THE INFLUENCE OF TAPE DIAGRAMS AND BAR MODELS ON MIDDLE SCHOOL STUDENTS' PROPORTIONAL REASONING

by

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The following individuals read and discussed the dissertation submitted by student Katie Paulding, and they evaluated her presentation and response to questions during the final oral examination. They found that the student passed the final oral examination.

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DEDICATION

I dedicate this dissertation to the following family and friends:

To my grandparents, Les and Naomi Luby, and Earl and Rose Paulding, who have displayed a love of lifelong learning and never give up in times of adversity.

To my parents, John and Sue, who have provided me with every advantage and encouraged me to pursue my dreams.

To my brother, Jimmy, and sister-in-law, Kendra, who encouraged me to take a new adventure and gave me great studying ideas.

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ABSTRACT

Proportional reasoning is an integral component of adolescent mathematical cognitive development and a foundational concept for students to understand in order to be successful in higher level mathematics and science courses. Yet research indicates students struggle to proportionally reason. Task features of proportional reasoning problems are known to influence student cognition and success in problem solving, including familiarity with problem context, problem type, numerical content, and mode of task representation. The purpose of this study was to examine the influence of two iconic representations (tape diagrams and bar models) and three ratio relationships (6:3, 8:2, and 5:2) on student cognition in proportional reasoning via individual cognitive interviews. Data was analyzed through a combination of protocol analysis and verbal analysis. Results suggest there is some evidence to support the claim the bar model provides more scaffolding than the tape diagram in terms of helping students visualize the multiplicative comparison relationship.

Keywords: proportional reasoning, task features, cognition, mode of representation, ratio relationships, cognitive interviews, protocol analysis, verbal analysis

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LIST OF ABBREVIATIONS

BM	Bar Model
CU	Composed Unit Conception
IAS	Incorrect Additive Strategy
IC	Incorrect Mathematical Calculation
INC	Inconclusive
IND	Indeterminate
INV	Inverse Relationship
MC	Multiplicative Comparison Conception
MR-INV	Inverse Multiplicative Relationship
NR	No Response
OPER	Operations on Given Numbers in Ratio Relationship – Addition, Subtraction, or Multiplication
QDR	Qualitative Description of the Relationship
RGR	Recreate Given Ratio
STEM	Science, Technology, Engineering, and Mathematics
TD	Tape Diagram
UR	Unit Rate

A complete list of codes can be found in the Codebook in Appendix E.

CHAPTER ONE: INTRODUCTION

Background

Mathematics is a vital component of the fields of science, technology, and engineering. In order for our society to continue to make advancements in these fields, students need to have a deep knowledge of mathematics and an awareness of how to apply mathematics to multiple areas (National Academy of Sciences, 2010; National Science Foundation, 2010). However, many students choose to not enter STEM (Science, Technology, Engineering, and Mathematics) fields because they did not experience success in STEM courses in high school, often due to lack of adequate academic, economic, and social support (ACT, 2006; Adelman, 1999; Wang, 2013). It is therefore critical to identify areas of mathematics that contribute to student struggle and develop resources to address them.

One area within the domain of secondary mathematics that affects success in future mathematics courses and which students generally find challenging to learn is proportional reasoning (Brahmia, Boudreaux & Kanim, 2016; Cohen, Anat Ben, & Chayoth, 1999; Lobato, Ellis, & Charles, 2010). A key aspect of proportional reasoning is the ability to reason about two quantities in a ratio relationship simultaneously (Ellis, 2013). From a mathematics perspective, this entails reasoning about scalar and functional relationships, while from a student cognition perspective, this entails the formation of a composed unit that can be iterated or partitioned or understanding and making use of the constant multiplicative comparison between quantities in the ratio (Carney & Smith, 2016; Lobato et al., 2010). A *composed unit* conception involves simultaneously coordinating the two quantities in a ratio to form a new unit with which to operate (Carney & Crawford, 2016; Carney et al., 2015; Ellis, 2013; Lobato et al., 2010). A *multiplicative comparison* conception compares two given quantities in a ratio by identifying the multiplicative, or functional, relationship between them (Carney & Crawford, 2016; Lobato et al., 2010; Tourniaire & Pulos, 1985). This conception is explained in more detail in Chapter 2.

The ability of students to successfully apply their understanding of the multiplicative relationships in proportional reasoning situations is influenced by familiarity with problem context and content, as students experience greater success in problem solving when presented with tasks that are familiar to them (Booth & Koedinger, 2012; Heller, Ahlgren, Post, Behr, & Lesh, 1989; Tourniaire & Pulos, 1985). Likewise, the format of the task, also called problem type, may elicit certain solution strategies (Lamon, 1993; Tourniaire & Pulos, 1985; Webb, 1984). The mode of task representation may also influence students to solve through different cognitive processes, such as whether students were provided a physical model, picture, or an algebraic equation to help them solve the problems (Bruner, 1966; Kaput, 1985). For example, the mode of representation may help students solve tasks by enabling them to rely more frequently on their intuition rather than on memorized algorithms (Liu & Shen, 2011).

Additionally, various characteristics of the numbers in a task also affect student cognition. Smaller numbers are generally easier for students to manipulate than larger numbers, and integers are easier to use than non-integers (Cramer, Post, & Currier, 1993;

Hart, Brown, Kuchemann, Kerslake, Ruddock, & McCartney, 1981; Pulos, Karplus, & Stage, 1981; Rupley, 1981; Tourniaire & Pulos, 1985). Middle school students generally have more difficulty solving problems in which the first integer provided in missing value tasks is smaller than the second integer provided (Rupley, 1981). Likewise, whether variables are presented as discrete units of measure (e.g., marbles) or as continuous units (e.g., time) is also influential, as discrete quantities are generally easier to mentally manipulate than continuous quantities (Hart et al., 1981; Tourniaire & Pulos, 1985).

This study sought to determine the influence of bar models and tape diagrams, along with ratio relationships, on eliciting the multiplicative comparison conception in middle school students via cognitive interviews. Additionally, I examined students' solution strategies to determine the influence of the models and ratio relationships on students' approaches to solving the given tasks.

Problem Statement

Proportional reasoning is a challenging topic for adolescents (Brahmia et al., 2016; Cohen et al., 1999; Lobato et al., 2010). Proportional reasoning entails "attending to and coordinating two quantities" in a ratio relationship (Lobato et al., 2010, p.12). Iconic mathematical models, specifically tape diagrams and bar models, may influence how to reason proportionally. It is important to note students should learn to fluently and flexibly use both composed unit (via scalar relationships) and multiplicative comparison conceptions (via functional relationships); however, research has shown students have difficulty identifying and using the multiplicative comparison conception (Carney & Crawford, 2016; Steinthorsdottir & Sriraman, 2009).

Purpose Statement

The purpose of this study was to determine the influence of models (tape diagrams and bar models) and additional factors, such as ratio relationships (6:3, 8:2, and 5:2), on students' conceptions and associated solution strategies regarding the composed unit and multiplicative comparison conceptions in missing value proportional reasoning tasks. Previous research suggests task features of proportional reasoning influence student cognition, but no study has specifically dealt with the impact of tape diagrams and bar models on students' conceptions of the composed unit (scalar relationship) and multiplicative comparison (functional relationship).

Research Questions

This research was guided by the following questions:

1. In what ways do the mathematical models of tape diagram and bar model influence students' use of the multiplicative comparison conception and associated solution strategies in proportional reasoning situations?
2. What additional factors, such as the ratio relationships 6:3, 8:2, and 5:2, influence student solution strategies in relation to the bar models and tape diagrams?

Definitions of Terms

The operational definitions for this research are as follows:

- *Bar Model*

An iconic mathematical representation that compares the linear magnitude of two or more quantities (Booth & Koedinger, 2012). See Figure 1 for an example of a bar model.

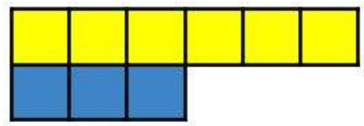
Prompt: Given 6 cans of yellow paint and 3 cans of blue paint...
Bar Model


Figure 1. Bar Model

- *Cognitive Interview*

  A common method in qualitative studies for gathering data on a participant's thought processes through *verbalization* (Desimone & Carlson Le Floch, 2004; Ericsson & Simon, 1980).

- *Composed Unit Conception*

  Involves simultaneously coordinating the two quantities in a ratio to form a new unit with which to operate (Ellis, 2013; Lobato et al., 2010). See Figure 2 for an example of a composed unit conception.

Prompt: Given 6 cans of yellow paint and 3 cans of blue paint, how many blue cans of paint are needed for 18 cans of yellow paint?
Student's Verbalization
"Well, I would say I can take the six yellow cans and the three blue cans and multiply them each by three. Since six times three equals eighteen, three times three equals nine. You would need nine cans of blue paint for eighteen cans of yellow paint."

Figure 2. Composed Unit Conception

- *Functional Relationship*

Involves finding a constant multiplicative relationship within the two quantities that form a ratio (Carney et al., 2015; Karplus, Pulos, & Stage, 1983a; Noelting, 1980b; Tourniaire & Pulos, 1985; Vergnaud, 1983). See Figure 3 for an example of a functional relationship.

Prompt: Given 6 cans of yellow paint to 3 cans of blue paint, how many blue cans of paint are needed for 18 cans of yellow paint?

Functional Relationship

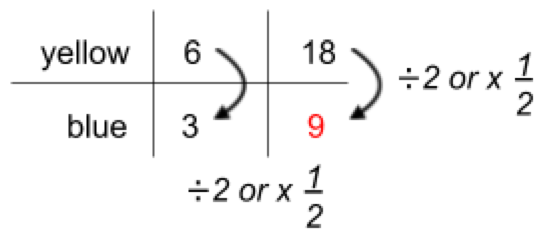


Figure 3. Functional Relationship

- *Modes of Representation*

Bruner (1966) explicates three modes of representation used in modeling with mathematics (i.e., physical (enactive), iconic (pictorial), and symbolic). This study focused on iconic models which denote pictorial or diagrammatic representations of a physical object (Bruner, 1966).

- *Multiplicative Comparison Conception*

Compares two given quantities in a ratio by identifying the multiplicative, or functional, relationship between them (Carney & Crawford, 2016; Lobato et al., 2010; Tourniaire & Pulos, 1985). See Figure 4 for an example of a multiplicative comparison conception.

Prompt: Given 6 cans of yellow paint and 3 cans of blue paint, how many blue cans of paint are needed for 18 cans of yellow paint?
Student's Verbalization
"Well, three is half of six, so there are half as many blue cans of paint as yellow cans of paint. So eighteen divided by two is nine. So you would need nine cans of blue paint for eighteen cans of yellow paint."

Figure 4. Multiplicative Comparison Conception

- *Proportion*

"A relationship of equality between two ratios" (Lobato et al., 2010, p. 33). Proportional situations are commonly written in symbolic notation as $a/b = c/d$ (Heller et al., 1989).

- *Proportional Reasoning*

Entails simultaneously reasoning about two quantities that exist in a ratio relationship, between which there exists a constant multiplicative relationship (Lobato et al., 2010). Also "denotes reasoning in a system of two variables between which there exists a linear functional relationship" (Karplus et al., 1983a, p. 219).

- *Protocol Analysis*

A form of data analysis that focuses on determining the extent to which the student's thinking followed predetermined steps to obtain the correct solution (Ericsson, 2006; Ericsson & Simon, 1980, 1984/1993; Willis, 2005).

- *Ratio*

A "multiplicative comparison of two quantities, or… a joining of two quantities in a composed unit" (Lobato et al., 2010, p. 12). This definition differs from other definitions of ratio (e.g., part-to-whole relationships), but is necessary for explication of multiplicative relationships (multiplicative comparison and composed unit) in proportional situations.

- *Scalar Additive Strategies*

Strategies that create equivalent ratios by iterating an initial ratio using addition or subtraction to obtain the desired result (Carney et al., 2015; Lamon, 1993). See Figure 5 for an example of a scalar additive strategy.

Prompt: Given 6 cans of yellow paint with 3 cans of blue paint, how many cans of blue paint are needed for 18 cans of yellow paint?

Scalar Additive Strategy

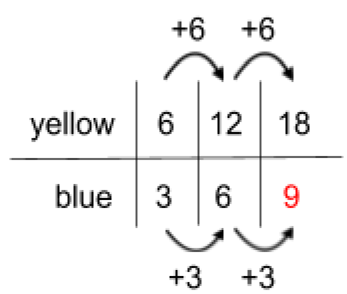


Figure 5. Scalar Additive Strategy

- *Scalar Multiplicative Strategies*

Strategies that create equivalent ratios by multiplying the initial ratio by a constant multiplicative factor to obtain the desired result (Carney & Smith, 2016). See Figure 6 for an example of a scalar multiplicative strategy.

Prompt: Given 6 cans of yellow paint with 3 cans of blue paint, how many cans of blue paint are needed for 18 cans of yellow paint?

Scalar Multiplicative Strategy

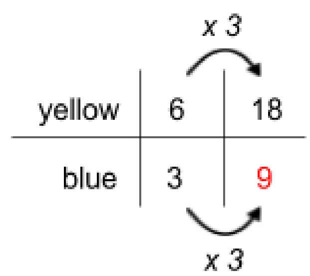


Figure 6. Scalar Multiplicative Strategy

- *Tape Diagram*

An iconic tape-like mathematical representation that visually displays part-whole relationships or part-part relationships (Booth & Koedinger, 2012; Murata, 2008). See Figure 7 for an example of a tape diagram.

Prompt: Given 6 cans of yellow paint and 3 cans of blue paint...

Tape Diagram

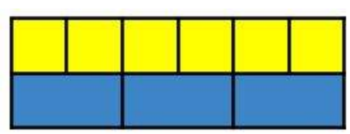


Figure 7. Tape Diagram

- *Think Aloud*

A form of concurrent verbalization produced by the student that provides the interviewer with data but does not interfere with the students' thought processes as they complete a task or form a response (Ericsson & Simon, 1980, 1984/1993).

- *Unit Ratio*

A ratio in which "one quantity in the ratio is divided by another to generate a per one relationship" (Carney & Smith, 2016, p. 8). A more sophisticated conception of composed unit (Lamon, 1993).

- *Verbal Analysis*

A form of data analysis that focuses on creating categories based upon what the students reveal during the interview process (Chi, 1997).

- *Verbal Probing*

A form of concurrent verbalization designed to elicit deeper explanations by students via specific questions, such as "How are you thinking about that?" (Beatty & Willis, 2007, p. 290) or "How did you come up with that answer?" (Blair & Burton, 1987, p. 283).

Significance of the Study

This study addresses a gap in the literature regarding the influence of tape diagrams and bar models on student cognition in proportional reasoning. While the mode of representation may influence students to solve the task through different cognitive processes, such as relying on intuition instead of memorized algorithms, results are currently inconclusive (Kozma, 2003; Liu & Shen, 2011; Seufert, 2003). For example, it is currently unknown if there is a correlation between display of problems (graphic or numerical) and problem solving strategies, as researchers have obtained conflicting results regarding the influence of modes of representation on cognitive processes (Liu &

Shen, 2011). Steffe & Parr (1968) found the design of concentration problem types did not influence students' ability to solve, but Liu & Shen (2011) concluded the opposite in a similar study involving science. This study sought to address this gap in the literature by studying the influence of iconic representations of tape diagrams and bar models on student cognition in proportional reasoning. The results of this study are relevant for curriculum designers, teacher educators, professional development providers, teachers, and mathematics coaches, as the results may help them to potentially identify influential instructional practices to aid students in deepening their understanding of proportional reasoning. For mathematics teacher educators, this study seeks to inform instructional practices teachers could use in both the learning and assessment of proportional reasoning.

CHAPTER TWO: LITERATURE REVIEW

Theoretical Framework

This study is framed by the concept of constructivism, as well as the constructs of using mathematical models and iconic modes of representation (Bruner, 1966; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010; Piaget, 1970, 1972). This chapter describes each of these in depth and explains how these concepts work together to frame this study. The concept of constructivism is discussed next.

The theoretical framework for this study is constructivism and how it influences the learning process (Piaget, 1970, 1972). While many definitions of learning exist, constructivism posits that learning occurs when one is given opportunities to construct knowledge for themselves, rather than being lead to a specific idea (Piaget, 1970, 1972; Simon, 1993). The extent to which a student learns is affected by a variety of factors, including the learner's physical environment, prior knowledge, and the ways in which they engage in the learning process (Piaget, 1970, 1972).

In order to construct knowledge, the learner must interact with a stimulus that either confirms or contradicts their prior knowledge (Inhelder & Piaget, 1958; Piaget, 1958, 1970, 1972, 1975). These interactions are directly observable in the physical environment, such as when a student uses manipulatives to solve a problem; however, they are only indirectly observable in the cognitive environment, such as when a student changes their manner of thinking (Von Glasersfeld, 1991). When the interactions are primarily cognitive, the learning that takes place can only be hypothesized or inferred by an observer (Kaput, 1991; von Glasersfeld, 1991).

As a student learns, they will try to connect what they are currently learning with their prior knowledge, and encounter iterative cycles of conflict and resolution, which are called equilibrium (disequilibrium), accommodation, and assimilation (Inhelder & Piaget, 1958; Piaget, 1970, 1972). Equilibrium occurs when the stimulus produces a thought or response that aligns the student's current knowledge with their prior knowledge and thus falls in line with something they already know to be true (Inhelder & Piaget, 1958; Piaget, 1958, 1970, 1972, 1975). However, when the stimulus produces a thought or response that does not fall in line with the student's prior knowledge, the student experiences a state of disequilibrium (Inhelder & Piaget, 1958; Piaget, 1958, 1970, 1972, 1975). In this state of disequilibrium, the student struggles because they have recognized a contradiction between what they knew to be true and what they are currently experiencing (Simon, 1993). This struggle is a key component of the learning process and the construction and development of new knowledge (DeVries & Zan, 1996). When the student is able to resolve the conflict, either by changing their prior knowledge or fixing a current misconception, they reorder and coordinate their thought processes and engage in a process called accommodation (Piaget, 1970, 1972; Thompson, 1985). Once the student can generalize their knowledge into a more abstract form by applying it to a different situation, they have reached assimilation (Inhelder & Piaget, 1958; Piaget, 1970, 1972).

In order to ascertain cognitive interactions, researchers seek to uncover the learner's thought processes, discover how they perceive the stimulus, and how they construct new knowledge (Noddings, 1990). Researchers commonly gather this data via individual clinical or cognitive interviews (Piaget, 1929). During interviews, students are often asked to verbalize their thinking, providing them opportunities to examine their thinking and identify contradictions or misconceptions (Von Glasersfeld, 1991).

In order to study how students construct knowledge and how well they understand a concept, researchers will often provide students with stimuli intended to produce a state of disequilibrium and accommodation (Confrey, 1990; Noddings, 1990). Often the stimulus can be manipulatives, diagrams, or pictures that function as a mathematical tool to help students visualize what is occurring in the given task (Davis, 1990; Murata, 2008). For example, to teach positive and negative integers, the Madison Project used pebbles being placed in and removed from a bag to help students visualize the ideas (Davis, 1990). It is vital these tools are intentionally chosen to provide students opportunities to interact with the tasks and help them find patterns, determine errors in thinking, and discover misconceptions (Noddings, 1990; Thompson, 1985).

When provided with a mathematical task, students in a state of disequilibrium often change their answers, ask rhetorical questions, seek confirmation from others as to the correctness of their responses, or press forward by operating with misconceptions (Gould, 1996; Piaget, 1958; Shifter, 1996; Simon, 1993). If answers are incorrect, students may be misapplying, conflating, or combining mathematical knowledge in their solution strategies (Noddings, 1990).

