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The Influence of the Perturbation of the Initial Data on the Analytic Approximate Solution of the Van der Pol Equation in the Complex Domain

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Abstract: In this paper, we substantiate the analytical approximate method for Cauchy problem of the Van der Pol equation in the complex domain. These approximate solutions allow analytical continuation for both real and complex cases. We follow the influence of variation in the initial data of the problem in order to control the computational process and improve the accuracy of the final results. Several simple applications of the method are given. A numerical study confirms the consistency of the developed method.

Keywords: nonlinear differential equation of the second order; movable singular point; analytical approximate solution

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1. Introduction

As noted in [1–12], the Van der Pol equation has numerous applications: for instance, self-oscillation theory, nonlinear and symplectic dynamics, aerodynamics, biology, modelling of the processes in the human body, and in models of artificial intelligence, neural networks, biophysics, etc. The equation is also used in chaos theory [13,14] and in seismology when modelling geological faults [15].

The methodology for studying the Van der Pol equation refers mainly to the methods of qualitative differential equations theory and asymptotic methods [7,16–18]. This equation does not have an exact, analytic solution [19].

Let us note that the Van der Pol equation and its generalizations have been studied for the presence of symmetries. In particular, symmetry analysis [20] and study of the impact of symmetries on the occurrence of periodic solutions in systems of Van der Pol Equations [21] can be applied to the research of this equation.

The Van der Pol equation is a non-linear differential equation, is not solvable in quadratures, and has movable singular points in the complex plane [22]. The fact of the existence of movable singular points and the uniqueness of a solution of the Van der Pol equation for complex domain have been proven [23].

The article considers a modification of the Cauchy method, which allows for the construction of an analytical approximate solution with a given accuracy, indicating an estimate of the error of the resulting solution. It has been shown that influencing the variation in the initial data of the problem can gain control over the computational process and improve the accuracy of the final results. In [24], substantiation of an analytical approximation in solving some nonlinear differential equation in vicinity of movable singular points was presented.

These ideas were developed for other nonlinear differential equations in [23,25]. In particular, the existence of movable singular points in the Van der Pole equation has been proven in [23]. A generalization of the modified majorant method for third-order nonlinear differential equation in the domain of analyticity was represented in [25]. An analytical approximate solution was built, taking into account the solution search domain.

This article contains a continuation of the research in article [24].

2. Methods of Research and Results

We consider the initial problem for the Van der Pol equation in the complex domain $G \subset C$

$$\frac{d^2 w}{dz^2} = a(w^2 - 1)\frac{dw}{dz} - w, \tag{1}$$

$$w(z_0) = w_0, \;\; w'(z_0) = w_1, \tag{2}$$

where $w(z) \in C^1(G)$ and $a = \text{const}$ is a parameter.

In [24], an approximate solution for the initial problem was set

$$w_N(z) = \sum_{n=0}^{N} C_n (z - z_0)^n \tag{3}$$

for some domain of initial data (2) and where C_n represents the coefficients. When implementing the analytical continuation [26,27] of the solution (3), we face a mathematical problem concerning the influence of variation in the initial data on the solution (3). In this case, Formula (3) will change to the form

$$\tilde{w}_N(z) = \sum_{n=0}^{N} \tilde{C}_n (z - z_0)^n, \tag{4}$$

where $\tilde{C}_n$ represents the coefficients related to the modified initial data

$$\tilde{w}(z_0) = \tilde{w}_0, \;\; \tilde{w}'(z_0) = \tilde{w}_1. \tag{5}$$

The following theorem allows us to obtain a prior error estimation for solutions (4) and (5).

