79. The Initial Value Problem for the Equations of Motion of Compressible Viscous and Heat-Conductive Fluids

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§ 1. **Introduction and theorem.** The motion of the general isotropic Newtonian fluids are described by the five conservation laws:

$$(1.1)\quad \begin{cases} \rho_t+(\rho u^j)_{x_j}=0 \\ u^i_t+u^j u^i_{x_j}+\dfrac{1}{\rho}p_{x_i}=\dfrac{1}{\rho}\{(\mu(u^i_{x_j}+u^j_{x_i}))_{x_j}+(\mu' u^j_{x_j})_{x_i}\}, & i=1,2,3 \\ \theta_t+u^j\theta_{x_j}+\dfrac{\theta p_\theta}{\rho c}u^j_{x_j}=\dfrac{1}{\rho c}\{(\kappa\theta_{x_j})_{x_j}+\Psi\}, \end{cases}$$

where ρ: density, $u=(u^1,u^2,u^3)$: velocity, θ: absolute temperature, $p=p(\rho,\theta)$: pressure, $\mu=\mu(\rho,\theta)$: viscosity coefficient, $\mu'=\mu'(\rho,\theta)$: second viscosity coefficient, $\kappa=\kappa(\rho,\theta)$: coefficient of heat conduction, $c=c(\rho,\theta)$: heat capacity at constant volume and $\Psi=\dfrac{\mu}{2}(u^j_{x_k}+u^k_{x_j})^2+\mu'(u^j_{x_j})^2$: dissipation function. We consider the initial value problem for (1.1) with the initial data

(1.2) $(\rho,u,\theta)(0,x)=(\rho_0,u_0,\theta_0)(x),\ x\in R^3.$

We seek the solutions in a neighbourhood of a constant state $(\rho,u,\theta)=(\bar\rho,0,\bar\theta)$, where $\bar\rho$, $\bar\theta$ are any positive constants. Thus we assume a natural condition on the system (1.1) of hyperbolic-parabolic type throughout this paper that

(i) p, c, μ, μ' and κ are smooth functions in $\mathcal{O}=\{(\rho,u,\theta):|\rho-\bar\rho|, |u|, |\theta-\bar\theta|<\bar\varepsilon\}$.

(ii) $\partial p/\partial\rho$, $\partial p/\partial\theta>0$, $c,\mu,\kappa>0$ and $\mu'+\dfrac{2}{3}\mu\geq 0$ in $\mathcal{O}$,

where $\bar\varepsilon<\min\{\bar\rho,\bar\theta\}$.

First rewrite the system (1.1) by the change of the unknown and known variables as follows: $\rho\to\bar\rho+\rho$, $u\to u$, $\theta\to\bar\theta+\theta$, $p(\bar\rho+\rho,\bar\theta+\theta)\to p(\rho,\theta)$, $\mu(\bar\rho+\rho,u,\bar\theta+\theta)\to\mu(\rho,u,\theta)$ and so on.

$$(1.3)\quad \begin{cases} L^0(\rho,u)\equiv\rho_t+(\bar\rho+\rho)u^j_{x_j}+u^j\rho_{x_j}=0 \\ L^i(u)\equiv u^i_t-\tilde\mu u^i_{x_jx_j}-(\tilde\mu+\tilde\mu')u^j_{x_ix_j}=G^i, & i=1,2,3 \\ L^4(\theta)\equiv\theta_t-\tilde\kappa\theta_{x_jx_j}=G^4, \end{cases}$$

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where

$$(1.4)\quad \begin{cases} G^i \equiv -\tilde{p}_\rho \rho_{x_i} - \tilde{p}_\theta \theta_{x_i} + g^i, \qquad G^4 \equiv -\tilde{p}_3 u^j_{x_j} + g^4, \\ g^i \equiv -u^j u^i_{x_j} + \{\mu_{x_j}(u^i_{x_j} + u^j_{x_i}) + \mu'_{x_i}(u^j_{x_j})\}/(\bar{\rho}+\rho) \\ g^4 \equiv -u^j \theta_{x_j} + (\kappa_{x_j}\theta_{x_j} + \Psi)/(\bar{\rho}+\rho)c. \end{cases}$$

