

THE INITIAL-VALUE PROBLEM FOR A STRETCHED STRING*

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1. Introduction. There have been many studies in the past thirty-five years of the motion of a stretched elastic string. In 1945 Carrier [1] developed an approximate equation for transverse motion for such a string which accounted, at least to second order, for the fact that such a motion was inherently nonlinear. These equations were reformulated by Narasimha [5] in 1967. In this paper, Narasimha introduced a term which accounted for external damping of the string.

The general theory of the Carrier-Narasimha equations has been partially worked out by Dickey and Nisida. Dickey [2, 3] and Nisida [6, 7] have investigated the initial-boundary value problem for these equations for a string of finite length and Dickey [4] has obtained partial results for the pure initial-value problem for these equations for a string of infinite length.

In [4], Dickey considered the initial value problem for the equation

$$U_{tt} = C^2(t)U_{xx}, \quad -\infty < x < \infty \text{ and } t > 0 \quad (1.1)$$

where the speed of sound $C(\cdot)$ is related to the vertical displacement $U(\cdot, \cdot)$ by

$$C^2(t) = 1 + \varepsilon \int_{-\infty}^{\infty} U_{\xi}^2(\xi, t) d\xi, \quad 0 < \varepsilon. \quad (1.2)$$

Through the use of certain differential inequalities Dickey was able to show that the initial-value problem for the system (1.1) and (1.2) was uniquely solvable on some interval $0 \leq t < T(\varepsilon)$ where $T(\varepsilon) = O(1/\varepsilon)$. He was also able to show, via direct substitution, that this system was capable of supporting traveling waves; that is, solutions of the form $U = f(x \pm C^*t)$ where C^* is a constant depending on f .

In this paper, we restrict our attention to the equations considered by Dickey in [4]. Our basic result is that if the parameter ε in Eq. (1.2) is sufficiently small and if the initial data for problem is sufficiently well-behaved, then the initial-value problem for (1.1) and (1.2) is uniquely solvable for all time t . Moreover, we establish the existence of functions $R_{\pm}(\cdot)$ and $L_{\pm}(\cdot)$ and a unique positive number C_* such that

$$\lim_{t \rightarrow \pm \infty} U(x + C_*t, t) = R_{\pm}(x), \quad \lim_{t \rightarrow \pm \infty} U(x - C_*t, t) = L_{\pm}(x). \quad (1.3, 1.4)$$

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Here,

$$C_*^2 = \lim_{t \rightarrow \pm\infty} \left(1 + \varepsilon \int_{-\infty}^{\infty} U_{\xi}^2(\xi, t) d\xi \right), \tag{1.5}$$

Our attack is similar to Dickey's. In Sec. 2, we Fourier-transform the original equations of motion and obtain integro-differential equations for the Fourier transform of the motion U . The form of these equations suggests a study of a certain linear problem. This is done in Sec. 3. The thrust of this section is to show that if a certain potential $q(\cdot)$, which is related to the sound speed $C(\cdot)$, has certain decay properties in $|t|$, then the Fourier transform U must itself have similar properties. Equipped with these results we are able to show (see Sec. 4) that it is possible to construct, through the identity (1.2), a well-defined mapping of functions $C(\cdot)$ into functions $C(\cdot)$; we also show that for $0 < \varepsilon \ll 1$ this map is a contraction. This fact yields the results claimed above.

2. Transformations. In this section we transform our original integro-differential equation into ordinary differential equations for the Fourier coefficients of U . Our original equation is

$$U_{tt} - C^2(t)U_{xx} = 0 \tag{2.1}$$

where U satisfies the initial conditions

$$U(x, 0) = f(x), \quad U_t(x, 0) = g(x) \tag{IC}$$

and C^2 is given by $C^2(t) = 1 + \varepsilon \int_{-\infty}^{\infty} U_x^2(x, t) dx$.

We assume that f' and g are twice continuously differentiable and that $(1 + x^2)|f'(x)| < \infty$ and $(1 + x^2)|g(x)| < \infty$. If we let $C_0^2 = 1 + \varepsilon \int_{-\infty}^{\infty} U_x^2(x, 0) dx = 1 + \varepsilon \int_{-\infty}^{\infty} (f'(x))^2 dx$ and $g(x) = C_0 g_1(x)$, the initial condition may be rewritten as

$$U(x, 0) = f(x), \quad U_t(x, 0) = C_0 g_1(x). \tag{IC'}$$

We shall use this fact later.

Our first task is to write Eq. (2.1) as a system. We let $A(x, t) = U_t(x, t) - C(t)U_x(x, t)$ and $B(x, t) = U_t(x, t) + C(t)U_x(x, t)$. The fact that Eq. (2.1) may be rewritten as

$$\begin{aligned} \left(\frac{\partial}{\partial t} + C(t) \frac{\partial}{\partial x} \right) \left(\frac{\partial}{\partial t} - C(t) \frac{\partial}{\partial x} \right) U &= -\dot{C}U_x, \\ \left(\frac{\partial}{\partial t} - C(t) \frac{\partial}{\partial x} \right) \left(\frac{\partial}{\partial t} + C(t) \frac{\partial}{\partial x} \right) U &= \dot{C}U_x \end{aligned} \tag{2.2}$$

yields the following equations for A and B :

$$A_t + C(t)A_x = \frac{-\dot{C}(t)}{2C(t)}(B - A), \quad B_t - C(t)B_x = \frac{\dot{C}(t)}{2C(t)}(B - A). \tag{2.3}$$

The initial conditions for A and B are computable in terms of f' and g . They are

$$\begin{aligned} A(x, 0) = A_0(x) &\stackrel{\text{def}}{=} C_0(g_1(x) - f'(x)), \\ B(x, 0) = B_0(x) &\stackrel{\text{def}}{=} C_0(g_1(x) + f'(x)). \end{aligned} \tag{IC}_1$$

The defining relation for C^2 becomes

$$C^2(t) = 1 + \frac{\varepsilon}{4C^2(t)} \int_{-\infty}^{\infty} (B - A)^2(x, t) dx. \tag{2.3a}$$

We now introduce the change of variable $\tau = \int_0^t C(s) ds$. Clearly, $d\tau/dt = C(t) > 1$ and thus τ is a strictly increasing function of t . We denote its inverse function by $t = T(\tau)$ and regard C as a function of τ , i.e. $C(\tau) = C(T(\tau))$. Then, $dC/dt = (dC/d\tau) \cdot (d\tau/dt) = C(\tau) \times (dC/d\tau)$.

If we apply the change of variable to Eq. (2.3), we obtain

$$A_\tau + A_x = -\frac{C_\tau}{2C} (B - A), \quad B_\tau - B_x = \frac{C_\tau}{2C} (B - A). \tag{2.4}$$

We are now regarding A and B as function of x and τ . The initial condition and the defining relationship for C are the same as described in (IC)₁ and (2.3a).

We now Fourier-transform the system (2.4). We let

$$\mathcal{A}_*(\omega, \tau) = \int_{-\infty}^{\infty} e^{-i\omega x} A(x, \tau) dx, \quad \mathcal{B}_*(\omega, \tau) = \int_{-\infty}^{\infty} e^{-i\omega x} B(x, \tau) dx.$$

The coefficients \mathcal{A}_* and \mathcal{B}_* satisfy

$$\mathcal{A}_{*\tau} + i\omega \mathcal{A}_* = -\frac{C_\tau}{2C} (\mathcal{B}_* - \mathcal{A}_*), \quad \mathcal{B}_{*\tau} - i\omega \mathcal{B}_* = \frac{C_\tau}{2C} (\mathcal{B}_* - \mathcal{A}_*) \tag{2.5}$$

and C is connected to \mathcal{A}_* and \mathcal{B}_* by

$$C^2(\tau) = 1 + \frac{\varepsilon}{8\pi C^2(\tau)} \int_{-\infty}^{\infty} (\mathcal{B}_*(\omega, \tau) - \mathcal{A}_*(\omega, \tau)) \cdot (\mathcal{B}_*(-\omega, \tau) - \mathcal{A}_*(-\omega, \tau)) d\omega. \tag{2.3b}$$

The initial condition for \mathcal{A}_* and \mathcal{B}_* are

$$\mathcal{A}_*(\omega, 0) = C_0(\hat{g}_1 - \hat{f}_1)(\omega) \stackrel{\text{def}}{=} C_0^\dagger \mathcal{A}(\omega), \quad \mathcal{B}_*(\omega, 0) = C_0(\hat{g}_1 - \hat{f}_1)(\omega) \stackrel{\text{def}}{=} C_0^\dagger \mathcal{B}(\omega), \tag{IC}_2$$

where $\hat{f}_1(\omega) = \int_{-\infty}^{\infty} e^{-i\omega x} f'(x) dx$.

Finally, if we let

$$\mathcal{A}(\omega, \tau) = \frac{\mathcal{A}_*(\omega, \tau)}{C^\dagger(\tau)} e^{i\omega\tau}, \quad \mathcal{B}(\omega, \tau) = \frac{\mathcal{B}_*(\omega, \tau)}{C^\dagger(\tau)} e^{-i\omega\tau},$$

then Eq. (2.5) transforms to

$$\mathcal{A}_\tau(\omega, \tau) = -\frac{C_\tau(\tau)}{2C(\tau)} e^{2i\omega\tau} \mathcal{B}(\omega, \tau), \quad \mathcal{B}_\tau(\omega, \tau) = -\frac{C_\tau(\tau)}{2C(\tau)} e^{-2i\omega\tau} \mathcal{A}(\omega, \tau) \tag{2.6}$$

and \mathcal{A} and \mathcal{B} satisfy the initial condition

$$\mathcal{A}(\omega, 0) = \tilde{\mathcal{A}}(\omega), \quad \mathcal{B}(\omega, 0) = \tilde{\mathcal{B}}(\omega). \tag{IC}_3$$

¹ Throughout, we exploit the identities $\mathcal{A}_*(-\omega, \tau) = \overline{\mathcal{A}_*(\omega, \tau)}$ and $\mathcal{B}_*(-\omega, \tau) = \overline{\mathcal{B}_*(\omega, \tau)}$.

These are equivalent to the integral equations

$$\begin{aligned} \mathcal{A}(\omega, \tau) &= \tilde{\mathcal{A}}(\omega) - \int_0^\tau e^{2i\omega s} \frac{C_s(s)}{2C(s)} \mathcal{B}(\omega, s) ds, \\ \mathcal{B}(\omega, \tau) &= \tilde{\mathcal{B}}(\omega) - \int_0^\tau e^{-2i\omega s} \frac{C_s(s)}{2C(s)} \mathcal{A}(\omega, s) ds, \end{aligned} \tag{2.7}$$

and C is connected to \mathcal{A} and \mathcal{B} by

$$\begin{aligned} C^2(\tau) &= 1 + \frac{\varepsilon}{8\pi C(\tau)} \int_{-\infty}^\infty (\mathcal{B}(\omega, \tau)e^{i\omega\tau} - \mathcal{A}(\omega, \tau)e^{-i\omega\tau}) \\ &\quad \times (\mathcal{B}(-\omega, \tau)e^{-i\omega\tau} - \mathcal{A}(-\omega, \tau)e^{i\omega\tau}) d\omega. \end{aligned} \tag{2.8}$$

Now our goal is to show that for $0 < \varepsilon \ll 1$ there exists a unique solution to (2.7) and (2.8). We shall also show there exists a function $(\mathcal{A}_{\pm\infty}, \mathcal{B}_{\pm\infty})(\omega)$ and a unique positive number $C_* > 0$ such that

$$\lim_{\tau \rightarrow \pm\infty} (\mathcal{A}(\omega, \tau), \mathcal{B}(\omega, \tau)) = (\mathcal{A}_{\pm\infty}, \mathcal{B}_{\pm\infty})(\omega)$$

and

$$\lim_{\tau \rightarrow \pm\infty} C(\tau) = C_* > 0.$$

These latter limit relations imply that the functions

$$A(x, \tau) = \frac{1}{2\pi} \int_{-\infty}^\infty e^{i\omega x} \mathcal{A}_*(\omega, \tau) d\omega, \quad B(x, \tau) = \frac{1}{2\pi} \int_{-\infty}^\infty e^{i\omega x} \mathcal{B}_*(\omega, \tau) d\omega$$

satisfy

$$\lim_{\tau \rightarrow \pm\infty} A(x + \tau, \tau) = A_{\pm\infty}(x), \quad \lim_{\tau \rightarrow \pm\infty} B(x - \tau, \tau) = B_{\pm\infty}(x)$$

where

$$A_{\pm\infty}(\xi) = \frac{C_*^\pm}{2\pi} \int_{-\infty}^\infty e^{i\omega\xi} \mathcal{A}_{\pm\infty}(\omega) d\omega, \quad B_{\pm\infty}(\xi) = \frac{C_*^\pm}{2\pi} \int_{-\infty}^\infty e^{i\omega\xi} \mathcal{B}_{\pm\infty}(\omega) d\omega.$$

That is, the system (2.3) and (IC)₁ has a well-defined scattering theory.