For example, Confrey (1990) used clinical interviews to discover the thought processes of a college student as they struggled to create a number line with numbers written in exponential notation. Via a series of interviews over the course of several weeks, the student encountered disequilibrium and made consecutive modifications to their number line in order to resolve their conflicting ideas.

In the classroom setting, teachers can intentionally create situations in which their students experience disequilibrium in order to help them identify and resolve errors or misconceptions in their thinking. To do so, teachers should clearly identify what they want the students to grapple with and choose specific questions and activities that will foster the construction of new knowledge (Thompson, 1985). For example, in order to introduce a unit on measurement, Ms. Hendry provided her second grade students with a scenario in which they had to find the size of a ship sent from the Pilgrims to the King of England (Shifter, 1996). When the students decided to measure the ship using a classmate Tom's height, Ms. Hendry asked her students, "How will the King know how long a "Tom" is?" Students then decided to measure using the length of a hand and later a foot, but soon discovered everyone's hands and feet were different sizes. By allowing students to experience the conflict of different sizes of hands and feet, Ms. Hendry probed students with questions that caused them to see potential errors in their thinking and guided them to the realization for a need of standard measurement based upon their own observations and suggestions for measurement (Shifter, 1996). This scenario highlights the importance of providing an environment that builds upon students' prior knowledge and uses students' own ideas as a catalyst for further exploration (Baroody & Ginsburg, 1990).

Constructivism is a useful framework for this study, as the interview protocol (e.g., bar model and tape diagram) has the potential to serve as a stimulus for students to construct new knowledge. In fact, the entire interview protocol may function as a learning environment and promote disequilibrium and subsequent accommodation within the students as they interact with the bar models and tape diagrams. An analysis of student responses to the interview protocol during cognitive interviews may indicate a change in students' thinking processes and provide insights into the influence of the bar model and tape diagram on students' cognition in proportional reasoning situations.

Conceptual Frameworks

Two key constructs inform this study along with constructivism as follows: using mathematical models and using iconic modes of representation (Bruner, 1966; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). Each construct is explicated next.

Using Mathematical Models

The Standards for Mathematical Practice in the Common Core State Standards for Mathematics suggest that using mathematical models involves teaching students how to effectively use models in problem-solving tasks to organize information, reason logically, and demonstrate understanding by providing evidence of their thinking (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). Using physical and pictorial mathematical models has been shown to support students in algebraic thinking by providing a powerful visual tool to represent quantities and their relationships involving comparisons, part-whole calculations, ratios, and proportions, as well as abstract concepts that may not be easily described with words (Brendefur, Carney, Hughes, & Strother, 2015; Hoven & Garelick, 2007; Patrick, Carter, & Wiebe, 2005; Stephens, Ellis, Blanton, & Brizuela, 2017). Modeling may take several modes of representation that influence student cognition (Bruner, 1966). The modes of representation are explicated next.

Modes of Representation

Bruner (1966) describes three modes of representation used in modeling with mathematics, called physical (enactive), iconic (pictorial), and symbolic. Enactive representations involve physical objects students can touch or handle in the problem-solving process, while iconic representations denote pictures or diagrams of a physical object (Bruner, 1966). Symbolic representations involve the use of mathematical symbols to represent meaning, such as algebraic equations (Bruner, 1966).

Two Iconic Models

Iconic models such as pictures and diagrams can be very effective in aiding student comprehension of mathematics (Bruner, 1966). Singaporean and Japanese textbooks extensively incorporate models such as double number lines, strip diagrams (Singapore), tape diagrams (Japan), and bar models in order to develop student thinking about relationships between quantities with great success (Beckmann, 2004; Cohen, 2013; Hoven & Garelick, 2007; Watanabe, 2015).

The manner in which relevant information is represented spatially may be a factor in diagram effectiveness (Booth & Koedinger, 2012). Two iconic representations that visually display the same information about relationships between two quantities but differ in visual appearance are tape diagrams and bar models. A tape diagram is an iconic tape-like mathematical representation that visually displays part-whole relationships or part-part relationships (Booth & Koedinger, 2012; Murata, 2008). A bar model compares the linear magnitude of the quantities (Booth & Koedinger, 2012). See Figure 8 for examples of a tape diagram and a bar model each relating 6 yellow cans of paint to 3 blue cans of paint.

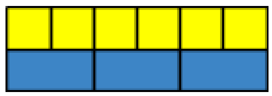
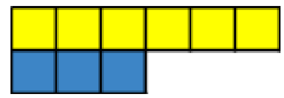
Scenario: Given 6 yellow cans of paint to 3 blue cans of paint...	
Tape Diagram	**Bar Model**
	

Figure 8. Examples of a Tape Diagram and Bar Model

Although research indicates iconic models are effective in aiding student development of conceptual understanding of mathematical ideas, there is a lack of research comparing iconic models such as tape diagrams and bar models specifically in the area of proportional reasoning (Beckmann, 2004; Bruner, 1966; Cohen, 2013; Hoven & Garelick, 2007; Watanabe, 2015). Therefore, this study sought to address this gap by examining the influence of iconic representations of tape diagrams and bar models in students' cognition in proportional reasoning.

Overview of Proportional Reasoning

Proportional reasoning is a multi-faceted topic in the domain of multiplicative structures (Heller et al., 1989; Vergnaud, 1983). The ability to effectively, fluently, and flexibly reason with ratios is an integral component of concepts of measure, fractions, and functions and graphing in algebra (Ellis, 2013; Carney et al., 2015; Carney & Smith, 2016; Lobato, et al., 2010). Yet research has shown many students struggle in their understanding and use of effective proportional reasoning strategies (Brahmia et al., 2016; Cohen et al., 1999; Lobato et al., 2010).

Proportional reasoning entails simultaneously reasoning about two quantities that exist in a ratio relationship and between which there exists a constant multiplicative relationship (Lobato et al., 2010). In order to explicitly define proportional reasoning, it is necessary to unpack the underlying concepts and definitions of what constitutes a ratio and proportion.

Lobato et al. (2010) provides a definition of ratio as a "multiplicative comparison of two quantities, or… a joining of two quantities in a composed unit" (p. 12). This definition differs from other definitions of ratio, but is helpful when focusing on student cognition in proportional reasoning. Building upon the above definition of ratio, Lobato et al. (2010) defines a proportion as "a relationship of equality between two ratios" (p. 33). Proportions are commonly written in symbolic notation as $a/b = c/d$ (Heller et al., 1989). Ratios can be scaled up or down additively or multiplicatively to create equivalent ratios (Lobato et al., 2010). For example, given the following proportional reasoning problem: "Annie is painting her room green. She mixed 6 cans of yellow paint with 3 cans of blue paint to create green paint. If Annie has 18 cans of yellow paint, how many cans of blue paint does she need to make the same color green as before?", one may form two equivalent ratios from the given quantities in the problem. The ratio of 6 yellow cans to 3 blue cans is equivalent to the ratio of 18 yellow cans to x blue cans of paint. This problem was modified from similar tasks by Lesh, Post, & Behr (1988) and Lobato et al., (2010) and will be referred to as the Annie Task throughout the remainder of this paper. See Figure 9 for a visual representation of two equivalent ratios forming a proportion.

Prompt: Annie is painting her room green. She mixed 6 cans of yellow paint with 3 cans of blue paint to create green paint. If Annie has 18 cans of yellow paint, how many cans of blue paint does she need to make the same color green as before?			
Colors of Paint Cans	**Initial Quantity**	**Final Quantity**	**Proportion**
Yellow	6	18	$\frac{6}{3} = \frac{18}{x}$
Blue	3	x	

Figure 9. Two Equivalent Ratios Form a Proportion

Cognition in Proportional Reasoning

Student cognition in proportional reasoning centers on the multiplicative nature of proportional reasoning and the manner in which students conceive of situations involving proportions (Cramer et al., 1993; Vergnaud, 1983). The influence of two mathematics perspectives of multiplicative relationships (i.e., scalar and functional) and two student conceptions of ratio relationships (i.e., composed unit and multiplicative comparison) as well as their associated solution strategies on student cognition are discussed next.

The Multiplicative Nature of Proportional Reasoning

Many mathematical relationships can be thought of as either additive or multiplicative; however, a defining characteristic of sophisticated proportional reasoning is the recognition of its multiplicative nature (Cramer et al., 1993; Vergnaud, 1983). In fact, the most important aspect of proportional reasoning is the understanding of the multiplicative relationships among the quantities in a proportional reasoning task (Cramer et al., 1993). Yet when students first start reasoning about proportions, they typically end up operating on the given ratios using additive relationships instead of multiplicative relationships, a developmental stage called pre-proportionality (Inhelder & Piaget, 1958). The ability to recognize and use the multiplicative relationships in proportional reasoning situations occurs at a later stage of cognitive development called formal operations (Inhelder & Piaget, 1958).

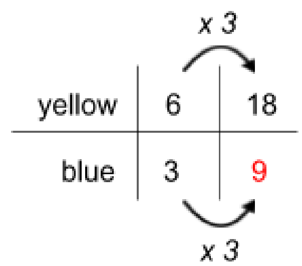
Two Mathematics Perspectives of Multiplicative Relationships

Two mathematics perspectives of relationships, called *scalar* and *functional*, influence the manner in which students conceive of the multiplicative relationships in proportional reasoning situations (Karplus et al., 1983a; Noelting, 1980b; Tourniaire & Pulos, 1985; Vergnaud, 1980, 1983). Researchers in both the fields of mathematics and science have studied these relationships and consequently view them from slightly different perspectives. Thus, scalar and functional relationships may be discussed using alternative terminology, as explicated next.

Scalar Relationships

From a mathematics perspective, a scalar relationship entails reasoning about the two quantities in a ratio by multiplying each quantity by a common factor (Carney & Crawford, 2016; Carney et al., 2015; Karplus et al., 1983a; Lobato et al., 2010; Noelting, 1980b; Tourniaire & Pulos, 1985; Vergnaud, 1983). In this manner, a ratio can be scaled up or down to create equivalent ratios (Carney & Crawford, 2016; Carney & Smith, 2016; Lobato et al., 2010). Freudenthal (1978) called this an "external" perspective, because the common scale factor operated upon the first ratio to form the second ratio and functions independently of the multiplicative relationship between the quantities in the ratio (see functional relationship next). See Figure 10 for a visual representation using the Annie Task.

Prompt: Annie is painting her room green. She mixed 6 cans of yellow paint with 3 cans of blue paint to create green paint. If Annie has 18 cans of yellow paint, how many cans of blue paint does she need to make the same color green as before?	
Scalar Multiplicative Perspective “External” or “Within Measure Space”	**Functional Perspective “Internal” or “Between Measure Spaces”**



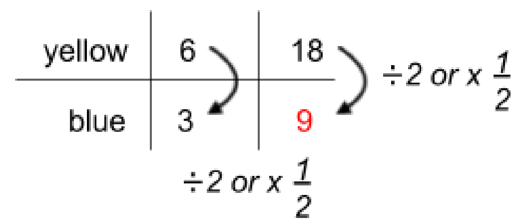


Figure 10. Two Mathematical Perspectives of Multiplicative Relationships

Noelting (1980b) chose to use the term “within” to explain this perspective due to his conception that the given quantities in the ratio relationship can be grouped according to their characteristics, which Vergnaud (1983) calls “measure spaces”. For example, in the Annie Task, if all quantities of yellow paint form a measure space and all quantities of blue paint form a second measure space the relationships of yellow and blue paint can be compared according to their measure spaces. When 6 cans of yellow paint are compared to 18 cans of yellow paint, the relationship is called “within” because both quantities of paint are within the same measure space. Many researchers refer to this as a scalar perspective (Cramer & Post, 1993; Karplus et al., 1983a; Karplus, Pulos, & Stage, 1983b; Vergnaud, 1983). See Figure 10 for a visual representation.

Functional Relationships

From a mathematics perspective, a *functional* relationship involves finding a constant multiplicative relationship within the two quantities that form a ratio (Karplus et al., 1983a; Noelting, 1980b; Tourniaire & Pulos, 1985; Vergnaud 1983). While the scalar relationship is called “external”, the functional relationship is called “internal,” due to the common multiplicative factor that exists between the two quantities in a ratio (Freudenthal, 1978). For example, given 6 yellow cans to 3 blue cans of paint, the constant multiplicative relationship between 6 and 3 is “times ½”. In the scalar relationship, the external multiplicative relationship can change, depending on the multiplier being used to generate equivalent ratios; however, for the functional relationship, the internal multiplicative relationship between the two quantities remains constant, even as the ratio is iterated or partitioned to create equivalent ratios (Carney et al., 2016).

While Noelting (1980b) calls the scalar relationship “within”, he refers to the functional relationship as “between” because the relationship that is being compared is between two different states. For example, in the Annie Task, if the measure space or state being compared is the color of the paint, the relationship of “times ½” between 6 yellow cans and 3 blue cans of paint is called “between” since the quantities being compared consist of different colors of paint.

Past research has indicated a scalar perspective is easier for students to identify and use than a functional perspective (Lamon, 1993; Tjoe & de la Torre, 2014; Vergnaud, 1980). However, current research suggests students’ choice to make use of a particular mathematical relationship is influenced by other factors such as problem context and by the location of the integer or non-integer multiplier (Carney & Crawford, 2016; Karplus et al., 1983a, 1983b; Tourniaire & Pulos, 1985). These task features are discussed in more detail later.

Two Student Conceptions of Ratio Relationships

Two student conceptions arise from the scalar and functional mathematics perspectives, called *composed unit* and *multiplicative comparison,* respectively.

Composed Unit Conception

A *composed unit* conception involves simultaneously coordinating the two quantities in a ratio to form a new unit with which to operate (Ellis, 2013; Lobato et al., 2010). While this new unit is "composed" of the two quantities in a ratio, it uniquely operates as a single entity. This new ratio is the *unit ratio*, which is a more sophisticated conception of composed unit (Lamon, 1993). A unit ratio is a ratio in which "one quantity in the ratio is divided by another to generate a per one relationship" (Carney & Smith, 2016, p. 8). For example, in the Annie Task, the ratio of 6 yellow cans to 3 blue cans of paint can be scaled down by a factor of 3 to obtain 2 yellow cans to 1 blue can of paint. Alternatively, the 6 yellow cans to 3 blue cans of paint can be iterated (repeated) by consecutively adding the given ratio to itself (a scalar additive strategy) or by multiplying the ratio by a constant multiplicative factor (a scalar multiplicative strategy) to obtain larger equivalent ratios as well (Carney & Smith, 2016). See Figure 11 for examples of a composed unit conception with 6 yellow cans to 3 blue cans of paint.

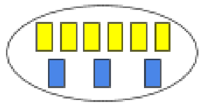
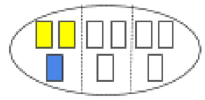
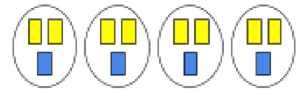
Ratio Relationship	One Whole	One Whole Scaled Down to Unit Ratio	Iterating the Composed Unit of 2:1 (Unit Ratio) by a Factor of 4
Visual (Iconic) Representation			
Symbolic Representation	yellow 6 blue 3	÷3 yellow 6 2 blue 3 1 ÷3	÷3 x4 yellow 6 2 8 blue 3 1 4 ÷3 x4

Figure 11. Composed Unit Conception

A student who has a composed unit conception typically also views the multiplicative relationships through a scalar perspective. However, multiple researchers caution that one's ability to form a ratio by creating a composed unit does not necessarily indicate one has a gained a deep understanding of the two multiplicative relationships that exist in proportional reasoning situations, although it is a meaningful step in conceptual development (Ellis, 2013; Hiebert & Lefevre, 1986; Lesh et al, 1988; Lobato et al., 2010).

Multiplicative Comparison Conception

A *multiplicative comparison* conception compares two given quantities in a ratio by identifying the multiplicative, or functional, relationship between them (Ellis, 2013; Lobato et al., 2010; Tourniaire & Pulos, 1985). A multiplicative comparison conception involves describing one component of the ratio as a multiplicative comparison of the other component; that is, the multiplicative relationship between the quantities is determined based upon which quantity is viewed as the base quantity or unit of measure

(Carney & Crawford, 2016). For example, given the ratio of 6 yellow cans to 3 blue cans of paint, if one identifies the blue cans of paint as the base quantity or unit of measure, one may say "There are twice the number of yellow paint cans as blue paint cans." Alternatively, if one identifies the yellow cans of paint as the base quantity or unit of measure, one may say "The number of blue paint cans is half the number of yellow paint cans." See Figure 12 for a visual representation of the multiplicative comparison conception.

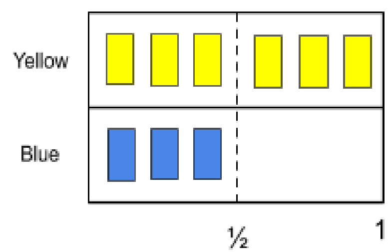
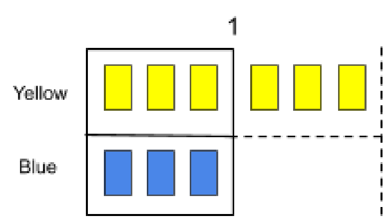
Scenario: Given 6 cans of yellow paint to 3 cans of blue paint...		
Questions	**Iconic Representation**	**Answers**
...if a group of 6 yellow cans is the unit of measure, what is the relationship between blue and yellow paint?		The number of blue paint cans is ½ of the number of yellow paint cans.
...if a group of 3 blue cans is the unit of measure, what is the relationship between yellow and blue paint?		The number of yellow paint cans is twice the number of blue paint cans.

Figure 12. Multiplicative Comparison Conception

It is important to note mastery of a multiplicative comparison conception includes understanding that the constant relationship between the yellow and blue paint does not change, even when the quantities of the ratio are operated on via multiplication, division, additive iteration, or partitioning (Lobato et al., 2010; Thompson & Saldanha, 2003).

This multiplicative comparison conception is very important, but it is one that students often do not easily perceive (Carney & Smith, 2016).

Associated Student Strategies

The way students view scalar and functional relationships will determine their solution strategies and whether they choose strategies that operate under a composed unit or multiplicative comparison conception. These strategies can be additive or multiplicative in nature, as discussed next.

Additive Strategies

Students in the pre-proportionality stage of Piaget's cognitive development will often operate on ratios additively instead of multiplicatively (Hart et al., 1981; Karplus et al., 1983b; Lamon, 2006; Lesh et al., 1988; Wollman & Karplus, 1974). For example, a student may scale up a given ratio by adding the same value to both quantities in the ratio, a scalar additive strategy, or by successively iterating the ratio until the desired value is reached, called additive iteration (Carney et al., 2015). Both scalar additive strategies and additive iteration can be labeled "building up" strategies, a term used to identify strategies in which students use either addition or multiplication to create equivalent ratios (Lamon, 1993). The building up strategy is explained in more detail as it relates to multiplication in the next section. While additive building up strategies may result in a correct answer, students who display additive reasoning typically do not demonstrate engagement in multiplicative thinking as well (Carney et al., 2015).

Multiplicative Strategies

When students operate on ratios multiplicatively, the strategies students employ to obtain a solution originate from the way they view the multiplicative relationships in the given task, whether from a composed unit conception (scalar perspective) or a multiplicative comparison conception (functional perspective) (Carney & Crawford, 2016; Lobato et al., 2010).

Solution Strategies Originating from a Composed Unit Conception (Scalar Perspective)

A composed unit conception typically entails reasoning from a scalar multiplicative perspective, and students will often solve by building up or by finding the unit rate and iterating (Carney & Smith, 2016; Ellis, 2013; Lamon, 1993; Lobato et al., 2010; Rupley, 1981).

