint^{y_3}\bar t_{2\ell}\Big)dy_3.
\end{aligned}
\tag{3.16}
$$

Condensing $G(\boldsymbol{f},\boldsymbol{g}^{\pm})$ into G and $H(\boldsymbol{f},\boldsymbol{g}^{\pm})$ into H, we obtain the following formulas for the first $\boldsymbol{v}^k$ (k even).

$$
\begin{aligned}
v_\alpha^2 &= \bar p_2\,\partial_\alpha \operatorname{div}_* \boldsymbol{\zeta}_*^0 + \bar p_3\,\partial_\alpha \Delta_* \zeta_3^0 + G_\alpha \\
v_3^2 &= \bar p_1 \operatorname{div}_* \boldsymbol{\zeta}_*^0 + \bar p_2\,\Delta_* \zeta_3^0\,,
\end{aligned}
\tag{3.17}
$$

$$
\begin{aligned}
v_\alpha^4 &= \bar p_2\,\partial_\alpha \operatorname{div}_* \boldsymbol{\zeta}_*^2 + \bar p_3\,\partial_\alpha \Delta_* \zeta_3^2 + \bar r_4\,\partial_\alpha \Delta_* \operatorname{div}_* \boldsymbol{\zeta}_*^0 + \bar r_5\,\partial_\alpha \Delta_*^2 \zeta_3^0 + (WG+H)_\alpha \\
v_3^4 &= \bar p_1 \operatorname{div}_* \boldsymbol{\zeta}_*^2 + \bar p_2\,\Delta_* \zeta_3^2 + \bar q_3\,\Delta_* \operatorname{div}_* \boldsymbol{\zeta}_*^0 + \bar q_4\,\Delta_*^2 \zeta_3^0 + (WG+H)_3\,.
\end{aligned}
\tag{3.18}
$$

4. The principles of construction of the inner expansion part. After the construction of the most general power series (3.1) solving (1.12)-(1.13), we see that the only remaining degrees of freedom can be given by traces of the Kirchhoff-Love generators $\boldsymbol{\zeta}^k$. As will be investigated for each case in particular, complementing traces of the Kirchhoff-Love generators $\boldsymbol{\zeta}^k$ can be determined along with the computation of the boundary layer terms $\boldsymbol{w}^k$.

The boundary layer Ansatz, namely $\sum_{k\geq 1} \varepsilon^k \boldsymbol{w}^k$ must satisfy the equations (1.12) inside Ω with vanishing right hand side and the boundary conditions (1.13) of zero traction on the horizontal faces of Ω, and must compensate for the lateral boundary conditions of the power series $\sum_{k\geq 0} \varepsilon^k \underline{\boldsymbol{u}}^k$ so that the lateral boundary conditions (1.14) are fulfilled. We present in this section some common features of all problems.

4.1. The equations of the inner expansion.

4.1.1. Lateral boundary conditions. In order to obtain the relations which have to be satisfied by the inner part of the expansion, we evaluate the boundary conditions for a displacement $\boldsymbol{u}$ of the form

$$
\boldsymbol{u}(\varepsilon)(x) = \underline{\boldsymbol{u}}(x) + (\boldsymbol{\varphi}_*, \varepsilon\varphi_3)(\frac{r}{\varepsilon}, s, x_3),
\tag{4.1}
$$

where $\underline{\boldsymbol{u}} = \sum_{k\geq 0} \varepsilon^k \underline{\boldsymbol{u}}^k$ and $\boldsymbol{\varphi} = \sum_{k\geq 1} \varepsilon^k \boldsymbol{\varphi}^k$. The form of the boundary layer term $(\boldsymbol{\varphi}_*, \varepsilon\varphi_3)$ is related to the covariant nature of displacements: indeed we return with $\boldsymbol{\varphi}$ to the homogeneity of the original unknown $\boldsymbol{u}^\varepsilon$. We denote by φ_t the normal component of $\boldsymbol{\varphi}$.

For $\boldsymbol{u}$ of the form (4.1), the formulas for the lateral Dirichlet conditions are obvious, and the lateral Neumann conditions can be written with the help of the following boundary operators acting on the profiles $\boldsymbol{\varphi}$

$$
\begin{aligned}
T_t^{(0)}(\boldsymbol{\varphi}) &= \lambda\,\partial_3\varphi_3 + (\lambda+2\mu)\partial_t\varphi_t, & T_t^{(1)}(\boldsymbol{\varphi}) &= \lambda(\partial_s\varphi_s - \tfrac{1}{R}\,\varphi_t), \\
T_s^{(0)}(\boldsymbol{\varphi}) &= \mu\,\partial_t\varphi_s, & T_s^{(1)}(\boldsymbol{\varphi}) &= \mu(\partial_s\varphi_t + \tfrac{2}{R}\,\varphi_s), \\
T_3^{(0)}(\boldsymbol{\varphi}) &= \mu(\partial_t\varphi_3 + \partial_3\varphi_t).
\end{aligned}
\tag{4.2}
$$

Thus, we can write the components of the lateral traction, *cf* (1.15), as

$$
T_n(\varepsilon) = \varepsilon\,T_t^{(0)}(\boldsymbol{\varphi}) + \varepsilon^2 T_t^{(1)}(\boldsymbol{\varphi}) + \lambda\,\partial_3\,\underline{u}_3 + \varepsilon^2\big(\lambda \operatorname{div}_* \underline{\boldsymbol{u}}_* + 2\mu\,\partial_n \underline{u}_n\big)
\tag{4.3a}
$$

$$
T_s(\varepsilon) = \varepsilon\,T_s^{(0)}(\boldsymbol{\varphi}) + \varepsilon^2 T_s^{(1)}(\boldsymbol{\varphi}) + \varepsilon^2\mu\big(\partial_s \underline{u}_n + \partial_n \underline{u}_s + \tfrac{2}{R}\,\underline{u}_s\big)
\tag{4.3b}
$$

$$
T_3(\varepsilon) = T_3^{(0)}(\boldsymbol{\varphi}) + \mu(\partial_n \underline{u}_3 + \partial_3 \underline{u}_n).
\tag{4.3c}
$$

4.1.2. Interior equations. In variables (t,s,x_3) and unknowns

$$\boldsymbol{\varphi} = (\varphi_t, \varphi_s, \varphi_3) \sim (\boldsymbol{w}_*, \frac{1}{\varepsilon}\, w_3)$$

the interior equations (1.12) for $\boldsymbol{w}$ have the form

$$\mathscr{B}(\varepsilon\,;\, t,s\,;\, \partial_t,\partial_s,\partial_3)\boldsymbol{\varphi} = 0,$$

where the three components $\mathscr{B}(\varepsilon)_t$, $\mathscr{B}(\varepsilon)_s$ and $\mathscr{B}(\varepsilon)_3$ of $\mathscr{B}(\varepsilon)$ can be written as polynomials of degree 2 in ε with coefficients involving partial derivative operators of degree ≤ 2 combined with integer powers of $R = R(s)$ and of $\frac{1}{\rho}$ with

$$\rho = R(s) - r = R(s) - \varepsilon t$$

which is the curvature radius in s of the curve $\{x_* \in \omega,\ \operatorname{dist}(x_*,\partial\omega) = r\}$. The thorough expression of $\mathscr{B}(\varepsilon)$ can be found in [11, §3]. A Taylor expansion at $t = 0$ of $\rho^{-1} = (R-\varepsilon t)^{-1}$ yields an asymptotic expansion of $\mathscr{B}$ in a power series of ε:

$$\mathscr{B} \sim \mathscr{B}^{(0)} + \varepsilon\mathscr{B}^{(1)} + \cdots \varepsilon^k\mathscr{B}^{(k)} + \cdots \tag{4.4}$$

where the $\mathscr{B}^{(k)}(t,s\,;\,\partial_t,\partial_s,\partial_3)$ are partial differential systems of order 2 with polynomial coefficients in t independent from ε. Here follow the expressions for $\mathscr{B}^{(0)}$ and $\mathscr{B}^{(1)}$:

$$\begin{aligned}
(\mathscr{B}^{(0)}\boldsymbol{\varphi})_t &= \mu\big(\partial_{tt}\varphi_t + \partial_{33}\varphi_t\big) + (\lambda+\mu)\,\partial_t\big(\partial_t\varphi_t + \partial_3\varphi_3\big)\\
(\mathscr{B}^{(0)}\boldsymbol{\varphi})_s &= \mu\big(\partial_{tt}\varphi_s + \partial_{33}\varphi_s\big)\\
(\mathscr{B}^{(0)}\boldsymbol{\varphi})_3 &= \mu\,\big(\partial_{tt}\varphi_3 + \partial_{33}\varphi_3\big) + (\lambda+\mu)\,\partial_3\big(\partial_t\varphi_t + \partial_3\varphi_3\big)
\end{aligned} \tag{4.5}$$

and, with the curvature $\kappa = \frac{1}{R}$:

$$\begin{aligned}
(\mathscr{B}^{(1)}\boldsymbol{\varphi})_t &= -\mu\,\kappa\,\partial_t\varphi_t + (\lambda+\mu)\,\partial_t\big(-\kappa\,\varphi_t + \partial_s\varphi_s\big)\\
(\mathscr{B}^{(1)}\boldsymbol{\varphi})_s &= \mu\,\kappa\big(\partial_{tt}(t\varphi_s) + \partial_{33}(t\varphi_s)\big) - \mu\,\kappa\,\partial_t\varphi_s + (\lambda+\mu)\,\partial_s\big(\partial_t\varphi_t + \partial_3\varphi_3\big)\\
(\mathscr{B}^{(1)}\boldsymbol{\varphi})_3 &= -\mu\,\kappa\,\partial_t\varphi_3 + (\lambda+\mu)\,\partial_3\big(-\kappa\,\varphi_t + \partial_s\varphi_s\big).
\end{aligned} \tag{4.6}$$

Thus, the interior equation $\mathscr{B}(\varepsilon)\boldsymbol{\varphi} = 0$ can be written as

$$\mathscr{B}^{(0)}\boldsymbol{\varphi} + \varepsilon\mathscr{B}^{(1)}\boldsymbol{\varphi} + \cdots \varepsilon^k\mathscr{B}^{(k)}\boldsymbol{\varphi} + \cdots \sim 0. \tag{4.7}$$

4.1.3. Horizontal boundary conditions. The boundary conditions on the horizontal sides $x_3 = \pm 1$ are, *cf* (1.13)

$$\mu(\partial_3\varphi_t + \partial_t\varphi_3) = 0, \tag{4.8a}$$

$$\mu\partial_3\varphi_s + \varepsilon\,\mu\,\partial_s\varphi_3 = 0, \tag{4.8b}$$

$$(\lambda+2\mu)\partial_3\varphi_3 + \lambda\,\partial_t\varphi_t + \varepsilon\,\lambda\big(-\tfrac{1}{\rho}\varphi_t + \tfrac{R}{\rho}\partial_s(\tfrac{R}{\rho}\varphi_s)\big) = 0. \tag{4.8c}$$

Similarly to the interior equations, we can develop the horizontal boundary conditions $\mathscr{G}$ (4.8) in powers of ε:

$$\mathscr{G} \sim \mathscr{G}^{(0)} + \varepsilon\mathscr{G}^{(1)} + \cdots \varepsilon^k\mathscr{G}^{(k)} + \cdots \tag{4.9}$$

where the $\mathscr{G}^{(k)}(t,s\,;\,\partial_t,\partial_s,\partial_3)$ are partial differential systems of order 1 with polynomial coefficients in t. The expressions for $\mathscr{G}^{(0)}$ and $\mathscr{G}^{(1)}$ are:

$$\begin{array}{lllllll} & (\mathscr{G}^{(0)}\boldsymbol{\varphi})_t &=& \mu(\partial_3\varphi_t+\partial_t\varphi_3), & (\mathscr{G}^{(1)}\boldsymbol{\varphi})_t &=& 0,\\ (4.10) & (\mathscr{G}^{(0)}\boldsymbol{\varphi})_s &=& \mu\partial_3\varphi_s, & (\mathscr{G}^{(1)}\boldsymbol{\varphi})_s &=& \mu\partial_s\varphi_3,\\ & (\mathscr{G}^{(0)}\boldsymbol{\varphi})_3 &=& (\lambda+2\mu)\partial_3\varphi_3+\lambda\,\partial_t\varphi_t, & (\mathscr{G}^{(1)}\boldsymbol{\varphi})_3 &=& \lambda(-\kappa\,\varphi_t+\partial_s\varphi_s).\end{array}$$

Thus, the horizontal boundary conditions $\mathscr{G}(\varepsilon)\boldsymbol{\varphi}=0$ can be written as

$$\mathscr{G}^{(0)}\boldsymbol{\varphi}+\varepsilon\mathscr{G}^{(1)}\boldsymbol{\varphi}+\cdots\varepsilon^k\mathscr{G}^{(k)}\boldsymbol{\varphi}+\cdots\sim 0. \tag{4.11}$$

4.2. The recursive equations. Assuming that $\sum_k \varepsilon^k\underline{\boldsymbol{u}}^k$ already fulfills the relations in Table 3.1, we determine now the equations satisfied by the profiles $\boldsymbol{\varphi}^k$ and the remaining conditions satisfied by the displacements $\underline{\boldsymbol{u}}^k$ so that

$$\sum_{k\,\geq\,0}\varepsilon^k\underline{\boldsymbol{u}}^k+\sum_{k\,\geq\,1}\varepsilon^k(\boldsymbol{\varphi}_*^k,\varepsilon\varphi_3^k) \tag{4.12}$$

satisfies equations (1.12)-(1.14).

4.2.1. Interior equations. (4.7) yields that

$$\forall k\geq 0,\qquad \sum_{\ell=0}^{k}\mathscr{B}^{(\ell)}\boldsymbol{\varphi}^{k-\ell}=0, \tag{4.13}$$

which guarantees (1.12) for the whole expansion (4.12).

4.2.2. Horizontal boundary conditions. (4.11) yields that

$$\forall k\geq 0,\qquad \sum_{\ell=0}^{k}\mathscr{G}^{(\ell)}\boldsymbol{\varphi}^{k-\ell}=0, \tag{4.14}$$

which guarantees (1.13) for the whole expansion (4.12).

4.2.3. Lateral Dirichlet boundary conditions. Let $\sum_k\varepsilon^kD_n^k$, $\sum_k\varepsilon^kD_s^k$ and $\sum_k\varepsilon^kD_3^k$ be the normal, tangential and vertical components of the lateral Dirichlet traces of the series (4.12). The lateral Dirichlet boundary conditions then read

$$\forall k\geq 0,\qquad D_n^k=0\ \text{ if }\ n\in A,\quad D_s^k=0\ \text{ if }\ s\in A,\quad D_3^k=0\ \text{ if }\ 3\in A, \tag{4.15}$$

which immediately yields the Dirichlet conditions for the whole expansion (4.12).

For the terms D^k, we have

$$D_n^0=\underline{u}_n^0,\quad D_s^0=\underline{u}_s^0,\quad D_3^0=\underline{u}_3^0,\quad D_3^1=\underline{u}_3^1, \tag{4.16}$$

and for $k\geq 1$

$$D_n^k=\varphi_t^k+\underline{u}_n^k, \tag{4.17a}$$
$$D_s^k=\varphi_s^k+\underline{u}_s^k, \tag{4.17b}$$
$$D_3^{k+1}=\varphi_3^k+\underline{u}_3^{k+1}\,. \tag{4.17c}$$

4.2.4. Lateral Neumann boundary conditions. Let $\sum_k \varepsilon^k T_n^k$, $\sum_k \varepsilon^k T_s^k$ and $\sum_k \varepsilon^k T_3^k$ be the normal, tangential and vertical components of the lateral Neumann traces of the series (4.12). The lateral Neumann boundary conditions then read

$$\forall k \geq 0, \qquad T_n^k = 0 \text{ if } n \in B, \quad T_s^k = 0 \text{ if } s \in B, \quad T_3^k = 0 \text{ if } 3 \in B, \tag{4.18}$$

which immediately yields the Neumann conditions for the whole expansion (4.12).