That such a result should have been anticipated follows from Dickey's observation [4] that the original equation $U_{tt} - C^2(t)U_{xx} = 0$ with $C^2(t) = 1 + \varepsilon \int_{-\infty}^\infty U_x^2(x, t) dx$ supports right- and left-facing traveling waves.

3. The linear problem. In this section we shall confine our attention to solutions of the integral equations:

$$\begin{aligned} \mathcal{A}(\omega, \tau) &= \tilde{\mathcal{A}}(\omega) - \int_0^\tau e^{2i\omega s} q(s) \mathcal{B}(\omega, s) ds \\ \mathcal{B}(\omega, \tau) &= \tilde{\mathcal{B}}(\omega) - \int_0^\tau e^{-2i\omega s} q(s) \mathcal{A}(\omega, s) ds \end{aligned} \tag{3.1}$$

when $q(\cdot)$ is prescribed.

In fact it will suffice to examine carefully solutions of

$$\alpha(\omega, \tau) = 1 - \int_0^\tau e^{2i\omega s} q(s) \beta(\omega, s) ds, \quad \beta(\omega, \tau) = - \int_0^\tau e^{-2i\omega s} q(s) \alpha(\omega, s) ds \quad (3.2)$$

because the solution of (3.1) may be expressed in terms of the functions α and β by

$$\begin{pmatrix} \mathcal{A} \\ \mathcal{B} \end{pmatrix}(\omega, \tau) = \begin{pmatrix} \alpha(\omega, \tau), \beta(-\omega, \tau) \\ \beta(\omega, \tau), \alpha(-\omega, \tau) \end{pmatrix} \begin{pmatrix} \tilde{\mathcal{A}}(\omega) \\ \tilde{\mathcal{B}}(\omega) \end{pmatrix}. \quad (3.3)$$

This last result is a consequence of the variation of constants formula for the system (3.1) and the fact that $W(\omega, \tau) \stackrel{\text{def}}{=} \alpha(\omega, \tau)\alpha(-\omega, \tau) - \beta(\omega, \tau)\beta(-\omega, \tau)$ satisfies $dW/d\tau = 0$.

3.1. *Boundedness and continuity lemmas.* In what follows we shall let

$$\|\cdot\| = \sup_{\omega, \tau} |\cdot|, \quad \|\cdot\|_1 = \int_{-\infty}^{\infty} |\cdot|(s) ds \quad (3.4)$$

and shall employ the working hypothesis that *the function $q(\cdot)$ in (3.1) satisfies $\|q\|_1 < 1$* . This assumption guarantees that the functions α and β of (3.2) are well defined. When comparing solutions of (3.1) we shall let $(\mathcal{A}_i, \mathcal{B}_i)$, $i = 1$ and 2 , denote the unique solutions of (3.1) corresponding to the potential $q_i(\cdot)$, $i = 1$ and 2 . The following notation will be used throughout the remainder of this section:

$$M_{n,i}(\tau) = \max \left[\sup_{\omega} |\omega^n \mathcal{A}_i(\omega, \tau)|, \sup_{\omega} |\omega^n \mathcal{B}_i(\omega, \tau)| \right], \quad (3.5)$$

$$N_{n,i}(\tau) = \max \left[\int_{-\infty}^{\infty} |\omega^n \mathcal{A}_i(\omega, \tau)| d\omega, \int_{-\infty}^{\infty} |\omega^n \mathcal{B}_i(\omega, \tau)| d\omega \right], \quad (3.6)$$

$$M_n^*(\tau) = \max_{i=1,2} M_{n,i}, \quad N_n^*(\tau) = \max_{i=1,2} N_{n,i}(\tau), \quad (3.7)$$

$$\Delta_n(\tau) = \max \left[\sup_{\omega} |\omega^n (\mathcal{A}_2 - \mathcal{A}_1)(\omega, \tau)|, \sup_{\omega} |\omega^n (\mathcal{B}_2 - \mathcal{B}_1)(\omega, \tau)| \right], \quad (3.8)$$

$$D_n(\tau) = \max \left[\int_{-\infty}^{\infty} |\omega^n (\mathcal{A}_2 - \mathcal{A}_1)(\omega, \tau)| d\omega, \int_{-\infty}^{\infty} |\omega^n (\mathcal{B}_2 - \mathcal{B}_1)(\omega, \tau)| d\omega \right]. \quad (3.9)$$

We shall now give some estimates which tie down the dependence of the functions (α_i, β_i) and $(\mathcal{A}_i, \mathcal{B}_i)$ on the potentials $q_i(\cdot)$.

LEMMA 3.1. (a) The functions α_i and β_i satisfy

$$\|\alpha_i\| \leq \frac{1}{1 - \|q_i\|_1^2}, \quad \|\beta_i\| \leq \frac{\|q_i\|_1}{1 - \|q_i\|_1^2}; \quad (3.10)$$

(b) the functions $M_{n,i}$ and $N_{n,i}$ satisfy

$$M_{n,i}(\tau) \leq \frac{M_{n,i}(0)}{1 - \|q_i\|_1} \quad N_{n,i}(\tau) \leq \frac{N_{n,i}(0)}{1 - \|q_i\|_1}; \quad (3.11)$$

(c) the functions Δ_n and D_n satisfy

$$\Delta_n(\tau) \leq \frac{M_n^*(0) \|q_2 - q_1\|_1}{\left(1 - \max_{m=1,2} \|q_m\|_1\right)^2}, \quad (3.12)$$

$$D_n(\tau) \leq \frac{N_n^*(0) \|q_2 - q_1\|_1}{\left(1 - \max_{m=1,2} \|q_m\|_1\right)^2}. \quad (3.13)$$

Proof. We establish (3.10) first. We discuss only the case where $\tau > 0$ and suppress the indices i . Eq. (3.2) yields the identity

$$\alpha(\omega, \tau) = 1 + \int_0^\tau e^{2i\omega s} q(s) \int_0^s e^{-2i\omega y} q(y) \alpha(\omega, y) dy ds$$

and from this we obtain the inequalities:

$$|\alpha(\omega, \tau)| \leq 1 + \int_0^\tau |q(s)| \int_0^s |q(y)| |\alpha(\omega, y)| dy ds \leq 1 + \|q\|_1^2 \sup_{0 \leq y \leq \tau} |\alpha(\omega, y)|,$$

$$\|\alpha\| \leq 1 + \|q\|_1^2 \|\alpha\|, \quad \|\alpha\| \leq \frac{1}{1 - \|q\|_1^2}.$$

We also have

$$\beta(\omega, \tau) = - \int_0^\tau e^{-2i\omega s} q(s) ds + \int_0^\tau e^{-2i\omega s} q(s) \int_0^s e^{2i\omega y} q(y) \beta(\omega, y) dy ds$$

which in turn yields

$$\begin{aligned} |\beta(\omega, \tau)| &\leq \int_0^\tau |q(s)| ds + \int_0^\tau |q(s)| \int_0^s |q(y)| |\beta(\omega, y)| dy ds \\ &\leq \|q\|_1 + \|q\|_1^2 \sup_{0 \leq y \leq \tau} |\beta(\omega, y)|, \end{aligned}$$

$$\|\beta\| \leq \|q\|_1 + \|q\|_1^2 \|\beta\|, \quad \|\beta\| \leq \frac{\|q\|_1}{1 - \|q\|_1^2}.$$

The identity (3.3) connecting the solution of (3.1) to the solution of (3.2) and the results of (3.10) yield (3.11).

To obtain (3.12) and (3.13) we first note that

$$\begin{aligned} (\mathcal{A}_2 - \mathcal{A}_1)(\omega, \tau) &= - \int_0^\tau (q_2 - q_1)(s) e^{2i\omega s} \mathcal{B}_1(\omega, s) ds \\ &\quad - \int_0^\tau (q_2(s) e^{2i\omega s} (\mathcal{B}_2 - \mathcal{B}_1)(\omega, s) ds \end{aligned}$$

and

$$\begin{aligned}
 (\mathcal{B}_2 - \mathcal{B}_1)(\omega, \tau) &= - \int_0^\tau (q_2 - q_1)(s) e^{-2i\omega s} \mathcal{A}_1(\omega, s) ds \\
 &\quad - \int_0^\tau q_2(s) e^{-2i\omega s} (\mathcal{A}_2 - \mathcal{A}_1)(\omega, s) ds.
 \end{aligned}$$

From the above we obtain

$$\begin{aligned}
 \Delta_n(\tau) &\leq \left(\int_{-|\tau|}^{|\tau|} |q_2 - q_1|(s) ds \right) \sup_{|s| \leq |\tau|} M_n^*(s) + \max_{m=1, 2} \left(\int_{-|\tau|}^{|\tau|} |q_m|(s) ds \right) \sup_{|s| \leq |\tau|} D_n(s), \\
 D_n(\tau) &\leq \left(\int_{-|\tau|}^{|\tau|} |q_2 - q_1|(s) ds \right) \sup_{|s| \leq |\tau|} N_n^*(s) + \max_{m=1, 2} \left(\int_{-|\tau|}^{|\tau|} |q_m|(s) ds \right) \sup_{|s| \leq |\tau|} \Delta_n(s),
 \end{aligned}$$

and these, when combined with $\max_{m=1, 2} \|q_m\|_1 < 1$, yield

$$\Delta_n(\tau) \leq \frac{\sup_{\tau \in (-\infty, \infty)} M_n^*(\tau)}{1 - \max_{m=1, 2} \|q_m\|_1} \|q_2 - q_1\|_1, \quad D_n(\tau) \leq \frac{\sup_{\tau \in (-\infty, \infty)} N_n^*(\tau)}{1 - \max_{m=1, 2} \|q_m\|_1} \|q_2 - q_1\|_1.$$

The desired result then follows from the above estimates and the inequalities

$$M_n^*(\tau) \leq \frac{M_n^*(0)}{1 - \max_{m=1, 2} \|q_m\|_1}, \quad N_n^*(\tau) \leq \frac{N_n^*(0)}{1 - \max_{m=1, 2} \|q_m\|_1},$$

the latter being a direct consequence of (3.11).

