

Research Article

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The influence of the noise on the exact solutions of a Kuramoto-Sivashinsky equation

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Abstract: In this article, we take into account the stochastic Kuramoto-Sivashinsky equation forced by multiplicative noise in the Itô sense. To obtain the exact stochastic solutions of the stochastic Kuramoto-Sivashinsky equation, we apply the $\frac{G'}{G}$ -expansion method. Furthermore, we extend some previous results where this equation has not been previously studied in the presence of multiplicative noise. Also, we show the influence of multiplicative noise on the analytical solutions of the stochastic Kuramoto-Sivashinsky equation.

Keywords: stochastic Kuramoto-Sivashinsky, multiplicative noise, stochastic exact solutions, $\left(\frac{G'}{G}\right)$ -expansion method

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1 Introduction

Nonlinear partial differential equations (NLPDEs) are applied to describe a wide range of phenomena in biology, fluid mechanics, chemical physics, chemical kinematics, solid-state physics, optical fibers, plasma physics, geochemistry, and a lot of other fields. The research of analytical solutions for NLPDEs is important in the investigation of nonlinear physical phenomena. Throughout the past several decades, the discovery of new phenomena has been aided by new exact solutions. Thus, the seeking of exact solutions to those equations of NLPDEs has long been a feature of mathematics and science. To obtain exact solutions of NLPDEs, a variety of effective techniques have been applied, for instance, the Exp-function method [1,2], the $\left(\frac{G'}{G}\right)$ -expansion method [3,4], the tanh–sech method [5,6], the improved tanh-function method [7], the $\exp(-\varphi(\eta))$ -expansion method [8], the perturbation method [9–12], the extended tanh method [13,14], the sine-cosine method [15,16], the Adomian decomposition method [17–20].

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Until the 1950s, deterministic models of differential equations were commonly used to describe the dynamics of the system in implementations. However, it is evident that the phenomena that exist in today's world are not always deterministic.

Noise has now been shown to be important in many phenomena, also called randomness or fluctuations. Therefore, random effects have become significant when modeling different physical phenomena that take place in oceanography, physics, biology, meteorology, environmental sciences, and so on. Equations that consider random fluctuations in time are referred to as stochastic differential equations.

Here, we treat the stochastic Kuramoto-Sivashinsky (SKS) equation in one dimension with multiplicative noise in the Itô sense as follows:

$$du + [au\partial_x u + p\partial_x^2 u + q\partial_x^4 u]dt = \sigma u d\beta, \quad (1)$$

where α , p , and q are nonzero real constants, σ is a noise strength, and $\beta(t)$ is the standard Wiener process and it depends only on t .

The Kuramoto-Sivashinsky (KS) equation (1) with $\sigma = 0$ was first proposed in the mid-1970s. Kuramoto was the first to derive the equations for the Belousov-Zhabotinskii reaction using reaction-diffusion equations. Also, Sivashinsky used it to describe tiny thermal diffusive instabilities in laminar flame Poiseuille flow of a film layer on an inclined surface in higher space dimensions. It may also be used to represent Benard convection in an elongated box in one space dimension, and it can be utilized to illustrate long waves at the interface between two viscous fluids and unstable drift waves in plasmas. The KS equation can be applied to control surface roughness in the growth of thin solid films by sputtering, step dynamics in epitaxy, amorphous film formation, and population dynamics models [21–25].

The deterministic Kuramoto-Sivashinsky equation (1) (i.e., $\sigma = 0$) has been studied by a number of authors to attain its exact solutions by different methods such as the modified tanh-coth method [26], the tanh method and the extended tanh method [27], homotopy analysis method [28], the truncated expansion method [29], the $\left(\frac{G'}{G}\right)$ -expansion [30], the polynomial expansion method [31–34], the perturbation method [35], the Painlevé expansions methods [36]. However, the analytical stochastic solutions of the stochastic Kuramoto-Sivashinsky have never been obtained till this moment.

Our motivation of this article is to obtain the analytical stochastic solutions of the SKS (1) with multiplicative noise by using the $\left(\frac{G'}{G}\right)$ -expansion method. The results introduced here extend earlier studies, for instance, those reported in [27]. Also, we address the effects of multiplicative noise on these solutions.