Theorem 1. *If $|a| \geq 1$, for the approximate solution (4) of the problem (1), (5) in the domain*

$$|z - z_0| < \rho_1, \;\; \rho_1 = const \neq 0,$$

for the error there exists an estimation

$$\Delta\tilde{w}_N(z) \leq \Delta_1 + \Delta_2,$$

where

$$\Delta_1 \leq \Delta M |1 + (z - z_0)|$$
$$+ \frac{|a| M \, \Delta M}{2} (M + \Delta M + 2)^2 |z - z_0|^2 \frac{1}{1 - |a| M (M + \Delta M + 2)|z - z_0|},$$

$$\Delta_2 \leq \frac{|a|^N M^N (M+2)^{N+1} |z - z_0|^{N+1}}{N(N+1)(1 - |a| M (M+2)|z - z_0|)},$$

$$\Delta M = \max\{\Delta\tilde{w}_0, \, \Delta\tilde{w}_1\}, \; M = \max\{|\tilde{w}_0|, \, |\tilde{w}_1|\},$$

$$\rho_1 = \min\left\{\frac{1}{|a| M (M+2)}, \frac{1}{|a| M (M + \Delta M + 2)}\right\}.$$

Proof. Applying the majorant method [24,27], we obtain

$$\Delta\tilde{w}_N(z) = |w(z) - \tilde{w}_N(z)| \leq |w(z) - \tilde{w}(z)| + |\tilde{w}(z) - \tilde{w}_N(z)| = \Delta_1 + \Delta_2.$$

Estimation for Δ_2 was done in [24]

$$\Delta_2 \leq \frac{|a|^N M^N (M+2)^{N+1} |z-z_0|^{N+1}}{N(N+1)(1-|a|M(M+2)|z-z_0|)}, \tag{6}$$

it is valid in the domain

$$|z-z_0| < \frac{1}{|a|M(M+2)}.$$

Let us consider the quantity Δ_1:

$$\Delta_1 = |w(z) - \tilde{w}(z)| = |\sum_{n=0}^{\infty} C_n (z-z_0)^n - \sum_{n=0}^{\infty} \tilde{C}_n (z-z_0)^n| \leq \sum_{n=0}^{\infty} \Delta\tilde{C}_n |z-z_0|^n,$$

where $\Delta\tilde{C}_n = |C_n - \tilde{C}_n|$. Let us prove the accuracy of the estimation for $\Delta\tilde{C}_n$ when $n \geq 2$

$$\Delta\tilde{C}_n \leq \frac{|a|^{n-1} M^{n-1} \Delta M}{n(n-1)} (M + \Delta M + 2)^n. \tag{7}$$

To prove inequality (7), we will use the recurrence relation for coefficients C_n [24]

$$n(n-1)C_n = a\, C^{***}_{n-2} - C_{n-2}, \tag{8}$$

where the following definitions are used

$$(w^2-1)\, w' = \sum_{n=0}^{\infty} C_n^{***} (z-z_0)^n; \;\; w^2 - 1 = \sum_{n=0}^{\infty} C_n^{**} (z-z_0)^n;$$

$$w^2 = \sum_{n=0}^{\infty} C_n^{*} (z-z_0)^n; \;\; C_n^* = \sum_{i=0}^{n} C_i\, C_{n-i};$$

$$C_0^{**} = C_0^* - 1; \;\; C_n^{**} = C_n^*, \;\; \forall n = 1,2,\ldots; \; C_n^{***} = \sum_{i=0}^{n} C_i^{**} (n+1-i)\, C_{n+1-i}.$$

Thus, we obtain

$$\Delta\tilde{C}_{n+1} = |C_{n+1} - \tilde{C}_{n+1}| = |\frac{1}{n(n+1)}\left(C^{***}_{n-1} - C_{n-1}\right) - \frac{1}{n(n+1)}\left(\tilde{C}^{***}_{n-1} - \tilde{C}_{n-1}\right)|$$

$$= |\frac{1}{n(n+1)}\left((C^{***}_{n-1} - \tilde{C}^{***}_{n-1}) - (C_{n-1} - \tilde{C}_{n-1})\right)|$$

$$= |\frac{1}{n(n+1)}\left(a\left(\sum_{i=0}^{n-1} C_i^{**}(n-i)C_{n-i} - \sum_{i=0}^{n-1} \tilde{C}_i^{**}(n-i)\tilde{C}_{n-i}\right) - (C_{n-1} - \tilde{C}_{n-1})\right)|$$