3.2. *Decay estimates for a single potential $q(\cdot)$.* Our task now is to obtain decay estimates for the solution of (3.1) for a given potential $q(\cdot)$. Recalling that \mathcal{A} and \mathcal{B} satisfy:

$$\mathcal{A}_\tau(\omega, \tau) = -q(\tau) e^{2i\omega\tau} \mathcal{B}(\omega, \tau), \quad \mathcal{B}_\tau(\omega, \tau) = -q(\tau) e^{-2i\omega\tau} \mathcal{A}(\omega, \tau),$$

we see that

$$\frac{\partial}{\partial \tau} (\mathcal{A}\bar{\mathcal{A}} + \mathcal{B}\bar{\mathcal{B}})(\omega, \tau) = -2q(\tau) (e^{-2i\omega\tau} (\mathcal{A}\bar{\mathcal{B}})(\omega, \tau) + e^{2i\omega\tau} (\bar{\mathcal{A}}\mathcal{B})(\omega, \tau)),$$

$$\frac{\partial}{\partial \tau} (\mathcal{A}\bar{\mathcal{B}})(\omega, \tau) = -q(\tau) e^{2i\omega\tau} (\mathcal{A}\bar{\mathcal{A}} + \mathcal{B}\bar{\mathcal{B}})(\omega, \tau),$$

or equivalently that

$$\begin{aligned}
 &(\mathcal{A}\bar{\mathcal{A}} + \mathcal{B}\bar{\mathcal{B}})(\omega, \tau) \\
 &= (\tilde{\mathcal{A}}\bar{\tilde{\mathcal{A}}} + \tilde{\mathcal{B}}\bar{\tilde{\mathcal{B}}})(\omega) - 2 \int_0^\tau q(s) (e^{-2i\omega s} (\mathcal{A}\bar{\mathcal{B}})(\omega, s) + e^{2i\omega s} (\bar{\mathcal{A}}\mathcal{B})(\omega, s)) ds \quad (3.14)
 \end{aligned}$$

$$(\mathcal{A}\bar{\mathcal{B}})(\omega, \tau) = (\tilde{\mathcal{A}}\bar{\tilde{\mathcal{B}}})(\omega) - \int_0^\tau q(s) e^{2i\omega s} (\mathcal{A}\bar{\mathcal{A}} + \mathcal{B}\bar{\mathcal{B}})(\omega, s) ds. \quad (3.15)$$

Our first task is to obtain a decay estimate for

$$J(\tau) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \omega e^{-2i\omega\tau} (\mathcal{A}\bar{\mathcal{B}})(\omega, \tau) d\omega. \tag{3.16}$$

Eq. (3.15) implies that $J(\tau) = J^1(\tau) - J^2(\tau)$, where

$$J^1(\tau) = \int_{-\infty}^{\infty} \omega e^{-2i\omega\tau} (\tilde{\mathcal{A}}\bar{\tilde{\mathcal{B}}})(\omega) d\omega \tag{3.17}$$

and

$$J^2(\tau) = \int_0^{\tau} q(s) \int_{-\infty}^{\infty} \omega e^{-2i\omega(\tau-s)} (\mathcal{A}\bar{\mathcal{A}} + \mathcal{B}\bar{\mathcal{B}})(\omega, s) d\omega ds. \tag{3.18}$$

Our first result is an estimate for $J^1(\tau)$. This estimate depends exclusively on the data $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$ and is independent of the potential $q(\cdot)$. In what follows we adopt the notation

$$\|f\|^k \stackrel{\text{def}}{=} \sup_{\tau} (1 + \tau^{2k}) |f(\tau)|. \tag{3.19}$$

LEMMA 3.2. If

$$\lim_{|\omega| \rightarrow \infty} \frac{d^r}{d\omega^r} (\omega \tilde{\mathcal{A}} \bar{\tilde{\mathcal{B}}})(\omega) = 0 \text{ for } 0 \leq r \leq 2k, \tag{3.20}$$

then

$$(1 + \tau^{2k})J^1(\tau) = \int_{-\infty}^{\infty} e^{-2i\omega\tau} \left(1 + \frac{1}{(2i)^{2k}} \cdot \frac{d^{2k}}{d\omega^{2k}} \right) (\omega \tilde{\mathcal{A}} \bar{\tilde{\mathcal{B}}})(\omega) d\omega, \tag{3.21}$$

and

$$\|J^1\|^k \leq \int_{-\infty}^{\infty} \left| \left(1 + \frac{1}{(2i)^{2k}} \cdot \frac{d^{2k}}{d\omega^{2k}} \right) (\omega \tilde{\mathcal{A}} \bar{\tilde{\mathcal{B}}})(\omega) \right| d\omega. \tag{3.22}$$

Moreover, these latter terms are finite if the initial data $\tilde{\mathcal{A}}(\omega)$ and $\tilde{\mathcal{B}}(\omega)$ are sufficiently well-behaved.

Proof. Repeated integration by parts and repeated application of the limit relations (3.20) yield the identity

$$\tau^{2k}J^1(\tau) = \int_{-\infty}^{\infty} e^{-2i\omega\tau} \left(\frac{1}{(2i)^{2k}} \cdot \frac{d^{2k}}{d\omega^{2k}} (\omega \tilde{\mathcal{A}} \bar{\tilde{\mathcal{B}}})(\omega) \right) d\omega,$$

and from this we obtain

$$(1 + \tau^{2k})J^1(\tau) = \int_{-\infty}^{\infty} e^{-2i\omega\tau} \left(1 + \frac{1}{(2i)^{2k}} \cdot \frac{d^{2k}}{d\omega^{2k}} \right) (\tilde{\mathcal{A}}\bar{\tilde{\mathcal{B}}})(\omega) d\omega.$$

The inequality (3.22) is an immediate consequence of (3.21).

We now seek a similar estimate for $J^2(\tau)$ (see Eq. (3.18)). We first prove the following preliminary lemma.

LEMMA 3.3. Let $k \geq 1$ and $H(\cdot, \cdot)$ and $q(\cdot)$ be two functions satisfying

$$(1 + \tau^{2k})|H(\tau, s)| \leq \bar{H}_k < \infty \text{ for all } \tau \text{ and } s, \tag{3.23}$$

$$\|q\|^k < \infty. \tag{3.24}$$

Then the following inequality holds:

$$(1 + \tau^{2k}) \left| \int_0^\tau q(s)H(\tau - s, s) ds \right| \leq 2^{2k+1} \|q\|^k \bar{H}_k \int_0^\infty \frac{ds}{1 + s^{2k}}. \tag{3.25}$$

Proof. We shall consider only the case where $\tau > 0$. The case $\tau < 0$ is similar. For $\tau > 0$

$$\begin{aligned} \left| \int_0^\tau q(s)H(\tau - s, s) ds \right| &\leq \int_0^\tau |q(s)| |H(\tau - s, s)| ds \\ &= \int_0^\tau \frac{(1 + s^{2k})|q(s)|}{1 + s^{2k}} \frac{(1 + (\tau - s)^{2k})|H(\tau - s, s)|}{(1 + (\tau - s)^{2k})} ds \\ &\leq \|q\|^k \bar{H}_k \int_0^\tau \frac{ds}{(1 + s^{2k})(1 + (\tau - s)^{2k})} = 2 \|q\|^k \bar{H}_k \int_0^{\tau/2} \frac{ds}{(1 + s^{2k})(1 + (\tau - s)^{2k})} \\ &\leq \frac{2 \|q\|^k \bar{H}_k}{1 + \left(\frac{\tau}{2}\right)^{2k}} \int_0^{\tau/2} \frac{ds}{1 + s^{2k}} \leq \frac{2^{2k+1} \|q\|^k \bar{H}_k}{1 + \tau^{2k}} \int_0^\infty \frac{ds}{1 + s^{2k}} \end{aligned}$$

and this is the desired result.

By virtue of the preceding lemma it suffices to show that (3.23) holds for

$$H(\tau, s) \stackrel{\text{def}}{=} \int_{-\infty}^\infty e^{-2i\omega\tau} \omega(\mathcal{A}\bar{\mathcal{A}} + \mathcal{B}\bar{\mathcal{B}})(\omega, s) d\omega. \tag{3.26}$$

It is easily checked that H is odd in its first argument. Thus it suffices to establish (3.23) when $\tau > 0$. We shall limit our discussion to the situation $\tau > 0$ and $s > 0$. The case $\tau > 0$ and $s < 0$ is similar. To analyze H we shall make use of the following identity:

$$\begin{aligned} H(\tau, s) &= \int_{-\infty}^\infty e^{-2i\omega\tau} \omega(\tilde{\mathcal{A}}\bar{\tilde{\mathcal{A}}} + \tilde{\mathcal{B}}\bar{\tilde{\mathcal{B}}})(\omega) d\omega \\ &\quad - 2 \int_0^s q(\eta) \int_{-\infty}^\infty (e^{-2i\omega(\tau+\eta)}(\omega\tilde{\mathcal{A}}\bar{\tilde{\mathcal{B}}})(\omega) + e^{-2i\omega(\tau-\eta)}(\omega\tilde{\mathcal{A}}\bar{\tilde{\mathcal{B}}})(\omega)) d\omega d\eta \\ &\quad + 2 \int_0^s q(\eta) \int_0^\eta q(\zeta)(H(\tau + \eta - \zeta, \zeta) + H(\tau - \eta + \zeta, \zeta)) d\zeta d\eta. \end{aligned} \tag{3.27}$$

This is a direct consequence of (3.14) and (3.15).

LEMMA 3.4. Let $\tau \geq 0, s \geq 0, 0 \leq \eta \leq s$, and $k \geq 1$, and suppose that

$$\lim_{|\omega| \rightarrow \infty} \frac{d^r}{d\omega^r} (\omega(\tilde{\mathcal{A}}\bar{\tilde{\mathcal{A}}} + \tilde{\mathcal{B}}\bar{\tilde{\mathcal{B}}})(\omega)) = 0 \tag{3.28}$$

and

$$\lim_{|\omega| \rightarrow \infty} \frac{d^r}{d\omega^r} (\omega \tilde{\mathcal{A}} \tilde{\mathcal{B}})(\omega) = 0 \text{ for } 0 \leq r \leq 2k. \tag{3.29}$$

Then the following inequalities hold:

$$\begin{aligned} & \left| (1 + \tau^{2k}) \int_{-\infty}^{\infty} e^{-2i\omega\tau} \omega (\tilde{\mathcal{A}} \tilde{\mathcal{A}} + \tilde{\mathcal{B}} \tilde{\mathcal{B}})(\omega) d\omega \right| \\ & \leq \int_{-\infty}^{\infty} \left| \left(1 + \frac{1}{(2i)^{2k}} \cdot \frac{d^{2k}}{d\omega^{2k}} \right) (\omega \tilde{\mathcal{A}} \tilde{\mathcal{A}} + \tilde{\mathcal{B}} \tilde{\mathcal{B}})(\omega) \right| d\omega, \end{aligned} \tag{3.30}$$

$$\begin{aligned} & \left| (1 + \tau^{2k}) \int_0^s q(\eta) \int_{-\infty}^{\infty} e^{-2i\omega(\tau+\eta)} \omega \tilde{\mathcal{A}} \tilde{\mathcal{B}}(\omega) d\omega d\eta \right| \\ & \leq \|q\|^k \int_0^{\infty} \frac{d\eta}{1 + \eta^{2k}} \int_{-\infty}^{\infty} \left| \left(1 + \frac{1}{(2i)^{2k}} \frac{d^{2k}}{d\omega^{2k}} \right) (\omega \tilde{\mathcal{A}} \tilde{\mathcal{B}}(\omega)) \right| d\omega, \end{aligned} \tag{3.31}$$

$$\begin{aligned} & \left| (1 + \tau^{2k}) \int_0^s q(\eta) \int_{-\infty}^{\infty} e^{-2i\omega(\tau-\eta)} (\omega \tilde{\mathcal{A}} \tilde{\mathcal{B}})(\omega) d\omega d\eta \right| \\ & \leq \|q\|^k (1 + 2^{2k+1}) \int_0^{\infty} \frac{d\eta}{1 + \eta^{2k}} \int_{-\infty}^{\infty} \left| \left(1 + \frac{1}{(2i)^{2k}} \frac{d^{2k}}{d\omega^{2k}} \right) (\omega \tilde{\mathcal{A}} \tilde{\mathcal{B}})(\omega) \right| d\omega. \end{aligned} \tag{3.32}$$

