The Initial Value Problem for the Linearized Equations of Water Waves

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Introduction. Let $x = (x', y) = (x_1, \cdots, x_{n-1}, y)$ denote a variable point in the n-dimensional Euclidean space R^n. Let G be a domain in R^n bounded by an $(n-1)$-dimensional domain Γ on $y = 0$ and by a finite number of manifolds $\Gamma_b, \Gamma_1, \cdots, \Gamma_{j_0}$ lying in $y < 0$, where $\Gamma_b \cup \Gamma$ is the outer boundary of G and the Γ_j are compact manifolds. Set $\Gamma' = \Gamma_b \cup \Gamma_1 \cup \cdots \cup \Gamma_{j_0}$.

Consider the following problem: Given functions u^0, u_t^0 on Γ, find a solution $u(x, t)$ of

$$\Delta u(x, t) = 0 \quad \text{in} \quad G, \qquad t \geqq 0 \qquad \left(\Delta = \sum_{i=1}^{n-1} \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y^2}\right), \tag{0.1}$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \Gamma', \qquad t \geqq 0 \qquad (\nu = \text{outward normal}), \tag{0.2}$$

$$u_{tt} + u_y = 0 \quad \text{on} \quad \Gamma, \qquad t \geqq 0, \tag{0.3}$$

$$u(x', 0, 0) = u^0(x'), \qquad u_t(x', 0, 0) = u_t^0(x') \quad \text{on} \quad \Gamma. \tag{0.4}$$

The normalized equations of water waves are the Laplace equation (0.1), the condition (0.2) (if Γ' is a fixed boundary) and the nonlinear Bernoulli equation on the free surface of the water. If the amplitude of the waves is small, and if we make a first order approximation, then the domain occupied by water is replaced by a fixed domain G, as above, and the Bernoulli equation is replaced by (0.3). Thus, for $n = 3$, the system (0.1)–(0.4) represents an approximation to the initial value problem for the equations of water waves [8]. We refer to it as the *linearized system of water waves.*

The purpose of the present work is to solve (0.1)–(0.4). In this connection the only general result in the literature seems to be that of Finkelstein (see [8]).

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He considered (0.1)–(0.4), for $n = 3$, in case $\Gamma = R^{n-1}$ and Γ_b is empty (*i.e.*, Γ_b is given by $y = -\infty$). He constructed explicitly a Green's function and used it to reduce the problem of solving (0.1)–(0.4) to that of solving an integral equation. Whereas Finkelstein's construction extends to the case where Γ_b is given by $y =$ const., it certainly fails to extend to any more general Γ_b.

Our approach to proving existence is based on separating the variables x and t. This leads to an eigenvalue problem with the parameter occurring in the boundary condition. In order to deal with this problem we introduce the transformation T defined as follows: Given $f \in L^2(\Gamma)$, let φ be the solution of

$$\begin{aligned} \Delta\varphi &= 0 \quad \text{in} \quad G, \\ \frac{\partial \varphi}{\partial \nu} &= 0 \quad \text{on} \quad \Gamma', \\ \varphi_\nu + \alpha\varphi &= f \quad \text{on} \quad \Gamma \qquad (\alpha \text{ positive constant}). \end{aligned} \tag{0.5}$$

Then set $(Tf)(x') = \varphi(x', 0)$.

Chapter 1. In case G is bounded we prove that T is a self-adjoint compact operator. With the aid of its eigenfunctions we then construct a solution of (0.1)–(0.4). The solution satisfies (0.1), (0.2), (0.4) in the usual sense, but (0.3) is satisfied only in some "strong" sense. We also derive an energy equality and, further, show that except for a linear term, the solution is almost periodic in t. All these results hold also for more general bounded domains (*i.e.*, for "normal" domains, as defined in §1.1).

When G is unbounded, T fails to be compact. However this case can be treated by approximating G by a sequence of bounded domains and using a compactness argument based upon an energy inequality. In this way we obtain a solution of (0.1)–(0.4) for any u^0 in $H^{1/2}(\Gamma)$ and u_t^0 in $L^2(\Gamma)$. The equations (0.1), (0.2) are still satisfied in the classical sense, but (0.3) and (0.4) hold in some weak sense.

Chapter 2. The method of this chapter applies for either G bounded or unbounded. It is based on separating variables in a less explicit way than in Chapter 1 (and Chapter 3 below). If G is unbounded, T can be defined as before. It is still self-adjoint, but it is not compact. The solution $u(x, t)$ that we construct is such that its restriction, $u^\Gamma(x', t)$, to Γ is given by an expression of the form

$$\int_\alpha^\infty \cos\,((\lambda - \alpha)^{1/2} t)\, dF_\lambda u^0 + \int_\alpha^\infty \frac{\sin\,(\lambda - \alpha)^{1/2} t}{(\lambda - \alpha)^{1/2}}\, dF_\lambda u_t^0\,,$$

where $\{F_\lambda\}$ is the spectral family of the unbounded operator T^{-1}. For G bounded, the condition (0.3) is satisfied in a weaker sense than in Chapter 1.

The method we use applies also to problems where instead of (0.3), (0.4) we have

$$p\left(\frac{\partial}{\partial t}\right)u + u_\nu = 0 \quad \text{on} \quad \Gamma, \qquad t \geqq 0, \tag{0.6}$$

$$(0.7) \qquad \frac{\partial^j}{\partial t^j} u(x', 0, 0) = u_j^0(x') \quad \text{on} \quad \Gamma \qquad (j = 0, 1, \cdots, s).$$

Here $p(\lambda)$ is any polynomial of degree $s + 1$ all of whose zeros have non-positive real parts.

The method of Chapter 2 clarifies the sense in which (0.1)–(0.4) is a "hyperbolic" problem. We show in fact that the restriction of the solution to the free surface satisfies an abstract wave equation.

Chapter 3. If G is unbounded, there is still another method of solving (0.1)–(0.4). Let $\{E_\lambda\}$ denote the spectral decomposition of T. It is known (see §3.2) that under certain conditions, the E_λ are primitive functions of "generalized eigenfunctions" χ_λ of T. Furthermore, the expected expansion theorem holds with (χ_λ, φ) as the Fourier coefficients (see (3.2.8), (3.2.9)).

In §3.3 we prove that these assumptions are in fact satisfied for the operator T, provided Γ_b satisfies a certain condition (A_R), stating that Γ_b is flat in the region $|x'| > R$. In order to define a formal solution we need to extend the χ_λ, in a suitable way, into G. Such an extension is given with the aid of the Green function which is constructed in §3.1.

The next step is to give the formal solution a meaning. Thus, in §3.4 we show that this solution can be considered as a distribution in G. Since this distribution satisfies the Laplace equation, it must coincide with a C^∞ function u. In §3.5 we prove that u is smooth in $G \cup \Gamma'$ and in §3.6 we show that $\partial u/\partial \nu = 0$ on Γ_b.

In §3.7 we prove that (0.3), (0.4) hold in a "weak" sense (see (3.7.25) and (3.7.26)–(3.7.29)). If u happens to be smooth in $G \cup \Gamma$, then it follows that u satisfies (0.3), (0.4) in the usual sense.

Finally, in §3.8 we eliminate the condition (A_R) stated above as well as another condition on the multiplicity of the spectrum of T.

The method of Chapter 3 also extends, with minor modifications, to the system consisting of (0.1), (0.2), (0.6), (0.7).

We finally note that the senses in which u satisfies (0.3) in Chapters 1 and 2 are different from the sense in which u satisfies (0.3) in Chapter 3.

Chapter 4. In this chapter we prove a uniqueness theorem for functions satisfying (0.1)–(0.4) in the classical sense and which are bounded by $Ce^{\epsilon|x'|}$ for some $\epsilon > 0$ sufficiently small.

Appendix. Here we solve the problem

$$(0.8) \qquad \begin{cases} \Delta u = 0 \quad \text{in} \quad G, & t \geqq 0, \\ \dfrac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \Gamma', & t \geqq 0, \\ u_t + u_y = 0 \quad \text{on} \quad \Gamma, & t \geqq 0, \\ u(x', 0, 0) = u^0(x') \quad \text{on} \quad \Gamma & \end{cases}$$

by two different methods. These methods fail to extend to the case of (0.1)–(0.4).

We finally remark that the methods of this work can be extended without difficulty to solve analogous problems where instead of the Laplacian $\triangle$ we have any self-adjoint elliptic operator with constant coefficients.

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CHAPTER 1: EXISTENCE THEOREMS (FIRST METHOD)

1. Definitions and preliminary lemmas. Let Ω be a bounded domain in R^n whose boundary $\partial\Omega$ is in $C^{2+\rho}$, for some $0 < \rho < 1$. We assume that $\partial\Omega$ consists of an exterior boundary $\partial_0\Omega$ and of a finite number of compact manifolds $\partial_i\Omega$ $(i = 1, \cdots, i_0)$ which bound domains $\triangle_i$. Denote by $x = (x', y) = (x_1, \cdots, x_{n-1}, y)$ a variable point in R^n. We assume that, if $\overline{\triangle}_i$ intersects $y = 0$ then the intersection consists of a finite number of sets, each set being a connected manifold (with boundary) of some dimension k $(0 \leqq k \leqq n - 1)$.

We make the same assumption with regard to the intersection of the exterior of $\partial_0\Omega$ with $y = 0$.

Let Γ be the intersection of Ω with $y = 0$. Our final assumption is that Γ is a non-empty $(n - 1)$-dimensional domain.

Definition. A domain G is said to be *normal* if there is a domain Ω satisfying all the foregoing assumptions, such that

$$G = \Omega \cap \{(x', y); y < 0\}.$$

In this chapter we assume, except in §5, that the domain occupied by the water is normal. It should be noted that the concept of a normal domain is general enough to include most applications; in particular, obstacles (partially or totally submerged in the water), islands, docks, overhanging shelves and beaches are all allowed.

We denote by $H^1(G)$ the completion of the $C^\infty(\bar{\Omega})$ complex-valued functions with respect to the norm

$$\|\varphi\|_{H^1(G)} = \left\{\int_G (|\varphi|^2 + |\nabla\varphi|^2)\, dx\right\}^{1/2} = \{\|\varphi\|^2_{L^2(G)} + \|\nabla\varphi\|^2_{L^2(G)}\}^{1/2}.$$

Write

$$\Gamma_{y_0} = \Omega \cap \{(x', y); y = y_0\},$$

$$\Omega_{y_0} = \Omega \cap \{(x', y); y < y_0\}.$$

Lemma 1.1. *Let G be a normal domain. For any sufficiently small $\epsilon > 0$ there exists a smooth domain (say $C^{2+\rho}$) G_ϵ having the following properties: $G_\epsilon \supset \Omega_\epsilon$, $\partial G_\epsilon \supset \Gamma_\epsilon$,*

$$G_\epsilon \cap \{(x', y); y < 0\} = G,$$

$$G_\epsilon \subset \{(x', y); y < \epsilon\},$$

and the distance of any point of ∂G_ϵ to Ω_ϵ is less than ϵ. The G_ϵ can be constructed to be monotone increasing with ϵ.

The proof of this rather standard fact is omitted.

Lemma 1.2. *Let G be normal, and let G_ϵ be as in Lemma 1.1. Then there exists an $\epsilon_0 > 0$ and a constant $c_0 > 0$ such that every function $\varphi \in H^1(G)$ has an extension $\tilde{\varphi}$ in $H^1(G_{\epsilon_0})$ with*

$$\|\tilde{\varphi}\|_{H^1(G_{\epsilon_0})} \leqq c_0 \|\varphi\|_{H^1(G)} . \tag{1.1}$$

Proof. We introduce a family of simple curves $\gamma(x')$ which cover $G_{\epsilon_0} \cap \{(x', y); -\epsilon_0 < y < \epsilon_0\}$ and do not intersect each other, and such that $\gamma(x')$ passes through the point $(x', 0)$ of Γ. We can parametrize these curves by writing $\gamma = \gamma(x', s)$, where s is the arc length, so that $\gamma(x', s)$ is in $C^{1+\rho}$. We normalize s so that $s > 0$ if $y > 0$ and $s < 0$ if $y < 0$. Define

$$\varphi(x', y) = \varphi[x_0', s]$$

where (x', y) is the point $\gamma(x_0', s)$. If $\varphi \in C^1(\bar{G})$ we now set:

$$\tilde{\varphi}[x_0', s] = 4\varphi\left[x_0', -\frac{s}{2}\right] - 3\varphi[x_0', -s] \quad \text{for} \quad s > 0.$$

It is easily seen that

$$\tilde{\varphi}[x_0', 0] = \varphi[x_0', 0], \qquad \frac{\partial}{\partial s}\tilde{\varphi}[x_0', 0] = \frac{\partial}{\partial s}\varphi[x_0', 0],$$

so that $\tilde{\varphi} \in H^1(G_{\epsilon_0})$. (1.1) is also clearly satisfied. Since $C^1(\bar{G})$ is dense in $H^1(G)$, the lemma follows.

Lemma 1.3. *Let G be a normal domain. If $\varphi \in H^1(G)$ then the restriction of φ to Γ is in $L^2(\Gamma)$. Moreover, the restriction mapping of $H^1(G)$ into $L^2(\Gamma)$ is compact.*

Proof. By Lemma 1.2, every $\varphi \in H^1(G)$ has an extension $\tilde{\varphi}$ in $H^1(G_{\epsilon_0})$. If $n \geqq 3$, the lemma then follows from the Sobolev–Kondrašev theorem [7, p. 84]. For $n = 2$ we have to proceed differently. First we extend φ to $H^1(G_{\epsilon_0})$, and then (since ∂G_{ϵ_0} is smooth) to $H^1(G_{\epsilon_0}')$ (with $G_{\epsilon_0}' \supset \bar{G}_{\epsilon_0}$) such that the extended function $\tilde{\varphi}$ has a compact support in G_{ϵ_0}'. Let $G_{\epsilon_0}'' = G_{\epsilon_0}' \cap \{(x', y); y < 0\}$ and let G_* be a smooth domain such that $G \subset G_* \subset G_{\epsilon_0}''$ while $\Gamma \subset \partial G_*$. Since G_* is a smooth domain, by [5] the restriction mapping from $H^1(G_*)$ into $L^2(\partial G_*)$ is compact. The conclusion of the lemma now follows.

Definition. Let $\varphi(x') \in C_0^\infty(\Gamma)$ and let $\hat{\varphi}(\xi')$ denote the Fourier transform of φ. We introduce the norm

$$|\varphi|_{H^{1/2}(\Gamma)} = \left[\int_{R^{n-1}} (1 + |\xi'|)\, |\hat{\varphi}(\xi')|^2\, d\xi'\right]^{1/2}$$

and denote by $H^{1/2}(\Gamma)$ the completion of $C_0^\infty(\Gamma)$ with respect to this norm.

Lemma 1.4. *Let G be a normal domain. Every function $\varphi \in H^{1/2}(\Gamma)$ has an extension to a function in $H^1(G)$. Conversely, the restriction to Γ of every function in $H^1(G)$ is in $H^{1/2}(\Gamma)$.*

Proof. Suppose $\varphi \in H^{1/2}(\Gamma)$. Let $\{\varphi_k\} \subset C_0^\infty(\Gamma)$ be such that $\varphi_k \to \varphi$ in $H^{1/2}(\Gamma)$. Define

$$\hat{\psi}_k(\xi', y) = \hat{\varphi}_k(\xi')e^{y(1+|\xi'|)} \quad \text{for} \quad y < 0,$$

and let $R_-^n = \{(x', y); y < 0\}$. We shall show that $\hat{\psi}_k(\xi', y)$ is the Fourier transform of a function $\psi_k(x', y) \in H^1(R_-^n)$ and that $\{\psi_k\}$ form a Cauchy sequence in $H^1(R_-^n)$. From this the first part of the lemma follows by restricting the ψ_k to G.

By Parseval's theorem,

$$\begin{aligned}(2\pi)^n\, \|\psi_k\|_{H^1(R^n_-)}^2 &= \int_{R^{n-1}} \int_{-\infty}^0 \left[(1 + |\xi'|^2)\, |\hat{\psi}_k|^2 + \left|\frac{\partial \hat{\psi}_k}{\partial y}\right|^2\right] dy\, d\xi' \\ &\leqq c \int_{R^{n-1}} (1 + |\xi'|)^2\, |\hat{\varphi}_k(\xi')|^2 \int_{-\infty}^0 e^{2y(1+|\xi'|)}\, dy\, d\xi' \\ &= c \int_{R^{n-1}} (1 + |\xi'|)\, |\hat{\varphi}_k(\xi')|^2\, d\xi' = c\, |\varphi_k|_{H^{1/2}(\Gamma)}\,.\end{aligned}$$

Thus, $\psi_k \varepsilon H^1(R_-^n)$. A similar calculation shows that $\{\psi_k\}$ is a Cauchy sequence in $H^1(R_-^n)$.

To prove the second part of the lemma, we extend φ to a function $\tilde{\varphi}$ in $H^1(G_*)$ (as in the proof of Lemma 1.2) and note that we may take $\tilde{\varphi}$ such that

$$||\tilde{\varphi}||_{H^1(G_*)} \leqq c \, ||\varphi||_{H^1(G)} .$$

Since G_* is a smooth domain, [5] shows that the restriction of $\tilde{\varphi}$ to ∂G_* is in $H^{1/2}(\partial G^*)$, and the proof of the lemma is thereby completed.

Lemma 1.5. *Let G be a normal domain. There exists a constant c such that for any* $\varphi \varepsilon H^1(G)$,

$$\int_G |\varphi|^2 \, dx \leqq c \int_\Gamma |\varphi|^2 \, dx' + c \int_G |\nabla\varphi|^2 \, dx. \tag{1.2}$$

Proof. Let $\triangle$ be domain lying on $\partial G \backslash \Gamma$ and having a boundary in $C^{2+\rho}$. The normals to $\triangle$ are then $C^{1+\rho}$ curves which cover some neighborhood of $\triangle$ and do not intersect each other in this neighborhood. If the diameter of $\triangle$ is sufficiently small, we can construct a tube T which lies in G and has the following properties:

The base of T is $\triangle$. In a neighborhood of $\triangle$, the lateral boundary of T consists of normals to $\triangle$. The "top" of T is on Γ. Let dS denote the surface element on cross sections of T and let ds denote the length element along the "stream lines" of the tube. Then, for some positive constants α_0, α_1,

$$\alpha_0 \leqq \frac{dS \, ds}{dx} \leqq \alpha_1 . \tag{1.3}$$

Integrating $\partial\varphi/\partial s$ along a stream line, we get

$$|\varphi(\xi)|^2 \leqq 2 \, |\varphi(x_\xi , 0)|^2 + 2 \int \left|\frac{\partial\varphi}{\partial s}\right|^2 ds \tag{1.4}$$

where ξ is any point in T and $(x_\xi , 0)$ is the point at which the stream line through ξ intersects Γ. Integrating and using (1.3), we find that

$$\int_T |\varphi|^2 \, dx \leqq c \int_\Gamma |\varphi|^2 \, dx' + c \int_G |\nabla\varphi|^2 \, dx.$$

The same result can, of course, be derived when we begin with a surface $\triangle$ not lying on $\partial G \backslash \Gamma$, but lying in G.

We can cover G by a finite number of such tubes. Thus the inequality (1.2) follows.

Lemma 1.6. *Let G be a normal domain. There exists a constant c such that for any* $\varphi \varepsilon H^1(G_\epsilon)$ $(0 < \epsilon < \epsilon_0)$

$$\int_G |\varphi|^2 \, dx \leqq c \int_{\Gamma_\epsilon} |\varphi|^2 \, dx' + c \int_{G_\epsilon} |\nabla\varphi|^2 \, dx; \tag{1.5}$$

c is independent of ϵ.