Here we also use the abbreviations

$$\tilde{\mu} = \mu/(\bar{\rho}+\rho),\ \tilde{\mu}' = \mu'/(\bar{\rho}+\rho),\ \tilde{p}_\rho = p_\rho/(\bar{\rho}+\rho),\ \tilde{p}_\theta = p_\theta/(\bar{\rho}+\rho),$$
$$\tilde{p}_3 = (\bar{\theta}+\theta)p_\theta/(\bar{\rho}+\rho)c \quad \text{and} \quad \tilde{\kappa} = \kappa/(\bar{\rho}+\rho)c.$$

Let H^l $(l=1,\cdots,4)$ be the Sobolev space with the norm $\|\ \|_l$ of the L_2-functions having all the l-th derivatives of L_2-functions. Define ε by the Sobolev's lemma so that for $\|f\|_2<\varepsilon$ we have $\max|f|\leq C\|f\|_2<\bar{\varepsilon}$. Denote $D^l f = \{\partial^l f/\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}$ for all $\alpha, \alpha_1+\alpha_2+\alpha_3=l\}$, $l=1,\cdots,4$. The initial data for (1.3) are given by

(1.5) $(\rho, u, \theta)(0) = (\rho_0, u_0, \theta_0) \in H^l \cap L_1$ for $l=3$ or 4.

The solution is sought in the space of functions $X^l(0,\infty;E)$ for some $E<\varepsilon$, $l=3$ or 4, where for $0\leq t_1<t_2\leq\infty$

(1.6) $X^l(t_1,t_2;E) = \{(\rho,u,\theta)(t) : \rho(t,x) \in C^0(t_1,t_2;H^l)\cap C^1(t_1,t_2;H^{l-1})$, $u^i(t,x), \theta(t,x) \in C^0(t_1,t_2;H^l)\cap C^1(t_1,t_2;H^{l-2})\cap L_2(t_1,t_2;H^{l+1})$, $i=1,2,3$, and $\sup_{t_1\leq t\leq t_2}\|(\rho,u,\theta)(t)\|_l^2 + \int_{t_1}^{t_2}\|\rho(s)\|_l^2 + \|(u,\theta)(s)\|_{l+1}^2 ds \leq E^2\}$.

Theorem. *Consider the initial value problem* (1.3) (1.5) *and let the initial data have the norm for* $l=4$

(1.7) $E_l = \|(\rho,u,\theta)(0)\|_l + \|(\rho,u,\theta)(0)\|_{L_1} < \infty$.

Then there exist positive constants δ_0 *and* $C_0<\infty$ $(C_0\delta_0\leq\varepsilon)$ *such that if* $E_l<\delta_0$, *then the problem* (1.3) (1.5) *has the unique solution* $(\rho,u,\theta)(t)$ *in the large such that*

$$(\rho,u,\theta)(t) \in X^l(0,\infty;C_0E_l)$$

and it has the decay rate

(1.8) $\|(\rho,u,\theta)(t)\|_2 \leq C_0E_l/(1+t)^{3/4}$.

In particular,

(1.9) *if* μ, μ' *and* κ *do not depend on* ρ,

then the above assertion holds for $l=3$ *also.*

In [1] we obtain the same type of result in the more restricted case of a polytropic gas. We also refer the reader to [1] for the bibliography of other known results on the initial (and boundary) value problem of equations of motion for compressible viscous and heat-conductive fluids.

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§2. Proof of theorem. Theorem is proved by a combination of a local existence theorem and *a priori* estimates for the solution in X^l.

Theorem 2.1 (local existence). *Consider the initial value problem* (1.3) (1.5). *Let the initial data*

$$(\rho, u, \theta)(t_1) \in H^l \qquad \text{for } l=3 \text{ or } 4.$$

Then there exist three constants $\delta_1>0$, $C_1<\infty$ $(C_1\delta_1<\varepsilon)$ *and* $\tau>0$ *such that if* $\|(\rho, u, \theta)(t_1)\|_l<\delta_1$, *then the problem* (1.3) (1.5) *has the unique solution*

$$(\rho, u, \theta)(t) \in X^l(t_1, t_1+\tau\,;\, C_1\|(\rho, u, \theta)(t_1)\|_l),$$

where δ_1, C_1, τ *do not depend on* t_1.