Building Up. A building up strategy entails scaling a given ratio up or down via either additive or scalar multiplicative processes and is a common strategy for childhood and adolescence (Hart, 1984; Hart et al., 1981; Karplus & Peterson, 1970; Lamon, 1993; Tourniaire & Pulos, 1985; Vergnaud, 1983). Students will typically use building up strategies in tasks containing integers, but have more difficulty in tasks containing non-integers (Hart et al., 1981). Sometimes two common building up strategies, doubling and halving, can also be characterized as scalar additive or scalar multiplicative, depending on how the student conceives of the process.

Unit Rate. A *unit rate* strategy entails forming a unit rate from the given ratio by partitioning it into equal-sized parts until one quantity in the ratio is one (Carney & Smith, 2016; Ellis, 2013; Lobato et al., 2010). This strategy is also called the *unit* strategy or the *unit-measure* strategy (Lamon, 1993; Rupley, 1981). Some researchers categorize the unit rate strategy as a basic component in Piaget's developmental stage of pre-proportionality, while others consider it to be a more sophisticated conception of composed unit (Lamon, 1993; Lesh, et al., 1988). Factors contributing to the use of a unit rate solution strategy include the presence of integers and the ease of divisibility of those integers (Rupley, 1981).

Factor of Change/Scalar Method. When students are given a task containing two equivalent ratios, such as a missing value problem, and identify the multiplicative scale factor relating two quantities, they are using a method commonly called "factor of change" (Cramer et al., 1993; Post, Behr, & Lesh, 1988). This is also called the "scale factor method" or simply "scalar" method as students use a scalar relationship to identify the common scale factor (Ercole, Frantz, & Ashline, 2011). For example, given the Annie Task, a student may say, "I can multiply 6 cans of yellow paint times 3 to get 18 cans of yellow paint, while also multiplying 3 cans of blue paint times 3 to get 9 cans of blue paint." Or, alternatively, they might say, "I know to get from 6 yellow cans to 18 yellow cans, I have to multiply by 3, so I am going to multiply 3 blue cans times 3 to get 9 blue cans of paint for 18 yellow cans of paint." Students use this method more often with tasks containing integers than tasks containing non-integers (Cramer & Post, 1993).

Solution Strategy Originating from a Multiplicative Comparison Conception (Functional Perspective)

A multiplicative comparison conception entails reasoning from a functional perspective by determining the constant multiplicative factor between the two quantities in the ratio relationship (Carney & Crawford, 2016). For example, given the Annie Task, a student may say, "Well, 3 is half of 6, so I am going to take half of 18 and get 9 cans of blue paint." It is important to note this conception does not necessarily lend itself to a specific strategy.

Other Solution Strategies

Strategies not specifically affiliated with either a composed unit or multiplicative comparison conception are detailed next.

Random Operations. Students who lack any specific conception of proportional reasoning might compute random operations in the given task (Lamon, 1993). This type of reasoning is often called *pre-ratio reasoning* and is considered to be a component of pre-proportionality (Inhelder & Piaget, 1958; Lesh et al., 1988).

Cross-Products Algorithm. The cross-products algorithm employs cross-multiplication and is typically used to solve missing value problems set up in $a/b=c/d$ format, and is often chosen as a solution strategy by eighth grade students who have learned the algorithm from their teachers (Cramer & Post, 1993; Cramer et al., 1993). Yet students who use this strategy may not actually be engaging in multiplicative reasoning even though they do obtain the correct solution (Carney et al., 2016; Cramer et al., 1993). However, students may be deterred from using this strategy if the tasks they are given contain integer ratios as opposed to non-integer ratios (Rupley, 1981; Vergnaud, 1983).

Task Features Influencing Cognition

Task features of proportional reasoning problems are known to influence student cognition and success in problem solving. These features include familiarity with problem context, problem type, problem content and numerical quantities in ratio relationships, and mode of task representation. The influence of each of these task features on student cognition is discussed next.

Problem Context

Research indicates the context of the problem task influences one's ability to successfully solve a given task, as well as the chosen solution strategy (Booth & Koedinger, 2012; Tourniaire & Pulos, 1985). The context of the task provides students with the situation or story as a frame of reference to derive meaning from the problem statement (Kulm, 1979). The context also affords access to the prior knowledge needed to solve the task.

Often the term *context* is used interchangeably with *content* when describing the problem situation (Kilpatrick, 1975). Yet mathematical tasks are often discussed according to their numerical quantities (mathematical content) and the problem situation (context) (Kilpatrick, 1975). For example, two tasks may contain the same numerical quantities, and thus have the same mathematical content, but differ in the topic or situation, and thus have different context (Kilpatrick, 1975). Task features primarily pertaining to context are discussed next.

Familiarity of Context

Familiarity of context is an important component in students' success in understanding proportional reasoning (Heller et al., 1989; Lamon, 1993; Rupley, 1981). Previous research suggests tasks should be based on real-life experiences of students in order to aid students in accessing their prior knowledge and personal experiences with the context (Baranes, Perry, & Stigler, 1989; Koedinger & Nathan, 2004). Likewise, contexts that can be easily visualized or imagined also aid students by allowing them to access prior knowledge (Wollman & Karplus, 1974). However, while real-world contexts can make problems more accessible, they often include vocabulary that is more difficult to read, potentially increasing the difficulty of the task (Walkington, Clinton, & Shivraj, 2018).

In their study of child street vendors in Brazil, Carraher, Carraher, & Schliemann (1985) found students had greater success solving problems informally while selling fruit on the streets than in a formal testing environment. Carraher et al., (1985) found students' solution strategies were affected by the change of context. In the informal street setting, children relied on intuitive strategies like iterating and partitioning to find the price of multiple pieces of fruit, while in the formal school setting, they relied on less meaningful traditional algorithmic procedures (Carraher et al., 1985).

Problem Type

Proportional reasoning tasks are often categorized into problem type based upon their situational context, semantic characteristics, or type of solution required (Lamon, 1993; Tourniaire & Pulos, 1985; Webb, 1984).

Situational Context: Mixture. Mixture problems involve mixing two separate quantities to create a new entity, such as determining concentrations of orange juice and water or combining two colors of paint to form a third color (Lesh et al., 1988; Lobato et al., 2010; Noelting, 1980a, 1980b; Tourniaire & Pulos, 1985).

Mixture problems are traditionally difficult for students to solve successfully (Tourniaire & Pulos, 1985). The units of measurement of the quantities as well as problem context may contribute to problem difficulty (Noelting, 1980b; Tourniaire & Pulos, 1985). The nature of mixture problems, in that both the initial quantities and the new quantity maintain the same unit of measure (i.e., cans of paint to cans of paint) may contribute to problem difficulty, unlike rate problems which compare quantities in different units (i.e., ounces to dollars) (Tourniaire & Pulos, 1985). The context of the problem situation is also a contributing factor as it is important for students to understand what is occurring as the two quantities are combined; otherwise, they might be confused and not be able to move beyond their misconceptions (Noelting, 1980b).

Semantic Characteristics: Part-Part-Whole. Proportional reasoning tasks are also categorized according to their semantic characteristics, that is, by the manner in which students thought about and operated on the given task (Lamon, 1993). When students conceive of a task in terms of a whole entity and the various parts that comprise it, they are using *part-part-whole* reasoning (Lamon, 1993). Part-part-whole problem types consistently elicited primitive building up strategies, even in students who demonstrated more sophisticated reasoning when provided with different problem types (Lamon, 1993). However, part-part-whole problems also have been found to aide students in coordinating composite units (Lamon, 1993). The Annie Task, in which cans of yellow paint (a part) and cans of blue paint (a part) to create green paint (the whole) is a part-part-whole problem type.

Solution Type: Missing Value. Finally, proportional reasoning tasks may also be categorized based upon what the task is asking the solver to provide as their answer, such as missing value problem types (Tourniaire & Pulos, 1985). Missing value problems typically present three of the four quantities needed to form a proportion and ask students to solve for the fourth corresponding value of the missing quantity (Cramer & Post, 1993; Kaput & Maxwell-West, 1994; Lamon, 2006; Tourniaire & Pulos, 1985). While previous research using missing value problem types consistently demonstrated adolescents struggled to reason proportionally, the influence of missing value problem types on student solution strategies is fairly inconclusive (Hart, 1978; Karplus et al., 1983a; Rupley, 1981; Vergnaud, 1980). While Vergnaud (1980) found his missing value problems involving ratios of corn to flour elicited a scalar strategy, Karplus et al., (1983a) found their lemonade concentration problems elicited a functional strategy. This led Karplus et al., (1983a) to conclude other factors such as the context of the problem and numerical content (presence of integral or equal ratios) were more influential than the missing value problem type in and of itself. Nevertheless, research has consistently demonstrated students who have been taught the cross-products algorithm will often solve missing value problem types via cross-multiplication (Lobato et al., 2010).

Location of the Missing Number. The location of the missing number in missing value problem types has been found to contribute to problem difficulty (Tourniaire & Pulos, 1985). When the missing value is in the second ratio relationship, students will often divide the larger quantity by the smaller quantity, without actually reasoning about what the question is asking (Lobato et al., 2010). However, complete consensus on its influence to problem difficulty has not been reached, as other research suggests students are more influenced by other factors such as the location of the integer multiplier (Carney et al., 2016).

Problem Content

Along with problem context, another influential factor in student cognition is the mathematical content of the task (Kilpatrick, 1975). In my study, I refer to mathematical content as simply *content*. Aspects of problem content that influence student cognition in proportional reasoning include the size, order, and equivalence of the numerical quantities, and the presence of integers and continuous quantities (Cramer et al., 1993;

Hart et al., 1981; Horwitz, 1981; Rupley, 1981; Tourniaire & Pulos, 1985). Each of these factors is explicated next.

Size of Numerical Quantities

The size of the numbers in the task has been found to influence problem difficulty (Rupley, 1981). Rupley (1981) calls this *numerical complexity* and found middle school students found it more difficult to solve problems when the numbers were greater than 30. Noelting (1980b) found larger numbers are generally harder to use, especially in mental manipulations, than smaller numbers.

Presence of Integers

Tasks containing integers are easier for students to solve than tasks containing non-integers because integers are more accessible to students (Cramer et al., 1993; Fernandez, Llinares, van Dooren, DeBock, & Verschaffel, 2011; Hart et al., 1981; Tourniaire & Pulos, 1985).

The presence of integers also influences student solution strategies (Karplus et al., 1983b). When faced with the option to use calculations producing integers or fractions, students often prefer strategies that avoid fractions altogether, a practice called fraction avoidance syndrome (Karplus et al., 1983b; Rupley, 1981).

Another factor regarding the presence of integers is their location within the given task (Carney et al., 2016). In their study using missing value problems, Carney et al., (2016) intentionally manipulated the location of the integer relationship to cause one ratio to have an integer relationship and the other ratio to have a non-integer relationship. They found if the integer relationship was in a location designed to press the scalar perspective, students would initially use the scalar perspective, but if the integer relationship was in a location designed to press the functional perspective, students would initially use the functional perspective (Carney et al., 2016).

Presence of Continuous Quantities

The presence of continuous quantities has been found to influence students' success in proportional reasoning (Horwitz, 1981; Pulos et al., 1981). Continuous quantities are non-countable, such as time or distance, as opposed to discrete quantities, such as pieces of gum or numbers of pennies, which are countable and are easier to visualize with mental imagery (Horwitz, 1981; Pulos et al., 1981; Shepard, 1978). Horwitz (1981) found continuous quantities were more challenging for students to use than discrete quantities and Tourniaire & Pulos (1985) caution that the presence of continuous quantities in mixture problem types may contribute to their difficulty.

Overview of Cognitive Interviews

This study used cognitive interviews to gather data, a methodology Piaget first employed to build his theory of constructivism (Piaget, 1929). Additionally, multiple researchers have used cognitive interviewing in their proportional reasoning research (Freudenthal, 1978; Karplus, Pulos, & Stage, 1980; Karplus et al., 1983a, 1983b; Lamon, 1993; Noelting, 1980a, 1980b; Piaget, 1929; Vergnaud, 1983). Cognitive interviews provide a medium to elicit student thinking via think alouds and verbal probing that do not interfere with cognitive processes (Chi, 1997; Ericsson & Simon, 1980, 1984/1993). This is advantageous because any interference with cognitive processes could decrease the validity of data, as the rest of this chapter explains.

Format and Setting of Cognitive Interviews

A common application of cognitive interviewing is the individual interview, in which a researcher assumes the role of investigator and poses tasks and questions to a student in a face-to-face setting (Beatty & Willis, 2007; Willis, 1994, 2005). When interviewing students on school campuses, the interview should take place in a relatively natural environment, such as a quiet location and be as free from distractions as possible (Eder & Fingerson, 2002; Piaget, 1929; Scott, 1997). Interviews should be relatively short, task expectations should be clear and specific, and tasks should maintain students' interest and motivation and avoid ambiguity (Borgers, Leeuw, & Hox, 2000; Henningsen & Stein, 1997; Ginsburg, 1981; Scott, 1997).

Gathering Data via Verbalizations

In cognitive interviews, the method of gathering data on participant's cognitive processes is through a verbal report, called *verbalization* (Ericsson & Simon, 1980). There are two main classes of verbalizations, *concurrent verbalizations* and *retrospective verbalizations*, which are categorized based upon the time frame when the verbalization occurs (Ericsson & Simon, 1980, 1984/1993). Concurrent verbalizations occur while the interview is taking place, while retrospective verbalizations occur after the interview is finished (Ericsson & Simon, 1980; 1984/1993). This study is primarily concerned with concurrent verbalizations produced by middle school students during cognitive interviews, and thus did not use retrospective verbalizations.

Concurrent Verbalizations

There are two main methods of concurrent verbalizations used in cognitive interviews to elicit responses from students, called *think aloud* and *verbal probing*

(Ericsson & Simon, 1980, 1984/1993; Willis, 2015). A detailed explanation of think alouds and verbal probing are discussed next.

Think Alouds. A think aloud is a form of concurrent verbalization used to elicit student responses without interfering with their thought processes as they complete a task or form a response (Ericsson & Simon, 1980, 1984/1993). Research indicates thinking aloud does not place an additional cognitive load on students, allowing the verbalizations to be as free from any cognitive interference or extraneous influence as possible (Chi, 1997; Ericsson, 2006; Ericsson & Simon, 1980, 1984/1993; Willis, 2005, 2015). In a think aloud, the interviewer instructs the student to say out loud what they are thinking as they answer the questions, or to "tell everything they can remember or are thinking of while performing the task" (American Statistical Association, 1997; Ericsson & Simon, 1980, p. 222).

For clarification purposes of think aloud instructions, researchers suggest incorporating practice problems into the interview protocol prior to the actual interview questions to help students learn to think aloud (Beatty & Willis, 2007; Ericsson & Simon, 1980, 1984/1993). However, investigators should take into account that practice problems will increase the time length of the interview and in some cases, depending on the content of the practice problems, may interfere with students' thought processes on the actual interview questions (Carney, personal correspondence, 2018; Ericsson & Simon, 1993).

Proponents of the think aloud methodology support it for its standardized procedures which leave little room for interviewer bias (Beatty & Willis, 2007; Priede & Farrall, 2011). Interviewer bias is reduced or eliminated in a think aloud because the interviewer assumes a more passive role, leaving them free to observe and take notes, while also allowing the student to stay focused on the task without worrying about interacting with the interviewer (Priede & Farrall, 2011). Likewise, the fact that the think aloud occurs during the interview itself may promote authenticity of responses and aid students in accessing their short-term memory, a critical aspect of validity of cognitive interviews (Beatty & Willis, 2007; Ericsson & Simon, 1980). Typically, information accessed from short-term memory is easier for students to recall, and thus easier for them to verbalize, as accessing information from long-term memory may take more time and students may become distracted with other ideas (Ericsson & Simon, 1980, 1984/1993).

However, some people may perform poorly on the think aloud process in general, (Beatty & Willis, 2007). When students struggle to think aloud, the collected data has less validity, as students may verbalize ideas that do not relate to the task at hand.

Levels of Verbalizations. Verbalizations are divided into three levels based upon the information attended to, or heeded, in participants' thought processes and the changes that occur to those thought processes during the verbalization process (Ericsson & Simon, 1980; 1984/1993). Each level of verbalization is explicated next.

Level 1 Verbalization. In a Level 1 verbalization, the student simply says what they are thinking out loud. The information is not changed in any way, and the interviewer is hearing exactly what the student is thinking in the moment (Ericsson & Simon, 1993). Since the student is not recoding (e.g., trying to reword their thinking so the interviewer can understand it), no additional mental effort is expended in the transfer of information from the brain into vocalization (Ericsson & Simon, 1993). The pure think aloud methodology as explicated by Ericsson & Simon satisfies this level of verbalization

(Ericsson & Simon, 1980, 1984/1993). Verbal probes by the interviewer, such as “Please continue thinking aloud”, do not change what students are thinking or how they are thinking it, and are therefore permitted to be used by interviewers in a pure think aloud situation (Ericsson & Simon, 1980, 1984/1993).

Level 2 Verbalization. A Level 2 verbalization involves providing a description or analyzation of thought processes currently in short term memory by recoding (Ericsson & Simon, 1993). In a Level 2 verbalization, no new information is brought to the attention of the student's short term memory, as they are only describing or analyzing the information currently in short term memory (Ericsson & Simon, 1984/1993). Due to the recoding process, response time may be increased, but no change in the structure of cognitive processes should occur (Ericsson & Simon, 1980, 1984/1993).

Level 3 Verbalization. A Level 3 verbalization involves providing an explanation or rationale for a specific thought process (Ericsson & Simon, 1993). In this case, the information is not simply recoded and then verbalized; it must first be linked to previous thoughts and information heeded while initially solving the task (Ericsson & Simon, 1993). This process places an additional cognitive load on the student because they now have to think about and then explain to the interviewer what they were thinking (Ericsson & Simon, 1993). In this manner, Level 3 verbalizations could affect the validity of results, as students become more concerned with the explanation of their thinking processes.

Verbal Probing. If students need to be prompted to continue verbalizing their thoughts, the interviewer may use as second type of concurrent verbalization, called *verbal probing*. These prompts may be simple remarks such as “ah-ha” or “I see” or they may be probing questions designed to elicit deeper explanations by the student, such as "How are you thinking about that?" or "How did you come up with that answer?" (Beatty & Willis, 2007; Blair & Burton, 1987; Priede, Jokinen, Ruuskanen, & Farrall, 2014).

Verbal Probe: Explanation. Other verbal probes ask students to explain why they solved a problem in the manner they did, such as by asking, "Please explain your thinking as you solve the problem" or "How were you thinking about that?", or to request clarification by asking, "Can you explain that again for me?" (Chi, 1997; Ericsson, 2006). This explanatory probing is a Level 3 verbalization which may slow down students' cognitive processes by placing an additional cognitive load on students as they respond, (Chi, 1997; Ericsson & Simon, 1980, 1984/1993). This additional cognitive load occurs because the students tend to over think or infer their reasons for using a specific solution strategy instead of simply verbalizing the strategy they actually used (Chi, 1997; Ericsson, 2006; Ericsson & Simon, 1980, 1984/1993). For this reason, explanatory probes have often been misapplied to think aloud situations, an occurrence which in the past has interfered with the acceptance of the verbal report as valid data (Ericsson, 2006; Ericsson & Simon, 1980, 1984/1993).

Methods of Data Analysis of Cognitive Interviews

There are two common methods of analyzing data from concurrent verbalizations, called *protocol analysis* (Ericsson & Simon, 1980, 1984/1993) and *verbal analysis* (Chi, 1997). These methods perceive the interviewer's relationships with the data from essentially opposing viewpoints, as explicated next.