This is a consequence of Lemma 1.5, since, as is easily verified, $\int_\Gamma |\varphi|^2\, dx'$ is bounded by the right hand side of (1.5).

Note that the constants c in Lemmas 1.5, 1.6 depend on G and as G increases they may also increase. Since later on we shall deal with a sequence of domains with unbounded union, we need a bound on the c's. This we shall now derive.

Definition. Let an unbounded domain G be a union of an increasing sequence of normal domains G_j, such that $G_j \cap \{(x', y); |x'| < \mu_j\} = G \cap \{(x', y); |x'| < \mu_j\}$ for a sequence $\{\mu_j\}$ increasing to ∞. Assume that for some $R > 0$, $h_0 > 0$ the following is true:

The boundary of $G \cap \{(x', y); |x'| > R\}$ consists of three manifolds; the first one lies on $|x'| = R$, the second, Γ^*, lies on $y = 0$, and the third is given by $y = -h(x')$, where $h(x')$ is defined and is in $C^{2+\rho}$ for all x' with $(x', 0) \in \Gamma^*$, and $0 < h(x') \leqq h_0$ for all x' in the interior of Γ^*.

We then say that G is an *unbounded normal* domain.

Let Ω_j be the domain associated with G_j in the definition of a bounded normal domain. We then define $\Gamma = \bigcup_j [\Omega_j \cap \{(x', y); y = 0\}]$.

If G is an unbounded normal domain, then in proving Lemmas 1.5, 1.6 for each G_j we may use just one tube for the whole subdomain $G_j \cap \{(x', y); |x'| > R\}$, namely, that whose stream lines are the parallels to the y-axis. We then have:

Lemma 1.7. *Let G be an unbounded normal domain. Then for each G_j the assertion of Lemmas* 1.5, 1.6 *is valid with a constant c independent of j and ϵ. The assertion of Lemmas* 1.5, 1.6 *is also true for G.*

We state two related lemmas, which will be used in Chap. 2. Let ∇' denote the gradient with respect to $x' = (x_1, \cdots, x_{n-1})$.

Lemma 1.8. *Let G be an unbounded normal domain and assume that $|\nabla' h(x')| \leqq M$ where M is a constant. Then, for any $\varphi \in C^1(\bar{G}_j)$,*

$$\int_{\partial G_j \setminus \Gamma_j} |\varphi|^2\, dS \leqq c \int_{\Gamma_j} |\varphi|^2\, dS + c \int_{G_j} |\nabla\varphi|^2\, dx \tag{1.6}$$

where Γ_j is the part of ∂G_j lying on $y = 0$. The constant c is independent of j.

Lemma 1.9. *Let G be an unbounded normal domain. Assume that $|\nabla' h(x')| \leqq M$. Then, for any $\varphi \in C^1(\bar{G})$ with $\varphi \in L^2(\Gamma)$, $\nabla\varphi \in L^2(G)$,*

$$\int_{\partial G \setminus \Gamma} |\varphi|^2\, dS \leqq c \int_{\Gamma} |\varphi|^2\, dS + c \int_{G} |\nabla\varphi|^2\, dx. \tag{1.7}$$

The proofs are similar to the proof of Lemma 1.5. Thus, in (1.4) we take ξ to vary only on the base $\triangle$ of the tube T. For (x', y) with $|x'| > R$ we use the relation

$$|\varphi(x', y)|^2 \leqq 2\, |\varphi(x', 0)|^2 + 2 \int_0^y \left| \frac{\partial\varphi}{\partial y}(x', y) \right|^2 dy$$

with $y = -h(x')$, integrate with respect to $dS(x')$ and then use the assumption $|\nabla' h(x')| \leqq M$ in order to estimate $dS(x')/dx'$ by a constant.

2. An auxiliary problem. Let G be a normal domain. For any sufficiently small $\epsilon > 0$, say $\epsilon < \epsilon_0$, there is a 1–1 smooth correspondence between Γ and Γ_ϵ; denote the point on Γ_ϵ corresponding to $(x', 0)$ on Γ by $(\xi_\epsilon(x'), \epsilon)$.

For any $f \in L^2(\Gamma)$ we define a function

$$f_\epsilon(x) = \begin{cases} f(x') & \text{if } \ x = (\xi_\epsilon(x'), \epsilon), \\ 0 & \text{if } \ x \in \partial G_\epsilon / \Gamma_\epsilon . \end{cases} \tag{2.1}$$

Let ζ_ϵ be a C^∞ function defined on ∂G_ϵ such that $0 \leqq \zeta_\epsilon \leqq 1$, while

$$\zeta_\epsilon(x) = \begin{cases} 1 & \text{for } \ x \in \Gamma_\epsilon , \\ 0 & \text{for } \ x \in \partial G_\epsilon / \Gamma. \end{cases}$$

Consider the boundary value problem

$$\begin{aligned} \triangle \varphi_\epsilon &= 0 \quad \text{in} \quad G_\epsilon , \\ \frac{\partial \varphi_\epsilon}{\partial \nu} + \alpha \zeta_\epsilon \varphi_\epsilon &= f_\epsilon \quad \text{on} \quad \partial G_\epsilon , \end{aligned} \tag{2.2}$$

where $\partial/\partial\nu$ denotes the outward normal derivative and α is any fixed positive constant.

Let $\triangle_\epsilon = \partial G_\epsilon \backslash \partial G$, and denote by $(\, , \,)_{\Gamma_\epsilon}$, $(\, , \,)_{\triangle_\epsilon}$, $((\, , \,))_{G_\epsilon}$ the scalar products in $L^2(\Gamma_\epsilon)$, $L^2(\triangle_\epsilon)$ and $L^2(G_\epsilon)$, respectively.

Lemma 2.1. *Let G be a normal domain. Define f_ϵ by* (2.1). *Then there exists a unique solution φ_ϵ of* (2.2) *in the following sense: φ_ϵ is harmonic in G_ϵ, $\varphi_\epsilon \in H^1(G_\epsilon)$ and*

$$((\nabla \psi, \nabla \varphi_\epsilon))_{G_\epsilon} + \alpha(\psi, \zeta_\epsilon \varphi_\epsilon)_{\triangle_\epsilon} = (\psi, f_\epsilon)_{\Gamma_\epsilon} \tag{2.3}$$

for all $\psi \in H^1(G_\epsilon)$. Moreover,

$$||\varphi||_{H^1(G)} \leqq c \, |f|_{L^2(\Gamma)} \tag{2.4}$$

where c is a constant independent of ϵ.

Proof. Fix $\epsilon > 0$. Let $\{f_k\}$ be a sequence of smooth functions converging to f_ϵ in $L^2(\partial G_\epsilon)$ as $k \to \infty$. Since both f_k and G_ϵ are smooth, there exists a smooth solution φ_k of

$$\begin{aligned} \triangle \varphi_k &= 0 \quad \text{in} \quad G_\epsilon , \\ \frac{\partial \varphi_k}{\partial \nu} + \alpha \zeta_\epsilon \varphi_k &= \zeta_\epsilon f_k \quad \text{on} \quad \partial G_\epsilon . \end{aligned}$$

By Green's theorem,

$$||\nabla \varphi_k||^2_{L^2(G_\epsilon)} + \alpha \, |\zeta_\epsilon^{1/2} \varphi_k|^2_{L^2(\triangle_\epsilon)} = (\varphi_k , \zeta_\epsilon f_k)_{\triangle_\epsilon} .$$

It follows that

$$(2.5)\qquad \|\nabla\varphi_k\|^2_{L^2(G_\epsilon)} + \frac{\alpha}{2}\,|\zeta_\epsilon^{1/2}\varphi_k|^2_{L^2(\Delta_\epsilon)} \leqq \frac{1}{2\alpha}\,|\zeta_\epsilon^{1/2}f_k|^2_{L^2(\Delta_\epsilon)}$$
$$\leqq \frac{1}{2\alpha}\,|f_k|_{L^2(\Delta_\epsilon)} \leqq c\,|f_\epsilon|_{L^2(\Gamma_\epsilon)} \leqq c\,|f|^2_{L^2(\Gamma)}$$

where the constants c are independent of f, ϵ.

(2.5) shows that there is a subsequence of $\{\nabla\varphi_k\}$ converging weakly to an element $\nabla\varphi_\epsilon$ in $L^2(G_\epsilon)$. Furthermore, by choosing still another subsequence, we can assume that $\{\zeta_\epsilon\varphi_k\}$ is weakly convergent to $\zeta_\epsilon\varphi_\epsilon$ in $L^2(\Delta_\epsilon)$.

Now, again by Green's theorem, whenever $\psi \in H^1(G_\epsilon)$,

$$((\nabla\psi, \nabla\varphi_k))_{G_\epsilon} + \alpha(\psi, \zeta_\epsilon\varphi_k)_{\Delta_\epsilon} = (\psi, \zeta_\epsilon f_k)_{\Delta_\epsilon}\,.$$

Letting $k \to \infty$ and noting that $f_\epsilon = 0$ on $\Delta_\epsilon \backslash \Gamma_\epsilon$, we obtain (2.3).

From (2.5) we also obtain, by letting $k \to \infty$,

$$(2.6)\qquad \|\nabla\varphi_\epsilon\|^2_{L^2(G_\epsilon)} + \frac{\alpha}{2}\,|\zeta_\epsilon^{1/2}\varphi_\epsilon|^2_{L^2(\Delta_\epsilon)} \leqq c\,|f|^2_{L^2(\Gamma)}\,.$$

Using Lemma 1.6 and the definition of ζ_ϵ we then get

$$\|\varphi_\epsilon\|^2_{H^1(G)} \leqq c\,|f|^2_{L^2(\Gamma)}\,.$$

Finally, from (2.3) it follows that φ_ϵ is harmonic in (2.3). The proof of the lemma is thereby completed.

Theorem 2.1. *Let G be a normal domain and let $f \in L^2(\Gamma)$. Then the problem*

$$(2.7)\qquad \Delta\varphi = 0 \quad \text{in} \quad G,$$

$$(2.8)\qquad \varphi_\nu + \alpha\varphi = f \quad \text{on} \quad \Gamma,$$

$$(2.9)\qquad \frac{\partial\varphi}{\partial\nu} = 0 \quad \text{on} \quad \partial G\backslash\Gamma$$

has a unique solution φ in the sense that φ is harmonic in G, $\varphi \in H^1(G)$, and, for every $\psi \in H^1(G)$,

$$(2.10)\qquad ((\nabla\psi, \nabla\varphi))_G + \alpha(\psi, \varphi)_\Gamma = (\psi, f)_\Gamma\,.$$

Proof. (2.4) implies that there is a sequence ϵ_k approaching zero such that

$$(2.11)\qquad \varphi_{\epsilon_k} \rightharpoonup \varphi \quad \text{in} \quad H^1(G),$$

the half-arrow standing for weak convergence. (2.5) shows that

$$(2.12)\qquad |\varphi_\epsilon|_{L^2(\Delta_\epsilon)} \leqq c\,|f|_{L^2(\Gamma)}$$

for some constant c. Recalling the fact that Γ_ϵ and Γ are diffeomorphic and noting that bounds on the derivatives can be taken to be independent of ϵ, we get

$$|\varphi_\epsilon|_{L^2(\Gamma)} \leqq c\,|f|_{L^2(\Gamma)}$$

with a different c. Hence we can choose a subsequence of $\{\epsilon_k\}$ (which we again denote by $\{\epsilon_k\}$) such that

(2.13) $$\varphi_{\epsilon_k} \rightharpoonup \varphi \quad \text{in} \quad L^2(\Gamma).$$

To prove (2.10), let $\psi \in H^1(G)$ and extend ψ to $\tilde{\psi} \in H^1(G_{\epsilon_0})$ by Lemma 1.2. Then

(2.14) $$\begin{aligned} |((\nabla \tilde{\psi}, \nabla \varphi_{\epsilon_k}))_{G_{\epsilon_k} \backslash G}| &\leqq ||\nabla \tilde{\psi}||_{L^2(G_{\epsilon_k} \backslash G)} \, ||\nabla \varphi_{\epsilon_k}||_{L^2(G_{\epsilon_k})} \\ &\leqq c \, |f|_{L^2(\Gamma)} \, ||\nabla \tilde{\psi}||_{L^2(G_{\epsilon_k} \backslash G)} \to 0 \end{aligned}$$

as $k \to \infty$. Here (2.5) has been used.

A similar argument shows that

(2.15) $$(\tilde{\psi}, \zeta_{\epsilon_k} \varphi_{\epsilon_k})_{\Delta_{\epsilon_k} \backslash \Gamma_{\epsilon_k}} \to 0 \quad \text{as} \quad k \to \infty,$$

provided we have

$$|\tilde{\psi}|_{L^2(\Delta_\epsilon \backslash \Gamma_\epsilon)} \to 0 \quad \text{as} \quad \epsilon \to 0.$$

We therefore have to construct the G_ϵ in Lemma 1.1 in such a way that the last relation is satisfied for every $\tilde{\psi} \in H^1(G_0)$. It suffices to construct the G_ϵ in a way that will enable us to break $\Delta_\epsilon \backslash \Gamma_\epsilon$ into a finite number (independent of ϵ) of portions Δ for each of which, there is a tube beginning on Δ and ending on

$$\Gamma_\epsilon^* = \bigcup_{0<\eta<\epsilon} \partial \Gamma_\eta$$

with the length of the stream lines $\leqq c\epsilon$ and with the property (1.3). For then, we can apply the proof of Lemma 1.5 to deduce

$$|\tilde{\psi}|_{L^2(\Delta_\epsilon \backslash \Gamma_\epsilon)} \leqq c \, |\tilde{\psi}|_{L^2(\Gamma_\epsilon^*)} + c\epsilon \, ||\nabla \tilde{\psi}||_{L^2(G_0)} \to 0 \quad \text{as} \quad \epsilon \to 0.$$

But the construction of the G_ϵ with the above mentioned property is standard, and the details are therefore omitted.

Combining (2.14), (2.15) with (2.11), (2.13) we derive (2.10) from (2.3) by letting $\epsilon = \epsilon_k \to 0$.

We shall now prove more about the solution φ.

Theorem 2.2. *$\varphi \in C^1[G \cup (\partial G \backslash \bar{\Gamma})]$ and* (2.9) *holds in the usual sense. If f is in $C^2(\Gamma)$, then $\varphi \in C^1(\bar{G} \backslash \partial \Gamma)$ and* (2.8) *is satisfied in the usual sense.*

Proof. To prove the first part, let $x^0 \in \partial G \backslash \bar{\Gamma}$ and denote by B_δ the intersection of ∂G with the ball with center x^0 and radius δ. Let D_δ be a smooth domain lying in G such that $\partial G \cap \partial D_\delta = B_\delta$. Denote by N a fundamental solution of Laplace's equation in G satisfying $\partial N / \partial \nu = 0$ on B_δ. In D_δ we can then represent each φ_k appearing in the proof of Lemma 2.1 in the form

(2.16) $$\varphi_k(x) = \int_{B_\delta^0} N(x, \xi) \frac{\partial \varphi^k(\xi)}{\partial \nu} \, dS_\xi + \int_{B_\delta^0} \frac{\partial N(x, \xi)}{\partial \nu} \varphi_k(\xi) \, dS_\xi$$

where $B_\delta^0 = \partial D_\delta \backslash B_\delta$. (Note that the φ^k are smooth in $\bar{B}_\delta$).

Choosing now a family of domains D_δ which vary smoothly and monotonically with δ, and integrating (2.16) with respect to δ, $\delta_1 < \delta < \delta_2$, we get

(2.17) $$\varphi_k(x) = \int_{B^*} \left[N(x, \xi) \sum_i a_i(\xi) \frac{\partial \varphi^k(\xi)}{\partial \xi_i} + \varphi^k(\xi) \sum_i b_i(\xi) \frac{\partial N(x, \xi)}{\partial \xi_i} \right] d\xi$$

where the a_i, b_i are continuous functions and B_* is the domain transversed by the B_δ^0.

Recalling that $\|\varphi_k\|_{H^1(G)} \leqq c$ we conclude from (2.17) that the $\varphi_k(x)$ are as "equi-smooth" in some $\bar{G}$-neighborhood of x^0 as the function $\partial N/\partial\xi$. Since $\partial G \in C^{2+\rho}$ it follows, by known theorems, that $\partial N/\partial \xi$ is continuously differentiable in that $\bar{G}$-neighborhood. Hence, $\{\varphi_k\}$ and $\{\nabla\varphi_k\}$ are equicontinuous. Since $\varphi_k \rightharpoonup \varphi_\epsilon$, $\nabla\varphi_k \rightharpoonup \nabla\varphi$ we have proved:

Lemma 2.2. *The solution φ_ϵ in Lemma 2.1 is continuously differentiable in $G \cup (\partial G\backslash\bar{\Gamma})$, and $\partial\varphi_\epsilon/\partial\nu = 0$ in the usual sense on $\partial G\backslash\bar{\Gamma}$.*

We can now use the same argument as before with φ_k replaced by the φ_{ϵ_k} which appear in the proof of Theorem 2.1, and thus complete the proof of the first part of Theorem 2.2.

We turn to the second part of the theorem. From (2.10) and the fact that

$$((\nabla\psi, \nabla\varphi))_{\Omega_y} \to ((\nabla\psi, \nabla\varphi))_G \quad \text{as} \quad y \to 0$$

we get, by using Green's formula,

$$(\varphi_y\,, \psi)_{\Gamma_y} \to (f, \psi)_\Gamma - \alpha(\varphi, \psi)_\Gamma\,.$$

Noting that $(\varphi, \psi)_{\Gamma_y}$ is continuous in y, we conclude that

$$(\varphi_y + \alpha\varphi, \psi)_{\Gamma_y} \to (f, \psi)_\Gamma \quad \text{as} \quad y \to 0. \tag{2.18}$$

Introduce now a fundamental solution $R(x, \xi)$ of the Laplace equation in a closed cube B_- with edges parallel to the coordinate axes, lying in $\bar{G}$ and having its top on $y = 0$. We take R such that

$$\frac{\partial R}{\partial\nu} + \alpha R = 0 \quad \text{on} \quad B_- \cap \{(x', y); y = 0\}.$$

$R(x', y + \mu; \xi)$ satisfies an analogous condition on $y = -\mu$.