The proof for $l=4$ is the same as that for polytropic gas in [1]. We need an approximation of the initial data in H^4 and the L_2 energy estimate for the case $l=3$.

Theorem 2.2 (*a priori* estimates). *Suppose that for the initial data having the norm* $E_l<\infty$ *for* $l=4$, *there is a solution*

$$(\rho, u, \theta)(t) \in X^l(0, T\,;\, E)$$

for some $T>0$ *and some* $E<\varepsilon$. *Then there exist positive constants* ε_2 $(<\varepsilon)$, δ_2 *and* C_2 $(C_2\delta_2<\varepsilon)$ *such that if* $E<\varepsilon_2$ *and* $E_l<\delta_2$, *then the solution has the a priori estimates*

$$(\rho, u, \theta)(t) \in X^l(0, T\,;\, C_2E_l),$$

where ε_2, δ_2, C_2 *do not depend on* T. *In particular in the case of* (1.9) *the above estimates are true for* $l=3$ *also.*

Proof of theorem. Take $\delta_0=\min\{\delta_1, \delta_2, \varepsilon_2/C_1, \delta_1/C_2, \varepsilon_2/(1+C_1)C_2\}$ and $C_0=C_2$. We use the standard continuation argument of local solution on $[0, n\tau]$, $n=1, 2, \cdots$ to get the global solution. In fact by the local existence theorem, the definition of δ_0 and the assumption $(E_l<\delta_0)$ we have a positive constant τ and a local solution

$$(\rho, u, \theta)(t) \in X^l(0, \tau\,;\, C_1E_l).$$

By $C_1E_l<C_1\delta_0\leq\varepsilon_2$ and $E_l<\delta_0\leq\delta_2$, *a priori* estimates give

$$(\rho, u, \theta)(t) \in X^l(0, \tau\,;\, C_2E_l).$$

But by $C_2E_l<C_2\delta_0\leq\delta_1$ and the local existence theorem, we have again

$$(\rho, u, \theta)(t) \in X^l(\tau, 2\tau\,;\, C_1C_2E_l).$$

Now by $(1+C_1)C_2E_l<C_1C_2\delta_0\leq\varepsilon_2$ and $E_l<\delta_0\leq\delta_2$, *a priori* estimate shows

$$(\rho, u, \theta)(t) \in X^l(0, 2\tau\,;\, C_2E_l).$$

Thus we can continue the same arguments on $[n\tau, (n+1)\tau]$ and $[0, (n+1)\tau]$ successively $n=2, 3, \cdots$.

§ 3. *A priori* estimates.

We present here a general method to obtain *a priori* estimates for small solutions of equations with dissipation, which is a combination of the linear spectral theory and the L_2-energy method. First we rewrite the system (1.3) so that all the nonlinear terms appear at the right hand side of equations:

$$\text{(3.1)} \quad \begin{cases} \rho_t+\bar{\rho}u^j_{x_j}=f^0, \\ u^i_t+\bar{p}_1\rho_{x_i}+\bar{p}_2\theta_{x_i}-\bar{\mu}u^i_{x_ix_j}-(\bar{\mu}+\bar{\mu}')u^j_{x_ix_j}=f^i, \qquad i=1, 2, 3, \\ \theta_t+\bar{p}_3u^j_{x_j}-\bar{\kappa}\theta_{x_jx_j}=f^4, \end{cases}$$

where $f=\{f^i, i=0, \cdots, 4\}$ is at least quadratic functions of (ρ, u, θ) and their first and second derivatives, and $\bar{p}_1=\tilde{p}_\rho(0, 0)$, $\bar{p}_2=\tilde{p}_\theta(0, 0)$, $\bar{p}_3=\tilde{p}_3(0, 0)$, $\bar{\mu}=\tilde{\mu}(0, 0, 0)$, $\bar{\mu}'=\tilde{\mu}'(0, 0, 0)$, $\bar{\kappa}=\tilde{\kappa}(0, 0, 0)$ are positive constants.