Protocol Analysis

Protocol analysis is based on the theory of information processing in which data from the cognitive interview is analyzed via characteristics that correspond to an anticipated sequence of steps (called an ideal template) that is used to solve the task presented to the student (Chi, 1997; Ericsson, 2006; Ericsson & Simon, 1980, 1984/1993; Leighton, 2017; Willis, 2005). This ideal template is comprised of the knowledge needed to solve the given task, broken up into sequences which follow a predefined path to the solution (Chi, 1997).

Protocol analysis focuses on determining the extent to which the student's thinking followed predetermined steps to obtain the correct solution (Ericsson, 2006; Ericsson & Simon, 1980, 1984/1993; Willis, 2005). In order to figure out the exact solution steps, the researcher maps the exact words and phrases of the student onto coding categories which correspond to a scripted sequence of events indicated by the model, and determines if the exact solution steps match the anticipated coding categories (Chi, 1997). When applied to data on student cognition in proportional reasoning, protocol analysis would identify the mathematics perspectives (scalar or functional), and associated student conceptions (composed unit or multiplicative comparison) and solution strategies (scalar additive or multiplicative, doubling/halving, building up, finding the unit rate, cross-multiplication algorithm). For example, given the Annie Task, an information processing approach might indicate students first identify a "times 3" scalar relationship between 6 and 18 (a scalar mathematics perspective) and then multiply 3 times 3 to obtain the solution of 9 (scalar multiplicative solution strategy). Alternatively, a different solution path for the same question could be that students first identify the "divide by 2" or "times ½" multiplicative relationship between 6 and 3 (functional mathematics perspective) and then divide 18 by 2 (multiplicative comparison conception) in order to obtain a solution of 9. The specific steps the student uses to find the correct solution as well as the efficiency of the solution process is studied and mapped back on to the anticipated steps built by the information processing model.

Verbal Analysis

Verbal analysis, on the other hand, focuses on creating categories for data analysis based upon what the students reveal during the interview process (Chi, 1997). Instead of focusing on the ability of the student to find a correct solution, a researcher employing verbal analysis focuses on the knowledge the student has and how they represent that knowledge during verbalization (Chi, 1997). The primary goal of verbal analysis is to "seek the model that a…[student]…has, without creating an ideal template a priori" (Chi, 1997, p. 277). During verbalization, the student will give clues about their thinking by what they say, how they act, how they move their hands or point to an object or picture, which can then be studied to determine what the student knows and understands about a given topic, as well as how their knowledge influences they way they choose to solve the given tasks (Chi, 1997).

Advantages of Combining Protocol Analysis and Verbal Analysis

Protocol analysis can be used to see how students' solution strategies match researchers' anticipated strategies. Yet verbal analysis is advantageous because it allows researchers to analyze data without preconceived notions of student responses, which may help alleviate researcher bias (Chi, 1997). Additionally, verbal analysis provides a sense of freedom for the researcher to see if new categories develop in the data without feeling constrained by previously anticipated categories. When both protocol analysis and verbal analysis are employed simultaneously to analyze the same data set, the researcher reaps the benefits of knowing where certain questions might lead and how they might influence student responses, while also being open to receiving new knowledge as it arises and potentially being able to discover new information for their field of research.

Summary

Cognitive interviewing may be used to investigate the influence of iconic mathematical models (i.e., tape diagrams and bar models) on students' cognition in proportional reasoning situations. In order to maintain high validity of results, students should think aloud or explain their thinking while the interview is in progress. To analyze the results, data may be sorted into pre-determined categories (i.e., protocol analysis) and emergent categories (i.e., verbal analysis).

CHAPTER THREE: METHODOLOGY

This study reports the results of cognitive interviews regarding the influence of ratio relationships and iconic mathematical models on middle school students' cognition in proportional reasoning. The purpose was to determine the influence of tape diagrams and bar models given three ratio relationships (6: 3, 8: 2, 5: 2) in a proportional reasoning task within the context of mixing yellow and blue paint to create green paint. Cognitive interviews were conducted with 47 middle school students in two western states. The interviews provided a glimpse into students' thinking processes when solving problems designed to elicit the multiplicative comparison conception.

Research Questions

This research was guided by the following questions:

1. In what ways do the mathematical models of tape diagram and bar model influence students' use of the multiplicative comparison conception and associated solution strategies in proportional reasoning situations?
2. What additional factors, such as the ratio relationships 6:3, 8:2, and 5:2, influence student solution strategies in relation to the bar models and tape diagrams?

Research Design

Participants

Participants were 47 middle school students from a total of five schools in two western states. Please note that the following demographic information was obtained from each state's Department of Education website; however, exact sources are not listed in order to protect participants' identities. In one western state, seven students (6^{th} grade: $n = 7$) were from a small, non-rural, private school with a negligible economically disadvantaged and English Learner population, while 15 students (7^{th} grade: $n = 4$; 8^{th} grade: $n = 11$) were from a large, non-rural public school with >95% socioeconomically disadvantaged and >40% English Learner population. In another western state, five students (6^{th} grade: $n = 4$; 7^{th} grade: $n = 1$) were from a large, non-rural public school with a >99% low-income and <15% English Learner population, while another 12 students (7^{th} grade: $n = 6$; 8^{th} grade: $n = 6$) were from a small, non-rural, public charter school, with >40% low-income and <5% English Learner population. A final eight students (7^{th} grade: $n = 1$; 8^{th} grade: $n = 7$) were from a large, non-rural public school with a >30% low-income population and <5% English Learner population. Please note the percentages of socioeconomically disadvantaged, low-income, and English Learner populations have been rounded to the nearest whole number. See Table 1 for a table of population demographics.

Table 1: Sample Sizes by Grade Level, with School Demographics

School	Grade Level 6th	7th	8th	% of Socio-economically Disadvantaged or Low-Income Students	% of English Learners
School 1	7	0	0	negligible	negligible
School 2	0	4	11	>95%	>40%
School 3	4	1	0	>99%	<15%
School 4	0	6	6	>40%	<5%
School 5	0	1	7	>30%	<5%

Setting

The interviews occurred in a quiet location on the students' school campuses and were approximately 20 minutes each. I videotaped and transcribed all interviews. Since some students were hesitant to have their faces videotaped, I pointed all cameras at the students' papers so as to only capture their voices and gestures as they explained their work on the paper. Students were provided with the interview tasks on paper and were given a pen or pencil and extra paper if needed.

Measurement Instrument

The measurement instrument I used to investigate the influence of tape diagrams and bar models was an interview protocol created to ascertain the influence of mathematical models (tape diagrams and bar models) and ratio relationships (6:3, 8:2, 5:2) on students' cognition in proportional reasoning.

Design of the Interview Protocol

Context. The context of the interview protocol was mixing yellow and blue paint to create green paint according to specific ratio relationships (6 yellow cans: 3 blue cans; 8 yellow cans: 2 blue cans; 5 yellow cans to 2 blue cans). See Appendix A for a copy of the interview protocol. In order to limit the influence of a change of context, the problem context is held constant across all tasks (Carney et al., 2015).

Content

Mathematical Models. The interview protocol was comprised of two different problem sets, one which included a tape diagram (Tape Diagram Set), and the other which included a bar model (Bar Model Set). Students were randomly selected to receive either the Tape Diagram Set or Bar Model Set as their interview protocol. No student received both problem sets. See Figure 13 for the Mathematical Models in each problem set.

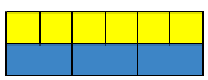
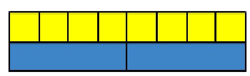
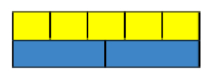
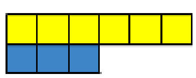
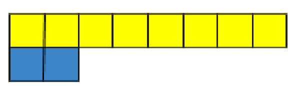
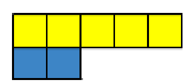
Context Paint	**Ratio Relationships**		
Problem Set	**6 to 3 yellow to blue**	**8 to 2 yellow to blue**	**5 to 2 yellow to blue**
Tape Diagram Set	 **Annie 1**	 **Bailey 1**	 **Carlos 1**
Bar Model Set	 **Annie 2**	 **Bailey 2**	 **Carlos 2**

Figure 13. Mathematical Models

Ratio Relationships. Each problem set was comprised of three parts according to ratio relationship (Part A: 6 to 3, Annie; Part B: 8 to 2, Bailey; Part C: 5:2, Carlos) (see Figure 14). In order to determine the influence of the ratio relationships independent of the mathematical model (tape diagram or bar model) provided in each task, both the Tape Diagram Set and Bar Model Set holds the mode of representation constant while varying the numbers in each task. For example, the Tape Diagram Set contains three tape diagram tasks with the varying ratio relationships of 6:3, 8:2, and 5:2, respectively. The Bar Model Set contains three bar model tasks with the same varying ratio relationships. Due to the design of the interview protocol, the influence of the numbers can be studied within the given models.

The ratio relationships of 6:3, 8:2, and 5:2 were chosen specifically because they form a trajectory of increasing difficulty across the three tasks. The ratio of 6:3 is provided first, because the multiplicative relationship between 6 yellow cans and 3 blue cans of paint is the easiest of the three given ratio relationships. If blue paint is the base quantity or unit of measure, the multiplicative relationship is that the yellow paint is two times more than the blue paint. If the yellow paint is the base quantity or unit of measure, the multiplicative relationship is that the blue paint is half the amount of the yellow paint.

The second ratio provided is 8:2, which has a slightly harder multiplicative relationship, as the yellow paint is four times more than the blue paint; or conversely, the blue paint is a quarter of the yellow paint, depending on which quantity is identified as the base quantity or unit of measure. Both ratio relationships of 6:3 and 8:2 are easier than the third ratio of 5:2, since both the ratios of 6:3 and 8:2 contain multiplicative relationships that can be expressed as integers (Rupley, 1981). The ratio 5:2 is significantly harder because the multiplicative relationship contains non-integers (Cramer et al., 1993; Hart et al., 1981; Rupley, 1981; Tourniaire & Pulos, 1985). The yellow paint is two and a half times the blue paint, or alternatively, the blue paint is two-fifths of the yellow paint. See Figure 14 for a visual of the multiplicative relationships in the tasks.

6 to 3	8 to 2	5 to 2
x2 $\frac{\text{6 yellow}}{\text{3 blue}}$ $\div 2$ or x $\frac{1}{2}$	x4 $\frac{\text{8 yellow}}{\text{2 blue}}$ $\div 4$ or x $\frac{1}{4}$	x2.5 $\frac{\text{5 yellow}}{\text{2 blue}}$ $\div 2.5$ or x $\frac{2}{5}$

Figure 14. Ratio Relationships

The location of the larger integer in each ratio relationship is held constant across all three relationships. The larger integer is provided first in each relationship to alleviate any effect created by providing the smaller integer first (Rupley, 1981), although the multiplicative relationship between the two quantities in the ratio remains constant no matter which integer is provided first.

<u>Organization of Parts A, B, and C</u>

Each part (A, B, or C) consisted of six questions divided into three sections (Pre-test, Intervention, and Post-test). Since each problem set was designed to assess the influence of a mathematical model (tape diagram or bar model), I created a pre-test to determine what students knew prior to being shown the model. The intervention portion of the interview provided the mathematical model along with two questions. I created a post-test to determine the influence of the model.

Pre-test and Post-test. The pre-test and post-test were identical for each part (A, B, and C). The pre-test and post-test consisted of two questions each, designed to elicit the multiplicative comparison conception. See Figure 15 for the design of the pre-test and post-test.

Intervention. The main body of the interview (consisting of the tape diagram or bar model in conjunction with Questions 3 and 4) was designed to be a learning intervention. Students were given either a bar model or a tape diagram and asked to solve two proportional reasoning questions. Question 3 provided students with the mathematical model and the given ratio and asked them to find the answer to a missing value problem. Question 4 was designed to see if students could generalize the ratio relationship to a broader setting. Question 4 was the same across all parts of the interview protocol. See Figure 15 for the design of the intervention.

	Interview Questions	Correct Responses		
		Part A: 6 to 3	Part B: 8 to 2	Part C: 5 to 2
Pre-Test	1.) The yellow paint is always _______ times the blue paint.	two	four	two and a half
	2.) The blue paint is always ______ times the yellow paint.	one half	one fourth	two-fifths
Intervention	3.) If __________ has ____ cans of yellow paint, how many cans of blue paint does she need to make the same color green as before?	Given: Annie and 18 Answer: 9 cans of blue paint	Given: Bailey and 16 Answer: 4 cans of blue paint	Given: Carlos and 20 Answer: 9 cans of blue paint
	4.) If you were going to the store to buy more paint for _________, how would you think of the relationship between yellow and blue paint to make sure you can make the same color of green?	The yellow paint is always twice the blue paint. OR The blue paint is always half of the yellow paint.	The yellow paint is always four times the blue paint. OR The blue paint is always one-fourth of the yellow paint.	The yellow paint is always two and a half times the blue paint. OR The blue paint is always two-fifths of the yellow paint.
Post-Test	5.) The yellow paint is always _______ times the blue paint.	two	four	two and a half
	6.) The blue paint is always ______ times the yellow paint.	one half	one fourth	two-fifths

Figure 15. Design of Interview Protocol

For ease of explanation from this point on, Question 1 from the pre-test and Question 5 from the post-test will be referred to as whole number relationship, while

Question 2 from the pre-test and Question 6 from the post-test will be referred to as the fractional relationship.

Data Collection

Prior to contacting schools about participating in my study, I gained approval from the Boise State University Institutional Review Board to conduct the study and received approval for all related forms (e.g., Principal Acknowledgement Form, Teacher Acknowledgment Form, Parent Informed Consent, Student Assent, and Interview Protocol).

Gaining School Participation

To find students to interview, for three schools, I contacted teachers and principals that I knew personally to request their participation in my study. For two other schools, my advisor reached out to teachers she thought would respond favorably to a request for participation.

Principal and Teacher Cooperation

Each principal and teacher who agreed their students could participate in the study were given an acknowledgment form to sign. The Principal Acknowledgment Form and Teacher Acknowledgment Form are in Appendices B and C, respectively. Via email communications, we determined an appropriate time for me to interview their students that would be least disruptive to their school schedules. All interviews took place in spring 2019.

Parent Informed Consent and Student Assent

In order to conduct the interviews with students, I provided teachers with a parent consent form and a student assent form, either a paper copy, electronic copy, or both.

Upon approval from the Boise State University Institutional Review Board, I combined the signature page for the parent consent and student assent forms and stapled both together. Since I was reaching out to schools with high populations of English Learners with parents who spoke Spanish, I provided both forms in English and Spanish. These forms are in Appendix D.

Population of Students

Although research suggests a representative sampling of students provides the most valid results, this was difficult to achieve in the context of this study. My student population was inherently narrow, as it was dependent on the students being present at school on the day of the interviews and also remembering to return the parent consent and student assent forms.

On the day of the interviews at a specific school site, I entered the teachers' classrooms and collected the parent consent and student assent forms that students had returned. Then I would either randomly select a student to interview in that class period, or the teacher would select a student who was available (maybe they had just finished an assignment, or the teacher knew the student would be leaving early that class period and needed to be interviewed sooner versus later). The interviews took place in a quiet room on campus, either a conference room or classroom of a teacher who was on prep.

Student Considerations

Some students were English Learners and their teachers suggested I conduct the interviews in Spanish. In the privacy of the interview setting, I asked these students, in Spanish, if they would like to speak in Spanish for the interview. However, all students said they wanted to speak in English. This may have implications for their understanding of the interview questions, which is discussed in Chapter 5.

Data Analysis

Transcription

I videotaped and transcribed all interviews. I also photographed the students' work and added the photographs to each individual transcription document for reference.

Approach to Analysis

For this study, I chose to analyze changes in students' responses across the interview protocol from pre-test to post-test that likely occurred as a result of the intervention (bar model or tape diagram). I did not analyze interview protocol parts in which students did not change answers (i.e. no changes in answers from pre-test to post-test). I did analyze interviews where answers changed from pre-test to post-test, as all students explained their thinking when they changed their answers (i.e., incorrect to correct, incorrect to a different incorrect, or correct to incorrect).

Types of Changes in Students' Responses

Since I analyzed changes in students' responses from pre-test to post-test, I created codes for the types of changes that occurred, based on the likelihood of the solution strategies to move students toward a multiplicative comparison conception. The codes that I used to determine if changes in students' responses (as a result of exposure to the different models) were positive, inconclusive, or negative. A positive change indicated a student moved from a strategy unlikely to move students toward a multiplicative comparison conception to a strategy more likely to move students toward a multiplicative comparison conception or to the multiplicative comparison conception directly. A negative change indicated a student moved from a multiplicative comparison or a likely strategy to a different or less likely strategy. An inconclusive change indicated a student moved from one unlikely strategy to another equally unlikely strategy. Responses in which answers did not change from the Pre-test to Post-test were not analyzed. The following analysis is based on looking at changes that occurred from pre-test to post-test, without analyzing the answers to the intervention questions.

Dual Coding

In order to consistently determine whether a change in students' responses was positive, inconclusive, or negative, each student's response was given two codes. The first code indicates whether the answer was correct (c-) or incorrect (inc-). The second code indicates the solution strategy used to solve the problem.

Coding Categories

Six coding categories were created, some of which were pre-determined by a review of proportional reasoning literature (via protocol analysis) and others which emerged from the data (via verbal analysis). The six categories are delineated according to the likelihood they indicated the student was moving towards a multiplicative comparison conception. The coding categories are as follows: strategies that elicited a multiplicative comparison conception, strategies that elicited a composed unit conception, strategies likely to move students towards a multiplicative comparison conception, strategies not likely to move students towards a multiplicative comparison conception, indeterminate response, and no response. The coding categories and the strategies that comprise them are explicated next.

Strategies that Elicited a Multiplicative Comparison Conception (MC)

A response was categorized as a strategy that elicited a multiplicative comparison the student described the same as the multiplicative relationship provided in the given problem. Each response that contained the multiplicative comparison conception was coded as correct (c-). See Figure 16 for a sample response of a multiplicative comparison conception.

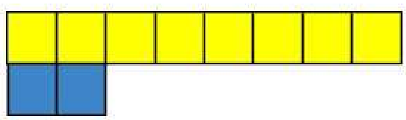
<table>
<tr><th colspan="2">Prompt: Intervention Question 3</th></tr>
<tr><td colspan="2">If Bailey has 16 cans of yellow paint, how many cans of blue paint does she need to make the same color green as before?</td></tr>
<tr><th>Model</th><th>Ratio Relationship</th></tr>
<tr><td></td><td>8 yellow: 2 blue</td></tr>
<tr><th colspan="2">Student’s Verbal Response</th></tr>
<tr><td colspan="2">So basically, on all the answers, you could times by four or divide by four, so I just figured if you have blue cans, you just times it by four to get the amount of paint to make the same tint of color, or you can get as many blue and make sure it’s divided by four to get the same amount of paint.</td></tr>
</table>

Figure 16. Multiplicative Comparison Conception

Strategies that Elicited a Composed Unit Conception (CU)

A response was categorized as a strategy eliciting a composed unit conception if the student scaled the given ratio up or down by a scalar multiplicative or scalar additive strategy. Each response that contained a composed unit conception was coded as correct (c-). See Figure 17 for a sample response for a composed unit scalar multiplicative conception and Figure 18 for a sample response for a composed unit scalar additive conception.