Let $x_0 = (x_0', 0)$ be any point on Γ, and let η be a C^∞ function which is equal to 1 for all x with $|x - x_0| < \delta/2$ and which is equal to 0 for any x with $|x - x_0| > \delta$; δ is a sufficiently small positive number. From Green's formula, applied to φ and ηR we obtain a representation

$$\varphi(x', y) = \int_{\Gamma_{\mu\delta}} \eta(\xi)R(x', y + \mu; \xi)[\varphi_\nu(\xi) + \alpha\varphi(\xi)]\, d\xi' + I_{\mu\delta} \tag{2.19}$$

where $\Gamma_{\mu\delta}$ is the intersection of Γ_μ with the ball B_δ of center x_0 and radius δ and $I_{\mu\delta}$ is an integral of the form

$$I_{\mu\delta} = \int_{B_{\delta\mu}^-} \left[R \sum a_i \frac{\partial\varphi}{\partial\xi_i} + \varphi \sum b_i \frac{\partial R}{\partial\xi_i}\right] d\xi$$

$$(B_{\delta\mu}^- = (B_\delta\backslash B_{\delta/2}) \cap \{(x', y); y < \mu\}).$$

We develop R into a power series:

$$R(x', y + \mu; \xi) = \sum_k R_k(x; \xi)\mu^k. \tag{2.20}$$

By (2.18), for any k,

$$\int_{\Gamma_{\mu\delta}} \eta R_k[\varphi_y + \alpha\varphi]\, d\xi' \to \int_{\Gamma_{0\delta}} \eta R_k f \quad \text{as} \quad \mu \to 0.$$

Setting $\tilde{R}_{m,\mu} = \sum_{k=m}^{\infty} R_k \mu^k$ we also have, for any $|y| < \mu_0 < \mu$,

$$\int_{\Gamma_{\mu\delta}} \varphi_y \eta \tilde{R}_{m,\mu} = \int_{\Gamma_\mu} \varphi_y \eta \tilde{R}_{m,\mu}$$

$$= \int_{\Gamma_{\mu_0}} \varphi_y \eta \tilde{R}_{m,\mu} + \int_{\Omega_\mu \setminus \Omega_{\mu_0}} \nabla\varphi \cdot \nabla(\eta \tilde{R}_{m,\mu})$$

and the right hand side clearly converges to zero as $m \to \infty$. Since $\varphi \in H^1(G)$ we also find that $\int \varphi\eta\tilde{R}_{m,\mu} \to 0$ as $m \to \infty$.

Combining these remarks we find that the first integral on the right hand side of (2.19) converges to

$$\int_{\Gamma_{0\delta}} \eta(\xi', 0) R(x', y; \xi', 0) f(\xi')\, d\xi' \tag{2.21}$$

as $\mu \to 0$. This integral is C^1 in a $\bar{G}$-neighborhood of x_0 if $f \in C^2$. Since, finally, $\varphi \in H^1(G)$, $\lim_{\mu\to 0} I_{\mu\delta}$ is a C^∞ function in some $\bar{G}$-neighborhood of x_0.

Having proved that $\varphi \in C^1(G \cup \Gamma)$, the assertion that (2.8) is satisfied in the usual sense now follows from (2.18).

Remark. If $f \in C^\infty(\Gamma)$ then the above proof shows that $\varphi \in C^\infty(G \cup \Gamma)$.

3. An eigenvalue problem.

Definition. Let $f \in L^2(\Gamma)$ and let φ be the solution of (2.7)–(2.9) whose existence is asserted in Theorem 2.1. By (2.13), $\varphi \in L^2(\Gamma)$. We define an operator T on $L^2(\Gamma)$ by

$$(Tf)(x') = \varphi(x', 0) \quad \text{where} \quad (x', 0) \in \Gamma.$$

Lemma 3.1. T is a bounded, positive and compact operator with zero nullspace.

Corollary. T has an infinite sequence of eigenvalues decreasing to zero. The corresponding eigenfunctions form a complete orthonormal sequence in $L^2(\Gamma)$.

Proof. Let $f \in L^2(\Gamma)$ and let φ be the corresponding solution of (2.7)–(2.9). By (2.11), $\varphi \in H^1(G)$. Setting $\psi = \varphi$ in (2.10) we obtain,

$$(Tf, f)_\Gamma = \|\nabla\varphi\|^2_{L^2(G)} + \alpha\, |Tf|^2_{L^2(\Gamma)} \geqq \alpha\, |Tf|^2_{L^2(\Gamma)}. \tag{3.1}$$

This shows that

$$|Tf|_{L^2(\Gamma)} \leqq \frac{1}{\alpha} |f|_{L^2(\Gamma)}, \tag{3.2}$$

i.e., T is a bounded operator. (3.1) also shows that T is positive. Hence it is also self adjoint.

The nullspace of T is trivial. For, if $Tf = 0$ then (3.1) shows that $\|\nabla\varphi\|_{L^2(G)} = 0$.

Thus, φ is constant. But the restriction of φ to Γ must be zero when $Tf = 0$. Consequently, $\varphi \equiv 0$, and (2.10) shows that f is zero.

Finally the compactness of T follows from Lemma 1.3.

Remark. Lemma 3.1 is essentially known [6], [4]. In the first reference it is assumed that ∂G is smooth, but in the second it is only assumed that ∂G is piecewise smooth, as is our case. The proof of Lemma 3.1 that we have given is more direct and gives additional results (*i.e.*, Theorems 2.1, 2.2). However, what is more important, our proof can easily be generalized to unbounded domains. Such a generalization will be needed in Chapter 2.

Denote the eigenvalues and the eigenfunctions of T by λ_k, f_k respectively. (3.2) shows that $\lambda_k \leqq 1/\alpha$. Moreover, the largest eigenvalue λ_1 equals $1/\alpha$, for the corresponding boundary value problem

$$\Delta\varphi = 0 \quad \text{in} \quad G,$$

$$\varphi_\nu = 0 \quad \text{on} \quad \partial G$$

clearly has a non-zero solution, namely, $\varphi = \text{const.}$

It will be convenient to write

$$\mu_k^2 = \frac{1}{\lambda_k} - \alpha.$$

By definition of T, then, for every μ_k there is a nontrivial solution of the problem

$$\Delta\varphi^k = 0 \quad \text{in} \quad G, \tag{3.3}$$

$$\varphi_\nu^k = \mu_k^2\varphi^k \quad \text{on} \quad \Gamma, \tag{3.4}$$

$$\frac{\partial\varphi^k}{\partial\nu} = 0 \quad \text{on} \quad \partial G\backslash\Gamma \tag{3.5}$$

in the sense that

$$((\nabla\psi, \nabla\varphi^k))_G = \mu_k^2(\psi, \varphi^k)_\Gamma \tag{3.6}$$

for any $\psi \in H^1(G)$.

By Theorem 2.2, φ^k satisfies (3.4) and (3.5) in the usual sense. Note also that $\varphi^k \in C^\infty(G \cup \Gamma)$.

For the sake of reference we sum up:

Theorem 3.1. *The boundary value problem* (3.3)–(3.5) *has a non-trivial solution for an infinite sequence of non-negative values of* μ_k *approaching infinity. These solutions exist in the sense of* (3.6), *they belong to*

$$H^1(G) \cap C^\infty(G \cup \Gamma) \cap C^1(G \cup \partial G\backslash\bar{\Gamma}),$$

and they satisfy (3.4) *and* (3.5). *The restrictions of the solutions to* Γ *are complete in* $L^2(\Gamma)$, *and may be chosen to be orthonormal in* $L^2(\Gamma)$*:*

$$(\varphi^k, \varphi^i)_\Gamma = \delta_{ki}\ ; \tag{3.7}$$

in this case, $\{(1/\mu_k)\nabla\varphi^k\}$ *are orthonormal in* $L^2(G)$*:*

$$(3.8)\qquad \left(\left(\frac{1}{\mu_k}\nabla\varphi^k, \frac{1}{\mu_j}\nabla\varphi^j\right)\right)_G = \delta_{kj}\,.$$

The only fact that still has to be proved is (3.8). But this follows from (3.6), (3.7).

4. Bounded domains. We are finally in position to consider the initial value problem of water waves. Let $\varphi(t)$ be a mapping from the non-negative reals into a Hilbert space H. We write $\varphi \in C^0([0, \infty); H)$ if $\|\varphi(t) - \varphi(s)\|_H \to 0$ as $t \to s$ for all $s \geqq 0$.

If $\varphi(x) = \varphi(x', y)$ is a function in $H^1(G)$, we denote its restriction to Γ by $\varphi^\Gamma(x')$.

We shall now establish the existence of a "weak" solution for the water waves problem.

Theorem 4.1. *Let G be a normal domain. Let $u^0(x') \in H^{1/2}(\Gamma)$, $u_t^0(x') \in L^2(\Gamma)$. Then there exists a unique function $u(x, t)$ $(x \in G,\ t \geqq 0)$ having the following properties:*

(i) $u(\cdot, t) \in C^0([0, \infty); H^1(G))$.

(ii) $u^\Gamma(\cdot, t) \in C^0([0, \infty); L^2(\Gamma))$ *and* $u^\Gamma(x', 0) = u^0(x')$ *almost everywhere on* Γ.

(iii) $u^\Gamma(\cdot, t)$ *has a strong derivative in* $C^0([0, \infty); L^2(\Gamma))$ *and* $u_t^\Gamma(x', 0) = u_t^0(x')$ *almost everywhere on* Γ.

(iv) *For all* $\psi \in H^1(G)$ *independent of* t, $(\psi^\Gamma, u_t^\Gamma)_\Gamma$ *is continuously differentiable in* t *and satisfies*

$$(4.1)\qquad ((\nabla\psi, \nabla u))_G + \frac{d}{dt}(\psi^\Gamma, u_t^\Gamma)_\Gamma = 0 \quad \text{for all} \quad t \geqq 0.$$

(v) $\|\nabla u(\cdot, t)\|_{L^2(G)}^2 + |u_t^\Gamma(\cdot, t)|_{L^2(\Gamma)}^2$ *is continuous in* $t \geqq 0$, *and*

$$(4.2)\qquad \|\nabla u(\cdot, t)\|_{L^2(G)}^2 + |u_t^\Gamma(\cdot, t)|_{L^2(\Gamma)}^2 = \|\nabla u(\cdot, 0)\|_{L^2(G)}^2 + |u_t^0|_{L^2(\Gamma)}^2\,.$$

Proof. In what follows we abbreviate $|\cdot|_{L^2(\Gamma)}$, $\|\cdot\|_{L^2(G)}$ by writing $|\cdot|$ and $\|\cdot\|$, respectively. Similarly for the scalar products.

Let φ^k be the functions whose existence was proved in Theorem 3.1, and define

$$(4.3)\qquad u(x, t) = \sum_{k=1}^{\infty}\left[(u^0, \varphi^k)\cos\mu_k t + \frac{1}{\mu_k}(u_t^0, \varphi^k)\sin\mu_k t\right]\varphi^k(x', y),$$

where, since $\mu_1 = 0$, we define

$$\frac{1}{\mu_1}\sin\mu_1 t = t.$$

By (3.7) the series is convergent in $L^2(\Gamma)$ when $y = 0$.

Let ψ be any function in $H^1(G)$. By (3.8) and Bessel's inequality,

$$(4.4)\qquad \|\nabla\psi\|^2 \geqq \sum_{k=1}^{\infty}\left|\left(\left(\nabla\psi, \frac{1}{\mu_k}\nabla\varphi^k\right)\right)\right|^2 = \sum_{k=1}^{\infty}\mu_k^2\,|(\psi, \varphi^k)|^2,$$

where (3.6) has been used. Now let Φ_m denote the m-th partial sum of (4.3).

If $m_2 > m_1$, we have

$$(4.5)\qquad \|\nabla\Phi_{m_2} - \nabla\Phi_{m_1}\|^2 = \sum_{k=m_1}^{m_2} |\mu_k(u^0, \varphi^k) \cos \mu_k t + (u_t^0, \varphi^k) \sin \mu_k t|^2$$
$$\leqq 2 \sum_{k=m_1}^{\infty} \mu_k^2 |(u^0, \varphi^k)|^2 + 2t \sum_{k=m_1}^{\infty} |(u_t^0, \varphi^k)|^2.$$

The last sum approaches zero, as $m_1 \to \infty$, because $u_t^0 \in L^2(\Gamma)$. Next, according to Lemma 1.4, u^0 has an extension to a function in $H^1(G)$. (4.4) then shows that the first sum on the right hand side of (4.5) approaches zero as $m_1 \to \infty$.

We conclude that $\nabla\Phi_m$ is convergent in $L^2(G)$. Since also $\{\Phi_m^\Gamma\}$ is convergent in $L^2(\Gamma)$, Lemma 1.5 shows that $\{\Phi_m\}$ is convergent in $L^2(G)$. Hence $\{\Phi_m\}$ is convergent in $H^1(G)$. Since finally the convergence is uniform with respect to t, and since each Φ_m is clearly in $C^0([0, \infty); H^1(G))$, it follows that

$$u \in C^0([0, \infty); H^1(G)).$$

This proves (i).

Since $\{\Phi_m^\Gamma(\cdot, t)\}$ converges in $L^2(\Gamma)$ uniformly with respect to t, it also follows that $u^\Gamma(\cdot, t) \in C^0([0, \infty); L^2(\Gamma))$. Also,

$$u^\Gamma(x', 0) = \sum_{k=1}^{\infty} (u^0, \varphi^k)\, \varphi^k(x', 0) = u^0(x')$$

almost everywhere on Γ. This proves (ii).

Consider

$$\frac{d}{dt}\Phi_m^\Gamma(x', t) = \sum_{k=1}^{m} [-\mu_k(u^0, \varphi^k) \sin \mu_k t + (u_t^0, \varphi^k) \cos \mu_k t]\varphi^k(x', 0).$$

A calculation similar to the one made before (in (4.5)) shows that $\{d\Phi_m^\Gamma(\cdot, t)/dt\}$ is convergent in $L^2(\Gamma)$, uniformly in any interval $0 \leqq t \leqq T$. Also

$$\left.\frac{d}{dt}\Phi_m^\Gamma(x', 0)\right|_{t=0} = \sum_{k=1}^{m} (u_t^0, \varphi^k)\varphi^k(x', 0) \to u_t^0(x')$$

almost everywhere in Γ. This proves (iii).

Now let $\psi \in H^1(G)$ be independent of t. Consider

$$\int_0^t ((\nabla\psi, \nabla u))\, dt = \sum_{k=1}^{\infty} [\mu_k(u^0, \varphi^k) \sin \mu_k t + (u_t^0, \varphi^k)(1 - \cos \mu_k t)](\psi, \varphi^k)$$
$$= (\psi^\Gamma, u_t^0 - u_t^\Gamma).$$

Since (ψ^Γ, u_t^0) is independent of t, it follows that $(\psi^\Gamma, u_t^\Gamma)$ is absolutely continuous and

$$(4.6)\qquad ((\nabla\psi, \nabla u)) + \frac{d}{dt}(\psi^\Gamma, u_t^\Gamma) = 0$$

for almost all $t \geqq 0$.

On the other hand, using (4.4), we find that $\{(d/dt)(\psi^\Gamma, \partial\Phi_m^\Gamma/\partial t)\}$ is uniformly convergent to $(d/dt)(\psi^\Gamma, u_t^\Gamma)(d/dt$ is taken as a strong derivative), so that the latter is a continuous function of t. It follows that this (continuous) function coincides with (the pointwise derivative) $(d/dt)(\psi^\Gamma, u_t^\Gamma)$ appearing in (4.6) almost everywhere. Since, finally, $((\nabla\psi, \nabla u))$ is continuous in t, (4.6) holds for all $t \geqq 0$. This proves (4.1).

To prove (v), we only have to prove (4.2). A straightforward calculation shows that (4.2) holds for Φ_m. Now let $m \to \infty$.

To prove uniqueness, suppose $u^0 = u_t^0 = 0$. If we can prove that $\|\nabla u(\cdot, 0)\| = 0$ then (4.2) would show that $\|\nabla u(\cdot, t)\| = 0$. Hence, we should have $u = g(t)$. Since u_t^Γ is also zero, $g(t)$ must be a constant. Since, finally, $u^\Gamma(\cdot, 0) = u^0(\cdot) = 0$, the constant would be zero, so that $u \equiv 0$. Thus, it remains to prove that $\|\nabla u(\cdot, 0)\| = 0$.

Now, by (ii), $u^\Gamma(\cdot, 0) = 0$. Therefore $(u^\Gamma(\cdot, 0), u_t^\Gamma(\cdot, t)) = 0$. Since the derivative of this function is also zero, we get from (4.6) (with $\psi(\cdot) = u(\cdot, 0)$), $\|\nabla u(\cdot, 0)\| = 0$. This completes the proof of the theorem.

We shall now prove that the solution found in Theorem 4.1 has additional properties.

Theorem 4.2. *The function $u(x, t)$ whose existence is asserted in Theorem 4.1 has the following additional properties:*

(vi) *For each $t \geqq 0$, $u(x, t) \in C^\infty(G)$ and $\triangle u = 0$.*

(vii) *For each $t \geqq 0$, $\nabla u(x, t) \in C^0(G \cup \partial G\backslash\bar{\Gamma})$ and $\partial u/\partial\nu = 0$ on $\partial G\backslash\bar{\Gamma}$. Further, $u(x, t)$ and $u_t(x, t)$ are continuous for*

$$(x, t) \in [G \cup (\partial G\backslash\bar{\Gamma})] \times [0, \infty).$$

(viii) *The restrictions of u_t and $(\partial/\partial y) \int_0^t u\, dt$ to Γ are in $L^2(\Gamma)$ and*

$$(u_t)^\Gamma + \left(\frac{\partial}{\partial y}\int_0^t u\, dt\right)^\Gamma = u_t^0 \tag{4.7}$$

almost everywhere on Γ.

(ix) *For every $\psi \in H^1(G)$ independent of t, $\lim_{\nu\nearrow 0} (\psi, u_\nu)_{\Gamma_\nu}$ exists and*

$$\lim_{\nu\nearrow 0} (\psi, u_\nu)_{\Gamma_\nu} + \frac{d}{dt}(\psi^\Gamma, u_t^\Gamma) = 0 \quad \text{for all} \quad t \geqq 0. \tag{4.8}$$

(x) *Suppose $u^0(x')$ has an extension $U^0(x)$ in $H^1(G)$ satisfying*

$$\triangle U^0 = 0 \quad \text{in} \quad G,$$
$$U^0 = u^0 \quad \text{on} \quad \Gamma,$$
$$\frac{\partial U^0}{\partial \nu} = 0 \quad \text{on} \quad \partial G\backslash\bar{\Gamma},$$

with $(U_\nu^0)^\Gamma \in L^2(\Gamma)$, and suppose $u_t^0 \in H^{1/2}(\Gamma)$. Then the restrictions of u_ν and u_{tt} to Γ are in $L^2(\Gamma)$ and (4.5) can be strengthened to read

$$(u_\nu)^\Gamma + (u_{tt})^\Gamma = 0 \tag{4.9}$$

almost everywhere on Γ.

(xi) *$u(x, t) - t(u_t^0, \varphi^1)\varphi^1(x)$ is almost periodic in t. If $\int_\Gamma u_t^0(x')\, dx' = 0$ then $u(x, t)$ is almost periodic in t.*

Proof. (vi) follows from (4.1) by taking ψ to be in $C_0^\infty(G)$.

To prove (vii) we first recall that the assertion is true for each φ^k and therefore for each Φ_m. We now use the proof of the first part of Theorem 2.2 to conclude that $\{\Phi_m\}$, $\{\partial\Phi_m/\partial t\}$ and $\{\nabla\Phi_m\}$ are uniformly convergent in some $\bar{G}$-neighborhood of each point $x_0 \in \partial G\backslash\bar{\Gamma}$. Since the convergence is uniform in any finite t-interval, (vii) follows.

We next prove (viii). We have, from (4.3), that

$$(u_t)^\Gamma = \sum [-\mu_k(u^0, \varphi^k) \sin \mu_k t + (u_t^0, \varphi^k) \cos \mu_k t]\varphi^k(x', 0),$$

$$\left(\frac{\partial}{\partial y}\int_0^t u\, dt\right)^\Gamma = \sum \left[-\frac{1}{\mu_k}(u^0, \varphi^k) \sin \mu_k t + (u_t^0, \varphi^k)\frac{1 - \cos \mu_k t}{\mu_k^2}\right]\varphi_y^k(x', 0)$$

where, to be consistent, we define

$$\frac{1 - \cos \mu_1 t}{\mu_1} = \frac{t^2}{2}.$$

Since (see Theorem 3.1) $\varphi_y^k(x', 0) = \mu_k^2\varphi^k(x', 0)$, (4.7) follows.