$$= \frac{1}{n(n+1)} |\left(a\left(\sum_{i=0}^{n-1} C_i^{*}(n-i)C_{n-i} - \sum_{i=0}^{n-1} \tilde{C}_i^{*}(n-i)\tilde{C}_{n-i}\right) - (C_{n-1} - \tilde{C}_{n-1})\right)|$$

$$\leq \frac{1}{n(n+1)} | a\left(\sum_{i=0}^{n-1}\left(\sum_{j=0}^{i} C_j C_{i-j}\right) C_{n-i} - \sum_{i=0}^{n-1}\left(\sum_{j=0}^{i} \tilde{C}_j \tilde{C}_{i-j}\right)\tilde{C}_{n-i}\right) - (C_{n-1} - \tilde{C}_{n-1})|$$

$$\leq \frac{1}{n(n+1)} | a\left(\sum_{i=0}^{n-1}\left(\sum_{j=0}^{i} (\tilde{C}_j + \Delta\tilde{C}_j)(\tilde{C}_{i-j} + \Delta\tilde{C}_{i-j})(\tilde{C}_{n-i} + \tilde{C}_{n-i})\right) - \sum_{i=0}^{n-1}\left(\sum_{j=0}^{i} \tilde{C}_j \tilde{C}_{i-j}\right)\tilde{C}_{n-i}\right)$$

$$-\Delta\tilde{C}_{n-i} | \leq \frac{1}{n(n+1)} | a \sum_{i=0}^{n-1}\left(\sum_{j=0}^{i} (\tilde{C}_j \Delta\tilde{C}_j)\tilde{C}_{n-i} + \left(\sum_{j=0}^{i} \Delta\tilde{C}_{i-j}\Delta\tilde{C}_j\right)\tilde{C}_{n-i}\right.$$

$$+\Big(\sum_{j=0}^{i}\Delta\tilde{C}_j\Delta\tilde{C}_{i-j}\Big)\tilde{C}_{n-i}+\Big(\sum_{j=0}^{i}\tilde{C}_j\tilde{C}_{i-j}\Big)\Delta\tilde{C}_{n-i}+\Big(\sum_{j=0}^{i}\tilde{C}_j\Delta\tilde{C}_{i-j}\Big)\Delta\tilde{C}_{n-i}$$

$$+\Big(\sum_{j=0}^{i}\tilde{C}_{i-j}\Delta\tilde{C}_j\Big)\Delta\tilde{C}_{n-i}+\Big(\sum_{j=0}^{i}\Delta\tilde{C}_j\Delta\tilde{C}_{i-j}\Big)\Delta\tilde{C}_{n-i}\Big)-\Delta\tilde{C}_{n-i}\mid.$$

Taking into account estimations for C_n [24] and $\Delta\tilde{C}_n$

$$|C_n|\leq\frac{\mid a\mid^{n-1}M^{n-1}}{n(n-1)}(M+2)^n,\;\;\Delta\tilde{C}_n\leq\frac{\mid a\mid^{n-1}M^{n-1}\Delta M}{n(n-1)}(M+\Delta M+2)^n,$$

after some transformations we derive the inequalities

$$\Delta\tilde{C}_{n+1}\leq\frac{1}{n(n+1)}|a|^{n-2}M^{n-3}\Delta M(M+\Delta M+2)^n\leq\frac{|a|^nM^n\,\Delta M(M+\Delta M+2)^{n+1}}{n(n+1)}.$$

Further, for Δ_1 we obtain

$$\Delta_1=|w(z)-\tilde{w}(z)|=\mid\sum_{n=0}^{\infty}\Delta\tilde{C}_n(z-z_0)^n\mid\leq$$

$$\mid\Delta\tilde{C}_0+\Delta\tilde{C}_1(z-z_0)+\sum_{n=2}^{\infty}\Delta\tilde{C}_n(z-z_0)^n\mid\leq$$

$$\Delta M|1+(z-z_0)|+\frac{|a|M\,\Delta M(M+\Delta M+2)^2|z-z_0|^2}{2(1-|a|M(M+\Delta M+2)|z-z_0|)}.$$