The format of this paper is as follows: In Section 2, we obtain the wave equation for SKS equation (1), while in Section 3, we have the exact stochastic solutions of the SKS (1) by applying the $\left(\frac{G'}{G}\right)$ -expansion method. In Section 4, we show several graphical representations to demonstrate the effect of multiplicative noise on the obtained solutions of SKS. Finally, the conclusions of this paper are shown.

2 Wave equation for SKS equation

To obtain the wave equation for SKS equation (1), we use the following wave transformation:

$$u(x, t) = \varphi(\eta)e^{(\sigma\beta(t) - \frac{1}{2}\sigma^2 t)}, \quad \eta = x - ct, \quad (2)$$

where c is the wave speed and φ is the deterministic function. Substituting equation (2) into equation (1) and using

$$\begin{aligned} du &= \left(-c\varphi' + \frac{1}{2}\sigma^2\varphi - \frac{1}{2}\sigma^2\varphi\right)e^{(\sigma\beta(t) - \frac{1}{2}\sigma^2 t)}dt + \sigma\varphi e^{(\sigma\beta(t) - \frac{1}{2}\sigma^2 t)}d\beta, \\ u_x &= \varphi'e^{(\sigma\beta(t) - \frac{1}{2}\sigma^2 t)}, \quad u_{xx} = \varphi''e^{(\sigma\beta(t) - \frac{1}{2}\sigma^2 t)} \\ u_{xxx} &= \varphi'''e^{(\sigma\beta(t) - \frac{1}{2}\sigma^2 t)}, \quad u_{xxxx} = \varphi''''e^{(\sigma\beta(t) - \frac{1}{2}\sigma^2 t)}, \end{aligned} \quad (3)$$

where $+\frac{1}{2}\sigma^2\varphi$ is the Itô correction term, we obtain

$$-c\varphi' + \alpha\varphi\varphi'e^{(\sigma\beta(t)-\frac{1}{2}\sigma^2t)} + p\varphi'' + q\varphi''' = 0. \quad (4)$$

Taking expectation on both sides and considering that φ is the deterministic function, we have

$$-c\varphi' + \alpha\varphi\varphi'e^{-\frac{1}{2}\sigma^2t}\mathbb{E}(e^{\sigma\beta(t)}) + p\varphi'' + q\varphi''' = 0, \quad (5)$$

Since $\beta(t)$ is the standard Gaussian random variable, then for any real constant γ , we have $\mathbb{E}(e^{\gamma\beta(t)}) = e^{\frac{\gamma^2}{2}t}$. Now equation (5) has the form

$$-c\varphi' + \alpha\varphi\varphi' + p\varphi'' + q\varphi''' = 0, \quad (6)$$

Integrating equation (6) once in terms of η yields

$$q\varphi''' + p\varphi' + \frac{\alpha}{2}\varphi^2 - c\varphi = 0, \quad (7)$$

where we put the constant of integration equal zero.

3 The stochastic exact solutions of SKS equation

In this section, we use the $\frac{G'}{G}$ -expansion method [3] to find the solutions of equation (7). As a result, we have the exact stochastic solutions of the SKS (1). First, we assume that the solution of equation (7) has the form:

$$\varphi = \sum_{k=0}^M \hbar_k \left[\frac{G'}{G} \right]^k, \quad (8)$$

where $\hbar_0, \hbar_1, \dots, \hbar_M$ are uncertain constants that must be calculated later, and G solves

$$G'' + \lambda G' + \mu G = 0, \quad (9)$$

where λ, μ are unknown constants. Let us now calculate the parameter M by balancing φ^2 with φ''' in equation (7) as follows:

$$2M = M + 3,$$

and hence,

$$M = 3. \quad (10)$$

From (10), we can rewrite equation (8) as follows:

$$\varphi = \hbar_0 + \hbar_1 \left[\frac{G'}{G} \right] + \hbar_2 \left[\frac{G'}{G} \right]^2 + \hbar_3 \left[\frac{G'}{G} \right]^3. \quad (11)$$