To prove (ix), let $\psi \in H^1(G)$ be independent of t. Then,

$$(\psi, u_y)_{\Gamma_y} = ((\nabla\psi, \nabla u))_{\Omega_y} \to ((\nabla\psi, \nabla u))_G \quad \text{as} \quad y \nearrow 0.$$

Because of (4.1), the proof is complete.

We proceed to prove (x). We have:

$$\begin{aligned} u(x, t) &- (u^0, \varphi^1)\varphi^1(x', y) - (u_t^0, \varphi^1)\frac{\sin \mu_1 t}{\mu_1}\varphi^1(x', y) \\ &= \sum_{k=2}^\infty \left[(u^0, \varphi^k) \cos \mu_k t + \frac{1}{\mu_k}(u_t^0, \varphi^k) \sin \mu_k t\right]\varphi^k(x', y) \\ &= \sum_{k=2}^\infty \left[\frac{1}{\mu_k^2}(u^0, \varphi_y^k) \cos \mu_k t + \frac{1}{\mu_k}(u_t^0, \varphi^k) \sin \mu_k t\right]\varphi^k(x', y) \\ &= \sum_{k=2}^\infty \left[\frac{1}{\mu_k^2}(U_y^0, \varphi^k) \cos \mu_k t + \frac{1}{\mu_k}(u_t^0, \varphi^k) \sin \mu_k t\right]\varphi^k(x', y). \end{aligned}$$

Thus we have gained a factor $1/\mu_k$ over what we had in Theorem 4.1. This allows us to prove (x) by repeating some of the calculations used in the proof of Theorem 4.1.

Finally, the first part of (xi) is obvious. The second part follows from the fact that the eigenfunction φ^1 is a constant.

5. Unbounded domains. Let G be an unbounded normal domain (see §1) with $G_j \nearrow G$, G_j normal, and denote by Γ_j the intersection of Ω_j with $y = 0$. Take $u^0 \in H^{1/2}(\Gamma)$ and $u_t^0 \in L^2(\Gamma)$. Then for each j there exists the solution $u_j(x, t)$ ($x \in G_j$, $t \geqq 0$) asserted in Theorem 4.1, and

$$(5.1) \quad \|\nabla u_j(\cdot, t)\|_{L^2(G_j)}^2 + \left|\frac{\partial}{\partial t}u_j^{\Gamma_j}(\cdot, t)\right|_{L^2(\Gamma_j)}^2 = \|\nabla u_j(\cdot, 0)\|_{L^2(G_j)}^2 + |u_t^0|_{L^2(\Gamma_j)}^2.$$

By (4.3),

$$u_j(\cdot, 0) = \sum (u^0, \varphi_j^k)_{\Gamma_j}\varphi_j^k.$$

Since the proof of the first part of Lemma 1.4 clearly extends to G unbounded, there is an extension U^0 in $H^1(G)$ of $u^0(x')$. Then, by (4.5),

$$(4.4), \qquad \|\nabla u_j(\cdot, 0)\|_{G_j}^2 = \sum_k \mu_{k,j}^2 \, |(u^0, \varphi_j^k)_{\Gamma_j}|^2 \leqq \|\nabla U^0\|_{L^2(G_j)}^2 .$$

Substituting this into (5.1), we obtain the inequality

$$(5.2) \qquad \|\nabla u_j(\cdot, t)\|_{L^2(G_j)}^2 + \left|\frac{\partial}{\partial t} u_j^{\Gamma_j}(\cdot, t)\right|_{L^2(\Gamma_j)}^2 \leqq \|\nabla U^0\|_{L^2(G)}^2 + |u_t^0|_{L^2(\Gamma)}^2 .$$

Noting (say, by (4.3)) that

$$|u_j^{\Gamma_j}(\cdot, t)|_{L^2(\Gamma_j)} \leqq \text{const.}$$

and using Lemma 1.7, we deduce from (5.2) that

$$(5.3) \qquad \|u_j(\cdot, t)\|_{H^1(G_j)} \leqq c.$$

Consider the functions

$$\chi_j(x, t) = \int_0^t u_j(x, \tau)\, d\tau.$$

From (5.3) it follows that for any $T > 0$,

$$(5.4) \qquad \|\chi_j(\cdot, t)\|_{H^1(G_j)} \leqq c \qquad (0 \leqq t \leqq T),$$

$$(5.5) \qquad \|\chi_j(\cdot, t) - \chi_j(\cdot, t')\|_{H^1(G_j)} \leqq c\, |t - t'| \qquad (0 \leqq t, t' \leqq T)$$

where c is a constant independent of j.

We can represent each χ_j in any compact subset M of $G_j \cup (\partial G_j \backslash \bar{\Gamma}_j)$ in terms of a fundamental solution, by a formula analogous to (2.17). If we then use (5.4), (5.5) we find that the $\chi_j(x, t)$ form an equicontinuous and uniformly bounded family of continuous functions in $M_0 \times [0, T]$ where M_0 is any compact subset of M with $\partial M_0 \backslash \partial G$ having a positive distance to $\partial M \backslash \partial G$. The same is true of $\nabla \chi_j(x, t)$.

We can therefore cover $[G \cup (\partial G \backslash \Gamma)] \times [0, \infty)$ by a countable number of sets of the form $M_0 \times [0, T)$ and thus conclude that there exists a subsequence of the χ_j which is convergent together with $\nabla \chi_j$ uniformly in compact subsets of $[G \cup (\partial G \backslash \Gamma)] \times [0, \infty)$. Denote this subsequence again by χ_j, and its limit by χ.

From (5.4), (5.5) it follows that $\chi \in C^\infty([0, \infty); H^1(G))$.

Next, by (4.1) for u_j, G_j:

$$((\nabla \psi, \nabla u_j))_{L^2(G_j)} + \frac{d}{dt}\left(\psi^{\Gamma_j}, \frac{\partial}{\partial t} u_j^{\Gamma_j}\right)_{\Gamma_j} = 0 \quad \text{for any} \quad \psi \in H^1(G).$$

Integrating this relation we get

$$(5.6) \qquad ((\nabla \psi, \nabla \chi_j))_{G_j} + \left(\psi^{\Gamma_j}, \frac{\partial}{\partial t} u_j^{\Gamma_j}\right)_{\Gamma_j} = (\psi^{\Gamma_j}, u_t^0)_{\Gamma_j} .$$

It follows that

$$\lim_{j\to\infty} \left(\psi^{\Gamma_j}, \frac{\partial^2}{\partial t^2} \chi_j^{\Gamma_j}\right)_{\Gamma_j}$$

exists. Since, by (5.2),

$$\left|\frac{\partial^2}{\partial t^2}\chi_j^{\Gamma_j}\right|_{L^2(\Gamma_j)} \leqq c,$$

it follows that $\{\partial^2\chi_j^{\Gamma_j}/\partial t^2\}$ converges weakly to some function $g_2(x', t)$ ($g_2(\cdot, t) \varepsilon L^2(\Gamma)$) in $L^2(\Gamma_0)$ for every compact subset Γ_0 of Γ.

Integrating the relation (5.6) once with respect to t and arguing in a similar manner, we find that $\{\partial\chi_j^{\Gamma_j}/\partial t\}$ converges weakly to some function $g_1(x', t)$ ($g_1(\cdot, t) \varepsilon L^2(\Gamma)$) in $L^2(\Gamma_0)$ for every compact subset Γ_0 of Γ. By still another integration we find that $\{\chi_j^{\Gamma_j}\}$ is weakly convergent to a function $g_0(x', t)$ ($g_0(\cdot, t) \varepsilon L^2(\Gamma)$) in $L^2(\Gamma_0)$ for any Γ_0 as before.

Consequently, if we define $\chi^\Gamma(\cdot, t)$ as an operator from $t \varepsilon [0, \infty)$ into $L^2(\Gamma)$ given by $g_0(\cdot, t)$, then $\chi^\Gamma(\cdot, t)$, as a functional on $L^2(\Gamma)$, has two derivatives given by $\partial\chi^\Gamma/\partial t = g_1$, $\partial^2\chi^\Gamma/\partial t^2 = g_2$.

Taking $j \to \infty$ in (5.6) we obtain:

$$((\nabla\psi, \nabla\chi))_G + \left(\psi^\Gamma, \frac{\partial^2}{\partial t^2}\chi^\Gamma\right)_\Gamma = (\psi^\Gamma, u_t^0)_\Gamma . \tag{5.7}$$

If we integrate (5.6) with respect to t and then let $j \to \infty$, we get $(\psi^\Gamma, \chi^\Gamma(\cdot, 0) - u^0)_\Gamma = 0$. Hence $\chi^\Gamma(\cdot, 0) = u^0$ as elements in $L^2(\Gamma)$.

Finally, letting $j \to \infty$ in (5.2). we find

$$||\nabla\chi_t(\cdot, t)||^2_{L^2(G)} + |\chi^\Gamma_{tt}(\cdot, t)|^2_{L^2(\Gamma)} \leqq ||\nabla U^0||^2_{L^2(G)} + |u_t^0|_{L^2(\Gamma)} . \tag{5.8}$$

We sum up:

Theorem 5.1. *Let G be an unbounded normal domain. Let $u^0 \varepsilon H^{1/2}(\Gamma)$, $u_t^0 \varepsilon L^2(\Gamma)$. Then there exists a unique function $\chi(x, t)$ having the following properties:*

(i) $\chi(\cdot, t) \varepsilon C^0([0, \infty); H^1(G))$.

(ii) *$\chi^\Gamma(\cdot, t)$ exists in the weak sense (of functionals over $L^2(\Gamma)$) described above and its first two weak t-derivatives also exist.*

(iii) $\chi^\Gamma(\cdot, 0) = 0$, $\chi^\Gamma_t(\cdot, 0) = u^0(\cdot)$.

(iv) *χ satisfies (5.7) for any $\psi \varepsilon H^1(G)$.*

(v) *$\chi(x, t)$ and $\nabla\chi(x, t)$ are continuous functions for $x \varepsilon G \cup (\partial G\backslash\Gamma)$, $t \geqq 0$, and $\partial\chi(x, t)/\partial\nu = 0$ on $\partial G\backslash\Gamma$.*

(vi) *For $x \varepsilon G$, $t \geqq 0$, $\triangle\chi = 0$.*

(vii) *For any $\psi \varepsilon H^1(G)$ with compact support in $G \cup \Gamma$,*

$$\lim_{y\nearrow 0} (\psi, \chi_\nu)_{\Gamma_y} + \left(\psi^\Gamma, \frac{\partial^2}{\partial t^2}\chi^\Gamma\right)_\Gamma = (\psi^\Gamma, u_t^0)_\Gamma . \tag{5.9}$$

(viii) (5.8) *is satisfied.*

The relation (5.9) follows from (5.7) by noting that

$$((\nabla\psi, \nabla\chi))_G = \lim_{y\nearrow 0} ((\nabla\psi, \nabla\chi))_{G_y} = \lim_{y\nearrow 0} (\psi, \chi_\nu)_{\Gamma_y}$$

where G_{y_0} is the intersection of G with $y < y_0$.

It remains to prove uniqueness. Suppose then that $u^0 = u_t^0 = 0$. We may use (5.8) with $U^0 = 0$. This gives

$$||\nabla\chi(\cdot, t)||_{L^2(G)} = 0.$$

Hence $\chi(x, t) = f(t)$. But $\chi \in H^1(G)$ so that, in particular, $\chi \in L^2(G)$. Since G is unbounded it follows that $f(t) = 0$. This completes the proof of uniqueness.

Remark. Let $\zeta(x')$ be any continuous function satisfying $\alpha \leqq \zeta(x') \leqq \beta$ where α, β are positive constants. The results of §§4, 5 can be extended to the situation where the condition on Γ is:

$$u_\nu + \zeta(x')u_{tt} = 0.$$

One has to replace (2.8) by

$$\varphi_\nu + \alpha\zeta\varphi = \zeta f.$$

T is then defined as before, and the φ^k in Theorem 3.1 satisfy, instead of (3.4):

$$\varphi_\nu^k = \mu_k^2\zeta(x')\varphi^k \quad \text{on} \quad \Gamma.$$

Chapter 2: Existence Theorems (Second Method)

In this chapter, we again consider the linearized equations of water waves and describe a very natural method that makes clear the sense in which the solutions of (0.1)–(0.4) are waves at all. The method has the additional virtue that it applies equally well to bounded and unbounded domains. If the domain is bounded, however, the results are contained in those of chapter 1.

1. Reduction to a wave equation. Let G be an unbounded normal domain (see §1.1). Let Γ denote the free surface and let Γ' denote the rest of the boundary of G. By hypothesis, if R is sufficiently large, the intersection of the cylinder $|x'| > R$ with Γ' is a surface with the equation $y = -h(x')$, with $h \in C^{2+\rho}$ $(0 < \rho < 1)$. We assume that there are positive constants h_0 and h_1 such that $h_0 \leqq h(x') \leqq h_1$. As usual, the problem we consider is

(1.1) $$\Delta u = 0 \quad \text{in} \quad G,$$

(1.2) $$u_\nu = 0 \quad \text{on} \quad \Gamma',$$

(1.3) $$u_\nu + u_{tt} = 0 \quad \text{on} \quad \Gamma,$$

(1.4) $$u|_{t=0} = u^0, \qquad u_t|_{t=0} = u_t^0 \quad \text{on} \quad \Gamma.$$

As in chapter 1, we consider an auxiliary problem: given $f \in C_0^\infty(\Gamma)$, find a function φ satisfying the equations

(1.5) $$\Delta\varphi = 0 \quad \text{in} \quad G,$$

(1.6) $$\varphi_\nu + \alpha\varphi = f \quad \text{on} \quad \Gamma \qquad (\alpha > 0),$$

(1.7) $$\varphi_\nu = 0 \quad \text{on} \quad \Gamma'.$$

φ can be constructed by first solving the analogous problems in a sequence of appropriately defined domains G_j converging to G. A convenient choice for the domains G_j is obtained by rounding off the corners of the domains G'_j defined by

$$G'_j = G \cap \{(x', y): |x'| < j\}.$$

The existence of a solution φ_j of the problem (1.5)–(1.7) with G replaced by G_j follows by the method of chapter 1, §2. We note that each φ_j satisfies the inequality

$$||\nabla\varphi_j||_{L^2(G)} + ||\varphi_j||_{L^2(G)} + ||\varphi_j||_{L^2(\Gamma)} \leqq c\, ||f||_{L^2(\Gamma)}\,,$$

where c is a positive constant. Because of this inequality, a compactness argument following the lines of chapter 1, §2 shows that subsequence of $\{\varphi_j\}$ has a weak limit, φ say, satisfying

$$||\nabla\varphi||_{L^2(G)} + ||\varphi||_{L^2(G)} + ||\varphi||_{L^2(\Gamma)} \leqq c\, ||f||_{L^2(\Gamma)} \tag{1.8}$$

and being, therefore, independent of the sequence $\{G_j\}$. φ is smooth in G.

We define an operator T from $C_0^\infty(\Gamma)$ into $L^2(\Gamma)$ by the equation

$$(Tf)(x') = \varphi(x', 0). \tag{1.9}$$

It follows from (1.8) that T is a bounded operator. Hence, it can be uniquely extended by continuity to bounded operator on all of $L^2(\Gamma)$. We again denote this extension by T.

For $f \in C_0^\infty(\Gamma)$,

$$\begin{aligned} ||Tf||_{L^2(\Gamma)}\, ||f||_{L^2(\Gamma)} &\geqq \int_\Gamma Tf\cdot \bar{f}\, dx' \\ &= \int_\Gamma \varphi(\bar{\varphi}_\nu + \alpha\bar{\varphi})\, dx' \\ &= \int_G |\nabla\varphi|^2\, dx + \alpha \int_\Gamma |\varphi|^2\, dx' \\ &\geqq \alpha\, ||\varphi||^2_{L^2(\Gamma)} \\ &= \alpha\, ||Tf||^2_{L^2(\Gamma)}\,, \end{aligned} \tag{1.10}$$

provided the second equality can be verified. To do so, denote by Γ_r and G_r, respectively, the intersection of Γ and G with $|x'| < r$. Then,

$$\int_{\Gamma_r} \varphi\bar{\varphi}_\nu\, dx' = \int_{G_r} |\nabla\varphi|^2\, dx - \int_{S_r} \varphi\bar{\varphi}_\nu\, dS, \tag{1.11}$$

where S_r is the intersection of G with $|x'| = r$. In view of (1.8), the last term in (1.11) is in $L^2(0, \infty)$ as a function of r. Hence, for some sequence $\{r_i\}$, diverging to infinity,

$$\int_{S_{r_i}} \varphi \bar{\varphi}_\nu \, dS \to 0.$$

Now, take $r = r_i$ in (1.11) and let $i \to \infty$.

From (1.10), we conclude that

$$\|Tf\| \leqq \frac{1}{\alpha} \|f\|, \tag{1.12}$$

$$(Tf, f)_{L^2(\Gamma)} \geqq 0, \tag{1.13}$$

$$\int_G |\nabla \varphi|^2 \, dx + \alpha \int_\Gamma |\varphi|^2 \, dx' = (Tf, f)_{L^2(\Gamma)} . \tag{1.14}$$

By extension, these relations remain true for any $f \in L^2(\Gamma)$. Hence, T is a bounded, self-adjoint operator, its spectrum lies in the interval $[0, 1/\alpha]$, and its nullspace is zero. (The proof of this last assertion is the same as in chapter 1.)

Definition. Let $u(x, t)$ be a function lying in $H^1(G)$ for each fixed $t \geqq 0$. Let u^Γ denote the restriction of u to Γ. We assume that $u^\Gamma \in C^0([0, \infty); L^2(\Gamma))$ and that the strong derivatives u_t^Γ and u_{tt}^Γ exist and belong to $L^2(\Gamma)$ for all $t \geqq 0$. Let $((\ ,\ ))_G$ denote the scalar product in $[L^2(G)]^n$. We say that a function u with the above properties is a *strong solution* of (1.1)–(1.4) if for every $\psi \in H^1(G)$,

$$((\nabla \psi, \nabla u))_G + (\psi^\Gamma, u_{tt}^\Gamma)_\Gamma = 0, \tag{1.15}$$

$$u^\Gamma(x', 0) = u^0(x') \text{ a.e. on } \Gamma, \tag{1.16}$$

$$u_t^\Gamma(x', 0) = u_t^0(x') \text{ a.e. on } \Gamma. \tag{1.17}$$

From the definition of T, it is clear (*cf.* Theorem 1.2.1) that if $f \in L^2(\Gamma)$, then $(Tf)(x') = \varphi(x', 0)$, where $\varphi(x', y)$ is the unique function in $H^1(G)$ satisfying

$$((\nabla \psi, \nabla \varphi))_G + \alpha(\psi^\Gamma, \varphi^\Gamma)_\Gamma = (\psi^\Gamma, f)_\Gamma$$

for every $\psi \in H^1(G)$. Hence, if u is a strong solution of (1.1)–(1.4), $T(u_{tt}^\Gamma - \alpha u^\Gamma) + u^\Gamma = 0$. As we have seen, T is one-to-one. Thus, setting $A = T^{-1}$, we find that u^Γ satisfies the abstract wave equation

$$u_{tt}^\Gamma + (A - \alpha I)u^\Gamma = 0 \tag{1.18}$$

with the initial conditions

$$u^\Gamma(0) = u^0, \qquad u_t^\Gamma(0) = u_t^0 . \tag{1.19}$$

We shall now prove the converse. Let $u^\Gamma(t)$ be a function with values in $L^2(\Gamma)$. Assume, moreover, that u^Γ is in the domain $D(A)$ of A and that (1.18) and (1.19) hold. For fixed t, let u denote the solution of the problem

$$\Delta u = 0 \quad \text{in} \quad G, \tag{1.20}$$

$$u_\nu = 0 \quad \text{on} \quad \Gamma', \tag{1.21}$$

$$u_\nu + \alpha u = Au^\Gamma \quad \text{on} \quad \Gamma. \tag{1.22}$$

By definition of T,

$$u|_{y=0} = TAu^\Gamma = u^\Gamma.$$

Thus, u^Γ is the restriction of u to Γ. Moreover, as we have seen, the problem (1.20)–(1.22) has a solution in $H^1(G)$ such that for every $\psi \in H^1(G)$,

$$\begin{aligned}((\psi, u))_G &= (\psi^\Gamma, u_\nu^\Gamma)_\Gamma \\ &= (\psi^\Gamma, Au^\Gamma - \alpha u^\Gamma)_\Gamma \\ &= -(\psi^\Gamma, u_{tt}^\Gamma)_\Gamma ,\end{aligned}$$

by (1.18). Thus, u is a strong solution of (1.1)–(1.4).