Because the estimation for Δ_1 is valid in the domain

$$|z-z_0|\leq\frac{1}{|a|M(M+\Delta M+2)},$$

the theorem holds in the domain

$$|z-z_0|<\rho,\quad\rho=\min\left\{\frac{1}{|a|M(M+2)},\frac{1}{|a|M(M+\Delta M+2)}\right\}.$$

□

Theorem 2. *If $|a|\leq 1$, a prior estimation for the solution (4) of problem (1), (5) in the domain*

$$|z-z_0|<\rho_2$$

has the form

$$\Delta\tilde{w}_N(z)\leq\Delta_1+\Delta_2,$$

where

$$\Delta_1\leq\Delta M(1+|z-z_0|)+\frac{M\,\Delta M(M+\Delta M+2)^2|z-z_0|^2}{2(1-M(M+\Delta M+2)|z-z_0|)},$$

$$\Delta_2\leq\frac{M^N(M+2)^{N+1}|z-z_0|^{N+1}}{N(N+1)(1-M(M+2)|z-z_0|)},$$

$$\Delta M=\max\{\Delta\tilde{w}_0,\,\Delta\tilde{w}_1\},\;M=\max\{|\tilde{w}_0|,\,|\tilde{w}_1|\},$$

$$\rho_2=\min\left\{\frac{1}{M(M+2)},\frac{1}{M(M+\Delta M+2)}\right\}.$$

Proof. Similarly to Theorem 1, we have

$$\Delta\tilde{w}_N(z) \le |w(z) - \tilde{w}(z)| + |\tilde{w}(z) - \tilde{w}_N(z)| = \Delta_1 + \Delta_2.$$

When $|a| \le 1$, in [24] an estimation for the C_n was found

$$|C_n| \le \frac{|a|M^{n-1}(M+1)^n + M}{n(n-1)}.$$

Let us strengthen this estimation

$$|C_n| \le \frac{|a|M^{n-1}(M+1)^n + M}{n(n-1)} \le \frac{M^{n-1}(M+2)^n}{n(n-1)}. \tag{9}$$

Taking into account estimation (9), we obtain

$$\Delta_2 = |\tilde{w}(z) - \tilde{w}_N(z)| =| \sum_{n=N+1}^{\infty} \tilde{C}_n (z-z_0)^n \,| \le$$

$$\sum_{n=2}^{\infty} \frac{M^{n-1}(M+2)^n}{n(n-1)} |z-z_0|^n = \frac{M^N(M+2)^{N+1}|z-z_0|^{N+1}}{N(N+1)(1-M(M+2)|z-z_0|)},$$

and it is valid in the domain

$$|z-z_0| < \frac{1}{M(M+2)}.$$

As in the case of Theorem 1, we obtain an estimation for $\Delta\tilde{C}_n$. Taking into account the condition $|a| \le 1$, we obtain

$$\Delta\tilde{C}_n \le \frac{M^{n-1}\Delta M(M+\Delta M+2)^n}{n(n-1)}.$$

Therefore, for Δ_1 we find the following estimation

$$\Delta_1 = |w(z) - \tilde{w}(z)| =| \sum_{n=0}^{\infty} (C_n - \tilde{C}_n)(z-z_0)^n \,|$$

$$\le \sum_{n=0}^{\infty} \frac{M^{n-1}\,\Delta M(M+\Delta M+2)^n|z-z_0|^n}{n(n-1)}$$

$$\le \Delta M + \Delta M|z-z_0| + \sum_{n=2}^{\infty} \frac{M^{n-1}\Delta M(M+\Delta M+2)^n|z-z_0|^n}{n(n-1)}$$

$$\le \Delta M(1+|z-z_0|) + \frac{M\,\Delta M(M+\Delta M+2)^2|z-z_0|^2}{2(1-M(M+\Delta M+2)|z-z_0|)}$$

in the domain

$$|z-z_0| < \frac{1}{M(M+\Delta M+2)},$$

where

$$\Delta M = \max\{\Delta\tilde{w}_0, \Delta\tilde{w}_1\}, \;\; M = \max\{|\tilde{w}_0|, |\tilde{w}_1|\}.$$

We will define

$$\rho_2 = \min\left\{\frac{1}{M(M+2)}, \frac{1}{M(M+\Delta M+2)}\right\}$$

and thereby complete the proof of Theorem 2. □

3. Numerical Study

Let us demonstrate the usage of the results of Theorems 1 or 2 depending on the value of the parameter a on the examples.