Proof. First we prove (3.30). The proof mimics the proof of Lemma 3.2. Repeated integration by parts and application of the limit relation (3.28) yields

$$\tau^{2k} \int_{-\infty}^{\infty} e^{-2i\omega\tau} \omega (\tilde{\mathcal{A}} \tilde{\mathcal{A}} + \tilde{\mathcal{B}} \tilde{\mathcal{B}})(\omega) d\omega = \int_{-\infty}^{\infty} e^{-2i\omega\tau} \left(\frac{1}{(2i)^{2k}} \cdot \frac{d^{2k}}{d\omega^{2k}} \right) (\omega \tilde{\mathcal{A}} \tilde{\mathcal{A}} + \tilde{\mathcal{B}} \tilde{\mathcal{B}})(\omega) d\omega,$$

and thus we have

$$\begin{aligned} & \left| (1 + \tau^{2k}) \int_{-\infty}^{\infty} e^{-2i\omega\tau} \omega (\tilde{\mathcal{A}} \tilde{\mathcal{A}} + \tilde{\mathcal{B}} \tilde{\mathcal{B}})(\omega) d\omega \right| \\ & = \left| \int_{-\infty}^{\infty} e^{-2i\omega\tau} \left(1 + \frac{1}{(2i)^{2k}} \cdot \frac{d^{2k}}{d\omega^{2k}} \right) (\omega \tilde{\mathcal{A}} \tilde{\mathcal{A}} + \tilde{\mathcal{B}} \tilde{\mathcal{B}})(\omega) d\omega \right| \\ & \leq \int_{-\infty}^{\infty} \left| \left(1 + \frac{1}{(2i)^{2k}} \cdot \frac{d^{2k}}{d\omega^{2k}} \right) (\omega \tilde{\mathcal{A}} \tilde{\mathcal{A}} + \tilde{\mathcal{B}} \tilde{\mathcal{B}})(\omega) \right| d\omega, \end{aligned}$$

as claimed.

If we apply the arguments used in Lemma 3.2 to the integral appearing in (3.31), we obtain

$$\begin{aligned} & (1 + \tau^{2k}) \int_0^s q(\eta) \int_{-\infty}^{\infty} e^{-2i\omega(\tau+\eta)} (\omega \tilde{\mathcal{A}} \tilde{\mathcal{B}})(\omega) d\omega d\eta \\ & = (1 + \tau^{2k}) \int_0^s \frac{q(\eta)(1 + (\tau + \eta)^{2k})}{1 + (\tau + \eta)^{2k}} \int_{-\infty}^{\infty} e^{-2i\omega(\tau+\eta)} (\omega \tilde{\mathcal{A}} \tilde{\mathcal{B}})(\omega) d\omega d\eta \\ & = (1 + \tau^{2k}) \int_0^s \frac{q(\eta)}{1 + (\tau + \eta)^{2k}} \int_{-\infty}^{\infty} e^{-2i\omega(\tau+\eta)} \left(1 + \frac{1}{(2i)^{2k}} \cdot \frac{d^{2k}}{d\omega^{2k}} \right) (\omega \tilde{\mathcal{A}} \tilde{\mathcal{B}})(\omega) d\omega d\eta, \end{aligned}$$

and this implies

$$\begin{aligned} & \left| (1 + \tau^{2k}) \int_0^s q(\eta) \int_{-\infty}^{\infty} e^{-2i\omega(\tau + \eta)} \omega (\tilde{\mathcal{A}} \tilde{\mathcal{B}})(\omega) \, d\omega \, d\eta \right| \\ & \leq (1 + \tau^{2k}) \int_0^s \frac{(1 + \eta^{2k}) |q(\eta)|}{(1 + \eta^{2k})(1 + (\tau + \eta)^{2k})} \int_{-\infty}^{\infty} \left| \left(1 + \frac{1}{(2i)^{2k}} \cdot \frac{1}{d\omega^{2k}} \right) (\omega \tilde{\mathcal{A}} \tilde{\mathcal{B}})(\omega) \right| d\omega \, d\eta \\ & \leq \|q\|^k \int_0^s \frac{d\eta}{1 + \eta^{2k}} \int_{-\infty}^{\infty} \left| \left(1 + \frac{1}{(2i)^{2k}} \cdot \frac{d^{2k}}{d\omega^{2k}} \right) (\omega \tilde{\mathcal{A}} \tilde{\mathcal{B}})(\omega) \right| d\omega, \end{aligned}$$

as claimed.

Finally, we turn to (3.32). We have

$$\begin{aligned} & (1 + \tau^{2k}) \int_0^s q(\eta) \int_{-\infty}^{\infty} e^{-2i\omega(\tau - \eta)} (\omega \tilde{\mathcal{A}} \tilde{\mathcal{B}})(\omega) \, d\omega \, d\eta \\ & = (1 + \tau^{2k}) \int_0^s \frac{q(\eta)(1 + (\tau - \eta)^{2k})}{1 + (\tau - \eta)^{2k}} \int_{-\infty}^{\infty} e^{-2i\omega(\tau - \eta)} (\omega \tilde{\mathcal{A}} \tilde{\mathcal{B}})(\omega) \, d\omega \, d\eta \\ & = (1 + \tau^{2k}) \int_0^s \frac{q(\eta)}{1 + (\tau - \eta)^{2k}} \int_{-\infty}^{\infty} e^{-2i\omega(\tau - \eta)} \left(1 + \frac{1}{(2i)^{2k}} \cdot \frac{d^{2k}}{d\omega^{2k}} \right) (\omega \tilde{\mathcal{A}} \tilde{\mathcal{B}})(\omega) \, d\omega \, d\eta, \end{aligned}$$

and thus the inequality

$$\begin{aligned} & \left| (1 + \tau^{2k}) \int_0^s q(\eta) \int_{-\infty}^{\infty} e^{-2i\omega(\tau - \eta)} (\omega \tilde{\mathcal{A}} \tilde{\mathcal{B}})(\omega) \, d\omega \, d\eta \right| \\ & \leq (1 + \tau^{2k}) \int_0^s \frac{(1 + \eta^{2k}) |q(\eta)|}{(1 + \eta^{2k})(1 + (\tau - \eta)^{2k})} \int_{-\infty}^{\infty} \left| \left(1 + \frac{1}{(2i)^{2k}} \cdot \frac{d^{2k}}{d\omega^{2k}} \right) (\omega \tilde{\mathcal{A}} \tilde{\mathcal{B}})(\omega) \right| d\omega \, d\eta \\ & \leq \|q\|^k \int_{-\infty}^{\infty} \left| \left(1 + \frac{1}{(2i)^{2k}} \cdot \frac{d^{2k}}{d\omega^{2k}} \right) (\omega \tilde{\mathcal{A}} \tilde{\mathcal{B}})(\omega) \right| d\omega \int_0^s \frac{(1 + \tau^{2k}) \, d\eta}{(1 + \eta^{2k})(1 + (\tau - \eta)^{2k})} \end{aligned}$$

The desired result, (3.32), now follows from the inequality above and the fact that

$$\int_0^s \frac{(1 + \tau^{2k}) \, d\eta}{(1 + \eta^{2k})(1 + (\tau - \eta)^{2k})} \leq (1 + 2^{2k+1}) \int_0^{\infty} \frac{d\eta}{1 + \eta^{2k}}. \tag{3.33}$$

Our final task is to show that the function H of (3.26) decays.

LEMMA 3.5. Let $k \geq 1$ and suppose there exist constants $0 < \delta_k < \infty$ and $0 < Q_k < \infty$ such that

$$\|q\|^k \leq \delta_k, \tag{3.34}$$

$$\int_{-\infty}^{\infty} \left| \left(1 + \frac{1}{(2i)^{2k}} \cdot \frac{d^{2k}}{d\omega^{2k}} \right) (\omega (\tilde{\mathcal{A}} \tilde{\mathcal{A}} + \tilde{\mathcal{B}} \tilde{\mathcal{B}})(\omega)) \right| d\omega \leq 2Q_k, \tag{3.35}$$

$$\int_{-\infty}^{\infty} \left| \left(1 + \frac{1}{(2i)^{2k}} \cdot \frac{d^{2k}}{d\omega^{2k}} \right) (\omega \tilde{\mathcal{A}} \tilde{\mathcal{B}})(\omega) \right| d\omega \leq Q_k, \tag{3.36}$$

and

$$\int_{-\infty}^{\infty} \left| \left(1 + \frac{1}{(2i)^{2k}} \cdot \frac{d^{2k}}{d\omega^{2k}} \right) (\omega \tilde{\mathcal{A}} \tilde{\mathcal{B}})(\omega) \right| d\omega \leq Q_k. \tag{3.37}$$

Then the function $H(\tau, s)$ defined in (3.26) satisfies the estimate

$$((1 + \tau^{2k})|H(\tau, s)|) \leq \frac{Q_k \left(2 + 4\delta_k(1 + 2^{2k}) \int_0^\infty \frac{d\eta}{1 + \eta^{2k}} \right)}{1 - 4(1 + 2^{2k+1})\delta_k^2 \left(\int_0^\infty \frac{d\eta}{1 + \eta^{2k}} \right)^2} \stackrel{\text{def}}{=} \bar{H}_k \quad (3.38)$$

provided

$$\delta_k < \frac{1}{2(1 + 2^{2k+1}) \int_0^\infty \frac{d\eta}{1 + \eta^{2k}}}. \quad (3.39)$$

Proof. By recalling that H is odd in its first argument it suffices to confine our attention to the case $\tau > 0$. We shall further limit our discussion to the case $s > 0$. The case $s < 0$ is similar. The results of Lemma 3.4 and the identity (3.27) yield

$$(1 + \tau^{2k})|H(\tau, s)| \leq 2 \left(1 + 2(1 + 2^{2k})\delta_k \int_0^\infty \frac{d\eta}{1 + \eta^{2k}} \right) Q_k \\ + 2(1 + \tau^{2k}) \int_0^s |q(\eta)| \int_0^\eta |q(\zeta)| |H(\tau + \eta - \zeta, \zeta) + H(\tau - \eta + \zeta, \zeta)| d\zeta d\eta. \quad (3.40)$$

We now let

$$h_k(\tau, s) \stackrel{\text{def}}{=} (1 + \tau^{2k})|H(\tau, s)|. \quad (3.41)$$

Clearly h_k is nonnegative. The fact that H is odd in τ implies that h_k is even in τ . Moreover, (3.40) implies that h_k satisfies

$$h_k(\tau, s) \leq 2 \left(1 + 2(1 + 2^{2k})\delta_k \int_0^\infty \frac{d\eta}{1 + \eta^{2k}} \right) Q_k \\ + 2\delta_k^2(1 + \tau^{2k}) \int_0^s \frac{1}{1 + \eta^{2k}} \int_0^\eta \frac{1}{1 + \zeta^{2k}} \left(\frac{h_k(\tau + \eta - \zeta, \zeta)}{1 + (\tau + \eta - \zeta)^{2k}} + \frac{h_k(\tau - \eta + \zeta, \zeta)}{1 + (\tau - \eta + \zeta)^{2k}} \right) d\zeta d\eta. \quad (3.42)$$