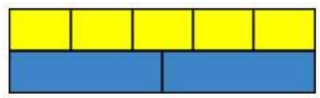
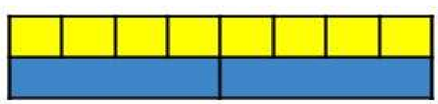
<table>
<tr><th colspan="2">Prompt: Intervention Question 3</th></tr>
<tr><td colspan="2">If Carlos has 20 cans of yellow paint, how many cans of blue paint does he need to make the same color green as before?</td></tr>
<tr><th>Model</th><th>Ratio Relationship</th></tr>
<tr><td></td><td>5 yellow: 2 blue</td></tr>
<tr><th colspan="2">Student’s Verbal Response</th></tr>
<tr><td colspan="2">It says that if he has twenty cans of yellow paint, how many cans of blue paint does he need to make the same color green before? Twenty is four times five, so if you multiply two by four, you get eight, and so he will need eight cans of blue paint to make the same color of green that he had before.</td></tr>
</table>

Figure 17. Composed Unit Conception (Scalar Multiplicative)

<table>
<tr><th colspan="2">Prompt: Intervention Question 3</th></tr>
<tr><td colspan="2">If Bailey has 16 cans of yellow paint, how many cans of blue paint does she need to make the same color green as before?</td></tr>
<tr><th>Model</th><th>Ratio Relationship</th></tr>
<tr><td></td><td>8 yellow: 2 blue</td></tr>
<tr><th colspan="2">Student’s Verbal Response</th></tr>
<tr><td colspan="2">Student: So if she wants sixteen you would have to, um, like you have to do four - one, two, three, four [student pointing at first set of yellow squares on left side of tape diagram], one, two, three, four [student pointing at second set of yellow squares on right side of tape diagram], one, two, three, four [counting four more invisible yellow squares if tape diagram were to continue] and then you do one, two, three, four [counting four more invisible yellow squares if tape diagram were to continue] which would equal sixteen but you have to add two more blue paints [pointing at two blue rectangles on tape diagram] to make green.

Investigator: Excellent. So what would your total answer be? If you had sixteen yellow, how many blue would you need?

Student: Four. Four blue.</td></tr>
</table>

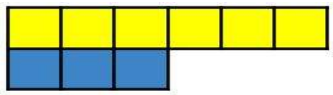
Figure 18. Composed Unit Conception (Scalar Additive)

Strategies Likely to Move Students toward a Multiplicative Comparison Conception

A response was categorized as likely to move students toward a multiplicative comparison conception if it utilized one of the following solution strategies: qualitative description of the relationship (QDR), using negative numbers (UNN), indentifying an inverse multiplicative relationship (MR-INV), or an incorrect additive strategy (IAS). While these solution strategies generally led students closer to a multiplicative comparison conception, their initial coding category is incorrect (inc-) since the answer did not explicitly satisfy the multiplicative comparison conception evidence statement.

Qualitative Description of the Relationship (QDR)

Some students responded to the given statements in the pre-test and post-test by adding words such as “more”, “less”, “larger”, or “smaller” into the statements to qualify their answers. Although they have the general idea of the multiplicative comparison conception and were trying to express it, they were unable to explain it in a format that satisfied the statement response of “one fourth”. See Figure 19 for a sample response.

<table>
<tr><th colspan="2">Prompt: Intervention Question 2</th></tr>
<tr><td colspan="2">The blue paint is always ______ times the yellow paint.</td></tr>
<tr><th>Model</th><th>Ratio Relationship</th></tr>
<tr><td></td><td>6 yellow: 3 blue</td></tr>
<tr><th colspan="2">Student’s Verbal Response</th></tr>
<tr><td colspan="2">Student: Uh, it’s like the same as the first [question], except um, they just switched out the words, so it’s two times less than the yellow paint, instead of more than the yellow paint, I mean, than the blue paint- wait, yeah.

Investigator: Okay, great. Now tell me what you mean by the word “less”. Explain that a little bit more.

Student: Um, like, when, like, cause it’s a- cause you have six and three, and um, there’s six- three plus three equals six and then, mmm, yep, like technically it’s like six divided by two, to which you get- which you subtract, so- well, which is kind of like subtracting, so you get, um, three cans instead of six cans of bl- of blue paint, and yeah.</td></tr>
</table>

Figure 19. Qualitative Description of the Relationship

<u>Using Negative Numbers (UNN)</u>

A response was categorized as using negative numbers if students added a negative sign to their written answer. For example, a student would say “the blue paint is negative two times the yellow paint”. See Figure 20 for a sample response.

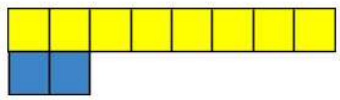
<table>
<tr><th colspan="2">Prompt: Intervention Question 2</th></tr>
<tr><td colspan="2">The blue paint is always _______ times the yellow paint.</td></tr>
<tr><th>Model</th><th>Ratio Relationship</th></tr>
<tr><td></td><td>8 yellow: 2 blue</td></tr>
<tr><th colspan="2">Student’s Verbal Response</th></tr>
<tr><td colspan="2">And then on this one I put negative four to try and divide it again.</td></tr>
</table>

Figure 20. Sample Response for Using Negative Numbers

<u>Identifying an Inverse Multiplicative Relationship (MR-INV)</u>

A response was categorized as an inverse multiplicative relationship if the answer provided by the student in the questions designed to elicit a fractional answer (Q2 or Q6) was the inverse multiplicative relationship of the answer provided in Questions 1 and 5 (see explanation provided previously). See Figure 21 for a sample response.

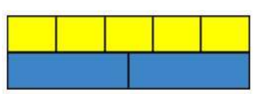
<table>
<tr><th colspan="2">Prompt: Post-test Questions 5 and 6</th></tr>
<tr><td colspan="2">Question 5: The yellow paint is always ______ times the blue paint.
Question 6: The blue paint is always ______ times the yellow paint.</td></tr>
<tr><th>Model</th><th>Ratio Relationship</th></tr>
<tr><td></td><td>5 yellow: 2 blue</td></tr>
<tr><th colspan="2">Student’s Verbal Response</th></tr>
<tr><td colspan="2">The yellow paint is always two point five times the blue paint because if you multiply two by two point five, it’s gonna equal five, and so he’ll need two and a half times the blue paint that he has for yellow paint.</td></tr>
</table>

Figure 21. Inverse Multiplicative Relationship

Incorrect Additive Strategy (IAS)

A response was categorized as an incorrect additive strategy if the student used addition or subtraction in situations that are actually multiplicative. Students who used this strategy were able to identify the multiplicative relationship, but then used addition instead of multiplication to solve the problem. See Figure 22 for a sample response.

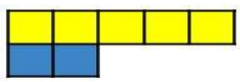
<table>
<tr><th colspan="2">Prompt: Intervention Question 3</th></tr>
<tr><td colspan="2">If Carlos has 20 cans of yellow paint, how many cans of blue paint does he need to make the same color green as before?</td></tr>
<tr><th>Model</th><th>Ratio Relationship</th></tr>
<tr><td></td><td>5 yellow: 2 blue</td></tr>
<tr><th colspan="2">Student’s Verbal Response</th></tr>
<tr><td colspan="2">Twenty minus three because five minus two equals three so you want to take away three from twenty which is seventeen.</td></tr>
</table>

Figure 22. Incorrect Additive Strategy

Incorrect Mathematical Calculation (IC)

Some students had the correct concept but struggled in their calculations. Given the 5:2 ratio relationship, many students struggled to divide five by two. See Figure 23 for a sample response.

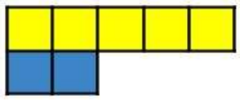
Prompt: Pre-test Question 1	
The yellow paint is always ________ times the blue paint.	
Model	**Ratio Relationship**
	5 yellow: 2 blue
Student's Verbal Response	
Student's Written Answer: 1.5 Okay, so number one, um, one- if you do two plus two that's four, and so half of two is one, and then you add that onto that and that equals five.	

Figure 23. Incorrect Mathematical Calculation

<u>Strategies that Were Not Likely to Move Students Toward a Multiplicative Comparison Conception</u>

Some strategies did not move students toward the multiplicative comparison conception and did not produce correct answers. These were initially coded as incorrect (inc-). These strategies are as follows: recreating the given ratio, recreating the given ratio with unit rate, inverse of recreating the given ratio, operations with the given numbers in the ratio (i.e., addition, subtraction, or multiplication), stating the quantities must be equal, or providing a non-mathematical response. Each strategy is explained next with corresponding examples from the data.

<u>Recreating the Given Ratio (RGR)</u>

Some students answered the two statements in the pre-test or the two statements in the post-test by providing answers that when combined, recreated the given ratio. When provided the statement "The yellow paint is _________ times the blue paint" a student would respond with the first number in the ratio relationship. When provided the statement "The blue paint is _________ times the yellow paint", the student would respond with the second number in the ratio relationship. Sometimes students would recreate the given ratio but use the unit rate instead. See Figure 24 for a sample response

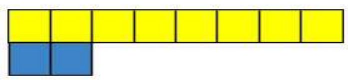
Prompt: Pre-test Questions 1 and 2	
Question 1: The yellow paint is always _______ times the blue paint. **Question 2:** The blue paint is always ______ times the yellow paint.	
Model	**Ratio Relationship**
	8 yellow: 2 blue
Student's Verbal Response	
Student: The yellow paint is always eight times the blue paint. And the blue paint is always two times the yellow paint. Investigator: Okay, so tell me how you got those numbers. Student: Well, I think, um, I got eight because she says- because it's saying that she has eight cans of yellow paint, so I would think it would be eight because you're adding eight into the two of the blue cans. So that's how I got the two times the yellow paint.	

Figure 24. Recreating the Given Ratio

Recreating the Given Ratio with Unit Rate (RGR-UR)

Some students recreated the given ratio as previously described, but they would state the unit rate for their answers. See Figure 25 for a sample response.

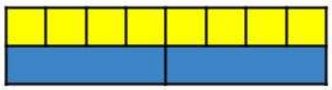
Prompt: Pre-test Questions 1 and 2	
Question 1: The yellow paint is always _______ times the blue paint. **Question 2:** The blue paint is always ______ times the yellow paint.	
Model	**Ratio Relationship**
	8 yellow: 2 blue
Student's Verbal Response	
The yellow paint is four times more than the blue cans of paint and the blue is like, only one time as the yellow cans of paint because it's, like, less.	

Figure 25. Recreating the Given Ratio with Unit Rate

<u>Inverse of Recreating the Given Ratio (RGR-INV)</u>

Some students recreated the given ratio as explained above, but they would describe the inverse relationship. See Figure 26 for a sample response.

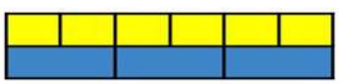
Prompt: Post-test Questions 5 and 6	
Question 5: The yellow paint is always _____ times the blue paint. **Question 6:** The blue paint is always _____ times the yellow paint.	
Model	**Ratio Relationship**
	6 yellow: 3 blue
Student's Verbal Response	
I think it might be one, cause of that, like, drawing of it [referencing tape diagram provided in Questions 3 and 4] cause it had, like, two yellows and then one blue, so that may be one times- maybe it's two, because like there's two of the yellows if you want- wait, the yellow paint is always, wait, hold on. [Student writing answer of 1 in blank for Question 5]. Um, so when I did the chart thing, it was one- so if I had one I believe it was blue, then it would be two yellows, so I got that for that, one over two [pointing to answer to question five and then answer to question six] like a ratio table, kind of.	

Figure 26. Inverse of Recreating the Given Ratio

Operations with the Given Numbers in the Ratio (OPER-ADD, OPER-SUB, OPER-MULT)

Some students operated on the given numbers in the ratio relationship by adding them together, subtracting them, or multiplying them in order to obtain an answer. This strategy is different than the incorrect additive strategy above, because in this case, students were not scaling additively when they should have been scaling multiplicatively. See Figure 27 for a sample response.

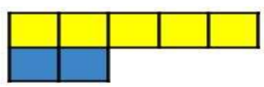
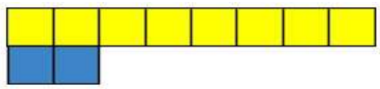
Prompt: Pre-test Question 1	
The yellow paint is always __________ times the blue paint.	
Model	**Ratio Relationship**
	5 yellow: 2 blue
Student's Verbal Response	
The yellow paint is always three times the blue paint, because there's five cans of yellow and two cans of blue, and when you subtract them, there's three more yellow cans than the blue paint.	

Figure 27. Subtracting Given Numbers in Ratio Relationship

Equal Quantities (EQ)

Some students thought the quantities of yellow paint and blue paint had to be "even" or equal. See Figure 28 for a sample response.

<table>
<tr><th colspan="2">Prompt: Intervention Question 4</th></tr>
<tr><td colspan="2">If you were going to the store to buy more paint for Bailey, how would you think of the relationship between yellow and blue paint to make sure you can make the same color of green?</td></tr>
<tr><th>Model</th><th>Ratio Relationship</th></tr>
<tr><td></td><td>8 yellow: 2 blue</td></tr>
<tr><th colspan="2">Student’s Verbal Response</th></tr>
<tr><td colspan="2">Student: I put you will have to see how much paint you would need and how much to make it even.

Investigator: Excellent. Okay, so let’s say you needed sixteen cans of yellow paint. How much blue would you need?

Student: Wouldn’t you need sixteen blue?</td></tr>
</table>

Figure 28. Equal Quantities

Non-Mathematical Content (NMC)

Some responses did not contain mathematical content, or the mathematical content provided did not match the ratio relationships and context in the given questions. See Figure 29 for a sample response.

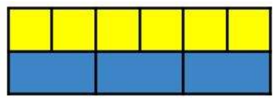
Prompt: Intervention Question 4	
If you were going to the store to buy more paint for Annie, how would you think of the relationship between yellow and blue paint to make sure you can make the same color of green?	
Model	**Ratio Relationship**
	6 yellow: 3 blue
Student's Verbal Response	
So what I wrote is I think I would think them as a well relationship between those two colors since they both are primary colors making a secondary color.	

Figure 29. Non-Mathematical Content

Indeterminate Response (IND)

A response was categorized as "indeterminate" if the student explanation did not provide enough evidence of a specific conception or solution strategy. See Figure 30 for a sample response.

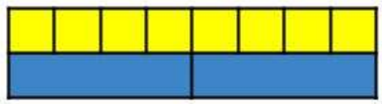
Prompt: Intervention Question 4	
If you were going to the store to buy more paint for Bailey, how would you think of the relationship between yellow and blue paint to make sure you can make the same color of green?	
Model	**Ratio Relationship**
	8 yellow: 2 blue
Student's Verbal Response	
Student: So about proportionality, if you want to make it the same color of green, you would have to do the same amount of blue and yellow paint [you] got the first round. Investigator: Excellent. Very good. So let's say, how much would you have gotten on the first round? Student: Well, it says it has eight cans of yellow paint, so you would get the same- you would get eight cans of the same yellow paint, like you can also get, like, the same brand if you really want it to be the same color. And same thing for the blue paint.	

Figure 30. Indeterminate Response

<u>No Response</u>

A response was categorized as "No Response" if the student left the question blank, wrote a question mark, or wrote or verbally stated "I don't know" as their response.

<u>Coding Flowchart of Solution Strategies</u>

A codebook for each of the solution strategies previously described is in Appendix E. See Figure 31 for a flowchart for the coding of solution strategies according to the type of change in the response from the pre-test to post-test (i.e., whole number relationship (Q1 to Q5) or fractional relationship (Q2 to Q6)).

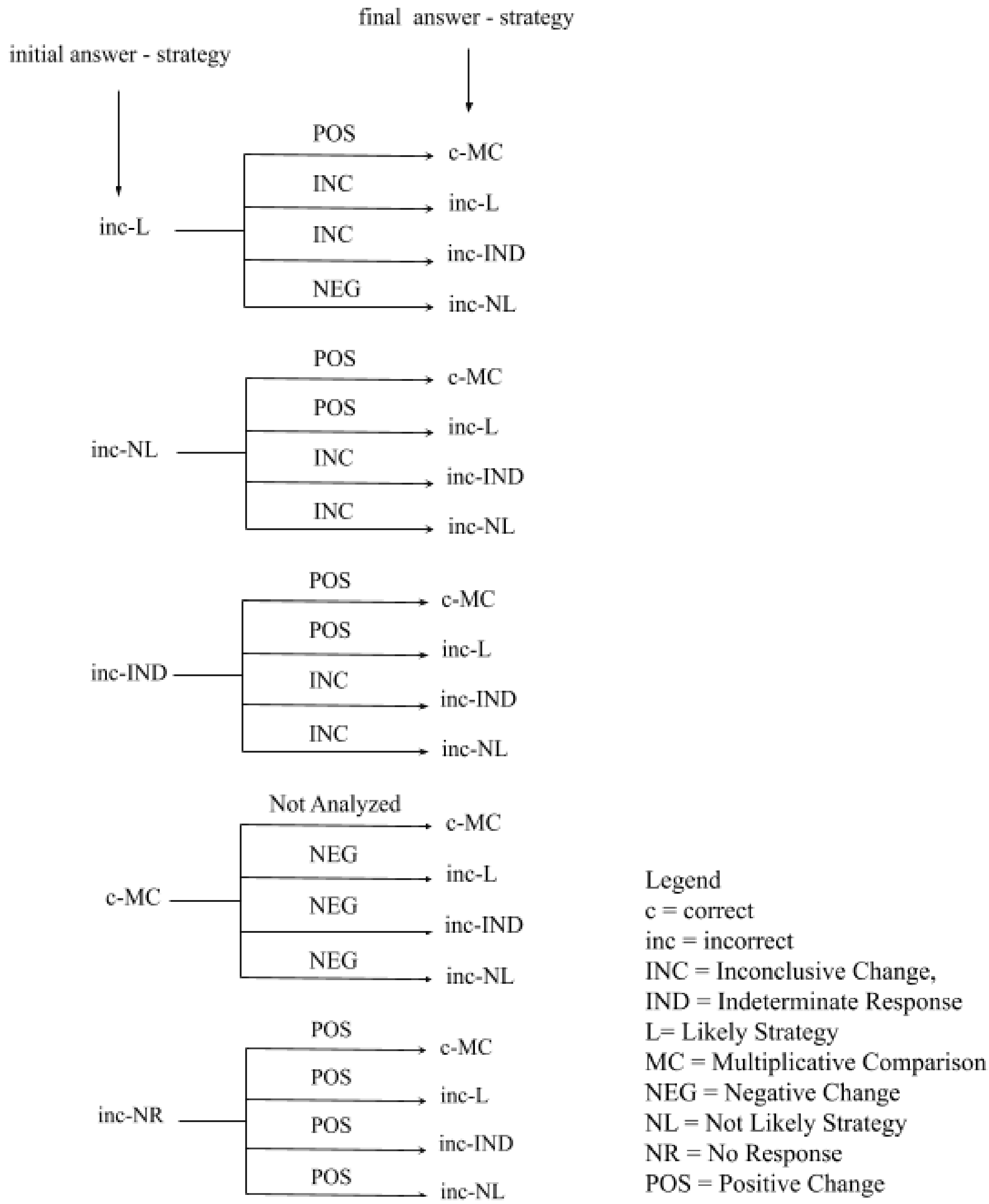


Figure 31. Flowchart of Solution Strategies according to Type of Change in Response

CHAPTER FOUR: RESULTS

The results presented in this study were analyzed based upon changes in responses from pre-test to post-test (i.e., whole number relationship (Q1 to Q5) and fractional relationship (Q2 to Q6)). The research questions I sought to answer as a result of this analysis are as follows:

Research Questions

1. In what ways do the mathematical models of tape diagram and bar model influence students' use of the multiplicative comparison conception and associated solution strategies in proportional reasoning situations?
2. What additional factors, such as the ratio relationships 6:3, 8:2, and 5:2, influence student solution strategies in relation to the bar models and tape diagrams?

Results

The results are organized according to the model (bar model or tape diagram) and the type of changes that occurred in student responses from pre-test to post-test.