We have proved

Theorem 1.1. If $u(x, t)$ is a strong solution of (1.1)–(1.4), *its restriction to Γ is a solution of* (1.18), (1.19). *Conversely, if $u^\Gamma(\cdot, t)$ is a solution of* (1.18), (1.19) *with values in $L^2(\Gamma)$ and having two strong derivatives in $L^2(\Gamma)$, then u^Γ is the restriction to Γ of a strong solution of* (1.1)–(1.4).

2. The existence theorem. In order to solve (1.18), (1.19), we shall require that the initial value, u^0, lies in the domain of the unbounded operator $A = T^{-1}$. Before going on to state the existence theorem, we should like to point out that the condition $u^0 \in D(A)$ is exactly the same as the hypothesis of Theorem 1.4.2(x). Specifically, let U^0 denote the solution of the following problem:

$$\begin{aligned}\triangle U^0 &= 0 \quad \text{in } G, \\ U^0 &= u^0 \quad \text{on } \Gamma, \\ \frac{\partial U^0}{\partial \nu} &= 0 \quad \text{on } \Gamma'.\end{aligned}$$

Then, $u^0 \in D(A)$ if and only if $u^0 \in L^2(\Gamma)$ and the restriction of $\partial U^0/\partial y$ to Γ is in $L^2(\Gamma)$. Indeed, in this case, Au^0 is exactly the restriction of $\partial U^0/\partial y + \alpha U^0$ to Γ.

We shall also require that $u_t^0 \in D(A^{1/2})$. Unfortunately, there does not seem to be an explicit way of characterizing the set $D(A^{1/2})$. However, it is obviously true that $D(A^{1/2}) \supset D(A)$. Thus, if the condition $u_t^0 \in D(A^{1/2})$ in the theorem below is replaced by the more definite condition $u_t^0 \in D(A)$, the theorem will surely remain true.

Theorem 2.1. Let $u^0 \in D(A)$, $u_t^0 \in D(A^{1/2})$. Then, there exists a strong solution of (1.1)–(1.4).

Proof. The operator A is, by definition, the inverse of the bounded, self-adjoint operator T. Thus, A is self-adjoint. Let $\{F_\lambda\}$ denote the spectral family of A, and define

$$u^\Gamma(t) = \int_\alpha^\infty \cos(t\sqrt{\lambda - \alpha})\, dF_\lambda u^0 + \int_\alpha^\infty \frac{\sin(t\sqrt{\lambda - \alpha})}{\sqrt{\lambda - \alpha}}\, dF_\lambda u_t^0 .$$

(The fact that a solution of (1.18), (1.19) could be defined in this way was pointed out to us by S. Kaniel.) It is a matter of straightforward computation to show that the function so defined is a solution of (1.18), (1.19) with two strong t-derivatives in $L^2(\Gamma)$. The theorem now follows immediately from Theorem 1.1.

Remark. The method of proof of Theorem 2.1 is not based on any *a priori* inequality, such as the energy inequality needed for Theorem 1.5.1. Consequently, Theorem 2.1 can be extended without difficulty to more general problems where the conditions on Γ are replaced by

$$u_y + p\left(\frac{\partial}{\partial t}\right)u = 0,$$

$$\left.\frac{\partial^j u}{\partial t^j}\right|_{t=0} = u_j^0, \qquad j = 0, 1, \cdots, s - 1,$$

where $p(\lambda)$ is a polynomial of degree s whose zeros all have non-positive real parts.

Chapter 3: Existence Theorems (Third Method)

1. Auxiliary construction of a Green's function. Consider an equation

$$\Delta u(x) + a(x)\cdot\nabla' u(x) + b(x)u(x) = 0 \tag{1.1}$$

where

$$\Delta = \sum_{i=1}^{n-1} \partial^2/\partial x_i^2 + \partial^2/\partial y^2, \qquad \nabla' = (\partial/\partial x_1, \cdots, \partial/\partial x_{n-1}),$$

$$x = (x', y) = (x_1, \cdots, x_{n-1}, y), \qquad a = (a_1, \cdots, a_{n-1}).$$

We take $n \geqq 3$, but all the results of this section extend, with obvious modifications, to the case $n = 2$. We denote by S the strip $\{(x', y); x' \varepsilon R^{n-1}, -H < y < 0\}$ where H is a finite number and assume, throughout this section, that $a \varepsilon C^1(\bar{S})$, $b \varepsilon C^\rho(\bar{S})$ for some $\rho > 0$ and, furthermore,

$$|a(x)| \leqq \frac{c_0(1 + |x'|)\epsilon}{1 + \epsilon\, |x'|^2}, \qquad |\nabla' a(x)| + |b(x)| \leqq \frac{c_0\epsilon}{1 + \epsilon\, |x'|^2} \tag{1.2}$$

where c_0 is a fixed constant and ϵ will be taken (later on) to be sufficiently small (depending only on c_0, H).

By an *upper Green's function* we mean a function $G(x; \xi)$ which satisfies all the properties of a Green's function of (1.1) in S except for the boundary condition $G(x', -H; \xi) = 0$ at the bottom of the strip.

The purpose of this section is to construct such a function G and derive bounds on its derivatives.

Let G_0 be the Green function for the Laplacian in the half space $y < 0$, *i.e.*,

$$G_0(x;\xi) \equiv G_0(x' - \xi', y, \eta)$$
$$= \gamma_0\{[|x' - \xi'|^2 + (y - \eta)^2]^{(2-n)/2} - [|x' - \xi'|^2 + (y + \eta)^2]^{(2-n)/2}\}$$

for some $\gamma_0 > 0$. Since $\triangle(G - G_0) = -a \cdot \nabla' G - bG$, we try to construct G in the form

$$G(x;\xi) = G_0(x;\xi) + \int_S G_0(x;\zeta)[a(\zeta)\cdot\nabla' G(\zeta;\xi) + b(\zeta)G(\zeta;\xi)]\, d\zeta. \tag{1.3}$$

Integrating by parts, formally, and using (1.2), we get

$$G(x;\xi) = G_0(x;\xi) + \int_S K_0(x;\zeta)G(\zeta;\xi)\, d\zeta \tag{1.4}$$

where

$$K_0(x;\zeta) = G_0(x;\zeta)\beta(\zeta) + \nabla' G_0(x;\zeta)\cdot\alpha(\zeta) \tag{1.5}$$

and

$$|\beta(\zeta)| \leqq \frac{c\epsilon}{1 + \epsilon\,|\zeta'|^2}, \qquad |\alpha(\zeta)| \leqq \frac{c(1 + |\zeta'|)\epsilon}{1 + \epsilon\,|\zeta'|^2}; \tag{1.6}$$

c is a constant.

In what follows, various positive constants depending only on c_0, H will be denoted by c.

From (1.6) we get the inequalities:

$$|\beta(\zeta)| \leqq \frac{c\epsilon^{1/2}}{|\zeta'|}, \qquad |\alpha(\zeta)| \leqq \frac{c\epsilon^{1/4}}{|\zeta'|^{1/2}}, \tag{1.7}$$

$$|\beta(\zeta)| \leqq c\epsilon, \qquad |\alpha(\zeta)| \leqq c\epsilon^{1/4}. \tag{1.8}$$

(Actually, we have $|\alpha(\zeta)| \leqq c\epsilon^{1/2}$, but it will be simpler to work with the weaker inequality $|\alpha(\zeta)| \leqq c\epsilon^{1/4}$.)

The following easily verified bounds on G_0 and on its derivatives will be needed later on:

$$|G_0(x;\xi)| \leqq \begin{cases} c\,|x - \xi|^{2-n} & \text{if } |x - \xi| \leqq 1, \\ c\,|x - \xi|^{1-n} & \text{if } |x - \xi| \geqq 1, \end{cases} \tag{1.9}$$

$$|\nabla' G_0(x;\xi)| \leqq \begin{cases} c\,|x - \xi|^{1-n} & \text{if } |x - \xi| \leqq 1, \\ c\,|x - \xi|^{-n} & \text{if } |x - \xi| \geqq 1, \end{cases} \tag{1.10}$$

$$|(\nabla')^2 G_0(x;\xi)| \leqq \begin{cases} c\,|x - \xi|^{-n} & \text{if } |x - \xi| \leqq 1, \\ c\,|x - \xi|^{-1-n} & \text{if } |x - \xi| \geqq 1. \end{cases} \tag{1.11}$$

Theorem 1.1. *If ϵ is sufficiently small (depending only on c_0, H) then there exists a unique solution of (1.4) and it is an upper Green's function. Furthermore,*

$$(1.12)\qquad |G(x;\xi)| \leqq \begin{cases} c\,|x-\xi|^{2-n} & \text{if} \quad |x-\xi| \leqq 1, \\ c\,|x-\xi|^{1-n} & \text{if} \quad |x-\xi| \geqq 1, \end{cases}$$

$$(1.13)\qquad |\nabla' G(x;\xi)| \leqq \begin{cases} c\,|x-\xi|^{1-n} & \text{if} \quad |x-\xi| \leqq 1, \\ c\,|x-\xi|^{2-n} & \text{if} \quad |x-\xi| \geqq 1. \end{cases}$$

Proof. We want to solve (1.4) by iteration, *i.e.*, in the form

$$(1.14)\qquad G(x;\xi) = G_0(x;\xi) + \sum_{m=0}^{\infty} \int_S K_m(x;\zeta)G_0(\zeta;\xi)\,d\zeta$$

where

$$(1.15)\qquad K_{m+1}(x;\xi) = \int_S K_0(x;\zeta)K_m(\zeta;\xi)\,d\zeta.$$

Setting

$$\gamma(\xi) = |\beta(\xi)| + |\alpha(\xi)|$$

we shall prove by induction that, for $m \geqq 1$,

$$(1.16)\qquad |K_m(x;\xi)| \leqq \begin{cases} d^{m+1}\epsilon^{m/4}\gamma(\xi)\,|x-\xi|^{2-n} & \text{if} \quad |x-\xi| \leqq 1, \\ d^{m+1}\epsilon^{m/4}\gamma(\xi)\,|x-\xi|^{1-n} & \text{if} \quad |x-\xi| \geqq 1, \end{cases}$$

where d is some constant depending only on c_0, H. For $m = 0$ (1.16) is false, but we have

$$(1.17)\qquad |K_0(x;\xi)| \leqq \begin{cases} c\gamma(\xi)\,|x-\xi|^{1-n} & \text{if} \quad |x-\xi| \leqq 1, \\ c\gamma(\xi)\,|x-\xi|^{1-n} & \text{if} \quad |x-\xi| \geqq 1. \end{cases}$$

We assume that (1.16) holds for some $m \geqq 1$, and prove it for $m + 1$; the proof for the case $m = 1$ is only slightly different from the proof from m to $m + 1$ and will therefore be omitted.

Consider first the case where $|x - \xi| \geqq 1$. Observing that if $|\zeta| > H + 1$ then $(1/|\zeta'|) \leqq c(1/|\zeta|)$ and using (1.7), (1.16) and (1.17), we get

$$(1.18)\qquad d^{-m-1}\,|K_{m+1}(x;\xi)|$$

$$\leqq \int_{S_1} \left[\frac{c\epsilon^{1/2}}{|x-\zeta|^{n-1}\,|\zeta|} + \frac{c\epsilon^{1/4}}{|x-\zeta|^{n}\,|\zeta|^{1/2}}\right] \frac{c\epsilon^{m/4}}{|\zeta-\xi|^{n-1}}\,\gamma(\xi)\,d\zeta$$

$$+ \int_{S_2} \frac{c\epsilon^{1/4}}{|x-\zeta|^{n-1}}\,\frac{c\epsilon^{m/4}}{|\zeta-\xi|^{n-2}}\,\gamma(\xi)\,d\zeta + \int_{S_3} \frac{c\epsilon}{|x-\zeta|^{n-1}}\,\frac{c\epsilon^{m/4}}{|\zeta-\xi|^{n-1}}\,\gamma(\xi)\,d\zeta$$

$$+ \int_{S_4} \frac{c\epsilon^{1/4}}{|x-\zeta|^{n-1}}\,\frac{c\epsilon^{m/4}}{|\zeta-\xi|^{n-2}}\,\gamma(\xi)\,d\zeta + \int_{S_5} = I_1 + I_2 + I_3 + I_4 + I_5$$

where

$$S_1 = S \cap \{\zeta;\, |x - \zeta| \geqq 1,\, |\xi - \zeta| \geqq 1,\, |\zeta| \geqq H + 1\},$$
$$S_2 = S \cap \{\zeta;\, |x - \zeta| \geqq 1,\, |\xi - \zeta| \leqq 1,\, |\zeta| \geqq H + 1\},$$
$$S_3 = S \cap \{\zeta;\, |x - \zeta| \leqq 1,\, |\xi - \zeta| \leqq 1,\, |\zeta| \geqq H + 1\},$$
$$S_4 = S \cap \{\zeta;\, |x - \zeta| \leqq 1,\, |\xi - \zeta| \leqq 1,\, |\zeta| \geqq H + 1\}$$

and

$$S_5 = S \cap \{\zeta;\, |\zeta| \leqq H + 1\}.$$

To evaluate I_1, consider the integral

$$J_1 = \int_{S_1} \frac{1}{|x - \zeta|^{n-1}\, |\zeta|\, |\zeta - \xi|^{n-1}}\, d\zeta = \int_{A'} + \int_{B'}$$

where $A' = S_1 \cap A$, $B' = S_1 \cap B$ and A, B are the complementary half spaces ($x \in A$, $\xi \in B$) whose common hyperplane is orthogonal to the segment $\overline{x\xi}$ at its midpoint. Clearly

$$\int_{A'} \leqq \frac{c}{|x - \xi|^{n-1}} \int_{A'} \frac{1}{|x - \zeta|^{n-1}\, |\zeta|}\, d\zeta \leqq \frac{c}{|x - \xi|^{n-1}} \tag{1.19}$$

where the last inequality is obtained by noting that the integrand is $\leqq |x - \zeta|^{-n} + |\zeta|^{-n}$ and recalling that $|x - \zeta| \geqq 1$, $|\zeta| \geqq 1$, $A' \subset S$. Since $\int_{B'}$ is estimated in the same way, we obtain

$$J_1 \leqq \frac{c}{|x - \xi|^{n-1}}.$$

Similarly we obtain:

$$\int_{S_1} \frac{1}{|x - \zeta|^{n}\, |\zeta|^{1/2}\, |\zeta - \xi|^{n-1}}\, d\zeta \leqq \frac{c}{|x - \xi|^{n-1}}.$$

Hence,

$$I_1 \leqq \frac{c\epsilon^{(m+1)/4}}{|x - \xi|^{n-1}}\, \gamma(\xi). \tag{1.20}$$

To estimate I_2, note that if $|x - \xi| \geqq 2$ then $|x - \zeta| \geqq c\, |x - \xi|$ for $\zeta \in S_2$. Therefore we get

$$I_2 \leqq \frac{c\epsilon^{(m+1)/4}}{|x - \xi|^{n-1}}\, \gamma(\xi). \tag{1.21}$$

If $|x - \xi| \leqq 2$ then we use the rule (with B any ball of radius c)

$$\int_B \frac{d\zeta}{|x - \zeta|^{\alpha}\, |\zeta - \xi|^{\beta}} \leqq \frac{c}{|x - \xi|^{\alpha+\beta-n}} \qquad (0 < \alpha,\, 0 < \beta,\, \alpha + \beta < n) \tag{1.22}$$

and get

$$I_2 \leqq \frac{c\epsilon^{(m+1)/4}}{|x-\xi|^{n-3}}\gamma(\xi) \leqq \frac{c\epsilon^{(m+1)/4}}{|x-\xi|^{n-1}}\gamma(\xi),$$

i.e., (1.21) is still valid.

We may replace, in S_3, $|x - \zeta|^{1-n}$ by $|x - \zeta|^{2-n}$, and we then obtain an integral which coincides with the integral I_2 if x and ξ are interchanged. Hence, by (1.21),

$$I_3 \leqq \frac{c\epsilon^{(m+1)/4}}{|x-\xi|^{n-1}}\gamma(\xi). \tag{1.23}$$

If $|x - \xi| \geqq 2$ then $S_4 = \emptyset$, whereas if $|x - \xi| \leqq 2$ then we can use (1.22) to obtain

$$I_4 \leqq \frac{c\epsilon^{(m+1)/4}}{|x-\xi|^{n-3}}\gamma(\xi) \leqq \frac{c\epsilon^{(m+1)/4}}{|x-\xi|^{n-1}}\gamma(\xi). \tag{1.24}$$

To estimate I_5, we distinguish four cases:

(a): $|x| > H + 2$, $|\xi| > H + 2$. Then $|x - \zeta| \geqq c\,|x|$, $|\zeta - \xi| \geqq c\,|\xi|$ and

$$\frac{1}{|x|^{n-1}}\frac{1}{|\xi|^{n-1}} \leqq \frac{c}{(|x| + |\xi|)^{n-1}} \leqq \frac{c}{|x-\xi|^{n-1}}.$$

Using (1.8), (1.16), (1.17) we then get

$$I_5 \leqq \frac{c\epsilon^{(m+1)/4}}{|x-\xi|^{n-1}}\gamma(\xi). \tag{1.25}$$

(b): $|x| > H + 2$, $|\xi| \leqq H + 2$. Since $|\zeta| \leqq H + 1$ and $|x - \xi| \geqq 1$, we have $|x - \zeta| \geqq c\,|x - \xi|$, and (1.25) again follows by using (1.8), (1.16), (1.17).

(c): $|x| \leqq H + 2$, $|\xi| > H + 2$. This case is treated in the same way as the previous one.

(d): $|x| \leqq H + 2$, $|\xi| \leqq H + 2$. Using (1.8), (1.16), (1.17) and (1.22) we get

$$I_5 \leqq \frac{c\epsilon^{(m+1)/4}}{|x-\xi|^{n-3}}\gamma(\xi) \leqq \frac{c\epsilon^{(m+1)/4}}{|x-\xi|^{n-1}}\gamma(\xi),$$

i.e., (1.25) holds.