We now let

$$h_k^*(T) \stackrel{\text{def}}{=} \sup_{\substack{0 \leq s, 0 \leq \tau \\ s + \tau \leq T}} h_k(\tau, s). \quad (3.43)$$

Our goal is to show that h_k^* obeys the upper bound (3.38). For $0 \leq \tau, 0 \leq \zeta \leq \eta \leq s$ and $s + \tau \leq T$ the following inequalities are valid:

$$0 \leq \tau \leq \tau + \eta - \zeta \leq \tau + \eta \leq \tau + s \leq T,$$

$$0 \leq |\tau - \eta + \zeta| + \zeta \leq \max[\tau + \zeta, \eta - \tau] \leq \tau + s \leq T.$$

The above relations, together with the fact that $h_k(\cdot, \cdot)$ is even in its first argument, then yield

$$h_k(\tau + \eta - \zeta, \zeta) \leq h_k^*(T), \quad h_k(\tau - \eta + \zeta, \zeta) \leq h_k^*(T),$$

and this in turn implies that the second integral on the right-hand side of (3.42) is bounded from above by

$$2\delta_k^2 h_k^*(T) \left[\left(\int_0^\infty \frac{d\eta}{1 + \eta^{2k}} \right)^2 + (1 + \tau^{2k}) \int_0^\infty \frac{d\eta}{1 + \eta^{2k}} \int_0^\eta \frac{d\zeta}{(1 + \zeta^{2k})(1 + (\tau - \eta + \zeta)^{2k})} d\eta \right].$$

Moreover, it is a relatively trivial matter to show that

$$(1 + \tau^{2k}) \int_0^\infty \frac{d\eta}{1 + \eta^{2k}} \int_0^\eta \frac{1}{(1 + \zeta^{2k})(1 + (\tau - \eta + \zeta)^{2k})} d\zeta d\eta \leq (1 + 2^{2k+2}) \left(\int_0^\infty \frac{d\eta}{1 + \eta^{2k}} \right)^2.$$

Combining the last two facts with the inequality (3.42), we see that

$$h_k(\tau, s) \leq 2 \left(1 + 2(1 + 2^k)\delta_k \int_0^\infty \frac{d\eta}{1 + \eta^{2k}} \right) Q_k + 4(1 + 2^{2k+1})\delta_k^2 \left(\int_0^\infty \frac{d\eta}{1 + \eta^{2k}} \right)^2 h_k^*(T) \quad (3.44)$$

for all $\tau \geq 0$ and $s \geq 0$ such that $s + \tau \leq T$. The last inequality yields the result that $h_k^*(T)$ obeys the upper bound (3.38) and, since this result is independent of T , we have the result claimed for $H(\cdot, \cdot)$.

From the last lemma, we know that for δ_k sufficiently small, the function H of (3.26) satisfies the hypotheses of Lemma 3.3 with \bar{H}_k given by the right-hand side of (3.38). This in turn implies

LEMMA 3.6. If $\|q\|^k \leq \delta_k$ and δ_k is sufficiently small and if the hypotheses of Lemmas 3.2, 3.4, and 3.5 hold, then the function $J^2(\cdot)$ of (3.18) satisfies

$$\|J^2\|^k \leq 2^{2k+1}\delta_k \bar{H}_k \int_0^\infty \frac{d\eta}{1 + \eta^{2k}}. \quad (3.45)$$

3.3 Additional continuity and decay estimates. Now suppose that $(\mathcal{A}_1, \mathcal{B}_1)$ and $(\mathcal{A}_2, \mathcal{B}_2)$ are the unique solutions of (3.1) corresponding to the same data $(\tilde{\mathcal{A}}(\omega), \tilde{\mathcal{B}}(\omega))$. For $m = 1$ and 2 we let

$$J_m(\tau) = \int_{-\infty}^\infty \omega e^{-2i\omega\tau} (\mathcal{A}_m \bar{\mathcal{B}}_m)(\omega, \tau) d\omega, \quad (3.46)$$

$$H_m(\tau, s) = \int_{-\infty}^\infty \omega e^{-2i\omega\tau} (\mathcal{A}_m \tilde{\mathcal{A}}_m + \mathcal{B}_m \bar{\mathcal{B}}_m)(\omega, s) d\omega, \quad (3.47)$$

$$K(\tau, s) = H_2(\tau, s) - H_1(\tau, s). \quad (3.48)$$

Arguments similar to those previously employed may now be used to obtain

LEMMA 3.7. If the data $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$ and the potentials q_m , $m = 1$ and 2, satisfy the conditions of Lemmas 3.2, 3.4, and 3.5, then K satisfies

$$(1 + \tau^{2k}) |K(\tau, s)| \leq \bar{K}_k \|q_2 - q_1\|^k \quad \text{for all } \tau \text{ and } s, \quad (3.49)$$

where

$$\bar{K}_k = \frac{4(1 + 2^{2k})Q_k \int_0^\infty \frac{d\eta}{1 + \eta^{2k}} + 8(1 + 2^{2k+1})\bar{H}_k \delta_k \left(\int_0^\infty \frac{d\eta}{1 + \eta^{2k}} \right)^2}{1 - 4(1 + 2^{2k+1})\delta_k^2 \left(\int_0^\infty \frac{d\eta}{1 + \eta^{2k}} \right)^2} \quad (3.50)$$

and $\delta_k, Q_k,$ and \bar{H}_k are the constants defined in Lemma 3.5. Moreover, for $m = 1$ and 2 and all $\tau,$

$$(1 + \tau^{2k}) \left| \int_0^\tau q_m(s)K(\tau - s, s) ds \right| \leq 2^{2k+1} \delta_k \bar{K}_k \|q_2 - q_1\|^k \int_0^\infty \frac{d\eta}{1 + \eta^{2k}}. \tag{3.51}$$

4. The nonlinear problem. In this section we turn to the nonlinear problem derived in Sec. 2:

$$\mathcal{A}_\tau = -\frac{C_\tau}{2C} e^{2i\omega\tau} \mathcal{B}, \quad \mathcal{B}_\tau = -\frac{C_\tau}{2C} e^{-2i\omega\tau} \mathcal{A} \tag{4.1}$$

and

$$(\mathcal{A}, \mathcal{B})(\omega, 0) = (\tilde{\mathcal{A}}, \tilde{\mathcal{B}})(\omega) = C_0^\dagger (\hat{g}_1 - \hat{f}_1, \hat{g}_1 + \hat{f}_1)(\omega), \tag{4.2}$$

where

$$(\hat{g}_1, \hat{f}_1)(\omega) = \int_{-\infty}^\infty e^{-i\omega x} (g_1, f')(x) dx, \tag{4.3}$$

$$C_0^2 = 1 + \frac{\varepsilon}{2\pi} \int_{-\infty}^\infty \hat{f}_1(\omega) \hat{f}_1(-\omega) d\omega, \tag{4.4}$$

$$C^2(\tau) = 1 + \frac{\varepsilon}{8\pi C(\tau)} \int_{-\infty}^\infty (\mathcal{B}(\omega, \tau) e^{i\omega\tau} - \mathcal{A}(\omega, \tau) e^{-i\omega\tau} \\ \times (\mathcal{B}(-\omega, \tau) e^{-i\omega\tau} - \mathcal{A}(-\omega, \tau) e^{i\omega\tau}) d\omega. \tag{4.5}$$

It is easily checked that if $(\mathcal{A}, \mathcal{B}, C)$ is a solution of (4.1)-(4.5), then

$$(\tilde{\mathcal{A}}, \tilde{\mathcal{B}})(\omega, \tau) = (\mathcal{A}, \mathcal{B})(-\omega, \tau) \quad \text{and} \quad C^2(0) = C_0^2. \tag{4.6}$$

For any integer $k \geq 1,$ we introduce the class of functions

$$\mathcal{C}_k = \left\{ C(\cdot) \mid C(\cdot) \in BC^1(-\infty, \infty) \text{ and } \sup_\tau (1 + \tau^{2k}) |\dot{C}(\tau)| < \infty \right\}^2 \tag{4.7}$$

For functions $C(\cdot)$ in $\mathcal{C}_k,$ we let

$$\|C\|_k \stackrel{\text{def}}{=} \max \left[\sup_\tau |C(\tau)|, \sup_\tau (1 + \tau^{2k}) |\dot{C}(\tau)| \right]. \tag{4.8}$$

It is easily checked that \mathcal{C}_k with the norm $\|\cdot\|_k$ is a Banach space. For functions $C(\cdot)$ in \mathcal{C}_k with $k \geq 1,$ we also have the inequality

$$\|\dot{C}\|_1 \stackrel{\text{def}}{=} \int_{-\infty}^\infty |\dot{C}(\tau)| d\tau \leq 2 \int_0^\infty \frac{d\eta}{1 + \eta^{2k}} \|C\|_k^k. \tag{4.9}$$

For positive numbers δ we let

$$\mathcal{C}_{k, \delta} = \left\{ C(\cdot) \in \mathcal{C}_k \mid C(0) = C_0 \geq 1, C(\tau) \geq 1, \text{ and } \sup_\tau (1 + \tau^{2k}) |\dot{C}(\tau)| \leq \delta \right\}. \tag{4.10}$$

² We employ the notation $\dot{f}(\tau) = (df/d\tau)(\tau).$

The number $C_0 \geq 1$ is the positive square root of the right side of Eq. (4.4). It is easily checked that $\mathcal{C}_{k, \delta}$ is a closed convex subset of \mathcal{C}_k under the norm $\|\cdot\|^k$.

For functions $C(\cdot)$ in $\mathcal{C}_{k, \delta}$ we let

$$q_C(\tau) = \dot{C}(\tau)/2C(\tau), \tag{4.11}$$

and we let $(\mathcal{A}_C, \mathcal{B}_C)(\omega, \tau)$ be the unique solution of

$$\mathcal{A}_\tau = -q_C(\tau)e^{2i\omega\tau}\mathcal{B}, \quad \mathcal{B}_\tau = -q_C(\tau)e^{-2i\omega\tau}\mathcal{A} \tag{4.1}$$

satisfying

$$(\mathcal{A}_C, \mathcal{B}_C)(\omega, 0) = C_0^{\frac{1}{2}}(\hat{g}_1 - \hat{f}_1, \hat{g}_1 + \hat{f}_1)(\omega). \tag{4.2}$$

We define

$$\begin{aligned} \Gamma_C^2(\tau) = & 1 + \frac{\varepsilon}{8\pi C(\tau)} \int_{-\infty}^{\infty} (\mathcal{B}_C(\omega, \tau)e^{i\omega\tau} - \mathcal{A}_C(\omega, \tau)e^{-i\omega\tau}) \\ & \times (\mathcal{B}_C(-\omega, \tau)e^{-i\omega\tau} - \mathcal{A}_C(-\omega, \tau)e^{i\omega\tau}) d\omega, \end{aligned} \tag{4.12}$$

and Γ_C as the positive square root of the right side of (4.12).

The estimates of Lemma 3.1 and the inequality

$$\|q_C\|_1 = \frac{1}{2} \int_{-\infty}^{\infty} \left| \frac{\dot{C}}{C} \right|(\tau) d\tau \leq \delta \int_0^{\infty} \frac{d\eta}{1 + \eta^{2k}} \tag{4.13}$$

for all $C(\cdot) \in \mathcal{C}_{k, \delta}$ insure that $\mathcal{A}_C, \mathcal{B}_C$, and Γ_C are well-defined for all

$$\delta < 1 / \int_0^{\infty} \frac{d\eta}{1 + \eta^{2k}},$$

provided the initial data $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}})(\omega)$ is well enough behaved.