Results for Whole Number Relationship (Q1 to Q5)

The results for the whole number relationship, Question 1 on the Pre-test to Question 5 on the Post-test are presented according to the mathematical model provided in the protocol.

Bar Model Results for Whole Number Relationship (Q1 to Q5)

There were a total of 61 instances in which students were given the bar model (Part A: $n = 22$, Part B: $n = 20$, Part C: $n = 19$). In Part A, there were four instances in which students' answers changed from pre-test to post-test. Three were positive changes, zero were inconclusive changes, and one was a negative change. In Part B, there were also four instances in which students' answers changed from pre-test to post-test. Two were positive changes, two were inconclusive changes, and zero were negative changes. In Part C, there were two instances in which students changed their answers from pre-test to post-test. One instance was a positive change and the other instance was an inconclusive change. See Table 2 for a chart of the results.

Tape Diagram Results for Whole Number Relationship (Q1 to Q5)

There were a total of 65 instances in which students were given the tape diagram (Part A: $n = 25$; Part B: $n = 21$; Part C: $n = 19$). In Part A, there were five instances in which students changed their answers from pre-test to post-test. One was a positive change, two were inconclusive changes, and one was a negative change. In Part B, there was one instance where a student changed their answers from pre-test to post-test, and this was a negative change. In Part C, there were five instances in which students changed their answers from pre-test to post-test. Two were positive changes, two were inconclusive changes, and one was a negative change. See Table 2 for the number of instances and type of change that occurred for each protocol part for the whole number relationship (Q1 to Q5).

Table 2: Numbers of instances and type of change when students changed their solution approach from pre- to post-test across the whole number relationship (i.e., changes from Q1 to Q5)

Whole Number Relationship Pre-test to Post-test (Q1 to Q5)						
Model	Protocol Part	*n*	Changed Answers	Change		
				Positive	Inconclusive	Negative
Bar Model	A	22	4	3	0	1
	B	20	4	2	2	0
	C	19	2	1	1	0
	Total	61	10	6 (60%)	3 (30%)	1 (10%)
Tape Diagram	A	25	5	1	2	2
	B	21	1	0	0	1
	C	19	5	2	2	1
	Total	65	11	3 (27%)	4 (36%)	4 (36%)

If a student changed answers on more than one Protocol Part (A, B, or C), each part was analyzed as a separate instance. Therefore, a single student could have provided two inconclusive responses; for example, one in Part B and one in Part C. See Table 3 for a list of how many students provided instances for each type of change across the whole number relationship.

Table 3: Numbers of students who provided the number of analyzed instances from pre-test to post-test across the whole number relationship (i.e., changes from Q1 to Q5)

Across Parts A, B, and C for Whole Number Relationship (Q1 to Q5)			
Model	Type of Change	# Students Who Provided Responses	# Instances Analyzed
Bar Model	Positive	6	6
	Inconclusive	2	3
	Negative	1	1
Tape Diagram	Positive	3	3
	Inconclusive	3	4
	Negative	3	4

Results for Fractional Relationship (Q2 to Q6)

The results for the fractional relationship, Question 2 on the Pre-test to Question 6 on the Post-test are presented according to the mathematical model provided in the protocol.

Bar Model Results for Fractional Relationship

There were a total of 61 instances in which students were given the bar model (Part A: n = 22, Part B: n = 20, Part C: n = 19). In Part A, there were eight instances in which students changed their answers from pre-test to post-test. Five of the changes were positive, three were inconclusive, and zero were negative. In Part B, there were three instances in which students changed their answers, and all three instances were inconclusive changes. In Part C, there were three instances in which students changed their answers. One instance was a positive change, while two instances were inconclusive. See Table 4 for these results in a table format.

Tape Diagram Results for Fractional Relationship

There were a total of 65 instances in which students were given the tape diagram (Part A: n = 25; Part B: n = 21; Part C: n = 19). In Part A, there were six instances in which students changed answers from pre-test to post-test. One change was positive, four changes were inconclusive, and one change was negative. In Part B, there were five instances in which students changed answers from pre-test to post-test. Two were positive changes, one was inconclusive, and two were negative changes. In Part C, there were eight instances in which students changed their answers from pre-test to post-test. Three were positive changes, four were inconclusive changes, and one was a negative change. See Table 4 for these results in a table format.

Table 4: Numbers of instances and type of change when students changed their solution approach from pre- to post-test across the fractional relationship (i.e., changes from Q2 to Q6)

Fractional Relationship Pre-test to Post-test (Q2 to Q6)						
Model	Protocol Part	*n*	Changed Answers	Change		
				Positive	Inconclusive	Negative
Bar Model	A	22	8	5	3	0
	B	20	3	0	3	0
	C	19	3	1	2	0
	Total	61	14	6 (43%)	8 (47%)	0
Tape Diagram	A	25	6	1	4	1
	B	21	5	2	1	2
	C	19	8	3	4	1
	Total	65	19	6 (32%)	9 (47%)	5 (26%)

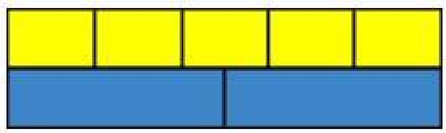
As with the whole number relationship, some students could have provided multiple instances that were analyzed. See Table 5 for a list of how many students provided instances for each type of change across the fractional relationship.

Table 5: Numbers of students who provided the number of analyzed instances from pre-test to post-test across the fractional relationship (i.e., changes from Q1 to Q5)

Across Parts A, B, and C for Fractional Relationship (Q2 to Q6)			
Model	Type of Change	# Students Who Provided Responses	# Instances Analyzed
Bar Model	Positive	5	6
	Inconclusive	6	8
	Negative	0	0
Tape Diagram	Positive	5	6
	Inconclusive	8	9
	Negative	3	4

Responses Related to Task Features of the Interview Protocol

Several responses that students provided dealt with task features of the interview protocol (i.e., problem context and format of mathematical models). Several students commented on the theoretical versus real-world paint context when given the 5:2 number relationship, in that the yellow paint cans are 2 ½ times the number of blue paint cans. Due to the realization they would need ½ cans of paint, three of 47 students discussed the idea that stores would not sell half cans of paint to customers. See Figure 32 for a sample response.

<table>
<tr><th colspan="2">Prompt: Pre-test Question 1</th></tr>
<tr><td colspan="2">The yellow paint is always _______ times the blue paint.</td></tr>
<tr><th>Model</th><th>Ratio Relationship</th></tr>
<tr><td></td><td>5 yellow: 2 blue</td></tr>
<tr><th colspan="2">Student’s Verbal Response</th></tr>
<tr><td colspan="2">Student: The first one is the yellow paint is always zero times the blue paint, and the second one is the blue paint is always zero times the yellow paint, that is why, because in this case you can’t really, well, if someone were to see this, someone else, they would probably say two and a half, but I think that you can’t do that because there’s no, in my opinion, there’s no paint of can that could just be like, partly, it has to be, like, full.

Investigator: Ahh, okay. Excellent. Now what if, theoretically, you could have a part?

Student: Well, then my answer would change and it would be two and a half.

Investigator: Ahh, okay. And why would you change it?

Student: Because, theoretically, it would be partly.</td></tr>
</table>

Figure 32. Theoretical versus Real-World Context

For several students, the structure of the tape diagram or bar model led to misconceptions. Given the tape diagram, two of 47 students provided responses that the relative sizes of the yellow and blue rectangles representing cans of paint made them think the yellow cans of paint were smaller than the blue cans. See Figure 33 for a sample response.

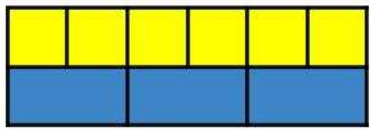
<table>
<tr><th colspan="2">Prompt: Intervention Question 3</th></tr>
<tr><td colspan="2">If Annie has 18 cans of yellow paint, how many cans of blue paint does she need to make the same color green as before?</td></tr>
<tr><th>Model</th><th>Ratio Relationship</th></tr>
<tr><td></td><td>6 yellow: 3 blue</td></tr>
<tr><th colspan="2">Student's Verbal Response</th></tr>
<tr><td colspan="2">Um, I thought about how like, um, obviously the two groups of yellow paint equal to one blue and make green, and so I did, like, how many times it would take to make, um, the same amount of blue with the half amount of yellow, but then you have to like multiply that by two, because there's two halves that equal a whole of blue and so that's basically what I did here. So I did eighteen because that's how many cans of yellow paint here divided by two to get each one for that [pointing at tape diagram] and then I did that, yeah.</td></tr>
</table>

Figure 33. Tape Diagram – Relative Size of Rectangles

Similarly, the structure of the bar model posed a challenge for some students. Given the bar model, two of 47 students provided responses which dealt with the idea that the bar model had "missing rectangles" in the second row. See Figure 34 for a sample response. One student also indicated the format of the bar model influenced them to think additively. See Figure 35 for a sample response.

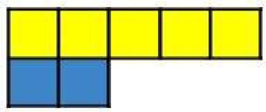
<table>
<tr><th colspan="2">Prompt: Intervention Question 4</th></tr>
<tr><td colspan="2">If you were going to the store to buy more paint for Carlos, how would you think of the relationship between yellow and blue paint to make sure you can make the same color of green?</td></tr>
<tr><th>Model</th><th>Ratio Relationship</th></tr>
<tr><td></td><td>5 yellow: 2 blue</td></tr>
<tr><th colspan="2">Student’s Verbal Response</th></tr>
<tr><td colspan="2">You would know each time you do get him paint that you would get three more of the yellow paint than the blue paint, and I know that because, for the five paint [pointing to five yellow rectangles in bar model] there was three of the yellow paints left [pointing at three yellow rectangles that stick out to the right] and like, the blue wasn’t there, like right there [pointing at empty space below the three yellow rectangles].</td></tr>
</table>

Figure 34. Bar Model – Idea of “Missing” Rectangles – Sample 1

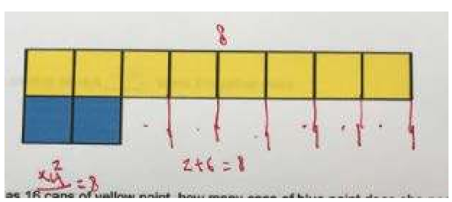
<table>
<tr><th colspan="2">Prompt: Intervention Question 3</th></tr>
<tr><td colspan="2">Prompt: If Bailey has 16 cans of yellow paint, how many cans of blue paint does she need to make the same color green as before?</td></tr>
<tr><th>Model</th><th>Ratio Relationship</th></tr>
<tr><td></td><td>8 yellow: 2 blue</td></tr>
<tr><th colspan="2">Student’s Verbal Response</th></tr>
<tr><td colspan="2">Student draws on the bar model (see image above) by adding six more sections to the second row of bar model. Student counts all eight of the yellow boxes in first row and makes notations on paper. “So I was thinking, I’m like, okay, I could either do it two times four equals eight, because that’s how many yellow paint of cans there is, and then I was like, oh wait, but I gotta also do two plus six equals eight, to get the same amount of yellow.”</td></tr>
</table>

Figure 35. Bar Model – Idea of “Missing” Rectangles – Sample 2

CHAPTER FIVE: CONCLUSION

Introduction

This research was viewed through the lens of constructivism, as it was anticipated that students would engage in the learning process through the interview protocol with proportional reasoning tasks. Upon analysis of student responses provided in the cognitive interviews, students did indeed interact with the mathematical models and construct new knowledge, indicating the mathematical models or the interview protocol in general provided opportunities for disequilibrium and subsequent accommodation in the students (Inhelder & Piaget, 1958; Piaget, 1970, 1972). A summary of my findings related to the influence of the bar model and tape diagram on students' cognition is presented next.

Summary of Findings

For the whole number relationship (Q1 to Q5), the bar model was more influential than the tape diagram at producing positive changes in responses (bar model: 60%; tape diagram: 27%). Both models were relatively equal in their influence on producing inconclusive changes in responses (bar model: 30%; tape diagram: 36%). The bar model was less influential in producing negative changes in responses than the tape diagram (bar model: 10%; tape diagram: 36%).

For the fractional relationship (Q2 to Q6), the bar model was more influential than the tape diagram at producing positive changes in responses (bar model: 43%; tape diagram: 32%). The models were equally influential in producing an inconclusive response (both at 47%). The bar model was less influential than the tape diagram in producing a negative response (bar model: 0%; tape diagram: 26%).

These results suggest there is some evidence to support the claim the bar model provides more scaffolding than the tape diagram in terms of helping students visualize the multiplicative comparison relationship and produced fewer negative changes across both the whole number relationship and the fractional relationship.

Along with the results for the mathematical models eliciting the multiplicative comparison conception, this study also had several other interesting findings. They involve the problem context and format of the tape diagram and bar model. These findings are discussed next.

<u>Theoretical versus Real World Context</u>

In accordance with research by Booth & Koedinger (2012) and Tourniaire & Pulos (1985), I found the context of paint influenced students' ability to solve the task. Since research suggests tasks should be based on real-life experiences of students, I chose the paint context because I thought it would be familiar to students and help them access their prior knowledge regarding paint (Baranes et al., 1989; Koedinger & Nathan, 2004). However, instead of being helpful, the paint context actually caused some students to be confused, as the context did not align with their experiences in real-life. For example, in questions involving the 5:2 ratio relationship, students struggled with the fractional relationships of paint cans. Since both the multiplicative comparison conception and the composed unit conception utilize fractional relationships (2 ½, or 2 ½ to 1, respectively), several students mentioned that in real life, stores would not sell half-empty cans of paint to customers. One student stated, "…if someone were to see this, someone else, they would probably say two and a half, but I think that you can't do that because there's no, in my opinion, there's no paint of can that could just be like, partly, it has to be, like, full." Another student resolved this dilemma by stating the customer should buy three cans of paint to make sure they have enough paint as follows: "I would know that if I bought one can of blue paint, I'd need to get- I'd need to get probably three cans of yellow, because you- cause you can't buy a half a can." When I asked the students to think theoretically instead of realistically, they responded with the correct answer of 2 ½ cans of paint. Nevertheless, it is important to note that the context of paint caused unintended confusion for some students due to the difference between theoretical and real-world situations.

Format of the Mathematical Models

As posited by Booth & Koedinger (2012), the format of the mathematical models (i.e., tape diagram and bar model) seemed to influence students' thinking by producing unintended disequilibrium. When given the tape diagram, the relative size of the yellow and blue rectangles representing yellow paint cans and blue paint cans caused several students to think that the yellow paint cans were half the size of the blue paint cans or had less paint in each can. This misconception was so strong that students continued to use it in solving the given tasks, even though it is in apparent contradiction to the given ratio relationship (6:3, 8:4, 5:2). When given the bar model, several students commented that the second row of the bar model was missing pieces. Given the 8:2 relationship, one student initially saw the multiplicative relationship of "one-fourth" or "divide by four", but then began to focus on an additive strategy of "2 blue rectangles + 6 missing rectangles = 8 total rectangles". While this student did eventually solve the task multiplicatively, the structure of the bar model produced confusion and disequilibrium that was not beneficial to the student's conceptions. On the other hand, the very nature of the bar model that made some students think rectangles were missing from the diagram could have also helped students see the paint cans were the same size. The bar model provided a linear visual representation of the difference in quantities of yellow and blue paint but maintained the relative sizes of the paint cans. Since more students made a positive change when given the bar model than the tape diagram, the very structure of the bar model may have helped students to move from disequilibrium to accommodation when answering the questions designed to elicit the multiplicative comparison conception.

Discussion and Conclusions

Regarding the influence of the mathematical models in the interview protocol, the bar model and tape diagram were designed to be tools provided to help students visualize what is occurring in the proportional reasoning tasks (Murata, 2008). According to my results, it appears the bar model was a more influential tool than the tape diagram in eliciting the multiplicative comparison conception or at least, moving students forward on a trajectory towards the multiplicative comparison conception. However, both the bar model and the tape diagram provided instances in which students experienced disequilibrium and accommodation. According to Von Glasersfeld (1991) asking students to verbalize their thinking allows them to examine their thought processes and potential contradictions. Since my interview protocol was designed to engage students in the learning process, I expected students to encounter disequilibrium and accommodation as they interacted with the tape diagram or bar model. This did indeed occur, as students who changed their answers from pre-test to post-test demonstrated they encountered disequilibrium and accommodation. It is worth noting that in this interview protocol, students did not have the opportunity to apply their ideas to a different context, and therefore reach the assimilation stage (Inhelder & Piaget, 1958; Piaget, 1970, 1972). While it could be argued that the progression of the different number relationships across the interview protocol could potentially function as a different situation, the fact the paint context remained constant somewhat nullifies this argument.

Several factors may have influenced students' thinking processes, including the structure of the interview protocol, problem context, format of the mathematical models, students' ability to speak English (as the interviews were conducted in English), and difficulty with working with fractions.

Structure of Interview Protocol

The structure of the interview protocol may have influenced students' thinking processes, specifically due to the design of the questions. Since my research questions targeted the multiplicative comparison conception, I designed the pre-test and post-test questions to elicit the multiplicative comparison conception. This is in accordance with research suggesting the tools used in the learning process be aligned with the learning objectives (Thompson, 1985). However, this decision to elicit the multiplicative comparison conception on the pre-test and post-test questions may have imposed constraints on students' responses in terms of the types of answers that would satisfy the given statements. For example, the design of the pre-test and post-test questions did not allow students the opportunity to demonstrate their understanding of the composed unit conception, which may have caused disequilibrium in their responses.

A second factor regarding the interview protocol on students' responses is the progression of the number relationships. Since the interview protocol began with easier number relationship of 6:3 and progressed to harder number relationships of 8:2 and 5:2, students may have been engaging in the learning process simply due to the progressive nature of the number relationships in interview protocol.

Another third influential factor regarding the interview protocol could be the probes I used. Since I anticipated the interview protocol could function as a learning environment, I chose to maintain a consistent set of probes designed to have students clarify their explanations instead of induce learning (Carney & Paulding, 2019). I could have chosen to augment their learning potential by probing more deeply into the students' thought processes and creating more opportunities for disequilibrium and accommodation (Piaget, 1967). However, since my study was focusing on the influence of the bar model and tape diagram, I anticipated it would be too difficult to tease out the influence of the bar model and tape diagram apart from the influence of my probing questions.

A fourth influential factor may be the instructions that I gave to students to think aloud. Since the literature states thinking aloud could potentially be distracting for students, I mitigated the potential effects of asking students to explain their thinking in several ways. First, I gave students the choice of either thinking aloud or explaining their thinking after solving the task, so they were able to decide for themselves which manner they felt more comfortable with. They also knew what was expected of them and were not surprised when I asked them to explain their thinking. Second, while students solved the problem, they were given a pen or pencil to write their work on the paper containing the interview questions, so when I asked students to explain their thoughts, they could look at their work to remember. Third, I asked students to explain immediately after they solved the task, so their responses were still considered concurrent verbalizations (Ericsson & Simon, 1980). In doing so, the information was still in students' short term memory, thus increasing the validity of the results (Ericsson & Simon, 1980).

Implications for Classroom Teachers

The above findings are important for classroom teachers because they provide a glimpse into the thinking processes of middle school students as they grapple with the challenging concepts of proportional reasoning. The findings reveal challenges that students encountered while working with fractional relationships and illuminate differences in task interpretations, as explicated next.