Let us evaluate the terms T^k. To that aim, we rely on the following formulas for $\underline{\boldsymbol{u}}^k$, *cf* Table 3.1, either $\underline{\boldsymbol{u}}^k = \boldsymbol{u}_{\mathrm{KL}}^k + \boldsymbol{v}^k$, i.e.

$$\underline{u}_n^k = \zeta_n^k - x_3\,\partial_n\zeta_3^k + v_n^k, \tag{4.19a}$$
$$\underline{u}_s^k = \zeta_s^k - x_3\,\partial_s\zeta_3^k + v_s^k, \tag{4.19b}$$
$$\underline{u}_3^k = \zeta_3^k + v_3^k, \tag{4.19c}$$

or $\underline{\boldsymbol{u}}^k = \boldsymbol{u}_{\mathrm{KL}}^k + V\boldsymbol{\zeta}^{k-2} + \boldsymbol{y}^{k-2}$, i.e.

$$\underline{u}_n^k = \zeta_n^k - x_3\,\partial_n\zeta_3^k + \bar p_2\,\partial_n\,\mathrm{div}_*\,\boldsymbol{\zeta}_*^{k-2} + \bar p_3\,\partial_n\Delta_*\zeta_3^{k-2} + \boldsymbol{y}_n^{k-2}, \tag{4.20a}$$
$$\underline{u}_s^k = \zeta_s^k - x_3\,\partial_s\zeta_3^k + \bar p_2\,\partial_s\,\mathrm{div}_*\,\boldsymbol{\zeta}_*^{k-2} + \bar p_3\,\partial_s\Delta_*\zeta_3^{k-2} + \boldsymbol{y}_s^{k-2}, \tag{4.20b}$$
$$\underline{u}_3^k = \zeta_3^k + \bar p_1\,\mathrm{div}_*\,\boldsymbol{\zeta}_*^{k-2} + \bar p_2\,\Delta_*\zeta_3^{k-2} + \boldsymbol{y}_3^{k-2}, \tag{4.20c}$$

where $\bar p_1$, $\bar p_2$, $\bar p_3$ are introduced in (3.4).

Thus, we find

$$T_n^0 = 0, \quad T_n^1 = 0, \quad T_s^0 = 0, \quad T_s^1 = 0, \quad T_3^0 = 0, \tag{4.21}$$

and for $k \geq 1$, *cf* (2.10), (2.12), (4.2):

$$\begin{aligned} T_n^{k+1} \;=\;& T_t^{(0)}(\boldsymbol{\varphi}^k) + T_t^{(1)}(\boldsymbol{\varphi}^{k-1}) + T_n^{\mathrm{m}}(\boldsymbol{\zeta}_*^{k-1}) - x_3\,M_n(\zeta_3^{k-1}) \\ &+ \lambda\,\partial_3\,\boldsymbol{y}_3^{k-1} + \lambda\,\mathrm{div}_*\,\boldsymbol{v}_*^{k-1} + 2\mu\,\partial_n v_n^{k-1} \end{aligned} \tag{4.22a}$$

$$\begin{aligned} T_s^{k+1} \;=\;& T_s^{(0)}(\boldsymbol{\varphi}^k) + T_s^{(1)}(\boldsymbol{\varphi}^{k-1}) + T_s^{\mathrm{m}}(\boldsymbol{\zeta}_*^{k-1}) - 2\mu x_3\,(\partial_n + \tfrac{1}{R})\partial_s\zeta_3^{k-1} \\ &+ \mu\big(\partial_s v_n^{k-1} + \partial_n v_s^{k-1} + \tfrac{2}{R}\,v_s^{k-1}\big) \end{aligned} \tag{4.22b}$$

$$\begin{aligned} T_3^{k} \;=\;& T_3^{(0)}(\boldsymbol{\varphi}^k) + \mu(\bar p_2 + \bar p_3')\,\partial_n\Delta_*\zeta_3^{k-2} \\ &+ \mu(\partial_n\boldsymbol{y}_3^{k-2} + \partial_3\boldsymbol{y}_n^{k-2}). \end{aligned} \tag{4.22c}$$

4.3. Solving the inner expansion. According to the calculations of the previous subsection, to solve the problem with the Ansatz (4.12), it remains to find a sequence of profiles $(\boldsymbol{\varphi}^k)_k$ and a sequence of Kirchhoff-Love generators $(\boldsymbol{\zeta}^k)_k$ such that (4.13), (4.14), (4.15) and (4.18) hold.

Let us consider now the profiles $\boldsymbol{\varphi}^k$ for $k \geq 1$ as main unknowns. In view of (4.13), (4.14), (4.17) and (4.22), we see that the sequence of problems satisfied by the $\boldsymbol{\varphi}^k$ can be written in a recursive way: for each $k \geq 1$ the profile $\boldsymbol{\varphi}^k$ has to solve the equation

$$\mathscr{B}_{\text{ⓘ}}(\boldsymbol{\varphi}^k) = (\mathfrak{f}^k; \mathfrak{g}^k; \mathfrak{h}^k), \tag{4.23}$$

where

- $\mathscr{B}_{\text{ⓘ}}$ is the operator $\mathscr{B}^{(0)}$ inside the domain, the traction operator $\mathscr{G}^{(0)}$ on the horizontal sides, the Dirichlet traces on the lateral side for $a \in A_{\text{ⓘ}}$ and the Neumann traces on the lateral side for $b \in B_{\text{ⓘ}}$,

- $\mathfrak{f}^k$ and $\mathfrak{g}^k$ are the following functions of the previous profiles

$$\mathfrak{f}^k = -\sum_{\ell=1}^{k} \mathscr{B}^{(\ell)} \boldsymbol{\varphi}^{k-\ell} \quad \text{and} \quad \mathfrak{g}^k = -\sum_{\ell=1}^{k} \mathscr{G}^{(\ell)} \boldsymbol{\varphi}^{k-\ell}, \tag{4.24}$$

so that (4.13)-(4.14) is solved, and $\mathfrak{h}^k$ involves previous profiles as well and certain traces of the Kirchhoff-Love generators $\boldsymbol{\zeta}^\ell$ according to (4.15)-(4.22).

An important point is now to note that neither $\mathscr{B}^{(0)}$, nor $\mathscr{G}^{(0)}$, nor the lateral trace operators of $\mathscr{B}_{\text{i}}$ contain any derivative with respect to the tangential variable s. Thus, the equations (4.23) can be solved in the variables $t \in \mathbb{R}^+$ and $x_3 \in (-1, 1)$, the role of s being only that of a parameter. So we introduce the half-strip

$$\Sigma^+ = \big\{(t, x_3); \quad 0 < t, \quad -1 < x_3 < 1\big\}. \tag{4.25}$$

Its boundary has two horizontal parts $\gamma_\pm = \mathbb{R}^+ \times \{x_3 = \pm 1\}$ and a lateral part

$$\gamma_0 = \big\{(t, x_3); \quad t = 0, \quad -1 < x_3 < 1\big\}. \tag{4.26}$$

Thus, we have

$$\mathscr{B}_{\text{i}}(\boldsymbol{\varphi}) = (\mathfrak{f}; \mathfrak{g}; \mathfrak{h}) \quad \Longleftrightarrow \quad \left\{ \begin{array}{rcll} \mathscr{B}^{(0)}(\boldsymbol{\varphi}) & = & \mathfrak{f}, & \text{in } \Sigma^+, \\ \mathscr{G}^{(0)}(\boldsymbol{\varphi}) & = & \mathfrak{g}, & \text{on } \gamma_\pm, \\ \varphi_a & = & \mathfrak{h}_a, & \text{on } \gamma_0, \quad \forall a \in A_{\text{i}}, \\ T_b^{(0)}(\boldsymbol{\varphi}) & = & \mathfrak{h}_b, & \text{on } \gamma_0, \quad \forall b \in B_{\text{i}}. \end{array} \right. \tag{4.27}$$

Essential is the possibility of finding *exponentially decreasing* solutions when $\mathfrak{f}$ and $\mathfrak{g}$ have the same property. This is what we start to investigate in the next section.

5. Exponentially decaying profiles in a half-strip.

5.1. General principles. The properties of the operators $\mathscr{B}_{\text{i}}$ are closely linked to those of the corresponding operator $\underline{\mathscr{B}}$ on the full strip $\Sigma := \mathbb{R} \times (-1, 1)$, defined as $\underline{\mathscr{B}}(\boldsymbol{\varphi}) = (\mathfrak{f}; \mathfrak{g})$ with $\mathfrak{f} = \mathscr{B}^{(0)}(\boldsymbol{\varphi})$ in Σ and $\mathfrak{g} = \mathscr{G}^{(0)}(\boldsymbol{\varphi})$ on $\mathbb{R} \times \{x_3 = \pm 1\}$, see also Nazarov & Plamenevskii [23, Ch. 5].

Let $\mathcal{P}$ be the space of polynomial displacements $\boldsymbol{Z}$ satisfying $\underline{\mathscr{B}}(\boldsymbol{Z}) = 0$. Computations like those of Mielke in [20] yield that $\mathcal{P}$ has eight dimensions and that a basis of $\mathcal{P}$ is given by the following polynomial displacements $\boldsymbol{Z}^{[1]}, \cdots, \boldsymbol{Z}^{[8]}$

$$\boldsymbol{Z}^{[1]} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \boldsymbol{Z}^{[2]} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \boldsymbol{Z}^{[3]} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \boldsymbol{Z}^{[4]} = \begin{pmatrix} -x_3 \\ 0 \\ t \end{pmatrix}$$

$$\boldsymbol{Z}^{[5]} = \begin{pmatrix} t \\ 0 \\ \bar{p}_1 \end{pmatrix} \qquad \boldsymbol{Z}^{[6]} = \begin{pmatrix} 0 \\ t \\ 0 \end{pmatrix} \qquad \boldsymbol{Z}^{[7]} = \begin{pmatrix} -2tx_3 \\ 0 \\ t^2 + 2\bar{p}_2 \end{pmatrix}$$

$$\boldsymbol{Z}^{[8]} = \begin{pmatrix} -3t^2 x_3 + 6\bar{p}_3 \\ 0 \\ t^3 + 6t\bar{p}_2 \end{pmatrix}$$

where $\bar{p}_1(x_3)$, $\bar{p}_2(x_3)$, $\bar{p}_3(x_3)$ are the polynomials previously introduced in (3.4).

Let us introduce weighted spaces H^m_η on the half-strip Σ^+: for $\eta > 0$, their elements are exponentially decreasing as $t \to \infty$:

DEFINITION 5.1. *Let $\eta \in \mathbb{R}$. For $m \geq 0$ let $H^m_\eta(\Sigma^+)$ be the space of functions v such that $e^{\eta t} v$ belongs to $H^m(\Sigma^+)$. We also denote $H^0_\eta(\Sigma^+)$ by $L^2_\eta(\Sigma^+)$. Similar definitions hold for $\mathbb{R}^+$.*

Like in [9, Lemmas 4.10 & 4.11], we have, with η_0 the smallest exponent arising from the Papkovich-Fadle eigenfunctions, compare PAPKOVICH [26] for early reference and GREGORY & WAN [17]:

LEMMA 5.2. *Let η, $0 < \eta < \eta_0$. Let $\mathfrak{f}$ belong to $L^2_\eta(\Sigma^+)^3$ and $\mathfrak{g}$ belong to $L^2_\eta(\mathbb{R}^+)^6$, let $\mathfrak{h}_a$ belong to $H^{1/2}(\gamma_0)$ for each $a \in A_{\text{ⓘ}}$ and $\mathfrak{h}_b$ belong to $H^{-1/2}(\gamma_0)$ for each $b \in B_{\text{ⓘ}}$. Then there exist $\boldsymbol{\varphi} \in H^1_\eta(\Sigma^+)^3$ and $\boldsymbol{Z} \in \mathcal{P}$ so that*

$$\mathscr{B}_{\text{ⓘ}}(\boldsymbol{\varphi} + \boldsymbol{Z}) = (\mathfrak{f}; \mathfrak{g}; \mathfrak{h}). \tag{5.1}$$

But the solution given by Lemma 5.2 is not unique. Let $\mathcal{T}_{\text{ⓘ}}$ denote the space of the polynomial displacements $\boldsymbol{Z}$ such that there exists $\boldsymbol{\varphi} = \boldsymbol{\varphi}(\boldsymbol{Z}) \in H^1_\eta(\Sigma^+)^3$ satisfying

$$\mathscr{B}_{\text{ⓘ}}(\boldsymbol{Z} + \boldsymbol{\varphi}(\boldsymbol{Z})) = 0.$$

Like in [9, Proposition 4.12], we can prove that the dimension of $\mathcal{T}_{\text{ⓘ}}$ is 4. Thus $\mathcal{P}$ can be split in the direct sum of two four-dimensional spaces $\mathcal{Z}_{\text{ⓘ}}$ and $\mathcal{T}_{\text{ⓘ}}$, and we have as corollary:

LEMMA 5.3. *Let $\mathfrak{f}$, $\mathfrak{g}$ and $\mathfrak{h}$ be as in Lemma 5.2. Then there exist $\boldsymbol{\varphi}$* unique *in $H^1_\eta(\Sigma^+)^3$ and $\boldsymbol{Z}$* unique *in the four-dimensional space $\mathcal{Z}_{\text{ⓘ}}$ so that* (5.1) *holds.*

At this stage, the conclusion is that we have a defect number equal to four for the solution of the sequence of the above equations (4.23) by exponentially decreasing displacements $\boldsymbol{\varphi}^k$, for each $s \in \partial\omega$. But four traces on $\partial\omega$ are still available, allowing to modify $\mathfrak{h}^k$. Note that this is coherent with the principle of 'matching asymptotics', according to which the behavior at infinity of the profiles is transformed into a function of the primitive variable x (which is a Kirchhoff-Love displacement).

5.2. The operators acting on profiles. We can immediately see that the operators $\mathscr{B}_{\text{ⓘ}}$ act separately on the couple of components (φ_t, φ_3) that we denote $\boldsymbol{\varphi}_\natural$, and on φ_s. On $\boldsymbol{\varphi}_\natural$ acts an elasticity operator with the Lamé constants λ and μ, and on φ_s a Laplace operator.

The interior elasticity operator in Σ^+ is

$$\mathscr{B}^{(0)}_\natural : \quad \boldsymbol{\varphi}_\natural \quad \longmapsto \quad \mathfrak{f}_\natural \;=\; \mu(\partial_{tt} + \partial_{33}) \begin{pmatrix} \varphi_t \\ \varphi_3 \end{pmatrix} + (\lambda + \mu) \begin{pmatrix} \partial_t \\ \partial_3 \end{pmatrix} (\partial_t \varphi_t + \partial_3 \varphi_3), \tag{5.2}$$

its horizontal boundary conditions $\mathscr{G}^{(0)}$ (4.10) on $\gamma_\pm$ are

$$\mathscr{G}^{(0)}_\natural : \quad \boldsymbol{\varphi}_\natural \quad \longmapsto \quad \mathfrak{g}_\natural \;=\; \begin{pmatrix} \mu(\partial_3 \varphi_t + \partial_t \varphi_3) \\ (\lambda + 2\mu)\partial_3 \varphi_3 + \lambda\, \partial_t \varphi_t \end{pmatrix} \tag{5.3}$$

and the lateral boundary conditions are either Dirichlet's or Neumann's acting on the traction $\boldsymbol{T}^{(0)}_\natural = (T^{(0)}_t, T^{(0)}_3)$, *cf* (4.25).