Combining (1.20), (1.21), (1.23), (1.24), (1.25), we obtain from (1.18)

$$d^{-m-1}\,|K_{m+1}(x;\xi)| \leqq \frac{c\epsilon^{(m+1)/4}}{|x-\xi|^{n-1}}\gamma(\xi) \quad \text{if} \quad |x-\xi| \geqq 1. \tag{1.26}$$

Consider now the case where $|x - \xi| < 1$. From the second inequality in (1.16) it follows that the first inequality in (1.16) remains true (with a different c) even if $|x - \xi| \leqq 4$. The same remark applies to the inequality (1.17). Hence, we get

$$(1.27)\qquad d^{-m-1}\,|K_{m+1}(x;\xi)| \leqq \int_{\Sigma_1} \frac{c\epsilon^{1/4}}{|x-\zeta|^{n-1}}\,\frac{c\epsilon^{m/4}}{|\zeta-\xi|^{n-2}}\,\gamma(\xi)\,d\zeta + \int_{\Sigma_2} \frac{c\epsilon^{1/4}}{|x-\zeta|^{n-1}}\,\frac{c\epsilon^{m/4}}{|\zeta-\xi|^{n-1}}\,\gamma(\xi)\,d\zeta = L_1 + L_2$$

where $\sum_1 = S \cap \{\zeta; |x-\zeta| < 3\}$, $\sum_2 = S \cap \{\zeta; |x - \zeta| > 3\}$. (Note that $|\zeta - \xi| < 4$ if $\zeta \in \sum_1$ and $|\zeta - \xi| > 1$ if $\zeta \in \sum_2$; this has been used in deriving (1.27).)

Using (1.22) we get

$$(1.28)\qquad L_1 \leqq \frac{c\epsilon^{(m+1)/4}}{|x-\xi|^{n-3}}\,\gamma(\xi).$$

Next, noting that $|\zeta - \xi| \geqq c\,|\zeta - x|$ if $\zeta \in \sum_2$ we find that

$$(1.29)\qquad L_2 \leqq \int_{\Sigma_2} \frac{c\epsilon^{(m+1)/4}}{|x-\zeta|^{2n-2}}\,\gamma(\xi)\,d\zeta \leqq c\epsilon^{(m+1)/4}\gamma(\xi).$$

Combining (1.28), (1.29) and using (1.27), we get

$$(1.30)\qquad d^{-m-1}\,|K_{m+1}(x;\xi)| \leqq \frac{c\epsilon^{(m+1)/4}}{|x-\xi|^{n-3}}\,\gamma(\xi) \quad \text{if} \quad |x-\xi| \leqq 1.$$

Taking d larger than the constants c which appear in (1.26), (1.30), the proof of (1.16) for $m + 1$ is completed.

For the case $|x - \xi| \leqq 1$ we have actually obtained a stronger inequality, *i.e.*, (1.30). This leads to the following result:

$$(1.31)\qquad |K_m(x;\xi)| \leqq \begin{cases} d^{m+1}\epsilon^{m/4}\gamma(\xi)\,|x-\xi|^{1-n-m}, \\ d^{m+1}\epsilon^{m/4}\gamma(\xi) \end{cases}$$

for $m \geqq 0$, $|x - \xi| \leqq 1$. (1.31) will not be needed in the future.

Having proved (1.16) we can now estimate the terms in the series in (1.14) to obtain:

$$(1.32)\qquad \left|\int_S K_m(x;\zeta)G_0(\zeta;\xi)\,d\zeta\right| \leqq \begin{cases} \dfrac{d^{m+1}\epsilon^{m/4}}{|x-\xi|^{n-2}} & \text{if} \quad |x-\xi| \leqq 1, \\[2ex] \dfrac{d^{m+1}\epsilon^{m/4}}{|x-\xi|^{n-1}} & \text{if} \quad |x-\xi| \geqq 1. \end{cases}$$

The proof follows by using a part of the calculations used in proving (1.16).

From (1.32) it follows that if $\epsilon d^4 < 1$ then the series in (1.14) is uniformly convergent if $|x - \xi| \geqq$ const. > 0, and $G(x; \xi)$ satisfies (1.12). Inserting G, from (1.14), into (1.4), and interchanging the order of integration (which is permissible by Fubini's theorem) we conclude that (1.4) is satisfied.

We proceed to prove (1.13). First we prove a lemma.

Lemma 1.1. *For any* $0 < \mu < 1$,

$$(1.33)\qquad |G(x;\xi) - G(x_* ;\xi)| \leqq \frac{c\,|x - x_*|^\mu}{|x - \xi|^{n-2+\mu}} \quad \textit{if} \quad |x - x_*| \leqq \tfrac{1}{3}\,|x - \xi|;$$

here $x = (x', y)$, $x_* = (x'_*, y)$.

Proof. We shall need the following inequalities which follow from (1.11) and the mean value theorem:

$$(1.34)\qquad |\nabla' G_0(x;\zeta) - \nabla' G_0(x_* ;\zeta)|$$

$$\leqq \begin{cases} \dfrac{c\,|x - x_*|^\mu}{|x - \zeta|^{n-1+\mu}} & \text{if} \quad |x - x_*| < \tfrac{1}{2}\,|x - \zeta|, \quad |x - \zeta| \leqq 1, \\[2ex] \dfrac{c\,|x - x_*|^\mu}{|x - \zeta|^{n+\mu}} & \text{if} \quad |x - x_*| < \tfrac{1}{2}\,|x - \zeta|, \quad |x - \zeta| > 1, \end{cases}$$

for any $0 < \mu \leqq 1$. Now, from (1.4), (1.5) we have

$$(1.35)\qquad G(x;\xi) = G_0(x;\xi) + W(x;\xi) + \int_S [\nabla' G_0(x;\zeta)\cdot\alpha(\zeta)]G(\zeta;\xi)\,d\zeta$$

where

$$(1.36)\qquad W(x;\xi) = \int_S G_0(x;\zeta)\beta(\zeta)G(\zeta;\xi)\,d\zeta.$$

We claim

$$(1.37)\qquad |\nabla' W(x;\xi)| \leqq \begin{cases} c\,|x - \xi|^{2-n} & \text{if} \quad |x - \xi| \leqq 1, \\ c\,|x - \xi|^{1-n} & \text{if} \quad |x - \xi| \geqq 1. \end{cases}$$

Indeed, we can write $\nabla' W = \int [\nabla' G_0]\beta G\,d\zeta$ and then estimate the integral by using a part of the calculations used in proving (1.16).

From (1.37) and the mean value theorem it follows that (1.33) holds with G replaced by W. (1.33) clearly also holds with G replaced by G_0. Hence, if we prove that

$$(1.38)\qquad I \equiv \left| \int_S [\nabla' G_0(x;\zeta) - \nabla' G_0(x_* ;\zeta)]\cdot\alpha(\zeta)G(\zeta;\xi)\,d\zeta \right| \leqq \frac{c\,|x - x_*|^\mu}{|x - \xi|^{n-2+\mu}}$$

whenever $|x - x_*| \leqq \frac{1}{3}\,|x - \xi|$, then, in view of (1.35), the proof of (1.33) is complete.

Write

$$(1.39)\qquad I = \int_{S\cap T'} + \int_{T''}$$

where T' is the ball of radius $2\,|x - x_*|$ about x. Then

$$\int_{S\cap T'} \leqq \int_{S\cap T'} |\nabla' G_0(x;\zeta)\cdot\alpha(\zeta)|\;|G(\zeta;\xi)|\,d\zeta$$

$$+ \int_{S\cap T'} |\nabla' G_0(x_* ;\zeta)\cdot\alpha(\zeta)|\;|G(\zeta,\xi)|\,d\zeta$$

$$= A_1 + A_2\,.$$

If $|x - \xi| \leqq 1$ then using (1.8), (1.10), (1.12) and $|\zeta - \xi| \geqq c\,|x - \xi|$, we get

$$A_1 \leqq \int_{S\cap T'} \frac{c}{|x - \zeta|^{n-1}\,|\zeta - \xi|^{n-2}}\,d\zeta$$

$$\leqq \frac{c}{|x - \xi|^{n-2}} \int_{T'} \frac{d\zeta}{|x - \zeta|^{n-1}} \leqq \frac{c\,|x - x_*|}{|x - \xi|^{n-2}} \leqq \frac{c\,|x - x_*|}{|x - \xi|^{n-1}}.$$

A_2 is estimated in the same way.

If $|x - \xi| > 1$ then $|\zeta - \xi| \geqq |x - \xi|/3 \geqq \frac{1}{3}$ for $\zeta \in S \cap T'$. Hence

$$A_1 \leqq \int_{S\cap T'} \frac{c}{|x - \zeta|^{n-1}\,|\zeta - \xi|^{n-1}}\,d\zeta \leqq \frac{c}{|x - \xi|^{n-1}} \int_{T'} \frac{d\zeta}{|x - \zeta|^{n-1}} \leqq \frac{c\,|x - x_*|}{|x - \xi|^{n-1}}.$$

Since A_2 is estimated in the same way, we conclude that

$$\int_{S\cap T'} \leqq \frac{c\,|x - x_*|}{|x - \xi|^{n-1}}. \tag{1.40}$$

It remains to estimate $\int_{T''}$. Using (1.34), (1.12) and (1.7), (1.8), we have:

$$\begin{aligned} \int_{T''} \leqq {} & \int_{T_1} \frac{c\,|x - x_*|}{|x - \zeta|^{n+1}\,|\zeta|^{1/2}} \frac{c}{|\zeta - \xi|^{n-1}}\,d\zeta + \int_{T_2} \frac{c\,|x - x_*|}{|x - \zeta|^{n+1}} \frac{c}{|\zeta - \xi|^{n-1}}\,d\zeta \\ & + \int_{T_3} \frac{c\,|x - x_*|}{|x - \zeta|^{n+1}} \frac{c}{|\zeta - \xi|^{n-2}}\,d\zeta + \int_{T_4} \frac{c\,|x - x_*|^{\mu}}{|x - \zeta|^{n+1-\mu}} \frac{c}{|\zeta - \xi|^{n-1}}\,d\zeta \\ & + \int_{T_5} \frac{c\,|x - x_*|^{\mu}}{|x - \zeta|^{n-1+\mu}} \frac{c}{|\zeta - \xi|^{n-2}}\,d\zeta = M_1 + M_2 + M_3 + M_4 + M_5 \end{aligned} \tag{1.41}$$

where

$$T_1 = S \cap \{\zeta;\, |\zeta - x| \geqq 1,\, |\zeta - \xi| \geqq 1,\, |\zeta| \geqq H + 1\},$$

$$T_2 = S \cap \{\zeta;\, |\zeta - x| \geqq 1,\, |\zeta - \xi| \geqq 1,\, |\zeta| \leqq H + 1\},$$

$$T_3 = S \cap \{\zeta;\, |\zeta - x| \geqq 1,\, |\zeta - \xi| \leqq 1\},$$

$$T_4 = S \cap \{\zeta;\, |\zeta - x| \leqq 1,\, |\zeta - \xi| \geqq 1\}$$

and

$$T_5 = S \cap \{\zeta;\, |\zeta - x| \leqq 1,\, |\zeta - \xi| \leqq 1\}.$$

Clearly,

$$M_1 \leqq c\,|x - x_*| \int_{T_1} \frac{1}{|x - \zeta|^{n-1}\,|\zeta|^{1/2}} \frac{1}{|\zeta - \xi|^{n-1}}\,d\zeta.$$

The integral on the right hand side can then be estimated by the calculations used in estimating I_1 in (1.18). Hence,

$$M_1 \leqq \frac{c\,|x - x_*|}{|x - \xi|^{n-1}}.$$

Next, either $|x - \zeta| \geqq \frac{1}{2}|x - \xi|$ or $|\xi - \zeta| \geqq \frac{1}{2}|x - \xi|$, and since $|x - \zeta| \geqq 1$, $|\xi - \zeta| \geqq 1$ for $\zeta \in T_2$, we get

$$M_2 \leqq \frac{c\,|x - x_*|}{|x - \xi|^{n-1}}.$$

We turn to M_3. If $|x - \xi| > 3$ then $|x - \zeta| \geqq c\,|x - \xi|$ and

$$M_3 \leqq \frac{c\,|x - x_*|}{|x - \xi|^{n+1}} \int_{T_3} \frac{d\zeta}{|\zeta - \xi|^{n-2}} \leqq \frac{c\,|x - x_*|}{|x - \xi|^{n-1}}.$$

On the other hand, if $|x - \xi| \leqq 3$ then

$$M_3 \leqq c\,|x - x_*| \int_{T_3} \frac{d\zeta}{|\zeta - \xi|^{n-2}} \leqq c\,|x - x_*| \leqq \frac{c\,|x - x_*|}{|x - \xi|^{n-1}}.$$

By (1.22),

$$M_4 \leqq \frac{c\,|x - x_*|^{\mu}}{|x - \xi|^{n-2+\mu}} \quad \text{if} \quad 0 < \mu < 1.$$

Finally, if $|x - \xi| > 3$ then $M_5 = 0$ since the domain of integration is empty, whereas if $|x - \xi| \leqq 3$ then, by (1.22),

$$M_5 \leqq \frac{c\,|x - x_*|^{\mu}}{|x - \xi|^{n-3+\mu}} \leqq \frac{c\,|x - x_*|^{\mu}}{|x - \xi|^{n-2+\mu}}.$$

Combining all the bounds on the M_i and using (1.41), (1.40), (1.39), the proof of (1.38) follows. This completes the proof of the lemma.

We return to the proof of (1.13). In view of (1.35), (1.37), we only have to consider the first derivatives of the integral on the right hand side of (1.35). Writing this integral as a sum $\sum_i \int (\partial G_0/\partial x_i)\alpha_i G\, d\zeta$, we see that its derivative with respect to x_j is the sum of $(\delta_{ij}/n)\alpha_i(x)G(x;\xi)$ and the integral

$$Q \equiv \int_S \frac{\partial^2 G_0(x;\zeta)}{\partial x_i\, \partial x_j} [\alpha_i(\zeta)G(\zeta;\xi) - \alpha_i(x)G(x;\xi)]\, d\zeta. \tag{1.42}$$

Thus, it suffices to prove that (1.13) holds with $\nabla' G$ replaced by Q.

Noting that $|\alpha_i(\zeta) - \alpha_i(x)| \leqq c\,|\zeta - x|$ and using Lemma 1.1, we obtain

$$|\alpha_i(\zeta)G(\zeta;\xi) - \alpha_i(x)G(x;\xi)| \leqq \frac{c\,|\zeta - x|^{\mu}}{|x - \xi|^{n-2+\mu}} \quad \text{if} \quad |\zeta - x| < \tfrac{1}{3}|x - \xi|. \tag{1.43}$$

Using (1.43) and (1.11) we can then write

$$
(1.44)\quad
\begin{aligned}
Q \leqq & \int_{V_1} \frac{c}{|x-\zeta|^{n-\mu}} \frac{1}{|x-\xi|^{n-2+\mu}}\, d\zeta \\
& + \int_{V_2} \frac{c}{|x-\zeta|^{n}} \left(\frac{1}{|\zeta-\xi|^{n-2}} + \frac{1}{|x-\xi|^{n-2}} \right) d\zeta \\
& + \int_{V_3} \frac{c}{|x-\zeta|^{n+1}} \left[\frac{|\alpha(\zeta)|}{|\zeta-\xi|^{n-1}} + \frac{c}{|x-\xi|^{n-1}} \right] d\zeta \\
& + \int_{V_4} \frac{c}{|x-\zeta|^{n+1}} \left(\frac{1}{|\zeta-\xi|^{n-2}} + \frac{1}{|x-\xi|^{n-2}} \right) d\zeta \\
= & \; Q_1 + Q_2 + Q_3 + Q_4,
\end{aligned}
$$

where

$$
\begin{aligned}
V_1 &= S \cap \{\zeta;\, |\zeta - x| < \tfrac{1}{3}|x-\xi|,\\
V_2 &= S \cap \{\zeta;\, |\zeta - x| < 1,\, |\zeta - x| > \tfrac{1}{3}|x-\xi|\},\\
V_3 &= S \cap \{\zeta;\, |\zeta - x| > \tfrac{1}{3}|x-\xi|,\, |\zeta - x| > 1,\, |\zeta-\xi| > 1\},\\
V_4 &= S \cap \{\zeta;\, |\zeta - x| > \tfrac{1}{3}|x-\xi|,\, |\zeta - x| > 1,\, |\zeta-\xi| < 1\}.
\end{aligned}
$$

Then,

$$Q_1 \leqq \frac{c}{|x-\xi|^{n-2+\mu}} \int_{V_1} \frac{1}{|x-\zeta|^{n-\mu}}\, d\zeta \leqq \frac{c}{|x-\xi|^{n-2}}.$$

Next, if $|x-\xi| > 3$ then the domain of integration of Q_2 is empty, so that $Q_2 = 0$. On the other hand, if $|x-\xi| \leqq 3$ then

$$Q_2 \leqq \frac{c}{|x-\xi|^{n-1}} \int_{V_2} \left[\frac{1}{|x-\zeta|\,|\zeta-\xi|^{n-2}} + \frac{1}{|x-\zeta|^{n-1}} \right] d\zeta \leqq \frac{c}{|x-\xi|^{n-1}}.$$

The part of Q_3 involving $|\alpha(\zeta)|$ can be estimated by breaking the integrand into two parts corresponding to $|\zeta| \geqq H+1$ and $|\zeta| \leqq H+1$ and using a part of the calculation used in estimating I_1 and I_5 of (1.18). The remaining part of Q_3 is bounded by

$$\frac{c}{|x-\xi|^{n-1}} \int_{V_3} \frac{c}{|x-\zeta|^{n+1}}\, d\zeta \leqq \frac{c}{|x-\xi|^{n-1}}.$$

Q_4 can be treated in the same way (using some of the calculation used in estimating I_2 and I_5).

Combining the estimates on the Q_i and using (1.44), it follows that (1.13) is satisfied with $\nabla' G$ replaced by Q. As has already been stated before, (1.13) is then completely proved.

We have proved that G, given by (1.14), satisfies (1.4) and (1.12), (1.13). We can now integrate by parts in (1.4) and obtain (1.3) (the boundary terms disappear because of (1.12)). In view of (1.12), (1.13), the integral in (1.3) is absolutely convergent if $x \neq \xi$. From (1.14) it follows that $G(x;\xi)$ is continuous in (x, ξ) if $x \neq \xi$ and from (1.4) it follows that $G(x', 0; \xi) = 0$. Next, for a given

x, break the integral in (1.3) into two parts:

$$\int_S = \int_{S'} + \int_{S''} \quad \text{where} \quad S' = S \cap \{(x', y);\ |x'| < R\} \tag{1.45}$$

for some $R > |x|$. Since the integrand is Hölder continuous in ζ, we have

$$\Delta \int_{S'} = -a(x)\cdot\nabla' G(x;\xi) - b(x)G(x;\xi). \tag{1.46}$$

Using (1.13) one can prove that $\Delta \int_{S''} = \int_{S''} \triangle$. Noting that the integrand of the last integral is zero, and recalling (1.45), (1.46), we conclude from (1.3) that $G(x;\xi)$, as a function of x, satisfies (1.1). This completes the proof of the theorem.

Later on we shall need some bounds on the derivatives of $G(x;\xi)$ up to some order (depending on n). We shall therefore derive such bounds now. These will not be too sharp, but their derivation is quite simple and they will suffice for our purposes. First we establish an auxiliary bound.