Substituting equation (11) into equation (7) and using equation (9), we obtain a polynomial with degree 6 of $\frac{G'}{G}$ as follows:

$$\begin{aligned} & \left(\frac{1}{2}\alpha\hbar_3^2 - 60q\hbar_3 \right) \left[\frac{G'}{G} \right]^6 + (-24q\hbar_2 + \alpha\hbar_2\hbar_3 - 144q\lambda\hbar_3) \left[\frac{G'}{G} \right]^5 \\ & + \left(\frac{1}{2}\alpha\hbar_2^2 - 3p\hbar_3 - 6q\hbar_1 + \alpha\hbar_1\hbar_3 - 111q\lambda^2\hbar_3 - 114q\mu\hbar_3 - 54q\lambda\hbar_2 \right) \left[\frac{G'}{G} \right]^4 \\ & + (-c\hbar_3 + 2p\hbar_2 + \alpha\hbar_0\hbar_3 + \alpha\hbar_1\hbar_2 - 3p\lambda\hbar_3 - 38q\lambda^2\hbar_2 - 40q\mu\hbar_2 - 27\lambda^3\hbar_3 - 12q\lambda\hbar_1 - 168q\lambda\mu\hbar_3) \left[\frac{G'}{G} \right]^3 \end{aligned}$$

$$\begin{aligned}
 &+ (-c\hbar_2 + \frac{1}{2}\alpha\hbar_1^2 - p\hbar_1 + \alpha\hbar_0\hbar_2 - 2p\lambda\hbar_2 - 3p\mu\hbar_3 - 7q\lambda^2\hbar_1 - 8q\mu\hbar_1 - 8q\lambda^3\hbar_2 - 52q\lambda\mu\hbar_2 - 60q\mu^2\hbar_3 \\
 &- 57q\lambda^2\mu\hbar_3)\left[\frac{G'}{G}\right]^2 + (-c\hbar_1 + \alpha\hbar_0\hbar_1 - p\lambda\hbar_1 - 2p\mu\hbar_2 - q\lambda^3\hbar_1 \\
 &- 16q\mu^2\hbar_2 - 8q\lambda\mu\hbar_1 - 14q\lambda^2\mu\hbar_2 - 36q\mu^2\lambda\hbar_3)\left[\frac{G'}{G}\right] \\
 &+ \left(-c\hbar_0 + \frac{1}{2}\alpha\hbar_0^2 - p\mu\hbar_1 - q\lambda^2\mu\hbar_1 - 6q\mu^2\lambda\hbar_2 - 2q\mu^2\hbar_1 - 6q\mu^3\hbar_3\right) = 0.
 \end{aligned}$$

Assuming coefficient of $\left[\frac{G'}{G}\right]^i$ ($i = 0, 1, 2, 3, 4, 5, 6$) to zero, we obtain a system of algebraic equations. Solving this system by using Maple, we obtain two cases:

First case:

$$\begin{aligned}
 \hbar_0 &= \pm \frac{30p}{19\alpha} \sqrt{\frac{-p}{19q}}, \quad \hbar_1 = \frac{90p}{19\alpha}, \quad \hbar_2 = 0, \quad \hbar_3 = \frac{120q}{\alpha}, \\
 c &= \pm \frac{30p}{19} \sqrt{\frac{-p}{19q}}, \quad \lambda = 0, \quad \mu = \frac{p}{76q}, \quad \text{if } \frac{p}{q} < 0.
 \end{aligned} \tag{12}$$

In this situation, the solution of equation (7) is

$$\varphi(\eta) = \hbar_0 + \hbar_1 \left[\frac{G'}{G}\right] + \hbar_3 \left[\frac{G'}{G}\right]^3. \tag{13}$$