Let us introduce the four elasticity operators that we need. For each of them $\mathfrak{f}_\natural = \mathscr{B}^{(0)}_\natural(\boldsymbol{\varphi}_\natural)$ and $\mathfrak{g}_\natural = \mathscr{G}^{(0)}_\natural(\boldsymbol{\varphi}_\natural)$. Only differs the definition of the lateral trace $\mathfrak{h}_\natural$:

- $E_{\rm Dir}$: $\boldsymbol{\varphi}_\natural \mapsto (\mathfrak{f}_\natural; \mathfrak{g}_\natural; \mathfrak{h}_\natural)$ with $\mathfrak{h}_\natural$ the trace of $\boldsymbol{\varphi}_\natural$ on γ_0,
- $E_{\rm Mix1}$: $\boldsymbol{\varphi}_\natural \mapsto (\mathfrak{f}_\natural; \mathfrak{g}_\natural; \mathfrak{h}_\natural)$ with $\mathfrak{h}_\natural$ the trace of $\big(T_t^{(0)}(\boldsymbol{\varphi}_\natural), \varphi_3\big)$ on γ_0,
- $E_{\rm Mix2}$: $\boldsymbol{\varphi}_\natural \mapsto (\mathfrak{f}_\natural; \mathfrak{g}_\natural; \mathfrak{h}_\natural)$ with $\mathfrak{h}_\natural$ the trace of $\big(\varphi_t, T_3^{(0)}(\boldsymbol{\varphi}_\natural)\big)$ on γ_0,
- $E_{\rm Free}$: $\boldsymbol{\varphi}_\natural \mapsto (\mathfrak{f}_\natural; \mathfrak{g}_\natural; \mathfrak{h}_\natural)$ with $\mathfrak{h}_\natural$ the trace of $\boldsymbol{T}_\natural^{(0)}(\boldsymbol{\varphi}_\natural)$ on γ_0,

whereas the Laplace operators are defined as:

- $L_{\rm Dir}$: $\varphi_s \mapsto (\mathfrak{f}_s; \mathfrak{g}_s; \mathfrak{h}_s)$ with $\mathfrak{f}_s = \mu\Delta\varphi_s$, $\mathfrak{g}_s = \mu\partial_3\varphi_s$ and $\mathfrak{h}_s = \varphi_s$ on γ_0,
- $L_{\rm Neu}$: $\varphi_s \mapsto (\mathfrak{f}_s; \mathfrak{g}_s; \mathfrak{h}_s)$ with $\mathfrak{f}_s = \mu\Delta\varphi_s$, $\mathfrak{g}_s = \mu\partial_3\varphi_s$ and $\mathfrak{h}_s = \mu\partial_t\varphi_s$ on γ_0.

Then we have the splittings:

$$\mathscr{B}_{①} = E_{\rm Dir}\oplus L_{\rm Dir} \quad \mathscr{B}_{②} = E_{\rm Dir}\oplus L_{\rm Neu} \quad \mathscr{B}_{③} = E_{\rm Mix1}\oplus L_{\rm Dir} \quad \mathscr{B}_{④} = E_{\rm Mix1}\oplus L_{\rm Neu}$$
$$\mathscr{B}_{⑤} = E_{\rm Mix2}\oplus L_{\rm Dir} \quad \mathscr{B}_{⑥} = E_{\rm Mix2}\oplus L_{\rm Neu} \quad \mathscr{B}_{⑦} = E_{\rm Free}\oplus L_{\rm Dir} \quad \mathscr{B}_{⑧} = E_{\rm Free}\oplus L_{\rm Neu}.$$

5.3. The Laplacian on the half-strip. The Neumann problem on the full strip Σ has a polynomial kernel of dimension two generated by 1 and t, corresponding to the elements $\boldsymbol{Z}^{[2]}$ and $\boldsymbol{Z}^{[6]}$ of the space $\mathcal{P}$ introduced at the beginning of the section.

5.3.1. Operator $L_{\rm Dir}$. The polynomial kernel of this problem is the function t and by integration by parts of $t\,\Delta(\varphi+\delta)$ on rectangles $\Sigma_L = (0,L)\times(-1,1)$ with $L \to +\infty$, we easily prove

Proposition 5.4. *For $\eta > 0$, let $f \in L^2_\eta(\Sigma^+)$, $g^\pm \in L^2_\eta(\mathbb{R}^+)^2$ and $h \in H^{1/2}(\gamma_0)$. If moreover $\eta < \pi/2$, then the problem*

$$L_{\rm Dir}(\psi) = (f; g^\pm; h)$$

has a unique solution $\psi = \varphi + \delta$ in $H^1_\eta(\Sigma^+) \oplus \operatorname{span}\{1\}$ with $\varphi \in H^1_\eta(\Sigma^+)$ and

$$\delta = \frac{1}{2\mu}\left(-\int_{\Sigma^+} t\, f(t,x_3)\,dt\,dx_3 + \int_{\mathbb{R}^+} t\big(g^+(t) - g^-(t)\big)\,dt + \mu\int_{-1}^{+1} h(x_3)\,dx_3 \right). \tag{5.4}$$

Later on we will use as model profile the exponentially decaying solution $\bar\varphi^{\rm s}_{\rm Dir}$ of a special problem involving $L_{\rm Dir}$:

Lemma 5.5. *Let $\bar\varphi^{\rm s}_{\rm Dir} \in H^1_\eta(\Sigma^+)$ be the exponentially decaying solution of the problem*

$$L_{\rm Dir}(\bar\varphi^{\rm s}_{\rm Dir}) = (0; 0; x_3)\,,$$

then it holds

$$\int_0^\infty \bar\varphi^{\rm s}_{\rm Dir}(t,1)\,dt > 0.$$

Proof. The function $\bar\varphi^{\rm s}_{\rm Dir}$ is an odd function with respect to x_3. Hence $\bar\varphi^{\rm s}_{\rm Dir}(t,0) = 0$ for $t \in \mathbb{R}^+$. Moreover, as $\bar\varphi^{\rm s}_{\rm Dir}$ is harmonic, it can be reflected by parity at the line $x_3 = 1$ according to the reflection principle of Schwarz for harmonic functions. Thus, we obtain a function $\tilde\varphi$, which is still harmonic, but now in $\widetilde\Sigma^+ = \mathbb{R}^+ \times (0,2)$. Hence $\tilde\varphi$ satisfies the Dirichlet problem $\Delta\,\tilde\varphi = 0$ in $\widetilde\Sigma^+$ and $\tilde\varphi = \tilde\Phi$ on $\partial\widetilde\Sigma^+$ with $\Phi(t,x_3) = 0$ for $x_3 = 0, 2$ and any t and $\Phi(0,x_3) = x_3$ for $0 < x_3 \le 1$ and $\Phi(0,x_3) = 2 - x_3$ for $1 \le x_3 < 2$. From the maximum principle for harmonic functions it follows $\tilde\varphi > 0$ in $\widetilde\Sigma^+$, hence the assertion. ∎

5.3.2. Operator L_{Neu}. The polynomial kernel of this problem is the function 1 and there holds similarly:

PROPOSITION 5.6. *For $\eta > 0$, let $f \in L^2_\eta(\Sigma^+)$, $g^\pm \in L^2_\eta(\mathbb{R}^+)^2$ and $h \in H^{-1/2}(\gamma_0)$. If moreover $\eta < \pi/2$, then the problem*

$$L_{\mathrm{Neu}}(\psi) = (f; g^\pm; h)$$

has a unique solution $\psi = \varphi + \delta\, t$ in $H^1_\eta(\Sigma^+) \oplus \mathrm{span}\{t\}$ with $\varphi \in H^1_\eta(\Sigma^+)$ and

$$\delta = \frac{1}{2\mu}\left(\int_{\Sigma^+} f(t,x_3)\, dt\, dx_3 - \int_{\mathbb{R}^+} \big(g^+(t) - g^-(t)\big)\, dt + \int_{-1}^{+1} h(x_3)\, dx_3\right). \tag{5.5}$$

We introduce the solution $\bar\varphi^{\mathrm{s}}_{\mathrm{Neu}}$ similarly as above, and using the second Green formula for the product $x_3\, \Delta\bar\varphi^{\mathrm{s}}_{\mathrm{Neu}}(t,x_3)$ on Σ_+ we prove:

LEMMA 5.7. *Let $\bar\varphi^{\mathrm{s}}_{\mathrm{Neu}} \in H^1_\eta(\Sigma^+)$ be the exponentially decaying solution of the problem*

$$L_{\mathrm{Neu}}(\bar\varphi^{\mathrm{s}}_{\mathrm{Neu}}) = (0; 0; 2\mu x_3)\,,$$

then it holds

$$\int_0^\infty \bar\varphi^{\mathrm{s}}_{\mathrm{Neu}}(t,1)\, dt = -\frac{2}{3}\,.$$

5.4. Elasticity on the half-strip. The problem (5.2)-(5.3) on the full strip has a polynomial kernel of dimension six generated by $\boldsymbol{Z}^{[1]}_\natural, \boldsymbol{Z}^{[3]}_\natural, \boldsymbol{Z}^{[4]}_\natural, \boldsymbol{Z}^{[5]}_\natural, \boldsymbol{Z}^{[7]}_\natural, \boldsymbol{Z}^{[8]}_\natural$, where the two components of $\boldsymbol{Z}^{[j]}_\natural$ are the first and third ones of $\boldsymbol{Z}^{[j]}$. In particular a basis of the 2D rigid motions is given by

$$\boldsymbol{Z}^{[1]}_\natural = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \boldsymbol{Z}^{[3]}_\natural = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \boldsymbol{Z}^{[4]}_\natural = \begin{pmatrix} -x_3 \\ t \end{pmatrix}.$$

5.4.1. Operator E_{Dir}. From [9, Proposition 4.12], we obtain that

PROPOSITION 5.8. *For $\eta > 0$, let $\mathfrak{f}_\natural \in L^2_\eta(\Sigma^+)^2$, $\mathfrak{g}^\pm_\natural \in L^2_\eta(\mathbb{R}^+)^4$ and $\mathfrak{h}_\natural \in H^{1/2}(\gamma_0)^2$. If moreover $\eta < \eta_0$, then the problem*

$$E_{\mathrm{Dir}}(\boldsymbol{\psi}) = (\mathfrak{f}_\natural; \mathfrak{g}^\pm_\natural; \mathfrak{h}_\natural)$$

has a unique solution in $H^1_\eta(\Sigma^+)^2 \oplus \mathrm{span}\{\boldsymbol{Z}^{[1]}_\natural, \boldsymbol{Z}^{[3]}_\natural, \boldsymbol{Z}^{[4]}_\natural\}$.

5.4.2. Other operators. Concerning the other operators E_{Mix1}, E_{Mix2} and E_{Free}, and in contrast to the case of E_{Dir}, they have a polynomial kernel generated by some of the $\boldsymbol{Z}^{[j]}_\natural$. Relying on the following duality relations (5.7) satisfied by the $\boldsymbol{Z}^{[j]}$, formulas for the coefficients in the asymptotics at infinity of the solutions can be obtained from integrations by parts.

LEMMA 5.9. *Let $\boldsymbol{T}^{(0)}$ denote the lateral inward traction operator $(T^{(0)}_t, T^{(0)}_s, T^{(0)}_3)$, see (4.2). With σ the permutation*

$$\begin{aligned} &\sigma(1) = 5, \quad \sigma(2) = 6, \quad \sigma(3) = 8, \quad \sigma(4) = 7,\\ &\sigma(5) = 1, \quad \sigma(6) = 2, \quad \sigma(7) = 4, \quad \sigma(8) = 3, \end{aligned}$$

the anti-symmetrized flux, which can be defined for any $L \in \mathbb{R}$ by

$$\underline{\Phi}(\boldsymbol{Z}^{[i]}, \boldsymbol{Z}^{[j]}) := \int_{-1}^{+1} \Big(\boldsymbol{T}^{(0)}(\boldsymbol{Z}^{[i]}) \cdot \boldsymbol{Z}^{[j]} - \boldsymbol{T}^{(0)}(\boldsymbol{Z}^{[j]}) \cdot \boldsymbol{Z}^{[i]} \Big)(L, x_3)\, dx_3 \tag{5.6}$$

is independent of L, compare [9, Lemma 3.1], and satisfies, for $i, j \in \{1, \cdots, 8\}$

$$\underline{\Phi}(\boldsymbol{Z}^{[i]}, \boldsymbol{Z}^{[j]}) = \bar{\gamma}_i\, \delta_{j\sigma(i)}, \tag{5.7}$$

with $\bar{\gamma}_i$ a non zero real number.

For $i = 2, 6$ we find again the simple relations on which rely Propositions 5.4 and 5.6. For the remaining values of i, the relations (5.7) apply to the bi-dimensional displacements $\boldsymbol{Z}_\natural^{[i]}$. Relying on (5.7) and integration by parts, we are able to present formulas for the coefficients in the asymptotics at infinity of the solutions to the problems concerning the operators E_{Mix1}, E_{Mix2} and E_{Free}.

Proposition 5.10. *For $\eta > 0$, let $\mathfrak{f}_\natural \in L^2_\eta(\Sigma^+)^2$, $\mathfrak{g}_\natural^\pm \in L^2_\eta(\mathbb{R}^+)^4$, $\mathfrak{h}_t \in H^{-1/2}(\gamma_0)$ and $\mathfrak{h}_3 \in H^{1/2}(\gamma_0)$. If moreover $\eta < \eta_0$, then the problem*

$$E_{\mathrm{Mix1}}(\boldsymbol{\psi}) = (\mathfrak{f}_\natural; \mathfrak{g}_\natural^\pm; \mathfrak{h}_\natural)$$

has a unique solution $\boldsymbol{\psi} = \boldsymbol{\varphi} + \delta_3 \boldsymbol{Z}_\natural^{[3]} + \delta_5 \boldsymbol{Z}_\natural^{[5]} + \delta_7 \boldsymbol{Z}_\natural^{[7]}$ with $\boldsymbol{\varphi} \in H^1_\eta(\Sigma^+)^2$ and

$$\bar{\gamma}_5 \delta_5 = \int_{\Sigma^+} f_t - \int_{\mathbb{R}^+} (g_t^+ - g_t^-) + \int_{-1}^{+1} \mathfrak{h}_t\,, \tag{5.8a}$$

$$\bar{\gamma}_7 \delta_7 = \int_{\Sigma^+} (-x_3 f_t + t f_3) + \int_{\mathbb{R}^+} \Big(g_t^+ + g_t^- - t(g_3^+ - g_3^-) \Big) - \int_{-1}^{+1} x_3 \mathfrak{h}_t\,, \tag{5.8b}$$

$$\bar{\gamma}_3 \delta_3 = \int_{\Sigma^+} \mathfrak{f}_\natural \cdot \boldsymbol{Z}_\natural^{[8]} - \int_{\mathbb{R}^+} \Big(\mathfrak{g}^+ \cdot \boldsymbol{Z}_\natural^{[8]}\big|_{\gamma^+} - \mathfrak{g}^- \cdot \boldsymbol{Z}_\natural^{[8]}\big|_{\gamma^-} \Big) + 6 \int_{-1}^{+1} \bar{p}_3\, \mathfrak{h}_t - \mu(\bar{p}_2 + \bar{p}_3') \mathfrak{h}_3. \tag{5.8c}$$