The corresponding calculations were performed in Mathematica system [28].

Taking into account the recurrence Formula (8), we find an explicit form of several coefficients $\tilde{C}_n$ $(n = 2, 3, \ldots, 10)$, which are necessary to calculate the approximate solution:

$$
\begin{aligned}
\tilde{C}_2 &= \frac{1}{2}\Big(a(\tilde{C}_0^2 - 1)\tilde{C}_1 - \tilde{C}_0\Big),\\
\tilde{C}_3 &= \frac{1}{6}\Big(2a\tilde{C}_0\tilde{C}_1^2 + 2a(\tilde{C}_0^2 - 1)\tilde{C}_2 - \tilde{C}_1\Big),\\
\tilde{C}_4 &= \frac{1}{12}\Big(a\Big(\tilde{C}_1^3 + 6\tilde{C}_0\tilde{C}_1\tilde{C}_2 + 3\Big(\tilde{C}_0^2 - 1\Big)\tilde{C}_3\Big) - \tilde{C}_2\Big),\\
\tilde{C}_5 &= \frac{1}{20}\Big(4a\Big(\tilde{C}_2\tilde{C}_1{}^2 + 2\tilde{C}_0\tilde{C}_3\tilde{C}_1 + \tilde{C}_0\tilde{C}_2{}^2 + (\tilde{C}_0{}^2 - 1)\tilde{C}_4\Big) - \tilde{C}_3\Big),\\
\tilde{C}_6 &= \frac{1}{30}\Big(5a\Big(\tilde{C}_3\tilde{C}_1^2 + (\tilde{C}_2^2 + 2\tilde{C}_0\tilde{C}_4)\tilde{C}_1 + 2\tilde{C}_0\tilde{C}_2\tilde{C}_3 + (\tilde{C}_0^2 - 1)\tilde{C}_5\Big) - \tilde{C}_4\Big),\\
\tilde{C}_7 &= \frac{1}{42}\Big(2a\Big(\tilde{C}_2^3 + 6(\tilde{C}_1\tilde{C}_3 + \tilde{C}_0\tilde{C}_4)\tilde{C}_2) + 3\tilde{C}_1^2\tilde{C}_4\\
&\quad -3\tilde{C}_6 + 3\tilde{C}_0\Big(\tilde{C}_3^2 + 2\tilde{C}_1\tilde{C}_5 + \tilde{C}_0\tilde{C}_6\Big)\Big) - \tilde{C}_5\Big),\\
\tilde{C}_8 &= \frac{1}{56}\Big(7a\Big(\tilde{C}_5\tilde{C}_1^2 + \Big(\tilde{C}_3^2 + 2\tilde{C}_0\tilde{C}_6\Big)\tilde{C}_1 + \tilde{C}_2^2\tilde{C}_3 + 2\tilde{C}_0\tilde{C}_3\tilde{C}_4\\
&\quad +2\tilde{C}_2(\tilde{C}_1\tilde{C}_4 + \tilde{C}_0\tilde{C}_5) + \Big(\tilde{C}_0^2 - 1\Big)\tilde{C}_7\Big) - \tilde{C}_6\Big),\\
\tilde{C}_9 &= \frac{1}{72}\Big(8a\Big(\tilde{C}_6\tilde{C}_1^2 + 2(\tilde{C}_3\tilde{C}_4 + \tilde{C}_0\tilde{C}_7)\tilde{C}_1 + \tilde{C}_2^2\tilde{C}_4 + \tilde{C}_0(\tilde{C}_4^2 + 2\tilde{C}_3\tilde{C}_5)\\
&\quad +\tilde{C}_2\Big(\tilde{C}_3^2 + 2\tilde{C}_1\tilde{C}_5 + 2\tilde{C}_0\tilde{C}_6\Big) + (\tilde{C}_0^2 - 1)\tilde{C}_8\Big) - \tilde{C}_7\Big),\\
\tilde{C}_{10} &= \frac{1}{90}\Big(3a\Big(\tilde{C}_3^3 + 6(\tilde{C}_2\tilde{C}_4 + \tilde{C}_1\tilde{C}_5 + \tilde{C}_0\tilde{C}_6)\tilde{C}_3 + 3\Big(\tilde{C}_7\tilde{C}_1^2\\
&\quad +(\tilde{C}_4^2 + 2\tilde{C}_2\tilde{C}_6 + 2\tilde{C}_0\tilde{C}_8)\tilde{C}_1 + \tilde{C}_2^2\tilde{C}_5 + 2\tilde{C}_0\tilde{C}_4\tilde{C}_5 + 2\tilde{C}_0\tilde{C}_2\tilde{C}_7 + (\tilde{C}_0^2 - 1)\tilde{C}_9\Big)\Big) - \tilde{C}_8\Big).
\end{aligned}
\tag{10}
$$