Solving equation (9) with $\lambda = 0, \mu = \frac{p}{76q}$ if $\frac{p}{q} < 0$, we obtain

$$G(\eta) = c_1 \exp\left(\sqrt{\frac{-p}{76q}} \eta\right) + c_2 \exp\left(-\sqrt{\frac{-p}{76q}} \eta\right), \tag{14}$$

where c_1 and c_2 are arbitrary constants. Substituting equation (14) into equation (13), we have

$$\begin{aligned}
 \varphi(\eta) &= \pm \frac{30p}{19\alpha} \sqrt{\frac{-p}{19q}} + \frac{90p}{19\alpha} \sqrt{\frac{-p}{76q}} \left[\frac{c_1 \exp\left(\sqrt{\frac{-p}{76q}} \eta\right) - c_2 \exp\left(-\sqrt{\frac{-p}{76q}} \eta\right)}{c_1 \exp\left(\sqrt{\frac{-p}{76q}} \eta\right) + c_2 \exp\left(-\sqrt{\frac{-p}{76q}} \eta\right)} \right] \\
 &+ \frac{120q}{\alpha} \left(\sqrt{\frac{-p}{76q}}\right)^3 \left[\frac{c_1 \exp\left(\sqrt{\frac{-p}{76q}} \eta\right) - c_2 \exp\left(-\sqrt{\frac{-p}{76q}} \eta\right)}{c_1 \exp\left(\sqrt{\frac{-p}{76q}} \eta\right) + c_2 \exp\left(-\sqrt{\frac{-p}{76q}} \eta\right)} \right]^3.
 \end{aligned}$$

Hence, the exact stochastic solution in this case of the SKS (1), by using (2), has the following form:

$$\begin{aligned}
 u_1(x, t) &= e^{(\sigma\beta(t) - \frac{1}{2}\sigma^2t)} \left\{ \pm \frac{30p}{19\alpha} \sqrt{\frac{-p}{19q}} + \frac{90p}{19\alpha} \sqrt{\frac{-p}{76q}} \left[\frac{c_1 \exp\left(\sqrt{\frac{-p}{76q}}(x - ct)\right) - c_2 \exp\left(-\sqrt{\frac{-p}{76q}}(x - ct)\right)}{c_1 \exp\left(\sqrt{\frac{-p}{76q}}(x - ct)\right) + c_2 \exp\left(-\sqrt{\frac{-p}{76q}}(x - ct)\right)} \right] \right. \\
 &\left. + \frac{120q}{\alpha} \left(\sqrt{\frac{-p}{76q}}\right)^3 \left[\frac{c_1 \exp\left(\sqrt{\frac{-p}{76q}}(x - ct)\right) - c_2 \exp\left(-\sqrt{\frac{-p}{76q}}(x - ct)\right)}{c_1 \exp\left(\sqrt{\frac{-p}{76q}}(x - ct)\right) + c_2 \exp\left(-\sqrt{\frac{-p}{76q}}(x - ct)\right)} \right]^3 \right\}, \tag{15}
 \end{aligned}$$

where $c = \pm \frac{30p}{19} \sqrt{\frac{-p}{19q}}$ and $\frac{p}{q} < 0$.

Second case:

$$\begin{aligned} \hbar_0 &= \pm \frac{30p}{19\alpha} \sqrt{\frac{11}{19q}}, & \hbar_1 &= \frac{-270p}{19\alpha}, & \hbar_2 &= 0, & \hbar_3 &= \frac{120q}{\alpha}, \\ c &= \pm \frac{30p}{19} \sqrt{\frac{11p}{19q}}, & \lambda &= 0, & \mu &= \frac{-11p}{76q}, & \text{if } \frac{p}{q} > 0. \end{aligned} \tag{16}$$

In this situation, the solution of equation (7) is expressed as follows:

$$\varphi(\eta) = \hbar_0 + \hbar_1 \left[\frac{G'}{G} \right] + \hbar_3 \left[\frac{G'}{G} \right]^3. \tag{17}$$

Solving equation (9) with $\lambda = 0$, $\mu = \frac{-11p}{76q}$, if $\frac{p}{q} > 0$, we obtain

$$G(\eta) = c_1 \exp\left(\sqrt{\frac{11p}{76q}} \eta\right) + c_2 \exp\left(-\sqrt{\frac{11p}{76q}} \eta\right). \tag{18}$$