Set $U={}^t(\sqrt{\bar{\rho}/\bar{p}_1}\rho, u, \sqrt{\bar{\theta}/c(0,0,0)}\theta)$ and write (3.1) in the form

(3.2) $U_t+AU=F(U)$.

The Fourier transform $\hat{A}(\xi)$ of the linear partial differential operator A is the 5×5 matrix

$$(3.3)\quad \hat{A}(\xi)=\begin{pmatrix} 0 & -ia\xi_k & 0 \\ -ia\xi_j & -\bar{\mu}\delta^{jk}|\xi|^2-(\bar{\mu}+\bar{\mu}')\xi_j\xi_k & -ib\xi_j \\ 0 & -ib\xi_k & -\bar{\kappa}|\xi|^2 \end{pmatrix},$$

where $a=\sqrt{p_\rho(0,0)}$, $b=\bar{p}_2\sqrt{\bar{\theta}/c(0,0,0)}$ and j,k run from 1 to 3. The eigenvalues λ_j, $j=1,\cdots,4$ of $\hat{A}$ and their projections P_j, $j=1,\cdots,4$, on the eigenspaces are analyzed by

Lemma 3.1. (i) *λ_j depends on $i|\xi|$ only and $\lambda_j=0$ if $|\xi|=0$, $j=1,\cdots,4$.*

(ii) *$\lambda_j\neq\lambda_k$, $j\neq k$, for all $|\xi|$ except at most four points of $|\xi|>0$.*

(iii) *There exist positive constants $r_1<r_2$ such that λ_j has a Taylor (Laurent) series expansion for $|\xi|<r_1$ ($|\xi|>r_2$, respectively). Especially the Taylor series has the form*

$$(3.4)\quad \begin{cases} \lambda_1=\sqrt{a^2+b^2}\,i|\xi|+\dfrac{(a^2+b^2)(2\bar{\mu}+\bar{\mu}')+b^2\bar{\kappa}}{2(a^2+b^2)}(i|\xi|)^2+\cdots \\ \lambda_2=\lambda_1^* \text{ (complex conjugate)} \\ \lambda_3=\dfrac{a^2\bar{\kappa}}{a^2+b^2}(i|\xi|)^2+\dfrac{a^2b^2\bar{\kappa}^2((a^2+b^2)(2\bar{\mu}+\bar{\mu}')-a^2\bar{\kappa})}{(a^2+b^2)^4}(i|\xi|)^4+\cdots \\ \lambda_4=\bar{\mu}(i|\xi|)^2 \end{cases}$$

(iv) *$\mathrm{rank}(\lambda_4-\hat{A})=3$ for all $|\xi|>0$ except at most one point of $|\xi|>0$.*

(v) *The matrix exponential has the spectral resolution*

$$(3.5)\quad e^{t\hat{A}(\xi)}=\sum_{j=1}^{4}e^{t\lambda_j(\xi)}P_j(\xi)$$

for all $|\xi|$ except at most four points of $|\xi|>0$.

(3.6) $\|P_j(\xi)\|\le C$ *for* $|\xi|\le r_1$.

It has the estimate by the modification of the right hand side of (3.5) *near the points of multiple eigenvalue*

(3.7) $\|e^{t\hat{A}(\xi)}\|\le C(1+t)^3e^{-\beta t}$

for $|\xi|>r_1$ and a positive constant β.

Lemma 3.2. *There is a constant $C=C(\varepsilon)$ such that*

$$(3.8)\quad \begin{aligned} &\|F(U)\|_{L_1},\ \|F(U)\|\le C\|U\|_2^2 \\ &\|D^kF(U)\|\le C\|U\|_2\|U\|_{k+2} \text{ for } k=1,2. \end{aligned}$$

In particular in the case of (1.9)

(3.9) $\|D^2F(U)\|\le C\|U\|_2(\|U\|_3+\|u,\theta\|_4)$

Proposition 3.3. *There exist δ_3, ε_3 and C_3 such that if $E_l<\delta_3$ and $E<\varepsilon_3$, then $U(t)$ satisfying* (3.2) *has the estimates*

$$(3.10)\quad \begin{cases} \|U(t)\|_2\le C_3E_l(1+t)^{-3/4} \\ \displaystyle\int_0^t\|U(t)\|_2^2ds\le C_3E_l, \end{cases}$$

where $l=4$ *in general and* $l=3$ *for the case* (1.9).