3.1. Example 1

Consider Equation (1), when $a = 2$. We set the initial conditions (5)

$$z_0 = 0.16, \;\; \tilde{w}_0 = 0.187933\, i, \;\; \tilde{w}_1 = 1.366035\, i. \tag{11}$$

The variation in the initial data is

$$\Delta\tilde{w}_0 = 10^{-5}, \;\; \Delta\tilde{w}_1 = 10^{-5}, \;\; N = 10.$$

Based on Theorem 1, we find the radius of the analytic continuation $\rho_1 = 0.108740$. Taking into account the value of the radius ρ_1, we choose the value $z_1 = 0.26$. Using Formulas (4), (10), and (11), we obtain the structure of the analytic continuation of the approximate solution

$$
\begin{aligned}
\tilde{w}_{10} &= -2.55122i(z-0.16)^{10} + 1.58662i(z-0.16)^{9} - 1.07093i(z-0.16)^{8} + 1.37633i(z-0.16)^{7}\\
&\quad - 1.552171i(z-0.16)^{6} + 0.894560i(z-0.16)^{5} - 0.211966i(z-0.16)^{4}\\
&\quad + 0.579544i(z-0.16)^{3} - 1.508248i(z-0.16)^{2} + 1.366035i(z-0.16).
\end{aligned}
\tag{12}
$$

The characteristics of the approximate solution (12) are presented in Table 1.

Table 1. Numerical characteristics of example 1.

No. Analytical Continuation	z_1	$\tilde{w}_{10}(z_1)$	$\Delta\tilde{w}_{10}$	Δ_1
1	0.26	0.122087i	0.016501	10^{-4}

Here, $\tilde{w}_{10}(z_1)$ is an approximate solution, $\Delta\tilde{w}_{10}(z_1)$ is a prior error estimation, and Δ_1 is a posterior error estimation. For a posterior estimation $\Delta_1 = 10^{-4}$, we choose $N = 42$ in the structure of the approximate solution (12). For an approximate solution (12) the values $\Delta\tilde{w}_{10}(z_1)$ has the calculation error 1.65×10^{-2} (Theorem 1). The summands from the 11th to 42nd in total do not exceed 10^{-4}. Therefore, $\tilde{w}_{10}(z_1)$ in the resulting domain has the accuracy 10^{-4}.