Proposition 5.11. *For $\eta > 0$, let $\mathfrak{f}_\natural \in L^2_\eta(\Sigma^+)^2$, $\mathfrak{g}_\natural^\pm \in L^2_\eta(\mathbb{R}^+)^4$, $\mathfrak{h}_t \in H^{1/2}(\gamma_0)$ and $\mathfrak{h}_3 \in H^{-1/2}(\gamma_0)$. If moreover $\eta < \eta_0$, then the problem*

$$E_{\mathrm{Mix2}}(\boldsymbol{\psi}) = (\mathfrak{f}_\natural; \mathfrak{g}_\natural^\pm; \mathfrak{h}_\natural)$$

has a unique solution $\boldsymbol{\psi} = \boldsymbol{\varphi} + \delta_1 \boldsymbol{Z}_\natural^{[1]} + \delta_4 \boldsymbol{Z}_\natural^{[4]} + \delta_8 \boldsymbol{Z}_\natural^{[8]}$ with $\boldsymbol{\varphi} \in H^1_\eta(\Sigma^+)^2$ and

$$\bar{\gamma}_8 \delta_8 = \int_{\Sigma^+} f_3 - \int_{\mathbb{R}^+} (g_3^+ - g_3^-) + \int_{-1}^{+1} \mathfrak{h}_3\,, \tag{5.9a}$$

$$\bar{\gamma}_1 \delta_1 = \int_{\Sigma^+} t f_t - \int_{\mathbb{R}^+} t(g_t^+ - g_t^-) - \int_{-1}^{+1} (\tilde{\lambda} + 2\mu) \mathfrak{h}_t - \frac{\tilde{\lambda}}{2\mu} \Big(\int_{\Sigma^+} x_3 f_3 - \int_{\mathbb{R}^+} (g_3^+ + g_3^-) + \int_{-1}^{+1} x_3 \mathfrak{h}_3 \Big), \tag{5.9b}$$

$$\bar{\gamma}_4 \delta_4 = \int_{\Sigma^+} \mathfrak{f}_\natural \cdot \boldsymbol{Z}_\natural^{[7]} - \int_{\mathbb{R}^+} \Big(\mathfrak{g}^+ \cdot \boldsymbol{Z}_\natural^{[7]} - \mathfrak{g}^- \cdot \boldsymbol{Z}_\natural^{[7]} \Big) + 2 \int_{-1}^{+1} \Big(\bar{p}_2 \mathfrak{h}_3 + (\tilde{\lambda} + 2\mu) x_3 \mathfrak{h}_t \Big). \tag{5.9c}$$

Proposition 5.12. *For $\eta > 0$, let $\mathfrak{f}_\natural \in L^2_\eta(\Sigma^+)^2$, $\mathfrak{g}_\natural^\pm \in L^2_\eta(\mathbb{R}^+)^4$ and $\mathfrak{h}_\natural \in H^{-1/2}(\gamma_0)^2$. If moreover $\eta < \eta_0$, then the problem*

$$E_{\mathrm{Free}}(\boldsymbol{\psi}) = (\mathfrak{f}_\natural; \mathfrak{g}_\natural^\pm; \mathfrak{h}_\natural)$$

has a unique solution $\boldsymbol{\psi} = \boldsymbol{\varphi} + \delta_5 \boldsymbol{Z}_\natural^{[5]} + \delta_7 \boldsymbol{Z}_\natural^{[7]} + \delta_8 \boldsymbol{Z}_\natural^{[8]}$ with $\boldsymbol{\varphi} \in H^1_\eta(\Sigma^+)^2$ and

$$\bar{\gamma}_5 \delta_5 = \int_{\Sigma^+} f_t - \int_{\mathbb{R}^+} (g_t^+ - g_t^-) + \int_{-1}^{+1} \mathfrak{h}_t\,, \tag{5.10a}$$

$$\bar{\gamma}_8 \delta_8 = \int_{\Sigma^+} f_3 - \int_{\mathbb{R}^+} (g_3^+ - g_3^-) + \int_{-1}^{+1} \mathfrak{h}_3\,, \tag{5.10b}$$

$$\bar{\gamma}_7 \delta_7 = \int_{\Sigma^+} (-x_3 f_t + t f_3) + \int_{\mathbb{R}^+} \big(g_t^+ + g_t^- - t(g_3^+ - g_3^-)\big) - \int_{-1}^{+1} x_3 \mathfrak{h}_t\,. \tag{5.10c}$$

6. Clamped plates.

6.1. Hard clamped plates: The first terms in the asymptotics. In [19, Ch. 16], Maz'ya, Nazarov & Plamenevskii prove estimates like (2.4) for isotropic clamped plates and in [8, 9], the analog of Theorem 2.2 is proved for monoclinic clamped plates.

Here we will show how the formulas relating to lateral boundary condition ① in Tables 2.1, 2.2, 2.4 and 2.5 can be derived.

From (4.16) it follows that boundary operators for the generators are the Dirichlet ones and that the four traces of $\boldsymbol{\zeta}^0$ are zero. We find again a fact known for long, *cf* [5, 13] for early reference.

Let us investigate $\boldsymbol{\zeta}^1$ and $\boldsymbol{\varphi}^1$ simultaneously. Condition (4.15) for $k = 1$ yields that $\zeta_3^1 = 0$, $\varphi_n^1 + \zeta_n^1 - x_3 \partial_n \zeta_3^1 = 0$ and $\varphi_s^1 + \zeta_s^1 - x_3 \partial_s \zeta_3^1 = 0$ on Γ_0. Moreover condition (4.15) for $k = 2$ with (4.17c) yields that $\varphi_3^1 + \zeta_3^2 + v_3^2 = 0$ on Γ_0.

Thus, the first profile $\boldsymbol{\varphi}^1(s) : (t, x_3) \mapsto \boldsymbol{\varphi}^1(t, s, x_3)$ has to solve for all $s \in \partial\omega$ — *cf* (4.23), the equation $\mathscr{B}_{①}(\boldsymbol{\varphi}^1(s)) = (0; 0; \mathfrak{h}^1(s))$ with the trace $\mathfrak{h}^1(s)$ equal to:

$$\mathfrak{h}_n^1(s) = -(\zeta_n^1 - x_3 \partial_n \zeta_3^1)(s), \quad \mathfrak{h}_s^1(s) = -(\zeta_s^1 - x_3 \partial_s \zeta_3^1)(s), \quad \mathfrak{h}_3^1(s) = -(\zeta_3^2 + v_3^2)(s).$$

Note that the unknowns are the profile $\boldsymbol{\varphi}^1$ and the traces of ζ_n^1, ζ_s^1, $\partial_n \zeta_3^1$ and ζ_3^2.

Since $\mathscr{B}_{①}$ splits into the direct sum $E_{\mathrm{Dir}} \oplus L_{\mathrm{Dir}}$, for each $s \in \partial\omega$ (fixed now, thus omitted),

- φ_s^1 is solution of the Poisson problem

$$L_{\mathrm{Dir}}(\varphi_s^1) = (0; 0; \mathfrak{h}_s^1), \tag{6.1}$$

- the couple $\boldsymbol{\varphi}_\natural^1$ is solution of the elasticity system

$$E_{\mathrm{Dir}}(\boldsymbol{\varphi}_\natural^1) = (0; 0; \mathfrak{h}_\natural^1). \tag{6.2}$$

We have to find the conditions on $\boldsymbol{\zeta}^1$ so that equations (6.1) and (6.2) admit exponentially decreasing solutions.

Concerning the Poisson problem, Proposition 5.4 yields that (6.1) admits an exponentially decreasing solution if the coefficient (5.4) is zero, i.e. if $\int_{-1}^{+1} \mathfrak{h}_s^1 = 0$. With the above expression of $\mathfrak{h}_s^1$, this yields that $\zeta_s^1 = 0$ on $\partial\omega$. Since we already found that $\zeta_3^1 = 0$ on $\partial\omega$, we obtain that $\mathfrak{h}_s^1 \equiv 0$, thus $\varphi_s^1 = 0$.

Concerning the Lamé problem, Proposition 5.8 yields a solution for (6.2) in $H^1_\eta(\Sigma^+)^2 \oplus \operatorname{span}\{\boldsymbol{Z}_\natural^{[1]}, \boldsymbol{Z}_\natural^{[3]}, \boldsymbol{Z}_\natural^{[4]}\}$. We first recall that, — *cf* (3.3)-(3.4)

$$v_3^2(x_*, x_3) = \bar p_1(x_3)\, \operatorname{div}_* \boldsymbol{\zeta}_*^0(x_*) + \bar p_2(x_3)\, \Delta_* \zeta_3^0(x_*)\,. \tag{6.3}$$

Let $\bar{\boldsymbol{\psi}}_\natural^{\mathrm{m}}$ be the solution in $H^1_\eta(\Sigma^+)^2 \oplus \operatorname{span}\{\boldsymbol{Z}_\natural^{[1]}, \boldsymbol{Z}_\natural^{[3]}, \boldsymbol{Z}_\natural^{[4]}\}$ of

$$E_{\mathrm{Dir}}(\bar{\boldsymbol{\psi}}_\natural^{\mathrm{m}}) = (0; 0; 0, -\bar p_1). \tag{6.4}$$

Since the right hand side of (6.4) has the parities of a membrane mode (the first component is even and the second odd with respect to x_3), the symmetries of the isotropic elasticity system yield that $\bar\psi_t^{\mathrm{m}}$ is even and $\bar\psi_3^{\mathrm{m}}$ odd. Thus the asymptotic behavior as $t \to \infty$ has the same parities: only $\boldsymbol{Z}_\natural^{[1]}$ is convenient.

Hence there exists a unique coefficient $c_1^{①}$ such that $\bar{\boldsymbol{\psi}}_\natural^{\mathrm{m}}$ splits into

$$\bar{\boldsymbol{\psi}}_\natural^{\mathrm{m}} = \bar{\boldsymbol{\varphi}}_\natural^{\mathrm{m}} + c_1^{①} \boldsymbol{Z}_\natural^{[1]} \quad \text{with } \bar{\boldsymbol{\varphi}}_\natural^{\mathrm{m}} \text{ exponentially decreasing.} \tag{6.5}$$

Similarly, let $\bar{\boldsymbol{\psi}}_\natural^{\mathrm{b}}$ be the solution in $H^1_\eta(\Sigma^+)^2 \oplus \operatorname{span}\{\boldsymbol{Z}_\natural^{[1]}, \boldsymbol{Z}_\natural^{[3]}, \boldsymbol{Z}_\natural^{[4]}\}$ of

$$E_{\mathrm{Dir}}(\bar{\boldsymbol{\psi}}_\natural^{\mathrm{b}}) = (0; 0; 0, -\bar p_2). \tag{6.6}$$

Since the right hand side of (6.6) has the parities of a bending mode, the symmetries of the problem yield that $\bar\psi_t^{\mathrm{b}}$ is odd and $\bar\psi_3^{\mathrm{b}}$ even with respect to x_3. Thus only $\boldsymbol{Z}_\natural^{[3]}$ and $\boldsymbol{Z}_\natural^{[4]}$ are present in the asymptotics at infinity of $\bar{\boldsymbol{\psi}}_\natural^{\mathrm{b}}$.

Hence there exist unique coefficients $c_3^{①}$ and $c_4^{①}$ such that $\bar{\boldsymbol{\psi}}_\natural^{\mathrm{b}}$ splits into

$$\bar{\boldsymbol{\psi}}_\natural^{\mathrm{b}} = \bar{\boldsymbol{\varphi}}_\natural^{\mathrm{b}} + c_3^{①} \boldsymbol{Z}_\natural^{[3]} + c_4^{①} \boldsymbol{Z}_\natural^{[4]} \quad \text{with } \bar{\boldsymbol{\varphi}}_\natural^{\mathrm{b}} \text{ exponentially decreasing.} \tag{6.7}$$

Then $\boldsymbol{\psi}_\natural^1$ defined as

$$\boldsymbol{\psi}_\natural^1(t, s, x_3) = \operatorname{div}_* \boldsymbol{\zeta}_*^0(s)\; \bar{\boldsymbol{\psi}}_\natural^{\mathrm{m}}(t, x_3) + \Delta_* \zeta_3^0(s)\; \bar{\boldsymbol{\psi}}_\natural^{\mathrm{b}}(t, x_3)$$

is solution for each $s \in \partial\omega$ of — *cf* (6.3), (6.4) and (6.6):

$$E_{\mathrm{Dir}}(\boldsymbol{\psi}_\natural^1) = (0; 0; 0, -v_3^2). \tag{6.8}$$

Thus, if we have for each $s \in \partial\omega$, *cf* (6.5) and (6.7)

$$\begin{pmatrix} \zeta_n^1(s) - x_3 \partial_n \zeta_3^1(s) \\ \zeta_3^2(s) \end{pmatrix} \quad = \quad \operatorname{div}_* \boldsymbol{\zeta}_*^0(s)\, c_1^{①} \boldsymbol{Z}_\natural^{[1]}\big|_{\gamma_0} + \Delta_* \zeta_3^0(s)\, (c_3^{①} \boldsymbol{Z}_\natural^{[3]} + c_4^{①} \boldsymbol{Z}_\natural^{[4]})\big|_{\gamma_0}$$

i.e.

$$\begin{pmatrix} \zeta_n^1(s) - x_3 \partial_n \zeta_3^1(s) \\ \zeta_3^2(s) \end{pmatrix} \quad = \quad \begin{pmatrix} \operatorname{div}_* \boldsymbol{\zeta}_*^0(s)\, c_1^{①} - x_3 \Delta_* \zeta_3^0(s)\, c_4^{①} \\ \Delta_* \zeta_3^0(s)\, c_3^{①} \end{pmatrix} \tag{6.9}$$

then $\boldsymbol{\varphi}_\natural^1$ defined as

$$\boldsymbol{\varphi}_\natural^1(t, s, x_3) = \operatorname{div}_* \boldsymbol{\zeta}_*^0(s)\; \bar{\boldsymbol{\varphi}}_\natural^{\mathrm{m}}(t, x_3) + \Delta_* \zeta_3^0(s)\; \bar{\boldsymbol{\varphi}}_\natural^{\mathrm{b}}(t, x_3) \tag{6.10}$$

is solution of $E_{\mathrm{Dir}}(\boldsymbol{\varphi}_\natural^1(s)) = (0; 0; \mathfrak{h}_\natural^1(s))$, see (6.2). Thus we have obtained all the results relating to $\boldsymbol{\zeta}^1$ and $\boldsymbol{\varphi}^1$.

6.2. The non-zero coupling constants. There holds

LEMMA 6.1. *The coefficients* $c_1^{①}$ *and* $c_4^{①}$ *are non-zero.*

Let us prove first that $c_4^{①}$ is not zero. Let us denote by $\boldsymbol{Z}_\natural$ the polynomial displacement $\frac{1}{2}\boldsymbol{Z}_\natural^{[7]}$. Thus $\boldsymbol{Z}_\natural$ satisfies:

$$E_{\mathrm{Dir}}(\boldsymbol{Z}_\natural) = (0; 0; 0, \bar{p}_2). \tag{6.11}$$

So, (6.11) joined with (6.6)-(6.7) yields that

$$\boldsymbol{K} := \boldsymbol{Z}_\natural + \bar{\boldsymbol{\varphi}}_\natural^{\mathrm{b}} + c_3^{①}\boldsymbol{Z}_\natural^{[3]} + c_4^{①}\boldsymbol{Z}_\natural^{[4]} \quad \in \quad \ker E_{\mathrm{Dir}}\,.$$

The proof proceeds by computation about the 'flux', see also (5.6):

$$\Phi_{t=t_0}(\boldsymbol{u}\,|\,\boldsymbol{v}) := \int_{-1}^{+1} \boldsymbol{T}_\natural^{(0)}(\boldsymbol{u})(t_0, x_3)\cdot \boldsymbol{v}(t_0, x_3)\,dx_3\,. \tag{6.12}$$

We have:

$$\boldsymbol{T}_\natural^{(0)}(\boldsymbol{Z}_\natural) = \begin{pmatrix} -4\,\frac{\mu(\lambda+\mu)}{\lambda+2\mu}\,x_3 \\ 0 \end{pmatrix}.$$

Thus

$$\Phi_{t=0}(\boldsymbol{Z}_\natural\,|\,c_3^{①}\boldsymbol{Z}_\natural^{[3]} + c_4^{①}\boldsymbol{Z}_\natural^{[4]}) = \frac{8}{3}\,\frac{\mu(\lambda+\mu)}{\lambda+2\mu}\,c_4^{①}. \tag{6.13}$$

We are going to prove that, *cf* (6.7):

$$\Phi_{t=0}(\boldsymbol{Z}_\natural\,|\,c_3^{①}\boldsymbol{Z}_\natural^{[3]} + c_4^{①}\boldsymbol{Z}_\natural^{[4]}) = \Phi_{t=0}(\boldsymbol{K}\,|\,c_3^{①}\boldsymbol{Z}_\natural^{[3]} + c_4^{①}\boldsymbol{Z}_\natural^{[4]} + \bar{\boldsymbol{\varphi}}_\natural^{\mathrm{b}}) \tag{6.14}$$

and that

$$\Phi_{t=0}(\boldsymbol{K}\,|\,c_3^{①}\boldsymbol{Z}_\natural^{[3]} + c_4^{①}\boldsymbol{Z}_\natural^{[4]} + \bar{\boldsymbol{\varphi}}_\natural^{\mathrm{b}}) > 0. \tag{6.15}$$

The fact that $c_4^{①} > 0$ is clearly a consequence of (6.13)-(6.15).