We also observe that if for some $k \geq 1$ and δ sufficiently small, the mapping $C \rightarrow \Gamma_C$ has a fixed point in $\mathcal{C}_{k, \delta}$, then the problem (4.1)-(4.5) has a globally defined solution for all time τ . Moreover, the estimate (4.13), the integral identities

$$\begin{aligned} \mathcal{A}(\omega, \tau) = & \tilde{\mathcal{A}}(\omega) - \frac{1}{2} \int_0^\tau \frac{\dot{C}}{C}(s) e^{2i\omega s} \mathcal{B}(\omega, s) ds, \\ \mathcal{B}(\omega, \tau) = & \tilde{\mathcal{B}}(\omega) - \frac{1}{2} \int_0^\tau \frac{\dot{C}}{C}(s) e^{-2i\omega s} \mathcal{A}(\omega, s) ds \end{aligned} \tag{4.14}$$

and the results of Lemma 3.1 will guarantee the existence of functions $(\mathcal{A}_{\pm\infty}, \mathcal{B}_{\pm\infty})(\omega)$ such that

$$\lim_{\tau \rightarrow \pm\infty} (\mathcal{A}, \mathcal{B})(\omega, \tau) = (\mathcal{A}_{\pm\infty}, \mathcal{B}_{\pm\infty})(\omega). \tag{4.15}$$

The fact that $C(\cdot) \in \mathcal{C}_{k, \delta}$ will guarantee the existence of numbers $C_{\pm\infty}$ such that

$$\lim_{\tau \rightarrow \pm\infty} C(\tau) = C_{\pm\infty}. \tag{4.16}$$

That these two numbers are equal will follow from (4.5), the conservation of energy identity which states that any solution of (4.1)-(4.5) satisfies

$$C^4(\tau) + \frac{\varepsilon}{4\pi} C(\tau) \int_{-\infty}^{\infty} (\mathcal{B}(\omega, \tau)e^{i\omega\tau} + \mathcal{A}(\omega, \tau)e^{-i\omega\tau}) \cdot (\mathcal{B}(-\omega, \tau)e^{-i\omega\tau} + \mathcal{A}(-\omega, \tau)e^{i\omega\tau}) d\omega = \text{constant}, \quad (4.17)$$

and the fact that such solutions will satisfy

$$\lim_{\tau \rightarrow \pm\infty} \int_{-\infty}^{\infty} \mathcal{A}(-\omega, \tau)\mathcal{B}(\omega, \tau)e^{2i\omega\tau} d\omega = 0. \quad (4.18)$$

By virtue of the last remarks, we shall confine our attention to showing that for some $k \geq 1$ and δ sufficiently small the mapping $C \rightarrow \Gamma_C$ has a fixed point in $\mathcal{C}_{k, \delta}$. For definiteness, we fix $k \geq 1$ and assume that the initial data

$$(\tilde{\mathcal{A}}, \tilde{\mathcal{B}})(\omega) = C \dagger (\hat{g}_1 - \hat{f}_1, \hat{g}_1 + \hat{f}_1)(\omega) \quad (4.2)$$

satisfies

$$M_n(0) = \max \left[\sup_{\omega} |\omega^n \tilde{\mathcal{A}}(\omega)|, \sup_{\omega} |\omega^n \tilde{\mathcal{B}}(\omega)| \right] < \infty, \quad n = 0, 1, \quad (4.19)$$

$$N_n(0) = \max \left[\int_{-\infty}^{\infty} |\omega^n \tilde{\mathcal{A}}(\omega)| d\omega, \int_{-\infty}^{\infty} |\omega^n \tilde{\mathcal{B}}(\omega)| d\omega \right] < \infty, \quad n = 0, 1, \quad (4.20)$$

and

$$\begin{aligned} Q_n(0) = \max & \left[\frac{1}{2} \int_{-\infty}^{\infty} \left| \left(1 + \frac{1}{(2i)^{2n}} \frac{d^{2n}}{d\omega^{2n}} \right) (\omega(\tilde{\mathcal{A}}\tilde{\mathcal{A}} + \tilde{\mathcal{B}}\tilde{\mathcal{B}})(\omega)) \right| d\omega, \right. \\ & \left. \int_{-\infty}^{\infty} \left| \left(1 + \frac{1}{(2i)^{2n}} \frac{d^{2n}}{d\omega^{2n}} \right) (\omega(\tilde{\mathcal{A}}\tilde{\mathcal{B}})(\omega)) \right| d\omega, \right. \\ & \left. \int_{-\infty}^{\infty} \left| \left(1 + \frac{1}{(2i)^{2n}} \frac{d^{2n}}{d\omega^{2n}} \right) (\omega(\tilde{\mathcal{A}}\tilde{\mathcal{B}})(\omega)) \right| d\omega \right] < \infty, \quad n = 0, 1, \dots, k. \quad (4.21) \end{aligned}$$

Our first task is to show that for fixed $k \geq 1$ and $0 < \delta$ sufficiently small, there is an $\tilde{\varepsilon}(k, \delta)$ such that for all $C(\cdot) \in \mathcal{C}_{k, \delta}$ and $0 < \varepsilon \leq \tilde{\varepsilon}(k, \delta)$ the function $\Gamma_C(\cdot)$ is in $\mathcal{C}_{k, \delta}$. Before starting to prove this assertion, we simplify our notation. We let $\mathcal{A}_C^\omega(\tau) = \mathcal{A}_C(\omega, \tau)$, $\mathcal{B}_C^\omega(\tau) = \mathcal{B}_C(\omega, \tau)$, $\tilde{\mathcal{A}}_C^\omega(\tau) = \mathcal{A}_C(-\omega, \tau)$ and $\tilde{\mathcal{B}}_C^\omega(\tau) = \mathcal{B}_C(\omega, \tau) = \mathcal{B}_C(-\omega, \tau)$.

THEOREM 4.1. For $k \geq 1$ fixed,

$$0 < \delta < \frac{1}{(1 + 2^{2k+1}) \dagger \int_0^\infty \frac{d\eta}{1 + \eta^{2k}}},$$

and

$$0 < \varepsilon \leq \tilde{\varepsilon}(k, \delta) \stackrel{\text{def}}{=} \frac{4\pi\delta}{Q_k(0) + 2^{2k}\delta\bar{H}_k \int_0^\infty \frac{d\eta}{1 + \eta^{2k}}}, \quad (4.22)$$

the mapping $C \rightarrow \Gamma_C$ of (4.12) takes $\mathcal{C}_{k, \delta}$ into $\mathcal{C}_{k, \delta}$.

Proof. From the definition of $\Gamma_C(\cdot)$ and Lemma 3.1 it is easy to check that $\Gamma_C(0) = C_0$, $\Gamma_C(\cdot) \in C^1(-\infty, \infty)$, and

$$1 \leq \Gamma_C^2(\tau) \leq 1 + \frac{\varepsilon(M_0 N_0)(0)}{2\pi \left(1 - \delta \int_0^\infty \frac{d\eta}{1 + \eta^{2k}}\right)^2}.$$

We need only show that, for $0 < \varepsilon \leq \tilde{\varepsilon}(k, \delta)$, $\|\dot{\Gamma}_C\|^k \leq \delta$.

The defining relationship (4.12) implies that

$$\begin{aligned} 2\Gamma_C \dot{\Gamma}_C &= -\frac{\varepsilon \dot{C}}{8\pi C^2} \int_{-\infty}^\infty (\mathcal{A}_C^\omega(\tau)e^{-i\omega\tau} - \mathcal{B}_C^\omega(\tau)e^{i\omega\tau})(\bar{\mathcal{A}}_C^\omega(\tau)e^{i\omega\tau} - \bar{\mathcal{B}}_C^\omega(\tau)e^{-i\omega\tau}) d\omega \\ &+ \frac{\varepsilon}{8\pi C} \int_{-\infty}^\infty (\mathcal{A}_{C,\tau}^\omega(\tau)e^{-i\omega\tau} - \mathcal{B}_{C,\tau}^\omega(\tau)e^{i\omega\tau})(\bar{\mathcal{A}}_C^\omega(\tau)e^{i\omega\tau} - \bar{\mathcal{B}}_C^\omega(\tau)e^{-i\omega\tau}) d\omega \\ &+ \frac{\varepsilon}{8\pi C} \int_{-\infty}^\infty (\mathcal{A}_C^\omega(\tau)e^{-i\omega\tau} - \mathcal{B}_C^\omega(\tau)e^{i\omega\tau})(\bar{\mathcal{A}}_{C,\tau}^\omega(\tau)e^{i\omega\tau} - \bar{\mathcal{B}}_{C,\tau}^\omega(\tau)e^{-i\omega\tau}) d\omega \\ &+ \frac{\varepsilon}{8\pi C} \int_{-\infty}^\infty (-i\omega)(\mathcal{A}_C^\omega(\tau)e^{-i\omega\tau} + \mathcal{B}_C^\omega(\tau)e^{i\omega\tau})(\bar{\mathcal{A}}_C^\omega(\tau)e^{i\omega\tau} - \bar{\mathcal{B}}_C^\omega(\tau)e^{-i\omega\tau}) d\omega \\ &+ \frac{\varepsilon}{8\pi C} \int_{-\infty}^\infty (i\omega)(\mathcal{A}_C^\omega(\tau)e^{-i\omega\tau} - \mathcal{B}_C^\omega(\tau)e^{i\omega\tau})(\bar{\mathcal{A}}_C^\omega(\tau)e^{i\omega\tau} + \bar{\mathcal{B}}_C^\omega(\tau)e^{-i\omega\tau}) d\omega \end{aligned}$$

and this when combined with Eq. (4.1) yields:

$$\Gamma_C \dot{\Gamma}_C = \frac{\varepsilon}{8\pi C} \int_{-\infty}^\infty i\omega(e^{-2i\omega\tau}(\mathcal{A}_C^\omega \bar{\mathcal{B}}_C^\omega)(\tau) - e^{2i\omega\tau}(\bar{\mathcal{A}}_C^\omega \mathcal{B}_C^\omega)(\tau)) d\omega.$$

Moreover, the fact that

$$-\int_{-\infty}^\infty i\omega e^{2i\omega\tau}(\bar{\mathcal{A}}_C^\omega \mathcal{B}_C^\omega)(\tau) d\omega = \int_{-\infty}^\infty i\omega e^{-2i\omega\tau}(\mathcal{A}_C^\omega \bar{\mathcal{B}}_C^\omega)(\tau) d\omega$$

reduces the last equation to

$$\dot{\Gamma}_C = \frac{\varepsilon}{4\pi(C\Gamma_C)(\tau)} \int_{-\infty}^\infty i\omega e^{-2i\omega\tau}(\mathcal{A}_C^\omega \bar{\mathcal{B}}_C^\omega)(\tau) d\omega \stackrel{\text{def}}{=} \frac{i\varepsilon}{4\pi\Gamma_C C} J(\tau), \tag{4.23}$$

where $J(\tau)$ is the function defined in (3.16). The fact that Γ_C and C are both greater than or equal to unity implies

$$|\dot{\Gamma}_C(\tau)| \leq \frac{\varepsilon}{4\pi} |J(\tau)| \tag{4.24}$$

and

$$(1 + \tau^{2k})|\dot{\Gamma}_C(\tau)| \leq \frac{\varepsilon}{4\pi} (1 + \tau^{2k})|J(\tau)|. \tag{4.25}$$

The results of Lemmas 3.2, 3.5 and 3.6 and (4.25) above imply that

$$\begin{aligned} (1 + \tau^{2k})|\dot{\Gamma}_C(\tau)| &\leq \frac{\varepsilon}{4\pi} (1 + \tau^{2k})|J(\tau)| \\ &\leq \frac{\varepsilon}{4\pi} \left(Q_k(0) + 2^{2k}\delta\bar{H}_k \int_0^\infty \frac{d\eta}{1 + \eta^{2k}} \right) \leq \delta \end{aligned} \tag{4.26}$$

provided $0 < \varepsilon \leq \tilde{\varepsilon}(k, \delta)$. In the last inequality

$$\bar{H}_k = \frac{\left(2 + 2(1 + 2^{2k})\delta \int_0^\infty \frac{d\eta}{1 + \eta^{2k}} \right) Q_k(0)}{1 - (1 + 2^{2k+1})\delta^2 \left(\int_0^\infty \frac{d\eta}{1 + \eta^{2k}} \right)^2} < \infty. \tag{4.27}$$

THEOREM 4.2. Let $k \geq 1$,

$$0 < \delta < \frac{1}{(1 + 2^{2k+1})^{\frac{1}{2}} \int_0^\infty \frac{d\eta}{1 + \eta^{2k}}},$$

and ε satisfy (4.22). Then there exists a constant D which may be chosen independently of ε such that

$$\|\Gamma_{C_2} - \Gamma_{C_1}\|^{2k} \leq \varepsilon D \|C_2 - C_1\|^{2k}.$$

It is now an immediate consequence of the contraction mapping principle that if the hypotheses of Theorems 4.1 and 4.2 hold then the map $C \rightarrow \Gamma_C$ has a unique fixed point in $C_{k, \delta}$ provided ε is small enough. This is the desired result.