Lemma 1.2. *Let ϵ be sufficiently small as in the assertion of Theorem* 1.1. *For any $0 < \delta < H$, if $-H \leqq y \leqq -\delta$ then $\partial G(x', y; \xi', \eta)/\partial\eta$ exists at $\eta = 0$ and is a continuous function in (x', y, ξ'), satisfying*

$$\left|\frac{\partial}{\partial\eta} G(x', y; \xi', 0)\right| \leqq \frac{c}{|x-\xi|^{n-2}}. \tag{1.46}$$

Proof. Let us consider first the general term in the series of (1.14) for $\xi = (\xi', \eta)$. Its η derivative exists at $\eta = 0$ and is given by

$$N_m(x, \xi') \equiv \int_S K_m(x;\zeta)\frac{\partial}{\partial\eta} G_0(\zeta;\xi', 0)\, d\zeta.$$

The continuity of $N_m(x, \xi')$ in all its variables (for $-H \leqq y \leqq -\delta$) is obvious (using (1.16) and $|\partial G_0/\partial\eta| \leqq c\,|\zeta - \xi|^{1-n}$). Thus, to complete the proof of the lemma it suffices to show that

$$|N_m(x, \xi')| \leqq \frac{cd^{m+1}\epsilon^{m/4}}{|x-\xi|^{n-2}} \tag{1.47}$$

for some constant c depending on c_0, H, δ (but independent of d).

Set $\zeta = (\zeta_1, \cdots, \zeta_{n-1}, \lambda) = (\zeta', \lambda)$ and let $A = \{\zeta;\ -H < \lambda < -\delta/2\}$, $B = \{\zeta;\ -\delta/2 < \lambda < 0\}$. Using (1.16) and $|\partial G_0/\partial\eta| \leqq c\,|\zeta-\xi|^{1-n}$, we can write

$$\begin{aligned} d^{-m-1}\epsilon^{-m/4}\,|N_m(x, \xi')| &\leqq \int_{A_1} \frac{c}{|x-\zeta|^{n-1}}\gamma(\zeta)\frac{c}{|\zeta-\xi|^{n-1}}\, d\zeta \\ &+ \int_{A_2} \frac{c}{|x-\zeta|^{n-2}}\frac{c}{|\zeta-\xi|^{n-1}}\, d\zeta + \int_{B_1} \frac{c}{|x-\zeta|^{n-1}}\gamma(\zeta)\frac{c}{|\zeta-\xi|^{n-1}}\, d\zeta \\ &+ \int_{B_2} \frac{c}{|x-\zeta|^{n-1}}\frac{c}{|\zeta-\xi|^{n-1}}\, d\zeta = W_1 + W_2 + W_3 + W_4, \end{aligned} \tag{1.48}$$

where $A_1 = A \cap \{\zeta; |x - \zeta| > 1\}$, $A_2 = A \cap \{\zeta; |x - \zeta| < 1\}$, $B_1 = B \cap \{\zeta; |\zeta - \xi| > 1\}$, $B_2 = B \cap \{\zeta; |\zeta - \xi| < 1\}$. In deriving (1.48) we have used the $|x - \zeta| \geqq \delta/2 > 0$ for $\zeta \in B$.

Since $|x - \zeta| \geqq \delta/2$ for $\zeta \in A$, W_1 can be estimated by calculations used in estimating I_1, I_5 of (1.18). Similarly, W_2 can be estimated by calculation similar to those used in estimating I_3, I_5. Since $|x - \zeta| \geqq \delta/2$ for $\zeta \in B_1$, W_3 can be estimated in the same way as W_1. Finally, by (1.22), $W_4 \leqq c\,|x - \xi|^{2-n}$. Combining the bounds on the W_i, and recalling that $|x - \xi| \geqq \delta/2$, we get (from (1.48)) the inequality (1.47).

We shall now assume that $a \in C^{k+1}(S)$, $b \in C^k(S)$ for some $k > 0$, and

$$(1.49) \qquad |\nabla^j b(x)| + |\nabla^{j+1} a(x)| \leqq c_1 \quad \text{for} \quad x \in S, \qquad 0 \leqq j \leqq k.$$

Then in every ball with center x^0 and radius δ, lying in $S \cap \{(x', y); x' \in R^{n-1}, -H < y < -\mu\}$ $(\mu > 0)$, there exists Green's function $\Gamma_{x^0,\delta}(x; \xi)$ of (1.1) and

$$(1.50) \qquad |\nabla_x^j \nabla_\xi \Gamma_{x^0,\delta}(x; \xi)| \leqq c \quad \text{if} \quad |x - \xi| > \frac{\delta}{2}, \qquad 0 \leqq j \leqq k;$$

c is independent of x^0, δ.

Setting $u(x) = \partial G(x', y; \xi', 0)/\partial\eta$, we can represent it in the form

$$(1.51) \qquad u(x) = \int_{|x - x^0| = \delta} \frac{\partial \Gamma_{x^0,\delta}(x, x^*)}{\partial \nu(x^*)}\, u(x^*)\, dS(x^*).$$

From (1.50), (1.46) we conclude,

$$|\nabla^j u(x)| \leqq c \quad \text{if} \quad |x - x^0| < \frac{\delta}{2}, \qquad 0 \leqq j \leqq k,$$

where c is independent of δ. Hence, we have

Theorem 1.2. *Let ε be sufficiently small (as in the assertion of Theorem 1.1) and assume, in addition to (1.2), that $a \in C^{k+1}(S)$, $b \in C^k(S)$ and (1.49) holds. Then, for any $0 < \delta < H$,*

$$(1.52) \qquad \left| \nabla_x^j \frac{\partial}{\partial \eta} G(x', y; \xi', 0) \right| \leqq c \quad \text{for} \quad 0 \leqq j \leqq k$$

for all $x' \in R^{n-1}$, $\xi' \in R^{n-1}$, $-H + \delta \leqq y \leqq -\delta$, where c is a constant depending only on c_0, c_1, H, δ, k.

Note that from (1.51) it also follows that $\nabla_x^j\, \partial G(x', y; \xi', 0)/\partial\eta$ is continuous in all its variables, when $y < 0$.

Remark. Theorems 1.1, 1.2 remain true under a weaker assumption than (1.2). In fact, (1.2) leads to the inequalities (1.7), (1.8), whereas actually all the previous calculations extend with minor modifications to the case where (1.7), (1.8) are replaced by:

$$|\beta(\zeta)| \leqq \frac{c\epsilon^\gamma}{|\zeta'|^\delta}, \qquad |\alpha(\zeta)| \leqq \frac{c\epsilon^\gamma}{|\zeta'|^\delta} \quad (\text{if } |\zeta'| > 1)$$

$$|\beta(\zeta)| < c\epsilon^\gamma, \qquad |\alpha(\zeta)| \leqq c\epsilon^\gamma$$

for some $\gamma > 0$, $\delta > 0$.

Let $f(\xi')$ be a continuous function in R^{n-1} satisfying

$$|f(\xi')| \leqq c(1 + |\xi'|)^{1-n-\delta} \qquad (\delta > 0). \tag{1.53}$$

Consider the function

$$u(x', y) = \int_{R^{n-1}} \frac{\partial}{\partial \eta} G(x', y; \xi', 0) f(\xi')\, d\xi'. \tag{1.54}$$

By well known calculations it follows that $u(x)$ is continuous in $\bar{S}$, and

$$u(x', 0) = f(x'). \tag{1.55}$$

If $k \geqq 2$ in Theorem 1.2, then we can differentiate the integral in (1.54) twice by differentiating under the integral. We conclude that $u(x)$ satisfies (1.1). For later references we state:

Theorem 1.3. *Let the assumptions of Theorem* 1.2 *hold with* $k \geqq 2$, *and let* $f(\xi')$ *be a continuous function satisfying* (1.53). *Then* $u(x)$, *given by* (1.54), *is a solution of* (1.1) *satisfying* (1.55).

2. Spectral resolution with generalized functions. Let $\zeta(x)$ be a function in $C^{m_0+\mu}(R^m)$, where $m_0 = m + [m/2] + 1$, $0 < \mu < 1$ ($[m/2]$ stands for the smallest integer $\leqq m/2$), and assume that

$$c_0 \leqq \zeta(x) \leqq c_1, \qquad |D^\beta \zeta(x)| \leqq c_2 \quad \text{for} \quad 0 \leqq |\beta| \leqq m, \tag{2.1}$$

where c_0, c_1, c_2 are positive constants. The simplest example is $\zeta(x) \equiv 1$. Let

$$M_1(x) = (1 + \sigma\, |x|^2)^{3m/4+\delta/2}, \qquad M(x) = (1 + \sigma\, |x|^2)^{3m/2+\delta} = (M_1(x))^2,$$

where σ, δ are fixed positive numbers, and $\delta < 1$. Then

$$M_1(x) \geqq 1, \frac{|x_1 \cdots x_m|^{1/2}}{M_1(x)} \in L^1(R^m).$$

Denote by H the Hilbert space consisting of all the measurable functions $\varphi(x)$, $x \in R^m$, with finite norm

$$||\varphi|| = \left\{\int_{R^m} \zeta(x)\, |\varphi(x)|^2\, dx\right\}^{1/2}. \tag{2.2}$$

We introduce two more norms:

$$||\varphi||_1 = \sum_{|\alpha| \leqq m} \sup_{x \in R^m} \zeta(x) M_1(x)\, |D^\alpha \varphi(x)|, \tag{2.3}$$

$$(2.4) \qquad \|\varphi\|_\Phi = \sum_{|\beta| \leq m_0} \left\{ \int_{R^m} \zeta(x) M(x) \, |D^\beta \varphi(x)|^2 \, dx \right\}^{1/2},$$

and denote by Φ the set of all functions $\varphi \in C^{m_0+\mu}(R^m)$ for which $\|\varphi\|_\Phi < \infty$. We shall consider Φ as a linear normed space with the norm (2.3). One then can show that $\|\varphi\|_1 < \infty$ for any $\varphi \in \Phi$. In fact we have:

Lemma 2.1. *If $\varphi \in \Phi$ then $\|\varphi\|_1 \leqq c \, \|\varphi\|_\Phi$ where c is a constant (independent of φ).*

Proof. By the Sobolev inequality, for any $x \in R^m$,

$$|\zeta(x) M_1(x) D^\alpha \varphi(x)|^2 \leqq c \sum_{|\gamma| \leq [m/2]+1} \int_{R^m} |D^\gamma (\zeta M_1 D^\alpha \varphi)|^2 \, dx.$$

Now use (2.1), the inequalities $|D^\beta M_1| \leqq c M_1$ and the fact that $M_1^2 = M$.

Note that the topology of Φ is stronger than that of H, *i.e.*, $\|\varphi\| \leqq c \, \|\varphi\|_1$. Indeed from (2.3) we obtain

$$|\varphi(x)| \leqq c \, \|\varphi\|_1 / M_1(x)$$

from which the result follows by using (2.1) and recalling that $1/M_1^2$ is in $L^2(R^m)$.

Clearly Φ is a dense subspace of H, and since its topology, given by (2.3), is stronger than that of H, we have $\Phi \subset H \subset \Phi'$ (Φ' = conjugate of Φ). Denote by Φ_1 the completion of Φ and by Φ_1' the conjugate of Φ_1. Then also $\Phi_1 \subset H \subset \Phi_1'$.

Let A be a linear self-adjoint operator in H with $A\Phi \subset \Phi$, and let E_λ be its spectral family. Take a fixed $e \in H$ with $\|e\| = 1$.

Lemma 2.2. *There exists a constant C such that for any monotone sequence $\{\lambda_j\}$ with λ_j real, $\lambda_j \to \pm\infty$ as $j \to \pm\infty$,*

$$(2.5) \qquad \sum_j \|E_{\lambda_{j+1}} e - E_{\lambda_j} e\|_{\Phi_1'} \leqq C.$$

For $\zeta \equiv 1$ the lemma coincides with Theorem 1, p. 288 of [2]. The proof for general ζ is similar. Thus, $G_{\Delta_j}(x)$ of [2] is now defined as $\int_0^x \zeta^{1/2}(\xi) E(\Delta_j) e(\xi) \, d\xi$. Further details may be omitted.

Consider now the measure

$$\sigma(\lambda) = (E_\lambda e, e).$$

Since Lemma 2.2 is valid, we can apply Theorem 1, p. 197 of [2] to the functional $E_\lambda e$ and conclude that it has a weak derivative with respect to any non-negative completely additive Borel measure, thus, in particular, with respect to $\sigma(\lambda)$. Denote this derivative by χ_λ, *i.e.*,

$$(2.6) \qquad \frac{d(E_\lambda e, \varphi)}{d\sigma(\lambda)} = (\chi_\lambda, \varphi) \quad \text{for every} \quad \varphi \in \Phi;$$

χ_λ is an element of Φ'.

By the proof of Theorem 1, p. 216 of [2] we also have $A\chi_\lambda = \lambda\chi_\lambda$ in the sense that

$$(\chi_\lambda, A\varphi) = \lambda(\chi_\lambda, \varphi) \tag{2.7}$$

where by (χ_λ, φ) we mean the application of the functional χ_λ to φ. Let $H(e)$ be the closed subspace of H generated by the $E_\lambda e$. The proof of Theorem 2, p. 212 of [2] shows that for every $\varphi \varepsilon H(e)$, we have

$$\varphi = \int \overline{(\chi_\lambda, \varphi)}\chi_\lambda \, d\sigma(\lambda), \tag{2.8}$$

$$||\varphi||^2 = \int |(\chi_\lambda, \varphi)|^2 \, d\sigma(\lambda), \tag{2.9}$$

where the integral in (2.8) is convergent in the norm of H.

$\chi_\lambda \varepsilon \Phi_1'$, and in view of Lemma 2.1 it may also be considered as a bounded linear functional on Φ when Φ is provided with the norm $|| \; ||_\Phi$. Hence, χ_λ has the form

$$(\chi_\lambda, \varphi) = \sum_{|\alpha| \leq m_0} \int_{R^m} \zeta^{1/2}(x) M^{1/2}(x) D^\alpha \varphi(x) f_{\alpha\lambda}(x) \, dx \tag{2.10}$$

where the $f_{\alpha\lambda}(x)$ are measurable functions and

$$\sum_{|\alpha| \leq m_0} \int_{R^m} |f_{\alpha\lambda}(x)|^2 \, dx < \infty. \tag{2.11}$$

In case $H(e) \neq H$, we decompose H into a direct sum $\sum H(e^\gamma)$ and, in analogy with (2.8), (2.9), we have

$$\varphi = \sum_\gamma \int \overline{(\chi_\lambda^{(\gamma)}, \varphi)}\chi_\lambda^\gamma \, d\sigma_\gamma(\lambda), \tag{2.12}$$

$$||\varphi||^2 = \sum_\gamma \int |(\chi_\lambda^{(\gamma)}, \varphi)|^2 \, d\sigma_\gamma(\lambda). \tag{2.13}$$

Consider now the case where $H(e) = H$. We wish to replace (2.10) by another representation which will be more useful for our applications, namely,

$$(\chi_\lambda, \varphi) = \int_{R^m} g_\lambda(x) L\varphi(x) \, dx \tag{2.14}$$

where L is some elliptic differential operator, say

$$L = (4\pi^2 - \Delta)^k \qquad (\Delta = \text{Laplace's operator}). \tag{2.15}$$

Lemma 2.3. *If $2k > 3m/2 + [m/2] + 1$, then the representation* (2.14) *holds for any $\varphi \varepsilon C^{2k}(R^m)$ satisfying*

$$|D^\alpha \varphi(x)| \leq \text{const.} \, (1 + |x|)^{-5m/2-\delta} \quad \textit{for} \quad 0 \leq |\alpha| \leq 2k.$$

The function $g_\lambda(x)$ is continuous in $x \varepsilon R^m$ and

$$|g_\lambda(x)| \leq C(1 + |x|)^{3m/2+\delta/2} \qquad (C \text{ constant}). \tag{2.16}$$

Proof. It suffices to prove (2.14) for any $\varphi \in C_0^\infty(R^m)$, for then (2.14) would follow, for any φ as in the assertion of the lemma, by approximating it by $\alpha_\nu \varphi$ (for which (2.14) holds); the α_ν are defined as $\alpha(x/\nu)$ where $\alpha \in C_0^\infty(R^m)$, $\alpha(x) = 1$ if $|x| \leqq 1$. In view of (2.10), (2.14) (for all $\varphi \in C_0^\infty(R^m)$) is equivalent to the fact that g_λ is a solution of

$$(2.17) \qquad Lg_\lambda = \sum_{|\alpha| \leqq m_0} (-1)^{|\alpha|} D^\alpha(\zeta^{1/2} M^{1/2} f_{\alpha\lambda}) \equiv f$$

where all the derivatives are taken in the sense of distributions.

A fundamental solution of L is

$$(2.18) \qquad E(x) = \gamma_0 \, |x|^{(2k-m)/2} K_{(m-2k)/2}(2\pi \, |x|) \qquad (\gamma_0 \text{ constant})$$

where K_μ is the modified Hankel function, decreasing exponentially to zero as $|x| \to \infty$ and analytic everywhere except at the origin (where it behaves like either $c \, |x|^{(2k-m)/2} \log |x|$ or like $|x|^{(2k-m)/2}$, $c \neq 0$).

We can now solve (2.17) explicitly by $g_\lambda = E * f$, *i.e.*,

$$(2.19) \qquad g_\lambda(x) = \sum_{|\alpha| \leqq m_0} (-1)^{|\alpha|} \int_{R^m} D^\alpha E(x - \xi) \zeta^{1/2}(\xi) M^{1/2}(\xi) f_{\alpha\lambda}(\xi) \, d\xi.$$

If $2k > 3m/2 + [m/2] + 1$ then

$$(2.20) \qquad |D^\alpha E(x)| \leqq c e^{-\gamma |x|} h_\alpha(x) \qquad (0 \leqq |\alpha| \leqq m_0)$$

where $h_\alpha \in L^2(R^m)$ and $\gamma > 0$. Dividing any integral in (2.19) into two parts, one with $|\xi| < 2\,|x|$ and the other with $|\xi| \geqq 2\,|x|$, we find that (2.16) is satisfied. Finally, it is obvious that $g_\lambda(x)$ is a continuous function.

We also have:

If $2k > 2k_0 + 3m/2 + [m/2] + 1$, then $g_\lambda(x)$ has k_0 continuous derivatives.

The results of this section remain true for $\zeta \in C^m(R^m)$ (instead of $\zeta \in C^{m_0+\mu}(R^m)$) and for various subspaces Φ of $C^{m_0}(R^m)$. However, in what follows we shall only take ζ, Φ as above.

3. Construction of a generalized solution. Let $x = (x', y) = (x_1, \cdots, x_{n-1}, y)$ be a variable point in R^n. Let Ω be a domain in R^n bounded by the hyplerpane $\Gamma : y = 0$, by a manifold Γ_b given by $y = -h(x')$ where $h(x')$ is defined over all of R^{n-1} and satisfies

$$h_0 \leqq h(x') \leqq h_1 \qquad (h_0, h_1 \text{ are positive constants}),$$

and, finally, by a finite number of compact manifolds Γ_j (lying between Γ and Γ_b) where $j = 1, \cdots, j_0$. We call Γ_b the *bottom* of Ω and we refer to the Γ_j as *obstacles*. Throughout this chapter we assume:

$$(3.1) \qquad \Gamma_b \text{ and the } \Gamma_j \text{ belong to } C^{2+\rho}, \text{ for some } 0 < \rho < 1.$$

For brevity, we set $\Gamma' = \Gamma_b \bigcup (\bigcup_{j=1}^{j_0} \Gamma_j)$.