Students Struggled to State the Multiplicative Comparison Conception

While in many cases students struggled to explicitly state the multiplicative comparison conception, they used a variety of strategies to explain their understanding. Although they could not provide an answer that fit the given pre-test and post-test statements, they were certainly trying to use all their prior knowledge to explain their thoughts. For example, while students' numerical answers on paper looked completely incorrect at first glance, careful analysis of the students' accompanying verbal responses demonstrates that they applied their prior knowledge regarding negative numbers and qualitative descriptive phrases such as "two times less than" to try to express their thoughts on the fractional relationships. Other students provided responses with negative signs. However, when pressed/probed to explain the meaning of the negative sign, students were unable to verbalize the inverse or "opposite"; instead, they said "it gets less…." or "it is subtraction". For these responses, students may simply lack familiarity in working with fractional relationships, or they may be avoiding the use of fractions altogether (Karplus et al., 1983b; Rupley, 1981). It may also be that their struggle is rooted in deeper misconceptions on the part-whole nature of fractions in general.

<u>My Interpretations were Different than the Students' Interpretations</u>

In order to promote a classroom environment using constructivism, it is important that teachers understand how their students are thinking about the given task or activity and to realize when their own mental representations and interpretations are different than the students' mental representations (Maher & Davis, 1990). In my study, I learned to ask myself if the bar model and tape diagram caused different mental representations and interpretations for me than for the students I was interviewing.

For example, as I was initially analyzing responses that I later decided to code as "recreating the given ratio (RGR), I first thought the students were simply placing the given numbers from the ratio into the blanks in the given statements in the pre-test and post-test. Given 8 yellow to 2 blue cans, students often answered, "The yellow cans are always 8 times the blue cans" and "The blue cans are always 2 times the yellow cans" (see Figure 24 in Chapter 3). However, after I closely analyzed a student's response, it became clear that this student was thinking about the initial quantities of yellow and blue paint as a ratio of one to one (1 yellow can to 1 blue can of paint). According to this individual student's thinking, in order to form the relationship of 8 yellow to 2 blue cans, the yellow paint must be multiplied by 8 and the blue paint must be multiplied by 2. It was only after I carefully analyzed this student's response that I was able to visualize their thinking and understand their thought processes.

No Students Used the Cross-Products Algorithm

Interestingly enough, I anticipated the students would use the cross-products algorithm to solve the given tasks, since according to the literature, cross-multiplication is a common strategy students use to solve proportional reasoning problems (Cramer & Post, 1993; Cramer et al., 1993). However in this study, zero of 47 students used cross-multiplication to solve any of the given tasks. This may be due to the fact the pre-test and post-test questions were designed to elicit the multiplicative comparison conception. It is also possible that the students had not yet been taught the cross-products algorithm.

Limitations

There are several limitations to my study. First, the students did not have the same prior knowledge or learning experiences regarding proportional reasoning; however, since the students were from three different grade levels, I expected their prior experiences in proportional reasoning to differ. I also did not gather data on the teaching styles of the classroom teachers or the curriculum that was used for each school and grade level.

Second, since my study used such a small population of students (n = 47), my results cannot be generalized out to a larger population. Third, students' responses to the interview protocol may have been influenced by the fact the interviews were conducted in English. Some students were native Spanish speakers, and although I asked if they wanted to complete the interview in Spanish, they chose to use English instead. If the interviews were conducted in the students' native language, some responses may have changed.

Implications for Future Research

In constructivism-based teaching approaches, learners are encouraged to construct their own models that represent how they see the ideas; in essence, construct their own learning (Piaget, 1970). However, for my study, I diverged from this theory because I provided the students with the given mathematical model (e.g. tape diagram or bar model) instead of having them construct their own representation of the given proportional reasoning situation. Since I sought to ascertain the influence of specific models, I had to provide the students with the specific models I was studying. Yet in doing so, I was essentially placing a constraint on the possibilities of responses I could receive if I had students construct their own diagrams of proportional reasoning situations. In future iterations, I could provide students with similar tasks and ask them to create a model that would work for them.

Due to the challenges students faced in working with the fractional relationships, future research could focus on measuring students' fraction knowledge prior to conducting another iteration of the study. The students may have struggled more with the idea of using fractions than with the concept of coordinating two quantities in a relationship in general. A future study could try to distinguish between these two areas.

A future iteration could also focus on teachers' cognition in proportional reasoning by providing them with the bar model and tape diagram or by asking them to construct their own models as well. Since teachers' interpretations may be different than what students perceive, it would be important to see how the representations and interpretations differ between students and teachers.

Additionally, I would like to explore a larger population of students and see if my results would be consistent across other groups. I would like to provide this study in Spanish to native Spanish speakers to alleviate the effect of language learning.

The data I collected for this study was rich and detailed. A different analysis on this same data could focus on responses to the intervention questions and the influence of those intervention questions on students' responses to the post-test questions. A different analysis could also determine the likelihood of the tape diagram to specifically elicit the composed unit conception. I hypothesized the tape diagram would elicit a composed unit conception more frequently than a multiplicative comparison conception due to the vertical partitions helping students to conceive of the unit rate. A deeper analysis of the intervention questions may shed light on this issue, as Question 3 of the Intervention was designed to elicit scalar reasoning and Question 4 was designed to elicit functional reasoning. It may be that the tape diagram influenced students towards a composed unit conception, but the structure of my pre-test and post-test questions did not allow this to emerge.

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APPENDIX A

IRB Interview Protocol 108-SB18-008

Investigator will collect consent/assent forms.

"Good morning, thank you for participating in this research. My name is Katie Paulding and I am a graduate student at Boise State University currently pursuing my doctorate in Education. I am conducting research on middle school students' thinking about proportional relationships given specific diagrams and ratio relationships. In the future I hope this research will help teachers provide more effective instruction to their students regarding proportional reasoning.

"This interview will last no more than twenty minutes. If at any time you want to stop this interview, simply let me know. To protect your privacy, I will not be recording your name, but will be assigning you a pseudonym instead."

"I ask that you not discuss the problems given to you in this interview with any other students."

"I am going to give you a proportional reasoning problem along with a diagram and ask you a few questions regarding your thinking and your solution to the problem."

"Are you willing to be videotaped for this interview? I will only be videoing your paper, and not your face at all. Thank you. In a moment I will turn on the recorder and will need to ask you the same question again for the record."

"Do you have any questions for me before we begin?"

Investigator will start the video recording device.

"For the record, are you willing to be videotaped for this interview? Thank you."

Investigator will provide student with Problem Set: Tape Diagram (Parts A1, B1, and C1) or Problem Set: Bar Model (Parts A2, B2, and C2) as well as a writing utensil and extra paper if necessary.

Each Problem Set and corresponding parts are delineated next.

Problem Set: Tape Diagram

Task A1: 6 to 3 Tape Diagram

Investigator will provide student with Task A1: Part 1 and say, "Please read the problem out loud. Then, as you solve the problem, say out loud what you are thinking as you work on it."

If student is solving the task and not thinking aloud, investigator may prompt with the following verbal probes:

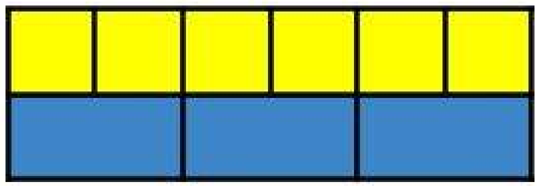
- "Remember to say what you are thinking out loud as you solve the problem."
- "Please explain your thinking as you solve the problem."
- "How were you thinking about that?"
- "Can you explain that again for me?"

Task A1: Part 1 Interview ID:

Task A1 **6 yellow cans of paint : 3 blue cans of paint**
Annie is painting her room green. She mixed 6 cans of yellow paint with 3 cans of blue paint to create green paint. 1.) The yellow paint is always _______ times the blue paint. 2.) The blue paint is always ______ times the yellow paint.

Investigator will collect Task A1: Part 1, give student Task A1: Part 2, and say, "Please read the problem out loud. Then, as you solve the problem, say out loud what you are thinking as you work on it."

Task A1: Part 2 Interview ID:

Task A1 **6 yellow cans of paint : 3 blue cans of paint**
Annie is painting her room green. She mixed 6 cans of yellow paint with 3 cans of blue paint to create green paint.  3.) If Annie has 18 cans of yellow paint, how many cans of blue paint does she need to make the same color green as before? 4.) If you were going to the store to buy more paint for Annie, how would you think of the relationship between yellow and blue paint to make sure you can make the same color of green?

Investigator will collect Task A1: Part 2 and say, "Now that you have had more time to think about these problems, I am going to give you the first two problems again. This does not mean your first answers were correct or incorrect. This is just part of my study. So, how would you answer these two questions again?"

Investigator will provide student with Task A1: Part 3.

Task A1: Part 3 Interview ID:

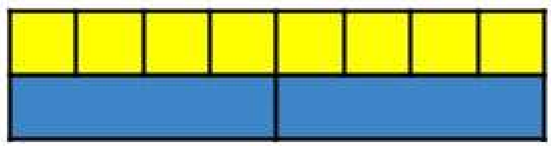
Task A1 6 yellow cans of paint : 3 blue cans of paint
Annie is painting her room green. She mixed 6 cans of yellow paint with 3 cans of blue paint to create green paint. 5.) The yellow paint is always _______ times the blue paint. 6.) The blue paint is always ______ times the yellow paint.

If student successfully solves Part A, investigator will ask student if they would like to continue the interview, provide student with Task B, and repeat the above process with Parts 1, 2, and 3 of Task B.

Task B1: Part 1 Interview ID:

Task B1 **8 yellow cans of paint : 2 blue cans of paint**
Bailey is painting her room green. She mixed 8 cans of yellow paint with 2 cans of blue paint to create green paint. 1.) The yellow paint is always _______ times the blue paint. 2.) The blue paint is always ______ times the yellow paint.

Task B1: Part 2 Interview ID:

Task B1 **8 yellow cans of paint : 2 blue cans of paint**
Bailey is painting her room green. She mixed 8 cans of yellow paint with 2 cans of blue paint to create green paint.  3.) If Bailey has 16 cans of yellow paint, how many cans of blue paint does she need to make the same color green as before? 4.) If you were going to the store to buy more paint for Bailey, how would you think of the relationship between yellow and blue paint to make sure you can make the same color of green?

Task B1: Part 3 Interview ID:

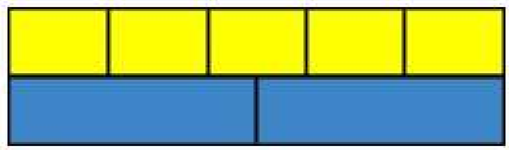
Task B1 **8 yellow cans of paint : 2 blue cans of paint**
Bailey is painting her room green. She mixed 8 cans of yellow paint with 2 cans of blue paint to create green paint. 5.) The yellow paint is always _______ times the blue paint. 6.) The blue paint is always ______ times the yellow paint.

If student successfully solves Part B, investigator will ask student if they would like to continue the interview, then provide student with Task C and repeat the above process with Parts 1, 2, and 3 of Task C.

Task C1: Part 1 Interview ID:

Task C1 **5 yellow cans of paint : 2 blue cans of paint**
Carlos is painting his room green. He mixed 5 cans of yellow paint with 2 cans of blue paint to create green paint. 1.) The yellow paint is always _______ times the blue paint. 2.) The blue paint is always ______ times the yellow paint.

Task C1: Part 2 Interview ID:

Task C1 **5 yellow cans of paint : 2 blue cans of paint**
Carlos is painting his room green. He mixed 5 cans of yellow paint with 2 cans of blue paint to create green paint.  3.) If Carlos has 20 cans of yellow paint, how many cans of blue paint does he need to make the same color green as before? 4.) If you were going to the store to buy more paint for Carlos, how would you think of the relationship between yellow and blue paint to make sure you can make the same color of green?

Task C1: Part 3 Interview ID:

Task C1 **5 yellow cans of paint : 2 blue cans of paint**
Carlos is painting his room green. He mixed 5 cans of yellow paint with 2 cans of blue paint to create green paint. 5.) The yellow paint is always _______ times the blue paint. 6.) The blue paint is always ______ times the yellow paint.

Investigator will say, "Thank you for taking the time to participate in this interview. I really appreciate your thoughtfulness in your responses. Thank you again!"

Investigator will turn off recording device.

<u>Problem Set: Bar Model</u>

<u>Task A2:</u> 6 to 3 Bar Model

Investigator will provide student with Task A2: Part 1 and say, "Please read the problem out loud. Then, as you solve the problem, say out loud what you are thinking as you work on it."

If student is solving the task and not thinking aloud, investigator may prompt with the following verbal probes:

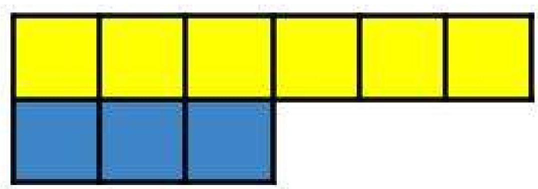
- "Remember to say what you are thinking out loud as you solve the problem."
- "Please explain your thinking as you solve the problem."
- "How were you thinking about that?"
- "Can you explain that again for me?"

Task A2: Part 1 Interview ID:

Task A2 **6 yellow cans of paint : 3 blue cans of paint**
Annie is painting her room green. She mixed 6 cans of yellow paint with 3 cans of blue paint to create green paint. 1.) The yellow paint is always _______ times the blue paint. 2.) The blue paint is always ______ times the yellow paint.

Investigator will collect Task A2: Part 1, give student Task A2: Part 2, and say, "Please read the problem out loud. Then, as you solve the problem, say out loud what you are thinking as you work on it."

Task A2: Part 2 Interview ID:

Task A2 **6 yellow cans of paint : 3 blue cans of paint**
Annie is painting her room green. She mixed 6 cans of yellow paint with 3 cans of blue paint to create green paint.  3.) If Annie has 18 cans of yellow paint, how many cans of blue paint does she need to make the same color green as before? 4.) If you were going to the store to buy more paint for Annie, how would you think of the relationship between yellow and blue paint to make sure you can make the same color of green?

Investigator will collect Task A2: Part 2 and say, "Now that you have had more time to think about these problems, I am going to give you the first two problems again. This does not mean your first answers were correct or incorrect. This is just part of my study. So, how would you answer these two questions again?"

Investigator will provide student with Task A2: Part 3.

Task A2: Part 3 Interview ID:

Task A2 **6 yellow cans of paint : 3 blue cans of paint**
Annie is painting her room green. She mixed 6 cans of yellow paint with 3 cans of blue paint to create green paint. 5.) The yellow paint is always _______ times the blue paint. 6.) The blue paint is always ______ times the yellow paint.

If student successfully solves Part A, investigator will ask student if they would like to continue the interview, provide student with Task B, and repeat the above process with Parts 1, 2, and 3 of Task B.

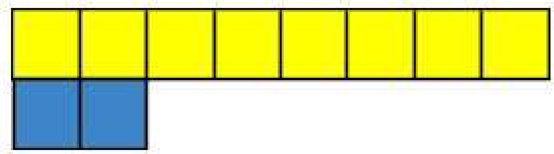
Task B2: Part 1 Interview ID:

Task B2 **8 yellow cans of paint : 2 blue cans of paint**
Bailey is painting her room green. She mixed 8 cans of yellow paint with 2 cans of blue paint to create green paint. 1.) The yellow paint is always _______ times the blue paint. 2.) The blue paint is always ______ times the yellow paint.

Task B2: Part 2 Interview ID:

Task B2
8 yellow cans of paint : 2 blue cans of paint

Bailey is painting her room green. She mixed 8 cans of yellow paint with 2 cans of blue paint to create green paint.



3.) If Bailey has 16 cans of yellow paint, how many cans of blue paint does she need to make the same color green as before?

4.) If you were going to the store to buy more paint for Bailey, how would you think of the relationship between yellow and blue paint to make sure you can make the same color of green?

Task B2: Part 3 Interview ID:

Task B2 **8 yellow cans of paint : 2 blue cans of paint**
Bailey is painting her room green. She mixed 8 cans of yellow paint with 2 cans of blue paint to create green paint. 5.) The yellow paint is always _______ times the blue paint. 6.) The blue paint is always ______ times the yellow paint.

If student successfully solves Part B, investigator will ask student if they would like to continue the interview, then provide student with Task C and repeat the above process with Parts 1, 2, and 3 of Task C.

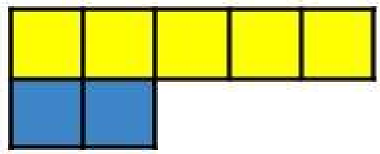
Task C2: Part 1 Interview ID:

Task C2 **5 yellow cans of paint : 2 blue cans of paint**
Carlos is painting his room green. He mixed 5 cans of yellow paint with 2 cans of blue paint to create green paint. 1.) The yellow paint is always _______ times the blue paint. 2.) The blue paint is always ______ times the yellow paint.

Task C2: Part 2 Interview ID:

Task C2
5 yellow cans of paint : 2 blue cans of paint

Carlos is painting his room green. He mixed 5 cans of yellow paint with 2 cans of blue paint to create green paint.



3.) If Carlos has 20 cans of yellow paint, how many cans of blue paint does he need to make the same color green as before?

4.) If you were going to the store to buy more paint for Carlos, how would you think of the relationship between yellow and blue paint to make sure you can make the same color of green?

Task C2: Part 3 Interview ID:

Task C2 **5 yellow cans of paint : 2 blue cans of paint**
Carlos is painting his room green. He mixed 5 cans of yellow paint with 2 cans of blue paint to create green paint. 5.) The yellow paint is always _______ times the blue paint. 6.) The blue paint is always ______ times the yellow paint.

Investigator will say, “Thank you for taking the time to participate in this interview. I really appreciate your thoughtfulness in your responses. Thank you again!”

Investigator will turn off recording device.

APPENDIX B

BOISE STATE UNIVERSITY
COLLEGE OF EDUCATION
Department of Curriculum, Instruction and Foundational Studies

Principal Data Collection Acknowledgement

School Name
School District
School Address
School Phone Number

MM/DD/2019

Dear Katie Paulding,

Based on my review of your proposed research, I give permission for you to conduct the study entitled "Influence of Models and Ratio Relationships on Middle School Student's Thinking of Proportional Reasoning" at School Name. As part of this study, I authorize you to conduct and videotape interviews with students on proportional reasoning activities during the students' math classes and provide School Name with the results of the study. Individuals' participation will be voluntary and at their own discretion.

We understand that School Name's responsibilities include providing students and their parent(s)/guardian(s) information on this study and consent and assent forms for research participation and providing access to a quiet, private, safe space on campus for the interviews to take place. We reserve the right to withdraw from the study at any time if our circumstances change.

The research will include a 20-minute interview individually with 10 students. This authorization covers the time period of Month, day, 2019 to Month, day, 2019.

I understand that any identifiable information obtained in connection with this study will remain confidential and will be disclosed only with my permission or as required by law. I understand that students' names will not be used in any written reports or publications which result from this research, as students will be given a pseudonym during this research to protect their identity.

I confirm that I am authorized to approve research in this setting.

Sincerely,
Name of Principal, Principal
School Name
School Address
School Phone Number

APPENDIX C

BOISE STATE UNIVERSITY
COLLEGE OF EDUCATION
Department of Curriculum, Instruction and Foundational Studies

Teacher Data Collection Acknowledgement

MM/DD/2019

Dear Mr./Mrs./Ms. ______________________,

I have obtained your principal's support to collect data for my research project entitled "Influence of Models and Ratio Relationships on Middle School Student's Thinking of Proportional Reasoning."

I am requesting your cooperation in the data collection process. I propose to conduct approximately 20 minute interviews with 5-10 of your students during math class. I will coordinate the exact times of data collection with you in order to minimize disruption to your instructional activities.

If you agree to be part of this research project, I would ask that you provide your students and their parent(s)/guardian(s) with information on this study via a consent and assent form. Study participants would miss 20 minutes of class time during their math class in order to complete the interview. I would ask that participants be allowed to makeup work that they miss while participating in my research study.