Substituting equation (14) into equation (13), we have

$$\begin{aligned} \varphi(\eta) &= \pm \frac{30p}{19\alpha} \sqrt{\frac{11p}{19q}} - \frac{270p}{19\alpha} \sqrt{\frac{11p}{76q}} \left[\frac{c_1 \exp\left(\sqrt{\frac{11p}{76q}} \eta\right) - c_2 \exp\left(-\sqrt{\frac{11p}{76q}} \eta\right)}{c_1 \exp\left(\sqrt{\frac{11p}{76q}} \eta\right) + c_2 \exp\left(-\sqrt{\frac{11p}{76q}} \eta\right)} \right] \\ &+ \frac{120q}{\alpha} \left(\sqrt{\frac{11p}{76q}} \right)^3 \left[\frac{c_1 \exp\left(\sqrt{\frac{11p}{76q}} \eta\right) - c_2 \exp\left(-\sqrt{\frac{11p}{76q}} \eta\right)}{c_1 \exp\left(\sqrt{\frac{11p}{76q}} \eta\right) + c_2 \exp\left(-\sqrt{\frac{11p}{76q}} \eta\right)} \right]^3. \end{aligned}$$

Therefore, by using (2), the exact stochastic solution in this case of the SKS (1) has the following form:

$$\begin{aligned} u_2(x, t) &= e^{(\sigma\beta(t) - \frac{1}{2}\sigma^2 t)} \left\{ \pm \frac{30p}{19\alpha} \sqrt{\frac{11p}{19q}} - \frac{270p}{19\alpha} \sqrt{\frac{11p}{76q}} \left[\frac{c_1 \exp\left(\sqrt{\frac{11p}{76q}}(x - ct)\right) - c_2 \exp\left(-\sqrt{\frac{11p}{76q}}(x - ct)\right)}{c_1 \exp\left(\sqrt{\frac{11p}{76q}}(x - ct)\right) + c_2 \exp\left(-\sqrt{\frac{11p}{76q}}(x - ct)\right)} \right] \right. \\ &\left. + \frac{120q}{\alpha} \left(\sqrt{\frac{11p}{76q}} \right)^3 \left[\frac{c_1 \exp\left(\sqrt{\frac{11p}{76q}}(x - ct)\right) - c_2 \exp\left(-\sqrt{\frac{11p}{76q}}(x - ct)\right)}{c_1 \exp\left(\sqrt{\frac{11p}{76q}}(x - ct)\right) + c_2 \exp\left(-\sqrt{\frac{11p}{76q}}(x - ct)\right)} \right]^3 \right\}, \end{aligned} \tag{19}$$

where $c = \pm \frac{30p}{19} \sqrt{\frac{11p}{19q}}$ and $\frac{p}{q} > 0$.

Special cases:

Case 1: If we choose $c_1 = c_2 = 1$, then equations (15) and (19) become

$$\begin{aligned} u_1(x, t) &= e^{(\sigma\beta(t) - \frac{1}{2}\sigma^2 t)} \left[\pm \frac{30p}{19\alpha} \sqrt{\frac{-p}{19q}} + \frac{90p}{19\alpha} \sqrt{\frac{-p}{76q}} \tanh\left(\sqrt{\frac{-p}{76q}}(x - ct)\right) \right. \\ &\left. + \frac{120q}{\alpha} \left(\sqrt{\frac{-p}{76q}} \right)^3 \tanh^3\left(\sqrt{\frac{-p}{76q}}(x - ct)\right) \right], \end{aligned} \tag{20}$$

where $c = \pm \frac{30p}{19} \sqrt{\frac{-p}{19q}}$ and $\frac{p}{q} < 0$, and

$$\begin{aligned} u_2(x, t) &= e^{(\sigma\beta(t) - \frac{1}{2}\sigma^2 t)} \left[\pm \frac{30p}{19\alpha} \sqrt{\frac{11p}{19q}} - \frac{270p}{19\alpha} \sqrt{\frac{11p}{76q}} \tanh\left(\sqrt{\frac{11p}{76q}}(x - ct)\right) \right. \\ &\left. + \frac{120q}{\alpha} \left(\sqrt{\frac{11p}{76q}} \right)^3 \tanh^3\left(\sqrt{\frac{11p}{76q}}(x - ct)\right) \right], \end{aligned} \tag{21}$$

where $c = \pm \frac{30p}{19} \sqrt{\frac{11p}{19q}}$ and $\frac{p}{q} > 0$.