In order to prove (6.14) and (6.15), we abbreviate the notations by

$$c_3^{①}\boldsymbol{Z}_\natural^{[3]} + c_4^{①}\boldsymbol{Z}_\natural^{[4]} := \boldsymbol{R} \quad \text{and} \quad \boldsymbol{\varphi} := \bar{\boldsymbol{\varphi}}_\natural^{\mathrm{b}}\,.$$

Proof. Of (6.14). We want to prove that $\Phi_{t=0}(\boldsymbol{Z}_\natural\,|\,\boldsymbol{R}) = \Phi_{t=0}(\boldsymbol{K}\,|\,\boldsymbol{R}+\boldsymbol{\varphi})$. Indeed, integrating by parts on the rectangle $\Sigma_L = (0, L)\times(-1, 1)$ we obtain

$$\begin{aligned}
&\int_{-1}^{+1}\Big[\boldsymbol{T}_\natural^{(0)}(\boldsymbol{K})\cdot(\boldsymbol{R}+\boldsymbol{\varphi}) - \boldsymbol{K}\cdot\boldsymbol{T}_\natural^{(0)}(\boldsymbol{R}+\boldsymbol{\varphi})\Big](0, x_3)\,dx_3 \,- \\
&\quad\int_{-1}^{+1}\Big[\boldsymbol{T}_\natural^{(0)}(\boldsymbol{K})\cdot(\boldsymbol{R}+\boldsymbol{\varphi}) - \boldsymbol{K}\cdot\boldsymbol{T}_\natural^{(0)}(\boldsymbol{R}+\boldsymbol{\varphi})\Big](L, x_3)\,dx_3 \;= \\
&\qquad\qquad\int_0^L\Big[\mathscr{G}_\natural^{(0)}(\boldsymbol{K})\cdot(\boldsymbol{R}+\boldsymbol{\varphi}) - \boldsymbol{K}\cdot\mathscr{G}_\natural^{(0)}(\boldsymbol{R}+\boldsymbol{\varphi})\Big](t, 1)\,dt \,- \\
&\qquad\qquad\quad\int_0^L\Big[\mathscr{G}_\natural^{(0)}(\boldsymbol{K})\cdot(\boldsymbol{R}+\boldsymbol{\varphi}) - \boldsymbol{K}\cdot\mathscr{G}_\natural^{(0)}(\boldsymbol{R}+\boldsymbol{\varphi})\Big](t, -1)\,dt \,- \\
&\qquad\qquad\qquad\qquad\int_{\Sigma_L}\mathscr{B}_\natural^{(0)}(\boldsymbol{K})\cdot(\boldsymbol{R}+\boldsymbol{\varphi}) - \boldsymbol{K}\cdot\mathscr{B}_\natural^{(0)}(\boldsymbol{R}+\boldsymbol{\varphi})\,.
\end{aligned}$$

As $\mathscr{B}^{(0)}_\natural(\boldsymbol{K}) = \mathscr{B}^{(0)}_\natural(\boldsymbol{Z}_\natural) = 0$ and $\mathscr{G}^{(0)}_\natural(\boldsymbol{K}) = \mathscr{G}^{(0)}_\natural(\boldsymbol{Z}_\natural) = 0$, the above right hand side is zero. Therefore

$$\Phi_{t=0}(\boldsymbol{K} \,|\, \boldsymbol{R}+\boldsymbol{\varphi}) = \Phi_{t=L}(\boldsymbol{K} \,|\, \boldsymbol{R}+\boldsymbol{\varphi}) - \int_{-1}^{+1} \boldsymbol{K}(L,x_3)\cdot \boldsymbol{T}^{(0)}_\natural(\boldsymbol{R}+\boldsymbol{\varphi})(L,x_3)\,dx_3.$$

Since $\boldsymbol{T}^{(0)}_\natural(\boldsymbol{R}) = 0$ ($\boldsymbol{R}$ is a rigid displacement) and since $\boldsymbol{\varphi}$ is exponentially decreasing, we deduce from the identity above that, for all $0<\eta<\eta_0$

$$\Phi_{t=0}(\boldsymbol{K} \,|\, \boldsymbol{R}+\boldsymbol{\varphi}) = \Phi_{t=L}(\boldsymbol{Z}_\natural \,|\, \boldsymbol{R}) + \mathcal{O}(e^{-\eta L}).$$

But for all L, we have the conservation of the flux against rigid displacements

$$\Phi_{t=L}(\boldsymbol{Z}_\natural \,|\, \boldsymbol{R}) = \Phi_{t=0}(\boldsymbol{Z}_\natural \,|\, \boldsymbol{R}),$$

whence the result. □

Proof. Of (6.15). We want to prove that $\Phi_{t=0}(\boldsymbol{K} \,|\, \boldsymbol{R}+\boldsymbol{\varphi}) > 0$. To see it, notice that, since $\boldsymbol{Z}_\natural\big|_{t=0} = -(\boldsymbol{R}+\boldsymbol{\varphi})\big|_{t=0}$ and since we easily check the equality $\Phi_{t=0}(\boldsymbol{Z}_\natural \,|\, \boldsymbol{Z}_\natural) = 0$, we have

$$\begin{aligned}
\Phi_{t=0}(\boldsymbol{K} \,|\, \boldsymbol{R}+\boldsymbol{\varphi}) &= \Phi_{t=0}(\boldsymbol{Z}_\natural \,|\, \boldsymbol{R}+\boldsymbol{\varphi}) + \Phi_{t=0}(\boldsymbol{R}+\boldsymbol{\varphi} \,|\, \boldsymbol{R}+\boldsymbol{\varphi})\\
&= -\Phi_{t=0}(\boldsymbol{Z}_\natural \,|\, \boldsymbol{Z}_\natural) + \Phi_{t=0}(\boldsymbol{\varphi} \,|\, \boldsymbol{R}+\boldsymbol{\varphi})\\
&= \Phi_{t=L}(\boldsymbol{\varphi} \,|\, \boldsymbol{R}+\boldsymbol{\varphi}) + \int_{\Sigma_L} A\,e(\partial_t,\partial_3)(\boldsymbol{\varphi}) : e(\partial_t,\partial_3)(\boldsymbol{R}+\boldsymbol{\varphi})\\
&= \int_{\Sigma_L} A\,e(\partial_t,\partial_3)(\boldsymbol{\varphi}) : e(\partial_t,\partial_3)(\boldsymbol{\varphi}) + \mathcal{O}(e^{-\eta L}).
\end{aligned}$$

Since $\boldsymbol{Z}_\natural + \boldsymbol{R}$ is clearly not zero on $\{t=0\}$, then $\boldsymbol{\varphi}\not\equiv 0$. The result follows from the positivity of the elasticity matrix A. □

The positivity of $c_1^{①}$ can be proved analogously to that of $c_4^{①}$, taking into account that $\boldsymbol{Z}^{[5]}_\natural$ satisfies problem $E_{\mathrm{Dir}}(\boldsymbol{Z}^{[5]}_\natural) = (0;0;0,\bar p_1)$, thus

$$\boldsymbol{K}^{\mathrm{m}} := \boldsymbol{Z}^{[5]}_\natural + \bar{\boldsymbol{\varphi}}^{\mathrm{m}}_\natural + c_1^{①}\boldsymbol{Z}^{[1]}_\natural \quad\in\quad \ker E_{\mathrm{Dir}}$$

and that moreover there hold

$$\boldsymbol{T}^{(0)}_\natural(\boldsymbol{Z}^{[5]}_\natural) = \begin{pmatrix} 4\,\frac{\mu(\lambda+\mu)}{\lambda+2\mu} \\ 0 \end{pmatrix} \quad\text{and}\quad \Phi_{t=0}(\boldsymbol{Z}^{[5]}_\natural \,|\, c_1^{①}\boldsymbol{Z}^{[1]}_\natural) = \frac{8\mu(\lambda+\mu)}{\lambda+2\mu}\,c_1^{①}.$$

6.3. Soft clamped plates: The first terms in the asymptotics. We have now to take care of the space $\mathcal{R}_{②}$, which is the space of rigid motions $\boldsymbol{v}$ satisfying the soft clamped plate conditions, i.e. v_n and $v_3 = 0$ on the lateral boundary Γ_0. If the mean surface ω is not a disk or an annulus, $\mathcal{R}_{②}$ is reduced to $\{0\}$. If ω is a disk or an annulus, that we may suppose centered in 0, $\mathcal{R}_{②}$ is one-dimensional, generated by the in-plane rotation $(x_2,-x_1,0)$ and the orthogonality condition (1.11) ensuring uniqueness can be transcribed in Ω into $\int_\Omega \boldsymbol{u}_*(\varepsilon)\cdot(x_2,-x_1)^\top = 0$.

Thus, in this situation, the compatibility conditions on ω for the membrane problems (2.13a) has to be checked and the coherence with the orthogonality condition (1.11) has to be realized by an orthogonality condition for the $\boldsymbol{\zeta}^k_*$ in ω. We refer to [11, §6] for details.

The behavior of the boundary layer terms is very similar to the hard clamped case because the boundary conditions involving the components $\boldsymbol{\varphi}_\natural$ are Dirichlet's as in ①, the only change concerns the lateral component φ_s, which is uncoupled from the previous ones, and subject now to lateral Neumann conditions instead of Dirichlet's.

6.3.1. The traces of $\boldsymbol{\zeta}^0$. Solving recursively equations (4.13)-(4.14), (4.15) and (4.18), we find first the Dirichlet traces at the order zero: $\zeta_n^0 - x_3\partial_n\zeta_3^0$ and ζ_3^0 are zero on $\partial\omega$. Thus, the Dirichlet conditions concerning $\boldsymbol{\zeta}^0$ are obtained.

The terms T_s^0 and T_s^1 are always zero. Next, condition $T_s^2 = 0$ yields, *cf* (4.22b)

$$T_s^{(0)}(\boldsymbol{\varphi}^1) = -T_s^{\mathrm{m}}(\boldsymbol{\zeta}_*^0) + 2\mu x_3\,(\partial_n + \tfrac{1}{R})\partial_s\zeta_3^0.$$

Taking account of the already known Dirichlet conditions for ζ_3^0, we obtain that φ_s^1 solves the Laplace Neumann problem on the half-strip:

$$L_{\mathrm{Neu}}(\varphi_s^1) = (0;0;-T_s^{\mathrm{m}}(\boldsymbol{\zeta}_*^0)). \tag{6.16}$$

Since, for each fixed $s \in \partial\omega$, $T_s^{\mathrm{m}}(\boldsymbol{\zeta}_*^0)$ is a constant, Proposition 5.6 yields that the only exponentially decreasing solution is $\varphi_s^1 \equiv 0$ obtained with $T_s^{\mathrm{m}}(\boldsymbol{\zeta}_*^0) = 0$ on $\partial\omega$. Then $\boldsymbol{\zeta}^0$ satisfies zero boundary conditions according to Table 2.1.

6.3.2. The traces of $\boldsymbol{\zeta}^1$. The equations (4.15) for $k = 1$ and for $k = 2$ yield the same condition as in case 1 for the trace of ζ_3^1 which must vanish, and the same equations (6.2) linking the couple $\boldsymbol{\varphi}_\natural^1$ and the traces of ζ_n^1, $\partial_n\zeta_3^1$, ζ_3^2. Thus, the result concerning these traces is the same for the hard and soft clamped situations.

As a consequence the coefficients $c_1^{(2)}$ and $c_4^{(2)}$ are equal to their homologues $c_1^{(1)}$ and $c_4^{(1)}$ for the hard clamped plate.

Concerning the tangential component, the condition $T_s^3 = 0$ yields, *cf* (4.22b)

$$T_s^{(0)}(\boldsymbol{\varphi}^2) = -T_s^{\mathrm{m}}(\boldsymbol{\zeta}_*^1) + 2\mu x_3\,(\partial_n + \tfrac{1}{R})\partial_s\zeta_3^1 - T_s^{(1)}(\boldsymbol{\varphi}^1).$$

Taking into account the already known trace condition $\zeta_3^1 = 0$, equation (4.23) leads to the following Neumann problem for the lateral part φ_s^2

$$L_{\mathrm{Neu}}(\varphi_s^2) = \Big(-\,(\mathscr{B}^{(1)}\boldsymbol{\varphi}^1)_s\,;-\,(\mathscr{G}^{(1)}\boldsymbol{\varphi}^1)_s\,;-\,T_s^{\mathrm{m}}(\boldsymbol{\zeta}_*^1) + 2\mu x_3\partial_{sn}\zeta_3^1 - T_s^{(1)}(\boldsymbol{\varphi}^1)\Big). \tag{6.17}$$

Proposition 5.6 yields that φ_s^2 is exponentially decreasing if and only if

$$\begin{aligned} T_s^{\mathrm{m}}(\boldsymbol{\zeta}_*^1) = -\frac{1}{2}\Bigg(\int_{\Sigma_+} &(\mathscr{B}^{(1)}\boldsymbol{\varphi}^1)_s(t,x_3)\;dt\,dx_3 \\ &- \int_{\mathbb{R}_+}\Big((\mathscr{G}^{(1)}\boldsymbol{\varphi}^1)_s(t,1) - (\mathscr{G}^{(1)}\boldsymbol{\varphi}^1)_s(t,-1)\Big)\,dt \\ &\qquad + \int_{-1}^{+1} T_s^{(1)}(\boldsymbol{\varphi}^1)(0,x_3) - 2\mu x_3\partial_{sn}\zeta_3^1(0)\;dx_3\Bigg). \end{aligned} \tag{6.18}$$

Since $\varphi_s^1 = 0$, the terms involved in (6.18) reduce to

$$(\mathscr{B}^{(1)}\boldsymbol{\varphi}^1)_s = (\lambda+\mu)\partial_s(\partial_t\varphi_t^1 + \partial_3\varphi_3^1),\quad (\mathscr{G}^{(1)}\boldsymbol{\varphi}^1)_s = \mu\,\partial_s\varphi_3^1,\quad T_s^{(1)}(\boldsymbol{\varphi}^1) = \mu\,\partial_s\varphi_t^1.$$

Since only the *even* terms in x_3 contribute to the integrals in (6.18) we see that we have only to take into consideration the membrane part of $\boldsymbol{\varphi}_\natural^1$, which is equal to $\mathrm{div}_*\,\boldsymbol{\zeta}_*^0(s)\;\bar{\boldsymbol{\varphi}}_\natural^{\mathrm{m}}(t,x_3)$, *cf* (6.10). Thus $T_s^{\mathrm{m}}(\boldsymbol{\zeta}_*^1) = c_2^{(2)}\,\partial_s\,\mathrm{div}_*\,\boldsymbol{\zeta}_*^0$, with $-\frac{2}{\mu}\,c_2^{(2)}$ equal to

$$\frac{\lambda+\mu}{\mu}\int_{\Sigma_+}(\partial_t\bar\varphi_t^{\mathrm{m}} + \partial_3\bar\varphi_3^{\mathrm{m}})\;dt\,dx_3 - \int_{\mathbb{R}_+}\Big(\bar\varphi_3^{\mathrm{m}}(t,1) - \bar\varphi_3^{\mathrm{m}}(t,-1)\Big)\,dt + \int_{-1}^{+1}\bar\varphi_t^{\mathrm{m}}(0,x_3)\;dx_3.$$

Formulas of Table 2.2 concerning case 2 are completely proved.

6.3.3. Recursivity. It can be proved like in [8], see also [11, §6].

7. Simply supported plates. The space of rigid motions $\mathcal{R}_{③}$ is reduced to $\{0\}$, whereas $\mathcal{R}_{④}$ is three-dimensional and spanned by the in-plane rigid motions. Here we only present the analysis for the hard simply supported plate. The main feature of the analysis of the soft simply supported plate is the treatment of compatibility conditions: we refer to [11, §8] for this.