Proof of Theorem 4.2. We shall employ the simplified notation:

$$\begin{aligned} \Gamma_m &= \Gamma_{C_m}, & q_m &= q_{C_m} = \dot{C}_m/2C_m, & \mathcal{A}_m^\omega(\tau) &= \mathcal{A}_{C_m}^\omega(\tau), \\ \mathcal{B}_m^\omega(\tau) &= \mathcal{B}_{C_m}^\omega(\tau), & m &= 1 \text{ and } 2. \end{aligned}$$

The functions \mathcal{A}_m^ω and \mathcal{B}_m^ω satisfy the identities

$$\begin{aligned} \mathcal{A}_m^\omega(\tau) &= \tilde{\mathcal{A}}(\omega) - \int_0^\tau \mathcal{A}_m(s) e^{2i\omega s} \mathcal{B}_m^\omega(s) ds \\ \mathcal{B}_m^\omega(\tau) &= \tilde{\mathcal{B}}(\omega) - \int_0^\tau q_m(s) e^{-2i\omega s} \mathcal{A}_m^\omega(s) ds \quad \text{for } m = 1, 2. \end{aligned}$$

The results of Lemma 3.1 yield

$$\sup_{\omega, \tau} |\omega^n \mathcal{A}_m^\omega(\tau)| \leq \frac{M_n(0)}{1 - \|q_m\|_1}, \tag{4.31}$$

$$\sup_{\omega, \tau} |\omega^n \mathcal{B}_m^\omega(\tau)| \leq \frac{M_n(0)}{1 - \|q_m\|_1}, \tag{4.32}$$

$$\sup_{\tau} \int_{-\infty}^\infty |\omega^n \mathcal{A}_m^\omega(\tau)| d\omega \leq \frac{N_n(0)}{1 - \|q_m\|_1}, \tag{4.33}$$

$$\sup_{\tau} \int_{-\infty}^{\infty} |\omega^n \mathcal{B}_m^\omega(\tau)| d\omega \leq \frac{N_n(0)}{1 - \|q_m\|_1}, \tag{4.34}$$

$$\sup_{\tau} \Delta_n(\tau) \leq \frac{M_n(0)}{\left(1 - \max_{m=1,2} \|q_m\|_1\right)^2} \|q_2 - q_1\|_1, \tag{4.35}$$

$$\sup_{\tau} D_n(\tau) \leq \frac{N_n(0)}{\left(1 - \max_{m=1,2} \|q_m\|_1\right)^2} \|q_2 - q_1\|_1 \tag{4.36}$$

where $\Delta_n(\tau)$ and $D_n(\tau)$ are defined in Eqs. (3.8) and (3.9) and $M_n(0)$ and $N_n(0)$ in Eqs. (4.19) and (4.20). We shall state without proof a crucial relation among the norms $\|\cdot\|_1$, $\|\cdot\|^k$, and $\|\cdot\|^{*k}$.

LEMMA 4.1. For $C_1, C_2 \in \mathcal{C}_{k, \delta}$, we have

$$\|q_2 - q_1\|_1 \leq (1 + \delta) \int_0^\infty \frac{d\eta}{1 + \eta^{2k}} \|C_2 - C_1\|^{*k},$$

$$\|q_2 - q_1\|^{*k} \leq \frac{1}{2}(1 + \delta) \|C_2 - C_1\|^{*k}.$$

From the definition of $\Gamma_m(\cdot)$, we have

$$\begin{aligned} \Gamma_m^2(\tau) &= 1 + \frac{\varepsilon}{8\pi C_m(\tau)} \int_{-\infty}^{\infty} (\mathcal{A}_m^\omega(\tau)e^{-i\omega\tau} - \mathcal{B}_m^\omega(\tau)e^{i\omega\tau})(\bar{\mathcal{A}}_m^\omega(\tau)e^{i\omega\tau} - \bar{\mathcal{B}}_m^\omega(\tau)e^{-i\omega\tau}) d\omega \\ &= 1 + \frac{\varepsilon}{8\pi C_m(\tau)} \int_{-\infty}^{\infty} |\mathcal{A}_m^\omega(\tau)e^{-i\omega\tau} - \mathcal{B}_m^\omega(\tau)e^{i\omega\tau}|^2 d\omega. \end{aligned} \tag{4.37}$$

The above identity and the fact that $\Gamma_m(\tau) \geq 1$ yield

$$\begin{aligned} |\Gamma_2 - \Gamma_1|(\tau) &= \left| \frac{\Gamma_2^2 - \Gamma_1^2}{\Gamma_2 + \Gamma_1} \right|(\tau) \\ &\leq \frac{1}{2} |\Gamma_2^2 - \Gamma_1^2|(\tau) \\ &\leq \frac{1}{2} \left| \frac{\varepsilon}{8\pi C_2(\tau)} \int_{-\infty}^{\infty} |\mathcal{A}_2^\omega(\tau)e^{-i\omega\tau} - \mathcal{B}_2^\omega(\tau)e^{i\omega\tau}|^2 d\omega \right. \\ &\quad \left. - \frac{\varepsilon}{8\pi C_1(\tau)} \int_{-\infty}^{\infty} |\mathcal{A}_1^\omega(\tau)e^{-i\omega\tau} - \mathcal{B}_1^\omega(\tau)e^{i\omega\tau}|^2 d\omega \right| \\ &\leq \frac{\varepsilon |C_2 - C_1|}{16\pi C_2 C_1} (\tau) \int_{-\infty}^{\infty} |\mathcal{A}_2^\omega(\tau)e^{-i\omega\tau} - \mathcal{B}_2^\omega(\tau)e^{i\omega\tau}|^2 d\omega \\ &\quad + \frac{\varepsilon}{16\pi C_1(\tau)} \left| \int_{-\infty}^{\infty} (|\mathcal{A}_2^\omega(\tau)e^{-i\omega\tau} - \mathcal{B}_2^\omega(\tau)e^{i\omega\tau}|^2 \right. \\ &\quad \left. - |\mathcal{A}_1^\omega(\tau)e^{-i\omega\tau} - \mathcal{B}_1^\omega(\tau)e^{i\omega\tau}|^2) d\omega \right| \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{\varepsilon |C_2 - C_1|(\tau)}{16\pi} \int_{-\infty}^{\infty} |\mathcal{A}_2^\omega(\tau)e^{-i\omega\tau} - \mathcal{B}_2^\omega(\tau)e^{i\omega\tau}|^2 d\omega \\
 &\quad + \frac{\varepsilon}{16\pi} \left| \int_{-\infty}^{\infty} ((\mathcal{A}_2^\omega \bar{\mathcal{A}}_2^\omega - \mathcal{A}_1^\omega \bar{\mathcal{A}}_1^\omega)(\tau) + (\mathcal{B}_2^\omega \bar{\mathcal{B}}_2^\omega - \mathcal{B}_1^\omega \bar{\mathcal{B}}_1^\omega)(\tau)) d\omega \right| \\
 &\quad + \frac{\varepsilon}{16\pi} \left| \int_{-\infty}^{\infty} (e^{-2i\omega\tau}(\mathcal{A}_2^\omega \bar{\mathcal{B}}_2^\omega - \mathcal{A}_1^\omega \bar{\mathcal{B}}_1^\omega)(\tau) \right. \\
 &\quad \left. + e^{2i\omega\tau}(\bar{\mathcal{A}}_2^\omega \mathcal{B}_2^\omega - \bar{\mathcal{A}}_1^\omega \mathcal{B}_1^\omega)(\tau)) d\omega \right|. \tag{4.38}
 \end{aligned}$$

We shall estimate the terms in (4.38) separately. For the first term we have

$$\begin{aligned}
 &\frac{\varepsilon}{16\pi} |C_2 - C_1|(\tau) \int_{-\infty}^{\infty} |\mathcal{A}_2^\omega(\tau)e^{-i\omega\tau} - \mathcal{B}_2^\omega(\tau)e^{i\omega\tau}|^2 d\omega \\
 &\leq \frac{\varepsilon}{16\pi} |C_2 - C_1|(\tau) \int_{-\infty}^{\infty} (|\mathcal{A}_2^\omega \bar{\mathcal{A}}_2^\omega| + |\mathcal{B}_2^\omega \bar{\mathcal{B}}_2^\omega| + |\mathcal{A}_2^\omega \bar{\mathcal{B}}_2^\omega| + |\bar{\mathcal{A}}_2^\omega \mathcal{B}_2^\omega|)(\tau) d\omega \\
 &\leq \frac{\varepsilon}{16\pi} |C_2 - C_1|(\tau) \cdot \frac{4(M_0 N_0)(0)}{(1 - \|q_2\|_1)^2} \leq \frac{\varepsilon(M_0 N_0)(0)}{4\pi(1 - \|q_2\|_1)^2} |C_2 - C_1|(\tau).
 \end{aligned}$$

The second term becomes

$$\begin{aligned}
 &\frac{\varepsilon}{16\pi} \left| \int_{-\infty}^{\infty} ((\mathcal{A}_2^\omega \bar{\mathcal{A}}_2^\omega - \mathcal{A}_1^\omega \bar{\mathcal{A}}_1^\omega)(\tau) + (\mathcal{B}_2^\omega \bar{\mathcal{B}}_2^\omega - \mathcal{B}_1^\omega \bar{\mathcal{B}}_1^\omega)(\tau)) d\omega \right| \\
 &\leq \frac{\varepsilon}{16\pi} \left| \int_{-\infty}^{\infty} (\mathcal{A}_2^\omega(\bar{\mathcal{A}}_2^\omega - \bar{\mathcal{A}}_1^\omega) + \bar{\mathcal{A}}_1^\omega(\mathcal{A}_2^\omega - \mathcal{A}_1^\omega) + \mathcal{B}_2^\omega(\bar{\mathcal{B}}_2^\omega - \bar{\mathcal{B}}_1^\omega) \right. \\
 &\quad \left. + \bar{\mathcal{B}}_1^\omega(\mathcal{B}_2^\omega - \mathcal{B}_1^\omega))(\tau) d\omega \right| \\
 &\leq \frac{\varepsilon}{16\pi} \int_{-\infty}^{\infty} (|\mathcal{A}_2^\omega| |\bar{\mathcal{A}}_2^\omega - \bar{\mathcal{A}}_1^\omega| + |\bar{\mathcal{A}}_1^\omega| |\mathcal{A}_2^\omega - \mathcal{A}_1^\omega| + |\mathcal{B}_2^\omega| |\bar{\mathcal{B}}_2^\omega - \bar{\mathcal{B}}_1^\omega| \\
 &\quad + |\bar{\mathcal{B}}_1^\omega| |\mathcal{B}_2^\omega - \mathcal{B}_1^\omega|)(\tau) d\omega \\
 &\leq \frac{\varepsilon}{16\pi} \Delta_0(\tau) \int_{-\infty}^{\infty} (|\mathcal{A}_2^\omega| + |\bar{\mathcal{A}}_1^\omega| + |\mathcal{B}_2^\omega| + |\bar{\mathcal{B}}_1^\omega|)(\tau) d\omega \\
 &\leq \frac{\varepsilon}{16\pi} \cdot \frac{M_0(0)\|q_2 - q_1\|_1}{\left(1 - \max_{m=1,2} \|q_m\|_1\right)^2} \cdot \frac{4N_0(0)}{1 - \max_{m=1,2} \|q_m\|_1} \\
 &\leq \frac{\varepsilon(M_0 N_0)(0)}{4\pi \left(1 - \max_{m=1,2} \|q_m\|_1\right)^3} \|q_2 - q_1\|_1.
 \end{aligned}$$