3.2. Example 2

We consider Equation (1), when $a = 1$. Using Theorem 2 for the initial conditions (5)

$$z_0 = 0.3, \quad \tilde{w}_0 = 0.345649\, i, \quad \tilde{w}_1 = 1.309878\, i \tag{13}$$

and variations in the initial data

$$\Delta\tilde{w}_0 = 10^{-5}, \quad \Delta\tilde{w}_1 = 10^{-5}, \quad N = 10,$$

we find the radius $\rho_2 = 0.230651$ and value $z_1 = 0.5$. Using Formulas (4), (10), and (13), we obtain an approximate solution

$$\begin{aligned} \tilde{w}_{10} = & -0.089172i(z-0.3)^{10} + 0.082906i(z-0.3)^9 + 0.040185i(z-0.3)^8 - 0.009819i(z-0.3)^7 \\ & - 0.232238i(z-0.3)^6 + 0.246389i(z-0.3)^5 + 0.115120i(z-0.3)^4 \\ & - 0.077914i(z-0.3)^3 - 0.906011i(z-0.3)^2 + 1.309878i(z-0.3). \end{aligned} \tag{14}$$

The characteristics of the approximate solution (14) are presented in Table 2.

Table 2. Numerical characteristics of example 2.

No. Analytical Continuation	z_1	$\tilde{w}_{10}(z_1)$	$\Delta\tilde{w}_{10}$	Δ_1
1	0.5	0.22536i	0.010915	10^{-4}

For a posterior estimation of $\Delta_1 = 10^{-4}$ we choose $N = 31$ in the structure of the approximate solution (14). For an approximate solution $\tilde{w}_{10}(z)$ of the form (14), we have the calculation error 10^{-2}. The summands from 11th to 31nd in total do not exceed 10^{-4}. Therefore, $\tilde{w}_{10}(z)$ in the resulting domain has an accuracy of 10^{-4}.

3.3. Example 3

We consider Equation (1), when $a = \frac{1}{50}$ (using Theorem 2). For initial conditions

$$z_0 = 0.3, \quad \tilde{w}_0 = 0.296422\, i, \quad \tilde{w}_1 = 0.961348\, i \tag{15}$$

and their variations

$$\Delta\tilde{w}_0 = 10^{-5}, \quad \Delta\tilde{w}_1 = 10^{-5}, \quad N = 10,$$

we calculate the radius $\rho_2 = 0.351260$ and value $z_1 = 0.6$. Using Formulas (4), (10), and (15), we obtain an approximate solution

$$\begin{aligned}\tilde{w}_{10} = {} & 7.21950 \times 10^{-6} i(z-0.3)^{10} + 0.000032 i(z-0.3)^9 - 0.000054 i(z-0.3)^8 \\ & - 0.000416 i(z-0.3)^7 - 0.000128 i(z-0.3)^6 + 0.008912 i(z-0.3)^5 \\ & + 0.013069 i(z-0.3)^4 - 0.160900 i(z-0.3)^3 - 0.158669 i(z-0.3)^2 + 0.961348 i(z-0.3).\end{aligned} \tag{16}$$

The values of the approximate solution (16) and its characteristics are presented in Table 3.

Table 3. Numerical characteristics of example 3.

No. Analytical Continuation	z_1	$\tilde{w}_{10}(z_1)$	$\Delta\tilde{w}_{10}$	Δ_1
1	0.6	$0.568236i$	0.011467	10^{-4}

For a posterior estimation $\Delta_1 = 10^{-4}$ we choose $N = 30$ in the structure of the approximate solution (16). For an approximate solution $\tilde{w}_{10}(z)$ of the form (16) we have the calculation error 1.15×10^{-2}. The summands from 11th to 30th in total do not exceed 10^{-4}. Therefore, $\tilde{w}_{10}(z)$ in the resulting domain has an accuracy of 10^{-4}.

The analytical approximate solutions (12), (14), and (16) obtained in the examples approximate the exact solution of the initial problems with the accuracy indicated in Tables 1–3, respectively.

4. Conclusions

In the present paper, we have presented results on the study of the analytical approximate solution of the Van der Pol equation in the analyticity domain. We have examined dependence of such an approximation on small changes in the initial data. This permits us to perform analytical continuation on the considered nonlinear equation. The theoretical studies are verified by numerical analysis. Optimization of an a priori estimate is carried out using an a posteriori estimate.

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