7.1. Hard simple support: The traces of $\boldsymbol{\zeta}^0$. According to (4.15), $D_3^0 = 0$ yields $\zeta_3^0 = 0$ on $\partial\omega$, then $D_s^0 = 0$ is equivalent to $\zeta_s^0 = 0$ on $\partial\omega$. Next, $D_3^1 = 0$ yields $\zeta_3^1 = 0$ on $\partial\omega$, and $D_s^1 = 0$ provides the equation $L_{\mathrm{Dir}}(\varphi_s^1) = (0;0;-\zeta_s^1)$. Then Proposition 5.4 yields that the only exponentially decreasing solution is $\varphi_s^1 \equiv 0$ obtained with $\zeta_s^1 = 0$ on $\partial\omega$.

Conditions $T_n^2 = 0$, *cf* (4.22b), and $D_3^2 = 0$ yield that $\boldsymbol{\varphi}_\natural^1$ has to solve

$$E_{\mathrm{Mix1}}(\boldsymbol{\varphi}_\natural^1) = \Big(0;\ 0;\ -T_n^{\mathrm{m}}(\boldsymbol{\zeta}_*^0) + x_3\,M_n(\zeta_3^0),\ -(\zeta_3^2 + v_3^2)\Big). \tag{7.1}$$

With formulas (5.8) we can compute the three coefficients δ_3, δ_5 and δ_7, and determine conditions on $T_n^{\mathrm{m}}(\boldsymbol{\zeta}_*^0)$, $M_n(\zeta_3^0)$ and ζ_3^2 so that these three coefficients are zero, ensuring that $\boldsymbol{\varphi}_\natural^1$ is exponentially decaying. We have

$$\bar\gamma_5\,\delta_5 = \int_{-1}^{+1} -T_n^{\mathrm{m}}(\boldsymbol{\zeta}_*^0) + x_3\,M_n(\zeta_3^0)\,dx_3 \tag{7.2a}$$

$$\bar\gamma_7\,\delta_7 = \int_{-1}^{+1} x_3\,T_n^{\mathrm{m}}(\boldsymbol{\zeta}_*^0) - x_3^2\,M_n(\zeta_3^0)\,dx_3 \tag{7.2b}$$

$$\bar\gamma_3\,\delta_3 = \int_{-1}^{+1} 6\bar p_3(-T_n^{\mathrm{m}}(\boldsymbol{\zeta}_*^0) + x_3\,M_n(\zeta_3^0)) + 6\mu(\bar p_2 + \bar p_3{}')(\zeta_3^2 + v_3^2)\,dx_3. \tag{7.2c}$$

With (7.2a) and (7.2b), the conditions $\delta_5 = 0$ and $\delta_7 = 0$ give immediately that $T_n^{\mathrm{m}}(\boldsymbol{\zeta}_*^0) = 0$ and $M_n(\zeta_3^0) = 0$ on $\partial\omega$ respectively. Then with the formula $v_3^2 = \bar p_1\,\mathrm{div}_*\,\boldsymbol{\zeta}_*^0 + \bar p_2\Delta_*\zeta_3^0$ we can compute from (7.2c)

$$\bar\gamma_3\,\delta_3 = -4(\tilde\lambda + 2\mu)\Big(\zeta_3^2 - \frac{\tilde\lambda}{30\mu}\,\Delta_*\zeta_3^0\Big),$$

whence the relation $30\mu\,\zeta_3^2 = \tilde\lambda\,\Delta_*\zeta_3^0$ on $\partial\omega$ ensuring the existence of a unique exponentially decreasing profile solution of (7.1).

But we have on $\partial\omega$

$$T_n^{\mathrm{m}}(\boldsymbol{\zeta}_*) = (\tilde\lambda + 2\mu)\,\mathrm{div}_*\,\boldsymbol{\zeta}_* + 2\mu(\kappa\,\zeta_n - \partial_s\zeta_s) \tag{7.3a}$$

$$M_n(\zeta_3) = (\tilde\lambda + 2\mu)\Delta_*\zeta_3 + 2\mu(\kappa\,\partial_n\zeta_3 - \partial_{ss}\zeta_3). \tag{7.3b}$$

Since ζ_s^0 and ζ_3^0 are zero on $\partial\omega$, then $\partial_s\zeta_s^0$ and $\partial_{ss}\zeta_3^0$ are also zero and since $T_n^{\mathrm{m}}(\boldsymbol{\zeta}_*^0) = 0$ and $M_n(\zeta_3^0) = 0$ we deduce from (7.3) the relations

$$\mathrm{div}_*\,\boldsymbol{\zeta}_*^0 = -\frac{2\mu}{\tilde\lambda + 2\mu}\,\kappa\,\zeta_n^0 \quad\text{and}\quad \Delta_*\zeta_3^0 = -\frac{2\mu}{\tilde\lambda + 2\mu}\,\kappa\,\partial_n\zeta_3^0. \tag{7.4}$$

Therefore, with $\bar{\boldsymbol{\varphi}}_\natural^{\mathrm{m}}$ the solution of $E_{\mathrm{Mix1}}(\bar{\boldsymbol{\varphi}}_\natural^{\mathrm{m}}) = \big(0;\ 0;\ 0,\ \frac{2\mu}{\tilde\lambda+2\mu}\,\bar p_1\big)$, and with $\bar{\boldsymbol{\varphi}}_\natural^{\mathrm{b}}$ the solution of $E_{\mathrm{Mix1}}(\bar{\boldsymbol{\varphi}}_\natural^{\mathrm{b}}) = \big(0;\ 0;\ 0,\ \frac{2\mu}{\tilde\lambda+2\mu}\,(\frac{\tilde\lambda}{30\mu} + \bar p_2)\big)$, we obtain the expression in Table 2.5 of the first boundary layer term.

7.2. The traces of $\boldsymbol{\zeta}^1$. The next relations are deduced from $T_n^3 = 0$ and $D_3^3 = 0$: $\boldsymbol{\varphi}_\natural^2$ has to solve

$$-E_{\mathrm{Mix1}}(\boldsymbol{\varphi}_\natural^2) = \Big((\mathscr{B}^{(1)}\boldsymbol{\varphi}^1)_\natural \ ; \ (\mathscr{G}^{(1)}\boldsymbol{\varphi}^1)_\natural \ ; \ T_t^{(1)}(\boldsymbol{\varphi}^1) + T_n^{\mathrm{m}}(\boldsymbol{\zeta}_*^1) - x_3\, M_n(\zeta_3^1),\ \zeta_3^3 + v_3^3 \Big).$$

Since $\varphi_s^1 = 0$, the terms in the right hand side reduce to

$$(\mathscr{B}^{(1)}\boldsymbol{\varphi}^1)_t = -(\lambda + 2\mu)\,\kappa\,\partial_t \varphi_t^1, \quad (\mathscr{G}^{(1)}\boldsymbol{\varphi}^1)_t = 0, \quad T_t^{(1)}(\boldsymbol{\varphi}^1) = -\lambda\,\kappa\,\varphi_t^1$$

The cancellation of the coefficients δ_5, δ_7 and δ_3, *cf* (7.2) is ensured by relations determining $T_n^{\mathrm{m}}(\boldsymbol{\zeta}_*^1)$, $M_n(\zeta_3^1)$ and ζ_3^3. In particular we have

$$\begin{aligned} T_n^{\mathrm{m}}(\boldsymbol{\zeta}_*^1) = -\frac{1}{2} \Bigg(\int_{\Sigma_+} & (\mathscr{B}^{(1)}\boldsymbol{\varphi}^1)_t(t, x_3)\; dt\, dx_3 \\ & - \int_{\mathbb{R}_+} \Big((\mathscr{G}^{(1)}\boldsymbol{\varphi}^1)_t(t, 1) - (\mathscr{G}^{(1)}\boldsymbol{\varphi}^1)_t(t, -1) \Big)\, dt \\ & \qquad + \int_{-1}^{+1} T_t^{(1)}(\boldsymbol{\varphi}^1)(0, x_3) - x_3 M_n(\zeta_3^1)(0)\; dx_3 \Bigg). \end{aligned}$$

Combining with the already known expression for $\boldsymbol{\varphi}^1$, we obtain the formula of Table 2.2 for $T_n^{\mathrm{m}}(\boldsymbol{\zeta}_*^1)$. The trace $M_n(\zeta_3^1)$ is determined similarly.

8. Sliding edge. Lateral condition ⑥ is the other one, with ③, which allows a reflexion across the boundary in any region $\mathcal{V}$ where it is flat. If the support of the data avoids $\mathcal{V}$, there are *no boundary layer* terms and $\boldsymbol{u}(\varepsilon)$ can be expanded in a power series in $\mathcal{V}$. In the special case when ω is a rectangle (in principle forbidden here!) and if the support of the data avoids the lateral boundary, the solution can be extended outside Ω in both in-plane directions into a periodic solution in $\mathbb{R}^2 \times I$: this link is indicated by Paumier in [27] where the periodic boundary conditions are addressed.

If the mid-plane of the plate ω is not a disk or an annulus, then the space $\mathcal{R}_{⑥}$ is one-dimensional and spanned by the vertical translation $(0, 0, 1)$. But if ω is a disk or an annulus, that we may suppose centered in 0, then $\mathcal{R}_{⑥}$ is two-dimensional generated by the vertical translation $(0, 0, 1)$ and the in-plane rotation $(x_2, -x_1, 0)$. Here we will only treat the generic case.

8.1. The traces of $\boldsymbol{\zeta}^0$. As the Dirichlet trace D_n^0 is zero, we have $\zeta_n^0 = 0$ and $\partial_n \zeta_3^0 = 0$ on $\partial\omega$. We deduce the problem for $\boldsymbol{\varphi}_\natural^1$ from $D_n^1 = 0$ and $T_3^1 = 0$:

$$E_{\mathrm{Mix2}}(\boldsymbol{\varphi}_\natural^1) = (0; 0; -\zeta_n^1 + x_3 \partial_n \zeta_3^1, 0)\,.$$

Proposition 5.11 then yields the conditions $\zeta_n^1 = 0$ and $\partial_n \zeta_3^1 = 0$ on $\partial\omega$ and thus $\boldsymbol{\varphi}_\natural^1 \equiv 0$.

The condition $T_s^2 = 0$ yields that φ_s^1 has to satisfy

$$L_{\mathrm{Neu}}(\varphi_s^1) = (0; 0; -T_s^{\mathrm{m}}(\boldsymbol{\zeta}_*^0) + 2\mu x_3 (\partial_n + \kappa) \partial_s \zeta_3^0)\,. \tag{8.1}$$

Proposition 5.6 yields that $T_s^{\mathrm{m}}(\boldsymbol{\zeta}_*^0) = 0$ on $\partial\omega$. Combining with $\partial_n \zeta_3^0 = 0$ on $\partial\omega$, this solution is given by, *cf* Lemma 5.7,

$$\varphi_s^1 = \kappa\, \partial_s \zeta_3^0(s)\, \bar{\varphi}_{\mathrm{Neu}}^{\mathrm{s}}(t, x_3). \tag{8.2}$$

With $T^3_s = 0$ we obtain that φ^2_s has to satisfy

$$L_{\mathrm{Neu}}(\varphi^2_s) = \Big(-\,(\mathscr{B}^{(1)}\boldsymbol{\varphi}^1)_s\,;\;-\,(\mathscr{G}^{(1)}\boldsymbol{\varphi}^1)_s\,;\;\mathfrak{h}_s\Big)\,, \tag{8.3}$$

where the terms in the right hand side are given by, since $\boldsymbol{\varphi}^1_\natural = 0$:

$$(\mathscr{B}^{(1)}\boldsymbol{\varphi}^1)_s = \mu\kappa\Big(\partial_{tt}(t\,\varphi^1_s) + \partial_{33}(t\,\varphi^1_s) - \partial_t\varphi^1_s\Big)\,,\quad (\mathscr{G}^{(1)}\boldsymbol{\varphi}^1)_s = 0,$$
$$\mathfrak{h}_s = -\Big(2\mu\kappa\varphi^1_s + T^{\mathrm{m}}_s(\boldsymbol{\zeta}^1_*) - 2\mu x_3(\partial_n+\kappa)\partial_s\zeta^1_3\Big)\,.$$

With the help of Proposition 5.6 and the fact that φ^1_s is odd with respect to x_3 we deduce that $T^{\mathrm{m}}_s(\boldsymbol{\zeta}^1_*) = 0$ on $\partial\omega$. Taking into account relation (8.2) and the already known condition $\partial_n\zeta^1_3 = 0$ on $\partial\omega$, this solution is given by

$$\varphi^2_s = -\kappa^2\partial_s\zeta^0_3\,\bar\psi^{\mathrm{s}}_{\mathrm{Neu}} + \kappa\,\partial_s\zeta^1_3\,\bar\varphi^{\mathrm{s}}_{\mathrm{Neu}}\,, \tag{8.4}$$

where $\bar\psi^{\mathrm{s}}_{\mathrm{Neu}}$ is the (odd) exponentially decreasing solution of

$$L_{\mathrm{Neu}}(\bar\psi^{\mathrm{s}}_{\mathrm{Neu}}) = \mu\Big(\Delta(t\,\bar\varphi^{\mathrm{s}}_{\mathrm{Neu}}) - \partial_t\bar\varphi^{\mathrm{s}}_{\mathrm{Neu}}\,;\;0\,;\;2\bar\varphi^{\mathrm{s}}_{\mathrm{Neu}}\Big). \tag{8.5}$$

Conditions $D^2_n = 0$ and $T^2_3 = 0$ lead to the following problem for $\boldsymbol{\varphi}^2_\natural$

$$E_{\mathrm{Mix2}}(\boldsymbol{\varphi}^2_\natural) = \Big(-\,(\mathscr{B}^{(1)}\boldsymbol{\varphi}^1)_\natural\,;\;-\,(\mathscr{G}^{(1)}\boldsymbol{\varphi}^1)_\natural\,;\;\mathfrak{h}_t\,,\;\mathfrak{h}_3\Big)\,, \tag{8.6}$$

where the terms in the right hand side are given by

$$(\mathscr{B}^{(1)}\boldsymbol{\varphi}^1)_t = (\lambda+\mu)\,\partial_t\partial_s\varphi^1_s\,,\qquad (\mathscr{G}^{(1)}\boldsymbol{\varphi}^1)_t = 0, \tag{8.7a}$$
$$(\mathscr{B}^{(1)}\boldsymbol{\varphi}^1)_3 = (\lambda+\mu)\,\partial_3\partial_s\varphi^1_s,\qquad (\mathscr{G}^{(1)}\boldsymbol{\varphi}^1)_3 = \lambda\,\partial_s\varphi^1_s, \tag{8.7b}$$
$$\mathfrak{h}_t = -\left(\zeta^2_n - x_3\partial_n\zeta^2_3 + \bar p_2\,\partial_n\,\mathrm{div}_*\,\boldsymbol{\zeta}^0_* + \bar p_3\,\partial_n\Delta_*\zeta^0_3 + (G(\boldsymbol{f},\boldsymbol{g}^\pm))_n\right)\,, \tag{8.7c}$$
$$\mathfrak{h}_3 = -\mu\left((\bar p_2+\bar p_3')\,\partial_n\Delta_*\zeta^0_3 + \partial_3(G(\boldsymbol{f},\boldsymbol{g}^\pm))_n\right)\,. \tag{8.7d}$$

Combining with (8.2), the condition $\delta_8 = 0$ from Proposition 5.11 yields:

$$2\mu\partial_s(\partial_n+\kappa)\partial_s\zeta^0_3\int_{\mathbb{R}^+}\bar\varphi^{\mathrm{s}}_{\mathrm{Neu}}(t,1)\,dt = \int_{-1}^{+1}\mathfrak{h}_3\,dx_3\,.$$

Using the expressions of G_n, *cf* Definition 3.5, and of $\bar p_2$ and $\bar p_3$, *cf* (3.4), we derive

$$\int_{-1}^{+1}\mathfrak{h}_3\,dx_3 = -\left[-\frac{2}{3}(\tilde\lambda+2\mu)\partial_n\Delta_*\,\zeta^0_3 + \int_{-1}^{+1}x_3\,f_n\,dx_3 + g^+_n + g^-_n\right]\Bigg|_{\partial\omega}\,.$$

Then Lemma 5.7 yields

$$\frac{2}{3}\underbrace{\Big((\tilde\lambda+2\mu)\partial_n\Delta_*\,\zeta^0_3 + 2\mu\partial_s(\partial_n+\kappa)\partial_s\zeta^0_3\Big)}_{=\,N_n(\zeta^0_3)} = \left(\int_{-1}^{+1}x_3\,f_n\,dx_3 + g^+_n + g^-_n\right)\Bigg|_{\partial\omega}\,,$$

hence the condition $N_n(\zeta^0_3) = \frac{3}{2}\int_{-1}^{+1}x_3\,f_n\,dx_3 + g^+_n + g^-_n$ on $\partial\omega$. Then the compatibility condition for the solvability of problem (2.13b) for ζ^0_3 reads:

$$\int_\omega R^0_{\mathrm{b}}(x_*)\,dx_* - \int_{\partial\omega}\frac{3}{2}\Big(\int_{-1}^{+1}x_3\,f_n\,dx_3 + g^+_n + g^-_n\Big)(0,s)\,ds = 0\,. \tag{8.8}$$

With the help of the divergence theorem and formula (3.9), we can rewrite (8.8) as

$$\frac{3}{2}\int_\omega \left\{ \int_{-1}^{+1} f_3\,dx_3 + g_3^+ - g_3^- \right\} dx_* = 0\,,$$

which is nothing else than the compatibility condition (1.10), whence (8.8).