The last term of (4.38) is similar to the second term; the result is

$$\frac{\varepsilon}{16\pi} \left| \int_{-\infty}^{\infty} (e^{-2i\omega\tau}(\mathcal{A}_2^\omega \mathcal{B}_2^\omega - \mathcal{A}_1^\omega \mathcal{B}_1^\omega)(\tau) - e^{2i\omega\tau}(\bar{\mathcal{A}}_2^\omega \bar{\mathcal{B}}_2^\omega - \bar{\mathcal{A}}_1^\omega \bar{\mathcal{B}}_1^\omega)(\tau)) d\omega \right| \leq \frac{\varepsilon(M_0 N_0)(0)}{4\pi \left(1 - \max_{m=1,2} \|q_m\|_1\right)^3} \|q_2 - q_1\|_1.$$

The last three inequalities imply that

$$|\Gamma_2 - \Gamma_1|(\tau) \leq \frac{\varepsilon(M_0 N_0)(0)}{4\pi(1 - \|q_2\|_1)^2} |C_2 - C_1|(\tau) + \frac{\varepsilon(M_0 N_0)(0)}{2\pi \left(1 - \max_{m=1,2} \|q_m\|_1\right)^3} \|q_2 - q_1\|_1,$$

and since $\|q_m\|_1 \leq \delta \int_0^\infty (d\eta/(1 + \eta^{2k}))$, for $m = 1, 2$, we can majorize the last inequality by

$$|\Gamma_2 - \Gamma_1|(\tau) \leq \frac{\varepsilon(M_0 N_0)(0)}{4\pi \left(1 - \delta \int_0^\infty \frac{d\eta}{1 + \eta^{2k}}\right)^2} |C_2 - C_1|(\tau) + \frac{\varepsilon(M_0 N_0)(0)}{2\pi \left(1 - \delta \int_0^\infty \frac{d\eta}{1 + \eta^{2k}}\right)^3} \|q_2 - q_1\|_1.$$

From the definition of $\|\cdot\|^k$ and Lemma 4.1 we have $|C_2 - C_1|(\tau) \leq \|C_2 - C_1\|^k$ and

$$\|q_2 - q_1\|_1 \leq (1 + \delta) \int_0^\infty \frac{d\eta}{1 + \eta^{2k}} \|C_2 - C_1\|^k,$$

and these combine to give

$$\begin{aligned} |\Gamma_2 - \Gamma_1|(\tau) &\leq \frac{\varepsilon(M_0 N_0)(0)}{4\pi \left(1 - \delta \int_0^\infty \frac{d\eta}{1 + \eta^{2k}}\right)^2} \|C_2 - C_1\|^k \\ &\quad + \frac{\varepsilon(1 + \delta) \left(\int_0^\infty \frac{d\eta}{1 + \eta^{2k}}\right) (M_0 N_0)(0)}{2\pi \left(1 - \delta \int_0^\infty \frac{d\eta}{1 + \eta^{2k}}\right)^3} \|C_2 - C_1\|^k \\ &\leq \frac{\varepsilon \left(1 + (2 + \delta) \int_0^\infty \frac{d\eta}{1 + \eta^{2k}}\right) (M_0 N_0)(0)}{4\pi \left(1 - \delta \int_0^\infty \frac{d\eta}{1 + \eta^{2k}}\right)^3} \|C_2 - C_1\|^k. \end{aligned} \tag{4.39}$$

We now turn to the derivatives. We have

$$|\dot{\Gamma}_2 - \dot{\Gamma}_1| = \left| \frac{\Gamma_2 \dot{\Gamma}_2}{\Gamma_2} - \frac{\Gamma_2 \dot{\Gamma}_1}{\Gamma_2} \right| \leq \frac{|\Gamma_2 \dot{\Gamma}_2 - \Gamma_1 \dot{\Gamma}_1|}{\Gamma_2} + \frac{|\dot{\Gamma}_1| |\Gamma_2 - \Gamma_1|}{\Gamma_2}$$

and therefore if we exploit $1 \leq \Gamma_2$ and $\sup_{\tau}(1 + \tau^{2k})|\dot{\Gamma}_1| \leq \delta$ we have

$$\sup_{\tau}(1 + \tau^{2k})|\dot{\Gamma}_2 - \dot{\Gamma}_1|(\tau) \leq \sup_{\tau}(1 + \tau^{2k})|\Gamma_2\dot{\Gamma}_2 - \Gamma_1\dot{\Gamma}_1|(\tau) + \delta \sup_{\tau}|\Gamma_2 - \Gamma_1|(\tau). \tag{4.40}$$

The inequality (4.39) yields

$$\sup_{\tau}(1 + \tau^{2k})|\dot{\Gamma}_2 - \dot{\Gamma}_1(\tau)| \leq \sup_{\tau}(1 + \tau^{2k})|\Gamma_2\dot{\Gamma}_2 - \Gamma_1\dot{\Gamma}_1|(\tau) + \varepsilon\delta D_1 \|C_2 - C_1\|^k, \tag{4.41}$$

where

$$D_1 = \frac{\left(1 + (2 + \delta)\int_0^{\infty} \frac{d\eta}{1 + \eta^{2k}}\right)(M_0 N_0)(0)}{4\pi\left(1 - \delta\int_0^{\infty} \frac{d\eta}{1 + \eta^{2k}}\right)^3},$$

and therefore it suffices to estimate the first term on the right side of (4.41).

Recalling the relation

$$\Gamma_m \dot{\Gamma}_m = \frac{\varepsilon i}{4\pi C_m} \int_{-\infty}^{\infty} \omega(\mathcal{A}_m^{\omega} \mathcal{B}_m^{\omega})(\tau) e^{-2i\omega\tau} d\omega$$

and the identities (3.46) and (3.47), we have

$$\begin{aligned} |\Gamma_2 \dot{\Gamma}_2 - \Gamma_1 \dot{\Gamma}_1|(\tau) &= \frac{\varepsilon}{4\pi} \left| \frac{J_2}{C_2} - \frac{J_1}{C_1} \right|(\tau) \\ &\leq \frac{\varepsilon}{4\pi} \left| \frac{J_2 - J_1}{C_2} - \frac{J_1(C_2 - C_1)}{C_1 C_2} \right|(\tau) \\ &\leq \frac{\varepsilon}{4\pi} \frac{|J_2 - J_1|}{C_2}(\tau) + \frac{\varepsilon}{4\pi} \cdot \frac{|J_1| |C_2 - C_1|}{C_1 C_2}(\tau). \end{aligned} \tag{4.42}$$

From the fact that $C_m \geq 1$, for $m = 1$ and 2 , we are able to replace (4.42) by

$$\begin{aligned} &(1 + \tau^{2k})|\Gamma_2 \dot{\Gamma}_2 - \Gamma_1 \dot{\Gamma}_1|(\tau) \\ &\leq \frac{\varepsilon}{4\pi} (1 + \tau^{2k})|J_2 - J_1|(\tau) + \frac{\varepsilon}{4\pi} (1 + \tau^{2k})|J_1|(\tau) \cdot |C_2 - C_1|(\tau) \\ &\leq \frac{\varepsilon}{4\pi} (1 + \tau^{2k}) \left| \int_0^{\tau} (q_2(s)H_2(\tau - s, s) - q_1(s)H_1(\tau - s, s)) ds \right| \\ &\quad + \frac{\varepsilon}{4\pi} (1 + \tau^{2k})|J_1|(\tau) |C_2 - C_1|(\tau) \\ &\leq \frac{\varepsilon}{4\pi} (1 + \tau^{2k}) \int_0^{\tau} |q_2 - q_1|(s) |H_2(\tau - s, s)| ds \\ &\quad + \frac{\varepsilon}{4\pi} (1 + \tau^{2k}) \int_0^{\tau} |q_1(s)| |(H_2 - H_1)(\tau - s, s)| ds \\ &\quad + \frac{\varepsilon}{4\pi} (1 + \tau^{2k})|J_1|(\tau) |C_2 - C_1|(\tau), \end{aligned} \tag{4.43}$$

where H_1 and H_2 are defined in Eq. (3.47).

The results of Lemmas 3.2-3.7 together with (4.26) and (4.27) yield

$$\begin{aligned}
 (1 + \tau^{2k}) |\Gamma_2 \dot{\Gamma}_2 - \Gamma_1 \dot{\Gamma}_1|(\tau) &\leq \frac{\varepsilon}{4\pi} 2^{2k+1} \bar{H}_k \|q_2 - q_1\|^k \\
 &+ \frac{\varepsilon}{4\pi} 2^{2k} \delta K_k \int_0^\infty \frac{d\eta}{1 + \eta^{2k}} \|q_2 - q_1\|^k \\
 &+ \frac{\varepsilon}{4\pi} \left(Q_k(0) + 2^{2k} \delta \bar{H}_k \int_0^\infty \frac{d\eta}{1 + \eta^{2k}} \right) |C_2 - C_1|(\tau)
 \end{aligned} \tag{4.44}$$

where

$$K_k = \frac{4(1 + 2^{2k}) Q_k(0) \int_0^\infty \frac{d\eta}{1 + \eta^{2k}} + 4(1 + 2^{2k+1}) \bar{H}_k \delta \left(\int_0^\infty \frac{d\eta}{1 + \eta^{2k}} \right)^2}{1 - (2^{2k+1} + 1) \delta^2 \left(\int_0^\infty \frac{d\eta}{1 + \eta^{2k}} \right)^2}$$

and \bar{H}_k is defined as in (4.27).

The theorem now follows from (4.39), (4.41), (4.44) and Lemma 4.1.

REFERENCES

- [1] G. F. Carrier, *On the nonlinear vibration problem of the elastic string*, Q. Appl. Math. 3, 157 (1945)
- [2] R. W. Dickey, *Infinite systems of nonlinear oscillation equations related to the string*, Proc. Amer. Math. Soc. 23, 459-468 (1969)
- [3] R. W. Dickey, *Infinite systems of nonlinear oscillation equations with linear damping*, SIAM J. Appl. Math. 19, 208-214 (1970)
- [4] R. W. Dickey, *The initial value problem for a nonlinear semi-infinite string*, University of Texas, Austin, Workshop, March 1977
- [5] R. Narasimha, *Nonlinear vibration of an elastic string*, J. Sound Vib. 8, 134-146 (1968)
- [6] T. Nisida, *A note on the nonlinear vibration of an elastic string*, Mem. Fac. Eng., Kyoto University 34 (1971)
- [7] T. Nisida, *Nonlinear vibration of an elastic string II*, preprint.