The Proposition is a consequence of Lemmas 3.1 and 3.2. In fact we have

$$\begin{cases} \|U(t)\| \leq C_0 E_l (1+t)^{-3/4} + C \int_0^t (1+t-s)^{-3/4} \|U(s)\|_2^2 ds \\ \|D^k U(t)\| \leq C_0 E_l (1+t)^{-5/4} + C \int_0^t (1+t-s)^{-5/4} \|U(s)\|_2 \cdot \\ \qquad \cdot (\|U(s)\|_2 + \|U(s)\|_4) ds, \qquad k=1,2. \end{cases}$$

Therefore for $M(t) = \sup_{0 \leq s \leq t} (1+s)^{3/4} \|U(s)\|_2$ we have $M(t) \leq C_0 E_l + CM(t)^2$ $CEM(t)$, where E is the norm (1.6) assumed on the solution. Thus we get the conclusion of Proposition 3.3 for $l=4$.

Next we have to obtain the estimates for the higher derivatives, which are given by

Proposition 3.4. *There exist* ε_4 *and* C_4 *such that if* $E < \varepsilon_4$ *and the solution* $U(t)$ *satisfies the estimates* (3.10), *then the following energy estimates hold*:

$$(3.11) \qquad \|D^k(u, \theta)(t)\|^2 + \int_0^t \|D^{k+1}(u, \theta)(s)\|^2 ds \leq C_4 E_l^2 \ \textit{for}\ 2 \leq k \leq l$$

$$(3.12) \qquad \|D^m \rho(t)\|^2 + \int_0^t \|D^m \rho(s)\|^2 ds \leq C_4 E_l^2 \ \textit{for}\ 3 \leq m \leq l.$$

Here we note that Theorem 2.2 is a direct consequence of Propositions 3.3 and 3.4. Using the estimates (3.10) for the lower order derivatives of the solution, the proof of Proposition 3.4 is given successively with respect to k and m in the same way as that for polytropic gases in [1]. In fact let us remind the operators L^i, $i=0, \cdots, 4$ in (1.3) and note the estimates for the nonlinear terms g in the right hand side of (1.3).

Lemma 3.5. *We have the estimates for* $k=0, 1, \cdots, 4$

$$(3.13) \qquad \|D^k g\| \leq C \|\rho, u, \theta\|_3 \|D(\rho, u, \theta)\|_k.$$

The estimate (3.11) for $k=2$ is given by the integration on $x \in R^3$, $0 \leq t \leq T$ of the equality

$$(3.14) \qquad D^k(L^i(u) - G^i) \cdot D^k u^i + D^k(L^4(\theta) - G^4) \cdot D^k \theta = 0.$$

Integrate by parts, use (3.10) and Lemma 3.5. The estimate (3.12) for $m=3$ is obtained by the integration on $x \in R^3$, $0 \leq t \leq T$ of the equality

$$D^{m-1}\{L^0(\rho, u)\}_{x_i} \cdot D^{m-1} \rho_{x_i} + \frac{\bar{\rho}+\rho}{2\tilde{\mu}+\tilde{\mu}'} \{D^{m-1}\{L^i(u) + \tilde{p}_\rho \rho_{x_i} - (g^i + \tilde{p}_\theta \theta_{x_i})\} \cdot D^{m-1} \rho_{x_i} = 0.$$

Integrate by parts, use the equations (1.3) and (3.10), (3.11) for $k=2$ and Lemma 3.5. We can proceed to get (3.11) for $k=3$ by (3.14) and so on. The detailed arguments using Friedrichs mollifier and the estimates for composite functions are the same as that in [1]. We omit them here.

Reference

[1] A. Matsumura and T. Nishida: The initial value problem for the equations of motion of viscous and heat-conductive gases (to appear in J. Math. Kyoto Univ., 1980).