8.2. The traces of $\boldsymbol{\zeta}^1$. The only remaining boundary condition is that for $N_n(\zeta_3^1)$. Therefore we only consider the problem for $\boldsymbol{\varphi}^3_\natural$, which is deduced from $D^3_n = 0$ and $T^3_3 = 0$ and reads

$$E_{\mathrm{Mix2}}(\boldsymbol{\varphi}^3_\natural) = \Big(-\,(\mathscr{B}^{(1)}\boldsymbol{\varphi}^2)_\natural - (\mathscr{B}^{(2)}\boldsymbol{\varphi}^1)_\natural\,;\, -(\mathscr{G}^{(1)}\boldsymbol{\varphi}^2)_\natural - (\mathscr{G}^{(2)}\boldsymbol{\varphi}^1)_\natural\,;\; \mathfrak{h}_t\,,\; \mathfrak{h}_3 \Big)\,.$$

The boundary condition prescribing $N_n(\zeta_3^1)$ is then found by the cancellation of the coefficient δ_8 (5.9a). For this, we need an expression for $\boldsymbol{\varphi}^2_\natural$, which is derived from the cancellation of the constants δ_1 and δ_4 (5.9b)-(5.9c) relating to problem (8.6). The details can be found in [12, §5].

Let us check the compatibility condition for ζ_3^1. Setting $\boldsymbol{\varphi} = \boldsymbol{\varphi}^1 + \varepsilon\boldsymbol{\varphi}^2$, we have by construction

$$\begin{aligned} N_n(\zeta_3^0 + \varepsilon\zeta_3^1) = \frac{3}{2}\bigg(\int_{\Sigma^+} \mathfrak{f}_3(\varepsilon) - \int_{\mathbb{R}^+} \Big(\mathfrak{g}_3^+(\varepsilon) - \mathfrak{g}_3^-(\varepsilon) \Big) + \int_{-1}^{+1} \mathfrak{h}_3(\varepsilon) \bigg) \\ + 2\mu\,\partial_s(\partial_n + \kappa)\partial_s(\zeta_3^0 + \varepsilon\zeta_3^1), \end{aligned} \tag{8.9}$$

where

$$\mathfrak{f}(\varepsilon) = \mathscr{B}\boldsymbol{\varphi} + \mathcal{O}(\varepsilon^2), \qquad \mathfrak{g}(\varepsilon) = \mathscr{G}\boldsymbol{\varphi} + \mathcal{O}(\varepsilon^2), \qquad \mathfrak{h}(\varepsilon) = \boldsymbol{T}\boldsymbol{\varphi} + \mathcal{O}(\varepsilon^2).$$

With $\boldsymbol{w}(\tilde{x}) = \chi(r)\,\boldsymbol{\varphi}(\frac{r}{\varepsilon}, s, \frac{\tilde{x}_3}{\varepsilon})$ on Ω^ε and integrating (8.9) along $\partial\omega$ we obtain for any rigid motion $\boldsymbol{v} = (0,0,a)$ in $\mathcal{R}_{⑥}$

$$\int_{\partial\omega} N_n(\zeta_3^0 + \varepsilon\zeta_3^1)\,v_3 = -\frac{3}{2}\int_{\Omega^\varepsilon} Ae(\boldsymbol{w}) : e(\boldsymbol{v}) + \mathcal{O}(\varepsilon^2) = \mathcal{O}(\varepsilon^2),$$

where we have used $\int_{\partial\omega} \partial_s(\partial_n + \kappa)\partial_s(\zeta_3^0 + \varepsilon\zeta_3^1)\,ds = 0$. The desired compatibility condition then follows.

9. Friction conditions. We only give a few precisions about the traces of the first Kirchhoff-Love generators $\boldsymbol{\zeta}^0$ and $\boldsymbol{\zeta}^1$ for conditions ⑤ and ⑦, referring to [12, §4 & §6] for the proofs, which make use in particular of Lemma 5.5.

The membrane boundary operators $\gamma^{\mathrm{m},j}$, $j = 1,2$, are Dirichlet's in both cases and the corresponding traces $\gamma^0_{\mathrm{m},j}$ and $\gamma^1_{\mathrm{m},j}$ are zero.

The spaces of rigid motions $\mathcal{R}_{⑤}$ and $\mathcal{R}_{⑦}$ are one-dimensional and both are generated by the vertical translation $(0,0,1)$. As a consequence, the first terms ζ_3^0 and ζ_3^1 have to satisfy the zero mean value condition on ω. The bending boundary operators $\gamma^{\mathrm{b},j}$, $j = 1,2$, are Dirichlet's for ⑤, and the trace operator on $\partial\omega$ and M_n for ⑦. Thus the corresponding problems (2.13b) are uniquely solvable. The way out is that the boundary conditions issued from the solution of the Ansatz include $\partial_s\zeta_3 = 0$ on $\partial\omega$. Thus the trace of ζ_3 can be fixed to any constant (we assume here for simplicity that $\partial\omega$ is connected), which can be chosen such that $\int_\omega \zeta_3 = 0$. The formula for this constant rely on the introduction of the solutions η_ω and ξ_ω of the following auxiliary problems:

TABLE 9.1
Auxiliary problems.

⑤	$\mathrm{mes}(\omega)\, L^{\mathrm{b}}(\eta_\omega) = 1 \quad \text{in } \omega$	$\eta_\omega = 0 \;\text{ and }\; \partial_n \eta_\omega = 0 \quad \text{on } \partial\omega$
⑦	$\mathrm{mes}(\omega)\, L^{\mathrm{b}}(\xi_\omega) = 1 \quad \text{in } \omega$	$\xi_\omega = 0 \;\text{ and }\; M_n(\xi_\omega) = 0 \quad \text{on } \partial\omega$

NOTATION 9.1. *If L is an integrable function on $\partial\omega$ such that $\int_{\partial\omega} L = 0$, then we denote by $\oint_{\partial\omega} L$ the unique primitive of L along $\partial\omega$ with zero mean value on $\partial\omega$ (that is $\int_{\partial\omega} \oint L\, ds = 0$). The second primitive $\oint_{\partial\omega}\oint_{\partial\omega} L$ then makes sense.*

For condition ⑤, $\partial_n \zeta_3^0 = 0$ and ζ_3^0 is equal to the constant $-\int_\omega R_{\mathrm{b}}^0 \eta_\omega$ on $\partial\omega$, whereas for condition ⑦, $M_n(\zeta_3^0) = 0$ and ζ_3^0 is equal to the constant $-\int_\omega R_{\mathrm{b}}^0 \xi_\omega$ on $\partial\omega$. Finally, here are the boundary conditions for ζ_3^1, with L given in (2.15):

TABLE 9.2
Boundary conditions.

⑤	$\zeta_3^1 = c_3^{⑤}\Big(\oint_{\partial\omega}\oint_{\partial\omega} L - \int_{\partial\omega}\Big(\oint_{\partial\omega}\oint_{\partial\omega} L\Big) N_n(\eta_\omega)\Big)$	$\partial_n \zeta_3^1 = 0$
⑦	$\zeta_3^1 = c_3^{⑦}\Big(\oint_{\partial\omega}\oint_{\partial\omega} L + 2\mu \int_{\partial\omega} L\, \partial_n \xi_\omega \; - \int_{\partial\omega}\Big(\oint_{\partial\omega}\oint_{\partial\omega} L\Big) N_n(\xi_\omega)\Big)$	$M_n(\zeta_3^1) = c_4^{⑦} L$

10. Free. The space $\mathcal{R}_{⑧}$ is six-dimensional and spanned by all rigid motions. We are only going to explain how the traces of $\boldsymbol{\zeta}^0$ can be determined by our method and refer to [12, §7] for the traces of $\boldsymbol{\zeta}^1$. The nonhomogeneity of the boundary condition $N_n(\zeta_3^0)$ is known, see CIARLET [4, Th. 1.7.2].

From the conditions $T_3^1 = 0$ and $T_n^2 = 0$ we obtain for $\boldsymbol{\varphi}_\natural^1$

$$E_{\mathrm{Free}}(\boldsymbol{\varphi}_\natural^1) = (0; 0; -T_n^{\mathrm{m}}(\boldsymbol{\zeta}_*^0) + x_3 M_n(\zeta_3^0), 0)\,. \tag{10.1}$$

From the cancellation of the constants δ_5 and δ_7 in Proposition 5.12, the conditions $T_n^{\mathrm{m}}(\boldsymbol{\zeta}_*^0) = 0$ and $M_n(\zeta_3^0) = 0$ on $\partial\omega$ are obtained. Thus $\boldsymbol{\varphi}_\natural^1 \equiv 0$.

The condition $T_s^2 = 0$ yields that φ_s^1 has to satisfy problem (8.1). Thus $T_s^{\mathrm{m}}(\boldsymbol{\zeta}_*^0) = 0$ on $\partial\omega$ and φ_s^1 is then given by, *cf* Lemma 5.7,

$$\varphi_s^1 = (\partial_n + \kappa)\partial_s \zeta_3^0(s)\, \bar\varphi_{\mathrm{Neu}}^{\mathrm{s}}\,. \tag{10.2}$$

With $T_s^3 = 0$ we obtain that φ_s^2 has to satisfy problem (8.3), hence the condition $T_s^{\mathrm{m}}(\boldsymbol{\zeta}_*^1) = 0$ on $\partial\omega$ ensures the existence of an exponentially decaying profile. Taking into account the relation (10.2), this solution is given by

$$\varphi_s^2 = -\kappa(\partial_n + \kappa)\partial_s \zeta_3^0\, \bar\psi_{\mathrm{Neu}}^{\mathrm{s}} + (\partial_n + \kappa)\partial_s \zeta_3^1\, \bar\varphi_{\mathrm{Neu}}^{\mathrm{s}}\,, \tag{10.3}$$

where $\bar\psi_{\mathrm{Neu}}^{\mathrm{s}}$ is the solution of problem (8.5).

The conditions $T_3^2 = 0$ and $T_n^3 = 0$ lead to the following problem for $\boldsymbol{\varphi}_\natural^2$:

$$E_{\mathrm{Free}}(\boldsymbol{\varphi}_\natural^2) = \Big(-\big(\mathscr{B}^{(1)}\boldsymbol{\varphi}^1\big)_\natural\,;\; -\big(\mathscr{G}^{(1)}\boldsymbol{\varphi}^1\big)_\natural\,;\; \mathfrak{h}_t\,,\; \mathfrak{h}_3\Big)\,, \tag{10.4}$$

where the terms in the right hand side of (10.4) are given by

$$\big(\mathscr{B}^{(1)}\boldsymbol{\varphi}^1\big)_t = (\lambda+\mu)\,\partial_t\partial_s\varphi_s^1\,,\; \big(\mathscr{G}^{(1)}\boldsymbol{\varphi}^1\big)_t = 0\,,\; \mathfrak{h}_t = -\big(\lambda\partial_s\varphi_s^1 + T_n^{\mathrm{m}}(\boldsymbol{\zeta}_*^1) - x_3 M_n(\zeta_3^1)\big)\,,$$

whereas $(\mathscr{B}^{(1)}\boldsymbol{\varphi}^1)_3$ and $(\mathscr{G}^{(1)}\boldsymbol{\varphi}^1)_3$ are still given by (8.7b) and $\mathfrak{h}_3$ by (8.7d). Thus, the cancellation of the constants δ_5, δ_7 and δ_8 from Proposition 5.12 is required. The cancellation of δ_5 leads to the boundary condition $T_n^{\mathrm{m}}(\boldsymbol{\zeta}_*^1) = 0$ on $\partial\omega$. Inserting the expressions involved, the condition $\delta_7 = 0$ reads

$$\Big[(\lambda+\mu)\int_{\Sigma^+}(x_3\,\partial_t\bar\varphi^{\mathrm{s}}_{\mathrm{Neu}} - t\,\partial_3\bar\varphi^{\mathrm{s}}_{\mathrm{Neu}})\,dt\,dx_3 + \lambda\int_0^\infty t\,(\bar\varphi^{\mathrm{s}}_{\mathrm{Neu}}(1,t)-\bar\varphi^{\mathrm{s}}_{\mathrm{Neu}}(1,t))\,dt$$
$$+\lambda\int_{-1}^{+1} x_3\,\bar\varphi^{\mathrm{s}}_{\mathrm{Neu}}(0,x_3)\,dx_3\Big]\,\partial_s(\partial_n+\kappa)\partial_s\zeta_3^0 - \int_{-1}^{+1} x_3^2\,M_n(\zeta_3^1)\,dx_3 = 0\,.$$

As the boundary layer term $\bar\varphi^{\mathrm{s}}_{\mathrm{Neu}}$ is odd, the above condition becomes

$$\frac{2}{3}\,M_n(\zeta_3^1) = \partial_s(\partial_n+\kappa)\partial_s\zeta_3^0\left[-\mu\int_{-1}^{+1} x_3\,\bar\varphi^{\mathrm{s}}_{\mathrm{Neu}}(0,x_3)\,dx_3 - 2\mu\int_0^\infty t\,\bar\varphi^{\mathrm{s}}_{\mathrm{Neu}}(1,t)\,dt\right].$$

Applying the second Green formula for Laplace to the functions $\bar\varphi^{\mathrm{s}}_{\mathrm{Neu}}(t,x_3)$ and $w(t,x_3) = t\,x_3$, yields the relation

$$2\int_0^\infty t\,\bar\varphi^{\mathrm{s}}_{\mathrm{Neu}}(t,1)\,dt = \int_{-1}^{+1} x_3\,\bar\varphi^{\mathrm{s}}_{\mathrm{Neu}}(0,x_3)\,dx_3\,.$$

Thus $M_n(\zeta_3^1) = c_3^{⑧}\partial_s(\partial_n+\kappa)\partial_s\zeta_3^0$ on $\partial\omega$ with $c_3^{⑧} = -3\mu\int_{-1}^{+1} x_3\,\bar\varphi^{\mathrm{s}}_{\mathrm{Neu}}(0,x_3)\,dx_3$.

The evaluation of the condition $\delta_8 = 0$ has been already done in §8.1, which yields in exactly the same way formula (2.14) for the trace $N_n(\zeta_3^0)$.