If you prefer not to be involved in this study, that is not a problem at all.

If circumstances change, please contact me via email at katiepaulding@boisestate.edu. Thank you for your consideration. I would be pleased to share the results of this study with you if you are interested.

I am requesting your signature to document that I have cleared this data collection with you.

Sincerely,

Katie Paulding
Graduate Student, Ed.D Curriculum and Instruction
Boise State University
katiepaulding@boisestate.edu

APPENDIX D

BOISE STATE UNIVERSITY
COLLEGE OF EDUCATION
Department of Curriculum, Instruction and Foundational Studies

INFORMED CONSENT

Study Title: Influence of Models and Ratio Relationships on Middle School Students' Thinking of Proportional Reasoning
Principal Investigator: Katie Paulding
Co-Principal Investigator/Faculty Advisor: Dr. Michele Carney

Dear Parent/Guardian:

My name is Katie Paulding and I am currently a doctoral student studying mathematics education at Boise State University. I am asking for your permission to include your child in my research. This consent form will give you the information you will need to understand why this study is being done and why your child is being invited to participate. It will also describe what your child will need to do to participate as well as any known risks, inconveniences or discomforts that your child may have while participating. I encourage you to ask questions at any time. If you decide to allow your child to participate, you will be asked to sign this form and it will be a record of your agreement to participate. You will be given a copy of this form to keep.

➢ PURPOSE AND BACKGROUND

Proportional reasoning is a foundational concept for higher-level mathematics and science. As part of my dissertation, I would like to videotape an interview with your child to determine the influence of models and ratio relationships on their thought processes regarding proportional reasoning. This information may lead to more effective teaching methods of instruction for proportional reasoning units in middle school mathematics curricula.

➢ PROCEDURES

This study will consist of an interview about your child's thinking on a proportional reasoning task as well as an analysis of their work on the task. The interview will take place during school hours in a private location on campus during your child's math class. The interview will last no longer than 20 minutes.

BOISE STATE UNIVERSITY
COLLEGE OF EDUCATION
Department of Curriculum, Instruction and Foundational Studies

CONSENTIMIENTO INFORMADO

Título de la investigación: La influencia de modelos de matemáticas y las relaciones de proporciones en los pensamientos de alumnos de la escuela secundaria sobre el tema de la razonamiento proporcional

Investigador Principal: Katie Paulding

Co-Investigator Principal/Profesorada de Tutor: Dr. Michele Carney

Queridos padres/tutores,

Soy la Srta. Katie Paulding y alumna doctoral del estudio de matemáticas en la Universidad de Boise State. Le pido su permiso de incluir su alumno en mi investigación de matemáticas de los pensamientos de alumnos sobre el tema de la razonamiento proporcional. Esta hoja de consentimiento contiene la información necesario para entender el propósito de la investigación de matemáticas y la razón que me gustaría incluir su alumno en la investigación de matemáticas. Se explica los pasos de participar y también cualquier riesgo, inconveniencia, o incomodidad de participar. En cualquier momento, si tiene alguna pregunta, por favor pregunta al/a la maestro/a de matemáticas de su alumno. Si me da su permiso de incluir su alumno en mi investigación de matemáticas, por favor firme su nombre en la página siguiente. Recibirá una copia de esta hoja para mantener como un registro de su permiso.

➢ EL PROPÓSITO Y LA INFORMACIÓN ANTECEDENTE

El conocimiento de la razonamiento proporcional es fundacional de tener el éxito en clases de niveles más altas de las matemáticas y las ciencias. En mi investigación de matemáticas, me gustaría filmar con una videocámara la entrevista con su alumno para investigar sus pensamientos sobre el tema de la razonamiento proporcional y buscar la influencia de los modelos de matemáticas y las relaciones proporcionales. Es posible que la información afectaría la manera de enseñar las matemáticas en el futuro.

➢ LOS PROCEDIMIENTOS

Esta investigación se consiste de una entrevista de los pensamientos de su alumno en unas preguntas del tema de la razonamiento proporcional. Analizaré la manera de pensar y sus soluciones a las preguntas. Ocurrirá la entrevista durante la clase de matemáticas en un salón sin otros alumnos y no durará más que veinte minutos.

➢ RISKS/DISCOMFORTS

Your child may feel uncomfortable being videotaped, but the camera will only be focused on your student's paperwork and not their face. Please note that you are able to remove your child from participation in this study at any time.

➢ EXTENT OF CONFIDENTIALITY

Reasonable efforts will be made to keep the personal information in your research record private and confidential. Any identifiable information obtained in connection with this study will remain confidential and will be disclosed only with your permission or as required by law.

Your child's name will not be used in any written reports or publications which result from this research, as your child will also be given a pseudonym during this research to protect their identity.

Direct quotations may be used in my dissertation or subsequent educational journal publications and presentations at educational conferences. The members of the research team and the Boise State University Office of Research Compliance (ORC) may access the data. The ORC monitors research studies to protect the rights and welfare of research participants. Data will be kept for three years (per federal regulations) after the study is complete and then destroyed.

➢ BENEFITS

There will be no direct benefit to your child from participating in this study. However, the information gained from this research may help education professionals better understand how students think about proportional reasoning problems and provide more effective means of instruction in proportional reasoning in middle school.

➢ PAYMENT

There will be no payment to you or your child as a result of your child taking part in this study.

➢ QUESTIONS

If you have any questions or concerns about participation in this study, please talk with your child's math teacher or email me at katiepaulding@boisestate.edu.
If you have questions about your child's rights as a research participant, you may contact the Boise State University Institutional Review Board (IRB), which is concerned with the protection of volunteers in research projects. You may reach the board office between 8:00 AM and 5:00 PM, Monday through Friday, by calling (208) 426-5401 or by writing: Institutional Review Board, Office of Research Compliance, Boise State University, 1910 University Dr., Boise, ID 83725-1138.

➢ EL RIESGO/LA INCONVENIENCIA

Es posible que la videocámara cause sentimientos incómodos, pero la videocámara solo se enfocará en el papel y no en la cara de su alumno. Es importante de saber que pueda retirar su permiso de participar de su alumno en cualquier momento que quiere.

➢ GRADO DE CONFIDENCIALIDAD

Se harán esfuerzos razonables para mantener privada y confidencial la información personal en el registro de la investigación de matemáticas. Cualquier información identificable obtenida en relación con esta investigación de matemáticas se mantendrá confidencial y se divulgará solo con su permiso o según lo exija la ley.
No se usará el nombre de su alumno en ningún informe escrito o publicación que resulte de esta investigación de matemáticas. También recibirá un seudónimo su alumno durante esta investigación de matemáticas para proteger su identidad.
Se pueden usar citas directas en mi disertación o publicaciones y presentaciones de revistas educativas en conferencias educativas subsiguientes. Los miembros del equipo de investigación y la Oficina de Cumplimiento de la Investigación (ORC) de la Universidad de Boise State pueden acceder a los datos. El ORC supervisa los alumnos de la investigación de matemáticas para proteger los derechos y el bienestar de los participantes de la investigación. Los datos se mantendrán durante tres años (según las regulaciones federales) una vez que se complete la investigación de matemáticas y luego se destruya.

➢ LOS BENEFICIOS

No habrá ningún beneficio directo para su alumno por participar en esta investigación de matemáticas. Sin embargo, la información obtenida de esta investigación puede ayudar a mejorar la comprensión de los alumnos de cómo pensar sobre el razonamiento proporcional y crear una manera de instrucción más efectiva en la escuela intermedia.

➢ LA RECOMPENSA

No habrá ninguna recompensa para Ud. o su alumno como un resultado de la participación de su alumno en esta investigación de matemáticas.

➢¿PREGUNTAS?

Si tiene alguna pregunta o preocupación de la participación de su alumno, por favor, hable con el/la maestro/a de su alumno o envíeme un mensaje por correo electrónico a katiepaulding@boisestate.edu.

Si tiene preguntas sobre los derechos de su alumno como participante en esta investigación de matemáticas, se puede comunicar con la Junta de Revisión Institucional (IRB) de Boise State University, que se ocupa de la protección de voluntarios en proyectos de investigaciones de matemáticas. Puede comunicarse con la oficina de la Junta de Revisión Institucional, entre las 8:00 AM y las 5:00 PM (MST), de lunes a viernes, llamando por teléfono al (208) 426-5401 o escribiendo a: Junta de Revisión Institucional, Oficina de Cumplimiento de Investigaciones, Boise State University, 1910 University Dr., Boise, ID 83725-1138.

BOISE STATE UNIVERSITY
COLLEGE OF EDUCATION
Department of Curriculum, Instruction and Foundational Studies

STUDENT ASSENT FORM

My name is Katie Paulding and I am currently a graduate student at Boise State University. I am conducting a research study titled "Influence of Models and Ratio Relationships on Middle School Students' Thinking of Proportional Reasoning." I am doing this study because I am trying to learn about how students think when given math problems involving proportional reasoning. I am asking you to be a part of this study because you are a student in middle school. This form will tell you a little bit about the study so you can decide if you want to be in the study or not.

If you want to be in this study, you will be asked to participate in a 20-minute interview. This study will take place on your school campus during your math class. I will be asking you to solve several math problems and explain how you are thinking about them as you solve them. You do not have to answer any question you don't want to. You can also stop being in this study at any time.

If you choose to participate, you will be helping me learn more about how students your age think about math problems. We might also find out information that will help teachers be able to teach more effective lessons on proportional reasoning as well.
Your name will not be recorded in this study, as I will be giving you a pseudonym (a fake name). I may use direct quotations of what you say in my research papers or conference presentations.

Please talk about this study with your parents before you decide if you want to be in it. I will also ask your parents to give their permission. Even if your parents say you can be in the study, you can still say that you don't want to. It is okay to say "no" if you don't want to be in the study. No one will be mad at you. If you change your mind later and want to stop, you can.

You can ask your math teacher any questions about this study. You can also talk to your parents or guardians about this study. After all your questions have been answered, you can decide if you want to be in this study or not.

If you want to be in this study, please sign your name on the next page.

If you don't want to, please do not sign.

Thank you!

Katie Paulding

FORMA DE ASENTIMIENTO DE ALUMNOS

Me llamo Katie Paulding, y soy alumna doctoral del estudio de matemáticas en la Universidad de Boise State. Estoy conduciendo una investigación de matemáticas que se llama “La influencia de modelos de matemáticas y las relaciones de proporciones en los pensamientos de alumnos de la escuela secundaria sobre el tema de la razonamiento proporcional”. Quiero investigar la manera de pensar de alumnos en la escuela secundaria cuando resuelven problemas de matemáticas sobre el tema de la razonamiento proporcional. Le pido a participar en esta investigación de matemáticas porque es un/a alumno/a en la escuela intermedia. Esta hoja explica mas información de la investigación de matemáticas para que pueda decidir si quiere participar o no.

Si acepta participar en esta investigación de matemáticas, participará en una entrevista de veinte minutos. La entrevista va a ocurrir durante su clase de matemáticas en un salón sin otros alumnos. Voy a pedirle que resuelva unos problemas de matemáticas y explicar sus pensamientos mientras las haga. No tiene que responder a una pregunta si no quiere. Se puede terminar la participación en cualquier momento.

Si quiere participar, va a ayudarme aprender más de los pensamientos de matemáticas de alumnos en la escuela secundaria. Es posible que se encuentre información que puede ayudarles a los maestros a enseñar lecciones del razonamiento proporcional en una manera más efectiva.

Su nombre no será registrado en esta investigación de matemáticas, y voy a darle un seudónimo (un nombre falso). Puedo usar citas directas de lo que dices en mis escrituras de la investigación de matemáticas o en presentaciones de conferencias.
Por favor, hable sobre esta oportunidad de participar en esta investigación de matemáticas con sus padres antes de decidir si quiere participar. También les pediré a sus padres que den su permiso. Es importante saber que aunque sus padres dicen que puede participar en el estudio, puede decir que no quiere. Está bien decir "no" si no desea participar en el estudio. Nadie se enojará con usted. Si cambia de opinión más tarde y quiere terminar su participación, puede hacerlo en cualquier momento.

Puede preguntar a su maestro/a de matemáticas cualquier pregunta sobre esta investigación de matemáticas. También puede hablar con sus padres o guardianes sobre esta investigación de matemáticas. Después de que todas sus preguntas hayan sido respondidas, puede decidir si desea participar en esta investigación de matemáticas o no.

Si desea participar en esta investigación de matemáticas, por favor, firme su nombre en la página.

Si no desea participar, no firme su nombre.

¡Muchisimas gracias!

Katie Paulding

➢ LA DOCUMENTACIÓN DEL CONSENTIMIENTO/DOCUMENTATION OF CONSENT

Yo he leído este hoja y decidí que mi alumno participará en la investigación de matemáticas descrito anteriormente. Sus propósitos generales, los detalles de la participación y los posibles riesgos se han explicado a mi satisfacción entera. Discutiré esta investigación de matemáticas con mi alumno y explicaré los procedimientos que se llevarán a cabo. Entiendo que puedo retirar a mi alumno en cualquier momento.

I have read this form and decided that my child will participate in the project described above. Its general purposes, the particulars of involvement, and possible risks have been explained to my satisfaction. I will discuss this research study with my child and explain the procedures that will take place. I understand I can withdraw my child at any time.

________________________ _____________________________

Nombre de Alumno/a Maestro/a de matemáticas

_______________________ _____________________ ________

Nombre de Padres/Tutores Firma de Padres/Tutores La Fecha
Printed Name of Parent/Guardian Signature of Parent/Guardian Date

➢ LA DOCUMENTACIÓN DE ASENTIMIENTO/DOCUMENTATION OF ASSENT

______________________ ____________

Nombre de Alumno/a/ La Fecha/Date
Student Name

______________________ ____________

Firma de Alumno/a/ La Fecha/Date
Student Signature

My math class is period/ Mi clase de matemáticas es período número 1 2 3 4 5 6 7

____________________________ ____________

Firma de Investigador Principal/ La Fecha/Date
Signature of Principal Investigator/
Firma de Persona que Recibe el Consentimiento/
Signature of Person Obtaining Consent

IRB: 108-SB18-008

APPENDIX E

Table 6 **Codebook**

Code	Meaning	Definition	Sample Using Student Explanation
CU	Composed Unit	A composed unit conception involves simultaneously coordinating the two quantities in a ratio to form a new unit with which to operate (Carney & Crawford, 2016; Carney et al., 2015; Ellis, 2013; Lobato et al., 2010).	Ratio Relationship: 5 yellow to 2 blue cans of paint Prompt: If Carlos has 20 cans of yellow paint, how many cans of blue paint does he need to make the same color green as before? Student Response: It says that if he has twenty cans of yellow paint, how many cans of blue paint does he need to make the same color green before? Twenty is four times five, so if you multiply two by four, you get eight, and so he will need eight cans of blue paint to make the same color of green that he had before.
EQ	Equal Quantities	The response indicated that the quantities of yellow paint and blue paint had to be "even" or equal.	Ratio Relationship: 8 yellow to 2 blue cans of paint Student Response: I put you will have to see how much paint you would need and how much to make it even. Investigator: Excellent. Okay, so let's say you needed sixteen cans of yellow paint. How much blue would you need? Student: Wouldn't you need sixteen blue?

IAS	Incorrect Additive Strategy	The student used addition or subtraction in situations that are actually multiplicative.	Ratio Relationship: 5 yellow to 2 blue cans of paint Prompt: If Carlos has 20 cans of yellow paint, how many cans of blue paint does he need to make the same color green as before? Student Response: Twenty minus three because five minus two equals three so you want to take away three from twenty which is seventeen.
IC	Incorrect Mathematical Calculation	The student struggled in their mathematical calculations.	Ratio Relationship: 5 yellow to 2 blue cans of paint Student’s Written Answer on Pre-test: 1.5 Student’s Written Answer on Post-test: 1/3 Student’s Verbal Response: Okay, that’s a three [pointing to answer in Question 6]. Cause I- and...I kind of realized that one point five was the same as one half, and it wasn’t really one half of that, it was kind of more of one third.
IND	Indeterminate Response	The response did not provide enough evidence of a specific conception or solution strategy.	Ratio Relationship: 8 yellow cans to 2 blue cans of paint Student Response: ...if you want to make it the same color of green, you would have to do the same amount of blue and yellow paint [you] got the first round. Investigator: Excellent. Very good. So let’s say, how much would you have gotten on the first round? Student: Well, it says it has eight cans of yellow paint, so you would get the same- you would get eight cans of the same yellow paint, like you can also get, like, the same brand if you really want it to be the same color. And same thing for the blue paint.

MC	Multiplicative Comparison Conception	The response provided sufficient evidence the student recognized the multiplicative relationship between the numbers in the given ratio and applied that relationship consistently across the questions in the task.	Ratio Relationship: 8 yellow cans to 2 blue cans of paint Student Response: If two is one-fourth of eight, then sixteen, for- if she has sixteen cans of yellow paint, then she will have to have one-fourth of the yellow paint, and blue is gonna be four cans because four is one-fourth of sixteen.
MR-INV	Inverse Multiplicative Relationship	The student identified the inverse relationship of the multiplicative comparison conception.	Ratio Relationship: 5 yellow to 2 blue cans of paint Student Response: The yellow paint is always two point five times the blue paint because if you multiply two by two point five, it’s gonna equal five, and so he’ll need two and a half times the blue paint that he has for yellow paint.
UNN	Using Negative Numbers	The student used a negative sign to their written answer.	Ratio Relationship: 8 yellow to 2 blue cans of paint Student Response: And then on this one I put negative four to try and divide it again.
NMC	Non-Mathematical Content	A response that did not contain mathematical content, or the mathematical content provided did not match the ratio relationships and context in the given questions.	Ratio Relationship: 6 yellow to 3 blue cans of paint Student Response: So what I wrote is I think I would think them as a well relationship between those two colors since they both are primary colors making a secondary color.
NR	No Response	The student did not provide a response or left the answer blank.	Student Response: I don't know.

OPER -ADD -SUB -MULT	Operations with Given Numbers in Ratio Relationship - Added	The student operated on the given numbers in the ratio relationship by adding them together, subtracting them, or multiplying them in order to obtain an answer.	Ratio Relationship: 5 yellow to 2 blue cans of paint Student Response: The yellow paint is always three times the blue paint, because there's five cans of yellow and two cans of blue, and when you subtract them, there's three more yellow cans than the blue paint.
RGR	Recreated Given Ratio	The student answered the two statements in the pre-test or the two statements in the post-test by providing answers that when combined, recreated the given ratio.	Ratio Relationship: 8 yellow to 2 blue cans of paint Student Response: The yellow paint is always eight times the blue paint. And the blue paint is always two times the yellow paint.
RGR-INV	Inverse Recreate Given Ratio	The student recreated the given ratio but described the inverse relationship.	Ratio Relationship: 6 yellow to 3 blue cans of paint Student Response: I think it might be one...so if I had one I believe it was blue, then it would be two yellows…

RGR-UR	Recreate Given Ratio Using Unit Rate	The student recreated the given ratio but stated the unit rate for their answers.	Ratio Relationship: 8 yellow to 2 blue cans of paint Student Response: The yellow paint is four times more than the blue cans of paint and the blue is like, only one time as the yellow cans of paint because it’s, like, less.
QDR	Qualitative Description of the Relationship	The student added words such as “more”, “less”, “larger”, or “smaller” into the statements to qualify their answers.	Ratio Relationship: 6 yellow to 3 blue cans of paint Student Response: Uh, it’s like the same as the first [question], except um, they just switched out the words, so it’s two times less than the yellow paint, instead of more than the yellow paint, I mean, than the blue paint- wait, yeah.