Case 2: If we choose $c_1 = 1$ and $c_2 = -1$, then equations (15) and (19) become

$$u_1(x, t) = e^{(\sigma\beta(t) - \frac{1}{2}\sigma^2 t)} \left\{ \pm \frac{30p}{19\alpha} \sqrt{\frac{-p}{19q}} + \frac{90p}{19\alpha} \sqrt{\frac{-p}{76q}} \coth\left(\sqrt{\frac{-p}{76q}}(x - ct)\right) + \frac{120q}{\alpha} \left(\sqrt{\frac{-p}{76q}}\right)^3 \coth^3\left(\sqrt{\frac{-p}{76q}}(x - ct)\right) \right\}, \tag{22}$$

where $c = \pm \frac{30p}{19} \sqrt{\frac{-p}{19q}}$ and $\frac{p}{q} < 0$, and

$$u_2(x, t) = e^{(\sigma\beta(t) - \frac{1}{2}\sigma^2 t)} \left\{ \pm \frac{30p}{19\alpha} \sqrt{\frac{11p}{19q}} - \frac{270p}{19\alpha} \sqrt{\frac{11p}{76q}} \coth\left(\sqrt{\frac{11p}{76q}}(x - ct)\right) + \frac{120q}{\alpha} \left(\sqrt{\frac{11p}{76q}}\right)^3 \coth^3\left(\sqrt{\frac{11p}{76q}}(x - ct)\right) \right\}, \tag{23}$$

where $c = \pm \frac{30p}{19} \sqrt{\frac{11p}{19q}}$ and $\frac{p}{q} > 0$.

Remark 1. If we put $\sigma = 0$ (i.e., equation (1) without noise) in equations (20)–(23), then we obtain the same results stated in [27].

4 The influence of noise on SKS solutions

Here, we discuss the influence of multiplicative noise on the exact solutions of the SKS equation (1). Fix the parameters $\alpha = p = q = 1$. We present a number of simulations for different values of σ (noise intensity). We utilize the MATLAB package to simulate our figures as follows:

In Figure 1, we can see that there is a kink solution, which indicates that the solution is not planar when $\sigma = 0$. But in Figure 2, when the noise appears and the intensity of the noise increases, we find that the surface becomes much more planar after small transit patterns. This means that the multiplicative noise affects and stabilizes the solutions.

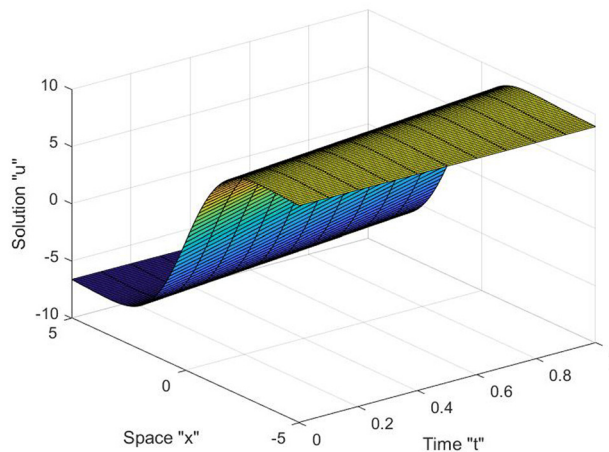


Figure 1: Graph of solution u_2 in equation (21) with $\sigma = 0$.