Now let us check the compatibility conditions ensuring the existence of the generator $\boldsymbol{\zeta}^0$. Concerning $\boldsymbol{\zeta}_*^0$, we have to show that the membrane right hand side $\boldsymbol{R}_{\mathrm{m}}^0$ of the limit problem is orthogonal to each of the two-dimensional rigid motions $(1,0)$, $(0,1)$ and $(x_2,-x_1)$, since we have homogeneous traction boundary conditions in the problem for $\boldsymbol{\zeta}_*^0$. These orthogonality conditions are clearly a consequence of the expression of the right hand side $\boldsymbol{R}_{\mathrm{m}}^0$ and of the three-dimensional compatibility conditions (1.10) for in-plane rigid motions.

The compatibility conditions for ζ_3^0 remains to be checked. They are related to the kernel of L^{b} with boundary conditions M_n and N_n, i.e. to the functions 1, x_1 and x_2. It has been already shown in §8.1 that the condition (8.8) relating to the element 1 of the kernel is fulfilled. Now let us check the condition for x_1, namely

$$\int_\omega x_1\,R_{\mathrm{b}}^0(x_*)\,dx_* - \frac{3}{2}\int_{\partial\omega} x_1\left(\int_{-1}^{+1} x_3\,f_n\,dx_3 + g_n^+ + g_n^-\right)(0,s)\,ds = 0\,.$$

With the help of the divergence theorem we can rewrite it as

$$\frac{3}{2}\left\{\int_\Omega (x_1\,f_3 - x_3\,f_1)\,dx_3\,dx_* + \int_\omega \left\{x_1\,(g_3^+ - g_3^-) - (g_1^+ + g_1^-)\right\}\,dx_*\right\} = 0\,,$$

which coincides with a compatibility condition (1.10) for the three-dimensional problem. Of course, the condition for x_2 can be proved analogously.

11. Error estimates. We provide in this section estimates in H^1 and L^2 norms.

11.1. In H^1 norm. In this section we prove Theorem 2.2, which yields an optimal estimation of the error between the scaled displacement $\boldsymbol{u}(\varepsilon)$ and the Ansatz of order N. This extends the results obtained in [8, §5] for the hard clamped situation to the eight 'canonical' boundary conditions on the lateral side. The proof relies on energy estimates and on a very simple argument consisting in pushing the development a few terms further.

We define the space $\mathcal{V}_{(\mathrm{i})}(\Omega)$ as the subspace of the admissible displacements $\boldsymbol{u}$ in $V_{(\mathrm{i})}(\Omega)$ which are orthogonal for the L^2 product to all the rigid motions $\boldsymbol{v} \in \mathcal{R}_{(\mathrm{i})}(\Omega)$. Thus $\boldsymbol{u}(\varepsilon)$ belongs to $\mathcal{V}_{(\mathrm{i})}(\Omega)$. Combining Korn's inequality without boundary conditions and the infinitesimal rigid displacement lemma we obtain a Korn inequality with boundary conditions for arbitrary $\boldsymbol{u} \in \mathcal{V}_{(\mathrm{i})}(\Omega)$, compare [25] and [4], which reads in terms of the scaled linearized strain tensor $\theta(\varepsilon)$

$$\left(\int_\Omega A\theta(\varepsilon)(\boldsymbol{u}) : \theta(\varepsilon)(\boldsymbol{u})\right)^{1/2} \geq C^* \|\theta(\varepsilon)(\boldsymbol{u})\|_{L^2(\Omega)^9} \geq C\|\boldsymbol{u}\|_{H^1(\Omega)} . \tag{11.1}$$

Defining the remainder at the order N of the asymptotics of $\boldsymbol{u}(\varepsilon)$ by $\overline{U}^N(\varepsilon) := \boldsymbol{u}(\varepsilon) - U^N(\varepsilon)$, where $U^N(\varepsilon)$ denotes the asymptotic expansion of order N, namely

$$U^N(\varepsilon) \;=\; \underbrace{\sum_{k=0}^{N} \varepsilon^k\, \underline{\boldsymbol{u}}^k}_{=:\, V^N(\varepsilon)} \;+\; \chi(r) \underbrace{\sum_{k=1}^{N} \varepsilon^k\, \boldsymbol{w}^k(\frac{r}{\varepsilon}, s, x_3)}_{=:\, W^N(\varepsilon)} \tag{11.2}$$

with $\underline{\boldsymbol{u}}^k := \boldsymbol{u}^k_{\mathrm{KL}} + \boldsymbol{v}^k$, compare §3.1 for notations, we only need to establish an a priori estimate for $\overline{U}^N(\varepsilon)$ in the norm of the space $H^1(\Omega)^3$.

Therefore, we split $U^N(\varepsilon)$ into its natural parts $U^N(\varepsilon) = V^N(\varepsilon) + \chi(r)\, W^N(\varepsilon)$. Considering carefully the construction algorithm, in particular the derivation of the boundary layer terms, we observe that for any $N \in \mathbb{N}$, $U^N(\varepsilon)$ belongs to the space $\mathcal{V}_{(\mathrm{i})}(\Omega)$. Thus, we have

$$\forall N \in \mathbb{N}\,, \quad \overline{U}^N(\varepsilon) \in \mathcal{V}_{(\mathrm{i})}(\Omega)$$

and the variational form of the problem for $\overline{U}^N(\varepsilon)$ can be written down, where we split the deviation to the true solution into an error generated by $V^N(\varepsilon)$ and an error coming from $W^N(\varepsilon)$, compare [8, (5.8)–(5.11)]. For the choice $\boldsymbol{v} = \overline{U}^N(\varepsilon)$ of the test function in the variational formulation of the problem for $\overline{U}^N(\varepsilon)$, we obtain as one side of the resulting equation the energy associated to the remainder, namely

$$\int_\Omega A\,\theta(\varepsilon)(\overline{U}^N(\varepsilon)) \,:\, \theta(\varepsilon)(\overline{U}^N(\varepsilon))\,.$$

Korn's inequality (11.1) and the coercivity of the operator of elasticity then provides the following rough estimate

$$\|\overline{U}^N(\varepsilon)\|_{H^1(\Omega)^3} \leq C\varepsilon^{N-3}$$

exactly in the same manner as in the proof of Lemma 5.3 in [8]. This estimate reads for $\|\overline{U}^{N+4}(\varepsilon)\|_{H^1(\Omega)^3} \le C\varepsilon^{N+1}$ at the rank $N+4$, whence

$$
(11.3)\qquad
\begin{aligned}
&\|\boldsymbol{u}(\varepsilon)(x) - \boldsymbol{u}^0_{\mathrm{KL}}(x) - \sum_{k=1}^{N} \varepsilon^k \boldsymbol{u}^k(x, \frac{r}{\varepsilon})\|_{H^1(\Omega)^3} \\
&\qquad \le C\,\varepsilon^{N+1} + \sum_{k=N+1}^{N+4} \varepsilon^k \left(\|\underline{\boldsymbol{u}}^k\|_{H^1(\Omega)^3} + \|\chi(r)\boldsymbol{w}^k(\frac{r}{\varepsilon}, s, x_3)\|_{H^1(\Omega)^3} \right) .
\end{aligned}
$$

With the help of the following H^1-estimates of each term in the asymptotics

$$
(11.4)\qquad \|\underline{\boldsymbol{u}}^k\|_{H^1(\Omega)^3} \le C \quad \text{and} \quad \|\chi(r)\boldsymbol{w}^k(\frac{r}{\varepsilon}, s, x_3)\|_{H^1(\Omega)^3} \le C\varepsilon^{-1/2} ,
$$

the estimate (2.4) directly follows from (11.3).

11.2. In other norms. The L^2-estimates of each term corresponding to (11.4)

$$
(11.5)\qquad \|\underline{\boldsymbol{u}}^k\|_{L^2(\Omega)^3} \le C \quad \text{and} \quad \|\chi(r)\boldsymbol{w}^k(\frac{r}{\varepsilon}, s, x_3)\|_{L^2(\Omega)^3} \le C\varepsilon^{1/2}
$$

lead in a straightforward way to the following estimates in L^2-norm

$$
(11.6)\qquad \|\boldsymbol{u}(\varepsilon) - \sum_{k=0}^{N} \varepsilon^k\, \underline{\boldsymbol{u}}^k - \chi(r) \sum_{k=1}^{N} \varepsilon^k\, \boldsymbol{w}^k(\frac{r}{\varepsilon}, s, x_3)\|_{L^2(\Omega)^3} \le C\,\varepsilon^{N+1} .
$$

The question of estimates in higher norms, H^2 for instance, is also considered in [9] for the clamped case. Such estimates require a splitting of the solution and of terms in the asymptotics, since in general the H^2 regularity is not attained. The situation is similar for all lateral conditions. Let us just emphasize that all the terms in the outer expansion are smooth, but also that the singularities along the edges $\partial\omega \times \{\pm 1\}$ of the plate are concentrated in the inner expansion: the model profiles are all non-smooth, with a regularity between $H^{3/2}$ and H^3. For example $\bar{\varphi}^{\mathrm{s}}_{\mathrm{Dir}}$ is almost H^2 and $\bar{\varphi}^{\mathrm{s}}_{\mathrm{Neu}}$ is almost H^3 whereas the profiles $\bar{\boldsymbol{\varphi}}^{\mathrm{m}}_{\mathrm{Dir},\natural}$ and $\bar{\boldsymbol{\varphi}}^{\mathrm{b}}_{\mathrm{Dir},\natural}$ occurring in the clamped plates have less regularity, *cf* [10].

12. Conclusions. Coming back to the family of thin domains Ω^ε, we will briefly address the question of the determination of a limit solution, and of the evaluation of the relative error between this limit and the 3D solution. The correct answer depends on the norm in which the error is evaluated and of the type of the loading.

12.1. H^1 **norm.** We have first to evaluate the behavior of the $H^1(\Omega^\varepsilon)$ norm denoted by $\|\cdot\|_{H1}$ of each of the four types of components of series (2.5), namely $\boldsymbol{u}^k_{\mathrm{KL,b}}$, $\boldsymbol{u}^k_{\mathrm{KL,m}}$, $\tilde{\boldsymbol{v}}^k$ and $\boldsymbol{\varphi}^k$. We find:

$$
\|\boldsymbol{u}^k_{\mathrm{KL,b}}\|_{H1} = \mathcal{O}(\varepsilon^{1/2}), \qquad \|\boldsymbol{u}^k_{\mathrm{KL,m}}\|_{H1} = \mathcal{O}(\varepsilon^{1/2}),
$$

$$
\|\tilde{\boldsymbol{v}}^k\|_{H1} = \mathcal{O}(\varepsilon^{-1/2}), \qquad \|\boldsymbol{\varphi}^k\|_{H1} = \mathcal{O}(1).
$$

In the case of a bending load such that R^0_{b}, *cf* (3.9), is non-zero, we have

$$
(12.1)\qquad \frac{\|\boldsymbol{u}^\varepsilon - \varepsilon^{-1}\boldsymbol{u}^0_{\mathrm{KL,b}}\|_{H1}}{\|\boldsymbol{u}^\varepsilon\|_{H1}} \le C\,\varepsilon,
$$

and this estimate is sharp for any lateral boundary condition, since the main contribution to the error comes from $\tilde{\boldsymbol{v}}^1$ which is equal to $(0, 0, \bar{p}_2(x_3)\,\Delta_*\zeta_3^0)$: indeed, since we assumed that $R_{\rm b}^0$ is non-zero, $\Delta_*^2\zeta_3^0$ is non-zero, and $\tilde{\boldsymbol{v}}^1 \not\equiv 0$.

In the case of a membrane load such that $\boldsymbol{R}_{\rm m}^0$, *cf* (3.6), is non-zero, we have to include $\tilde{\boldsymbol{v}}^1$ in the limit solution to have a convergence: we set

$$\boldsymbol{u}_{\rm m}^{\rm lim} = \boldsymbol{u}_{\rm KL,m}^0 + \varepsilon\tilde{\boldsymbol{v}}^1 = (\zeta_*^0,\ \bar{p}_1(\tilde{x}_3)\,{\rm div}_*\,\zeta_*^0). \tag{12.2}$$

Then

$$\frac{\|\boldsymbol{u}^\varepsilon - \boldsymbol{u}_{\rm m}^{\rm lim}\|_{H1}}{\|\boldsymbol{u}^\varepsilon\|_{H1}} \le C\,\varepsilon^{1/2}, \quad \text{in cases ① – ④}, \tag{12.3}$$

this estimate being generically optimal, in the sense that it is sharp when $\boldsymbol{\varphi}^1$ is non-zero, i.e. when ${\rm div}_*\,\zeta_*^0$ is non-zero on $\partial\omega$ in cases ①, ② and ④, and when $\kappa\zeta_n^0$ is non-zero on $\partial\omega$ in cases ③. On the other hand

$$\frac{\|\boldsymbol{u}^\varepsilon - \boldsymbol{u}_{\rm m}^{\rm lim}\|_{H1}}{\|\boldsymbol{u}^\varepsilon\|_{H1}} \le C\,\varepsilon, \quad \text{in cases ⑤ – ⑧}, \tag{12.4}$$

this estimate being generically optimal too, in the sense that it is sharp when $\tilde{\boldsymbol{v}}^2$ is non-zero, i.e. when ${\rm div}_*\,\zeta_*^0 \not\equiv 0$, compare also with [22] for a special membrane loading on a free plate.

12.2. Energy norm. We now set $\|\boldsymbol{u}\|_E = \big(\int_{\Omega^\varepsilon} Ae(\boldsymbol{u}) : e(\boldsymbol{u})\big)^{1/2}$. The energy of the four types of terms in the series (2.5) has the same behavior as their H^1 norm *except* the one concerning $\boldsymbol{u}_{\rm KL,b}^k$ whose energy is one order smaller:

$$\|\boldsymbol{u}_{\rm KL,b}^k\|_E = \mathcal{O}(\varepsilon^{3/2}).$$

We obtain exactly the same conclusions if we use this energy, or the L^2 norm of the strain tensor, or the complementary energy. We have to include the polynomial terms up to the order 2 to obtain a convergence: we set $\boldsymbol{u}_{\rm m}^{\rm lim}$ as above in (12.2) and moreover

$$\boldsymbol{u}_{\rm b}^{\rm lim} = \boldsymbol{u}_{\rm KL,b}^0 + \varepsilon\tilde{\boldsymbol{v}}^1 = (-\varepsilon x_3\nabla_*\zeta_3^0,\ \zeta_3^0 + \varepsilon\bar{p}_2(x_3)\,\Delta_*\zeta_3^0), \tag{12.5}$$

see also [28] and [30] in this context.

In the case of a bending load such that $R_{\rm b}^0$ is non-zero, we have

$$\frac{\|\boldsymbol{u}^\varepsilon - \boldsymbol{u}_{\rm b}^{\rm lim}\|_E}{\|\boldsymbol{u}^\varepsilon\|_E} \le C\,\varepsilon^{1/2}, \tag{12.6}$$

this estimate being generically optimal, in the sense that it is sharp when $\boldsymbol{\varphi}^1$ is non-zero, i.e. when $\ell^{\rm b}$ is non-zero on $\partial\omega$ in cases ① – ④, *cf* Table 2.5, and when $\ell^{\rm s}$ is non-zero on $\partial\omega$ in cases ⑤ – ⑧, *cf* Table 2.6.

In the case of a membrane load such that $\boldsymbol{R}_{\rm m}^0$ is non-zero, we have exactly the same behavior as with the H^1 norm, see (12.3) and (12.4). In particular, the condition for the optimality of the estimates is visibly sharp, which brings a conclusion to the work [2].

The observation of the first terms in the asymptotics also sheds light on the order of magnitude of the answer of the plate under the loading. The maximal answer rate (of order ε^{-2}) is obtained with a bending load such that R^0_{b} is non-zero and corresponds to the flexural nature of plates. In contrast, the membrane (or stretching) answer is of order 1 when $\boldsymbol{R}^0_{\mathrm{m}}$ is non-zero. Moreover, there are very many other types of loading (bending or membrane) whose answer rate is much lower, see [6].

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