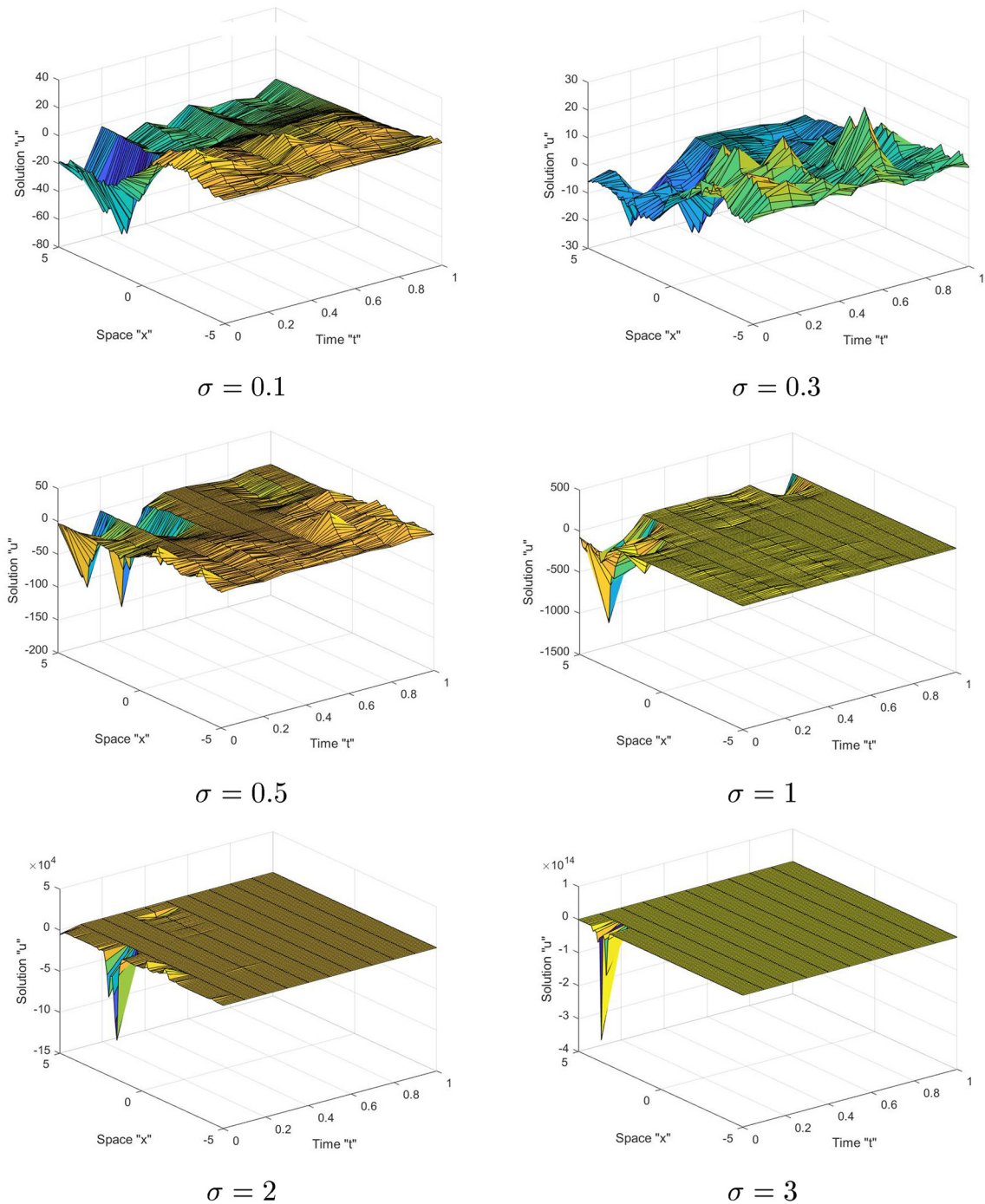


Figure 2: Graph of solution u_2 in equation (21) with $\sigma = 0.1, 0.3, 0.5, 1, 2, 3$.

5 Conclusion

In this paper, we presented a large variety of exact stochastic solutions of the Kuramoto-Sivashinsky equation (1) forced by multiplicative noise. Moreover, several results were extended such as those described in [27]. These types of solutions can be utilized to explain a variety of fascinating and complex physical phenomena. Finally, we used the MATLAB program to generate some graphical representations to show the

impact of multiplicative noise on the solutions of the SKS (1). In the future work, we can consider the multiplicative noise with more dimensions or we can take this equation with additive noise.

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