In this section we construct a generalized solution for the initial value problem

$$\triangle u = 0 \quad \text{in} \quad \Omega, \tag{3.2}$$

$$\zeta(x')p\left(\frac{d}{dt}\right)u + u_\nu = 0 \quad \text{on} \quad \Gamma, \tag{3.3}$$

$$u_\nu = 0 \quad \text{on} \quad \Gamma', \tag{3.4}$$

$$\left.\frac{\partial^j u}{\partial t^j}\right|_{t=0} = u_j^0 \quad \text{on} \quad \Gamma \quad \text{for} \quad j = 0, 1, \cdots, s-1, \tag{3.5}$$

where $p(\lambda) = \sum_{j=0}^{s} p_j\lambda^j$ is a given polynomial of degree s, ν is the outward normal to Γ', and $\zeta(x')$ is any positive function as in §2 (with $m = n - 1$). We shall employ results of §2 with $m = n - 1$. In particular, since ζ satisfies (2.1), the space Φ consists of all functions $\varphi(x')$ in $C^{n+[(n-1)/2]+\mu}(R^{n-1})$ $(0 < \mu < 1)$ satisfying:

$$\int_\Gamma (1 + |x'|^2)^{3(n-1)/2+\delta} \, |\nabla^k\varphi(x')|^2 \, dx' < \infty \quad \left(0 \leqq k \leqq n + \left[\frac{n-1}{2}\right]\right). \tag{3.6}$$

The initial data u_j^0 are assumed to belong to the space Φ.

As in Chapter 2, we consider first an auxiliary problem: given $f \in C_0^\infty(\Gamma)$, find φ which satisfies:

$$\triangle\varphi = 0 \quad \text{in} \quad \Omega, \tag{3.7}$$

$$\varphi_\nu + \alpha\zeta\varphi = 0 \quad \text{on} \quad \Gamma \qquad (\alpha \text{ positive constant}), \tag{3.8}$$

$$\varphi_\nu = 0 \quad \text{on} \quad \Gamma'. \tag{3.9}$$

φ can be constructed as in Chapter 2 (where the case $\zeta(x') \equiv 1$ was considered). We define T by $(Tf)(x') = \varphi(x', 0)$ and then extend it by continuity as a bounded linear operator in $L^2(\Gamma)$. It has the same properties as in Chapter 2 (for the case $\zeta \equiv 1$).

We shall now introduce a restriction on Γ_b :

Condition (A_R) : $h(x') = \text{const.}$ if $|x'| > R$.

This restriction will be removed later on, by approximating Γ_b by manifolds satisfying the condition (A_{R_m}) for a sequence $\{R_m\}$ with $R_m \to \infty$ as $m \to \infty$.

We shall need slightly stronger assumptions on ζ (than those of Sec. 2), namely:

$$\zeta \in C^{n_0+\mu}(R^{n-1}), \quad c_0 \leqq \zeta(x') \leqq c_1 , \quad |D^\beta\zeta(x')| \leqq c_2 \quad \text{for} \quad 0 \leqq |\beta| \leqq n_0 \tag{3.10}$$

where c_0 , c_1 , c_2 are positive constants and $n_0 = n + [(n-1)/2]$.

Finally, since

$$M(x') = (1 + \sigma\, |x'|^2)^{3(n-1)/2+\delta}, \tag{3.11}$$

for every integer k we have

$$\frac{|D^\beta M(x')|}{M(x')} \leqq c\sigma \qquad (0 \leqq |\beta| \leqq k) \tag{3.12}$$

where c is a constant depending on k.

Lemma 3.1. *If the conditions* (A_R) *and* (3.10) *are satisfied, and if* σ *in* (3.11) *is sufficiently small (independently of* R*), then* T *maps* Φ *into itself.*

Proof. Let $f \in \Phi$. Then $f \in C^{n_0+\mu}(R^{n-1})$ and, by well known theorems on the differentiability of solutions of elliptic equations, also $\varphi = Tf$ is in $C^{n_0+\mu}(R^{n-1})$. It thus remains to show that $||\varphi||_\Phi < \infty$, *i.e.*,

$$\int_\Gamma M(x')\,|D^\beta\varphi(x')|^2\,dx' < \infty \quad \text{for} \quad 0 \leqq |\beta| \leqq n_0\,. \tag{3.13}$$

Consider first the case $\beta = 0$. Formally we have:

$$\begin{aligned}\int_\Omega \varphi\nabla M\cdot\nabla\bar\varphi\,dx + \int_\Omega M\,|\nabla\varphi|^2\,dx &= \int_\Omega \nabla(M\varphi)\cdot\nabla\bar\varphi\,dx \\ &= \int_\Gamma M\varphi\bar\varphi_\nu\,dx' = -\alpha\int_\Gamma M\,|\varphi|^2\,dx' + \int_\Gamma M\varphi\bar f\,dx'.\end{aligned} \tag{3.14}$$

Using (3.12), Lemma 1 and Schwarz's inequality, we get after some calculation,

$$\frac{1}{2}\int_\Omega M\,|\nabla\varphi|^2\,dx + \frac{\alpha}{2}\int_\Gamma M\,|\varphi|^2\,dx' \leqq c\int_\Gamma M\,|f|^2\,dx' \tag{3.15}$$

provided σ is sufficiently small (independently of R). To prove (3.15) rigorously, we use approximations as in the construction of a solution to (3.7)–(3.9). Since each φ_i satisfies (3.15) with Ω, Γ replaced by Ω_i, Γ_i, (3.15) follows.

We next prove (3.13) for $|\beta| = 1$. Denote by Ω^*, Γ^*, Γ_b^* the intersections of Ω, Γ and Γ_b, respectively, with $|x'| > R^*$ and take $R^* > R$ such that all the obstacles Γ_i lie in $|x'| < R^*$. Denote by Γ_0^* the intersection of Ω with $|x'| = R^*$. It suffices to prove that

$$\int_{\Gamma^*} M(x')\,|D^\beta\varphi(x')|^2\,dx' < \infty \qquad (|\beta| = 1). \tag{3.16}$$

Consider the system

$$\begin{aligned} \triangle w &= 0 \quad \text{in} \quad \Omega^*, \\ w_\nu + \alpha\zeta w &= D^\beta\zeta\cdot f + \zeta D^\beta f - \alpha D^\beta\zeta\cdot\varphi \quad \text{on} \quad \Gamma^*, \\ w_\nu &= 0 \quad \text{on} \quad \Gamma_b^*, \\ w &= D^\beta\varphi \quad \text{on} \quad \Gamma_0^*. \end{aligned} \tag{3.17}$$

A solution w can be constructed by the method of the existence proof for (3.7)–(3.9), employing for the approximating sequence of solutions an inequality of the form (3.15). Thus we find that a solution w of (3.17) exists and satisfies

$$(3.18) \qquad \int_{\Omega^*} M\, |\nabla w|^2\, dx + \int_{\Gamma^*} M\, |w|^2\, dx' \leqq c \int_{\Gamma^*} M\, |f|^2\, dx' + c \int_{\Gamma^*} M\, |D^\beta f|^2\, dx' + cc_*$$

where c_* is a bound on $\int_{\Gamma_0{}^*} M w \bar{w}_\nu$ and c is a constant; use has been made of (3.10), (3.15).

If we prove that $w = D^\beta \varphi$, then the proof of (3.16) follows from (3.18). To prove this we may assume that $D^\beta = \partial/\partial x_1$ and introduce the finite differences

$$\varphi_h(x) = [\varphi(x_1 + h, x_2, \cdots, x_{n-1}, y) - \varphi(x_1, x_2, \cdots, x_{n-1}, y)]/h.$$

Consider the function $\psi^h = w - \varphi_h$. It satisfies a system similar to (3.17) (with $\psi_\nu^h = 0$ on Γ_0^* and $\psi^h \to 0$ on Γ_0^* as $h \to 0$). Since ψ^h, $\nabla\psi^h$ belong to $L^2(\Omega)$, we can derive for ψ^h an inequality of the form (3.18) (the proof is similar to the formal proof (3.15), using an argument similar to that following (2.1.10)). The inequality for ψ^h has the form

$$(3.19) \qquad \int_{\Omega^*} M\, |\nabla \psi^h|^2\, dx + \int_{\Gamma^*} M\, |\psi^h|^2\, dx' \leqq I^h + cc_*^h$$

where c_*^h is a bound on $\int_{\Gamma_0{}^*} M\psi^h \bar{\psi}_\nu^h$ and

$$(3.20) \qquad I_h \leqq c\, |h| \int_{\Gamma} M\, |f|^2\, dx' + c \int_{\Gamma^*} M \left| f_h - \frac{\partial f}{\partial x_1} \right|^2 dx'.$$

(We have used (3.10) and (3.16) with $\beta = 0$.)

Since $\psi^h \to 0$ uniformly in Γ_0^* as $h \to 0$, also $c_h^* \to 0$. Thus it remains to show that the last integral in (3.20) approaches 0 as $h \to 0$. Since the integrand converges to zero uniformly in compact subsets, as $h \to 0$, it suffices to prove that

$$(3.21) \qquad \sup_{|h| \leqq 1} \int_{\Gamma_\lambda{}^*} M\, |f_h|^2\, dx' \to 0 \quad \text{as} \quad \lambda \to \infty,$$

where Γ_λ^* is the intersection of Γ^* with $|x'| > \lambda$.

To prove (3.21) we use the mean value theorem and get:

$$|f_h(x', 0)|^2 \leqq \left| \frac{1}{h} \int_{x_1}^{x_1+h} \frac{\partial f}{\partial \xi_1} (\xi_1, x_2, \cdots, x_{n-1}, 0)\, d\xi_1 \right|^2 \leqq \int_{m-1}^{m+2} \left| \frac{\partial f}{\partial \xi_1} (\xi_1, x_2, \cdots, x_{n-1}, 0) \right|^2 d\xi_1$$

if $m \leqq x_1 \leqq m + 1$. Multiplying by $M(x')$ and integrating, we get

$$\int_{\Gamma_\lambda{}^*} M\, |f_h|^2\, dx' \leqq c \int_{\Gamma_{\lambda-1}{}^*} M \left| \frac{\partial f}{\partial x_1} \right|^2 dx',$$

and (3.21) follows.

We have thus proved that the right hand side of (3.19) converges to 0 if $h \to 0$. Since $\psi^h \to w - D^\beta \varphi$ uniformly in closed bounded subsets of Ω^*, it follows that $w \equiv D^\beta \varphi$. This completes the proof of (3.16).

We can now proceed step-by-step in the same way to establish (3.13) for $|\beta| = 2, 3, \cdots, n_0$.

Having completed the proof of Lemma 3.1, we can now apply the results of Sec. 1. We conclude that T has a complete set of generalized eigenfunctions $\chi_\lambda^{(\alpha)}$. More precisely, $\chi_\lambda^{(\alpha)} \in \Phi'$, $(\chi_\lambda^{(\alpha)}, T\varphi) = \lambda(\chi_\lambda^{(\alpha)}, \varphi)$, and (2.12), (2.13) hold for every $\varphi \in \Phi$. The $\chi_\lambda^{(\gamma)}$ can be represented in the form

$$(\chi_\lambda^{(\gamma)}, \varphi) = \int_{R^{n-1}} g_\lambda^{(\gamma)}(x') L\varphi(x')\, dx' \tag{3.22}$$

for any $\varphi \in C^{2k}$ satisfying

$$|D^\beta \varphi(x')| \leqq c(1 + |x'|)^{-5(n-1)/2-\delta} \qquad (0 \leqq |\beta| \leqq 2k)$$

where k is any fixed integer $> (3n - 1)/2 + [(n - 1)/2]$, $g_\lambda^{(\gamma)}(x')$ is continuous and

$$|g_\lambda^{(\gamma)}(x')| \leqq \text{const.}\ (1 + |x'|)^{3(n-1)/2+\delta/2}. \tag{3.23}$$

Since the spectrum of T lies in $[0, 1/\alpha]$, $\chi_\lambda^{(\gamma)} \equiv 0$, $g_\lambda^{(\gamma)} \equiv 0$ for $\lambda < 0$ and for $\lambda > 1/\alpha$.

Consider a particular $\chi_\lambda^{(\gamma)}$. For simplicity we write $\chi_\lambda = \chi_\lambda^{(\gamma)}$. The notation $\chi_\lambda(x')$ will be often used instead of χ_λ. We shall now continue $\chi_\lambda(x')$ into Ω. The continued function $\chi_\lambda(x', y)$ is required to satisfy:

$$\Delta \chi_\lambda(x) = 0 \quad \text{in} \quad \Omega, \tag{3.24}$$

$$\frac{\partial}{\partial \nu} \chi_\lambda(x) = 0 \quad \text{on} \quad \Gamma', \tag{3.25}$$

(3.26) as $y \to 0$, $\chi_\lambda(x', y) \to \chi_\lambda(x')$ in the sense of distributions.

Consider the equation

$$(1 + \epsilon |x'|^2)^{-p} \Delta[(1 + \epsilon |x'|^2)^p u] = 0 \qquad \left(p = n + \left[\frac{n-1}{4}\right] + 1\right). \tag{3.27}$$

It can be written in the form (1.1) with a, b satisfying the conditions of Theorems 1.2 (for any k) in a strip S with width $H > h_1$. Hence, if ϵ is sufficiently small then the upper Green's function $G(x; \xi)$ exists and (1.52) is valid. Set

$$\Gamma(x', y; \xi') = \frac{\partial}{\partial \eta} G(x', y; \xi', \eta)|_{\eta=0}\ .$$

Theorem 3.1. *Assume that the conditions* (A_R) *and* (3.10) *hold. There exists a function* $\chi_\lambda(x)$ *satisfying* (3.24)–(3.26). *It is given by*

$$\chi_\lambda(x) = Lg_\lambda(x) + w_\lambda(x) \tag{3.28}$$

where

$$g_\lambda(x', y) = (1 + |x'|^2)^p \int_{R^{n-1}} \Gamma(x', y; \xi') \frac{g_\lambda(\xi')}{(1 + |\xi'|^2)^p} d\xi' \tag{3.29}$$

and w_λ *is a solution of*

$$\Delta w_\lambda(x) = 0 \quad in \quad \Omega, \tag{3.30}$$

$$w_\lambda(x', 0) = 0 \quad on \quad \Gamma, \tag{3.31}$$

$$\frac{\partial}{\partial \nu} w_\lambda(x) = -\frac{\partial}{\partial \nu} Lg_\lambda(x) \quad on \quad \Gamma'. \tag{3.32}$$

Proof. From (3.23) and Theorem 1.3 it follows that $g_\lambda(x', y)$ satisfies

$$\Delta g_\lambda(x) = 0 \quad \text{in} \quad S = \{(x', y); x \in R^{n-1}, -H < y < 0\}, \tag{3.33}$$

$$g_\lambda(x', 0) = g_\lambda(x') \quad \text{on} \quad \Gamma, \tag{3.34}$$

and $g_\lambda(x)$ is continuous in $\bar{S}$. Furthermore, from Theorem 1.2 we get

$$|\nabla^j g_\lambda(x', y)| \leqq c_j(1 + |x'|^2)^p \qquad (j = 0, 1, 2, \cdots) \tag{3.35}$$

in every set $\bar{\Omega} \cap \{(x', y); x' \in R^{n-1}, -h_1 \leqq y < -\delta\}$ with $\delta > 0$, where c_j is a constant depending on δ. (Recall that $h_0 \leqq h(x') \leqq h_1 < H$.) Thus, in particular,

$$\left|\frac{\partial}{\partial \nu} Lg_\lambda(x)\right| \leqq c(1 + |x'|^2)^p \quad \text{on} \quad \Gamma'. \tag{3.36}$$

To construct w_λ we use a sequence Ω_j of smooth domains with $\Omega_j \nearrow \Omega$ such that, say, $\Omega_j \cap \{(x', y); |x'| < j\} = \Omega \cap \{(x', y); |x'| < j\}$. Denote by Γ_{bj} the intersection of Γ_b with $\bar{\Omega}_j$, and let $\Gamma_j' = \Gamma_{bj} \cup (\bigcup_{i=1}^{i_0} \Gamma_i)$ (Γ_i are the obstacles). We take j sufficiently large so that all the Γ_i lie in $|x'| < j$. Denote by Γ_j^* the complement of Γ_j' in $\partial\Omega_j$.

Consider the problem

$$\begin{aligned} \Delta v_j &= 0 \quad \text{in} \quad \Omega_j, \\ v_j &= 0 \quad \text{on} \quad \Gamma_j^*, \\ \frac{\partial}{\partial \nu} v_j &= -\frac{\partial}{\partial \nu}(Lg_\lambda) \quad \text{on} \quad \Gamma_j'. \end{aligned} \tag{3.37}$$

To prove that v_j exists and belongs to $H^1(\Omega_j)$ we truncate the boundary conditions, namely, we replace $\partial(Lg_\lambda)/\partial\nu$ by $\zeta_\epsilon\, \partial(Lg_\lambda)/\partial\nu$, where $\zeta_\epsilon \in C^\infty$, $\zeta_\epsilon = 1$ on the obstacles Γ_i and on the points of Γ_{bj} whose distance to Γ_j^* is $\geqq 2\epsilon$, whereas $\zeta_\epsilon = 1$ in an ϵ neighborhood of Γ_j^*. We also require that $0 \leqq \zeta_\epsilon \leqq 1$ at all the other points of Γ_{bj}.

The corresponding problem has a solution $v_{j\epsilon}$. If we prove that

$$\int_{\Omega_j} |v_{j\epsilon}|^2\, dx + \int_{\Omega_j} |\nabla v_{j\epsilon}|^2\, dx \leqq c \int_{\Gamma_j'} \left|\frac{\partial}{\partial \nu} Lg_\lambda\right|^2 dS \tag{3.38}$$

where c is independent of ϵ, then by a compactness argument similar to the one used in Chap. 1, Sec. 2 it follows that there is a sequence $\epsilon = \epsilon_m \to 0$ with $v_{j\epsilon}$ converging weakly to a solution v_j of (3.37), and

$$\int_{\Omega_j} |v_j|^2\,dx + \int_{\Omega_j} |\nabla v_j|^2\,dx \leqq c \int_{\Gamma_j'} \left|\frac{\partial}{\partial\nu} Lg_\lambda\right|^2 dS. \tag{3.39}$$

To prove (3.38) we use Green's identity:

$$\begin{aligned}\int_{\Omega_j} |\nabla v_{j\epsilon}|^2\,dx &= \int_{\Gamma_j'} \zeta_\epsilon v_{j\epsilon} \frac{\partial v_{j\epsilon}}{\partial\nu}\,dS \\ &\leqq \delta \int_{\Gamma_j'} |v_{j\epsilon}|^2\,dS + \frac{c}{\delta}\int_{\Gamma_j'} \left|\frac{\partial}{\partial\nu} Lg_\lambda\right|^2 dS.\end{aligned} \tag{3.40}$$

Noting that $v_{j\epsilon} = 0$ on Γ_j^* and using Lemma 1.1.8, we have

$$\int_{\Gamma_j'} |v_{j\epsilon}|^2\,dS \leqq c \int_{\Omega_j} |\nabla v_{j\epsilon}|^2\,dx.$$

Substituting this into (3.40) and taking δ sufficiently small, we get

$$\int_{\Omega_j} |\nabla v_{j\epsilon}|^2\,dx \leqq C \int_{\Gamma_j'} \left|\frac{\partial}{\partial\nu} Lg_\lambda\right|^2 dS.$$

Finally, using this inequality and Lemma 1.1.5, (3.38) follows.

Having constructed a solution v_j of (3.37), we remark that the inequality (3.40) will not be suitable in analyzing the case $j \to \infty$; indeed, the right hand side will then grow to ∞. We shall therefore derive now a different bound, involving a density function

$$\gamma(x') = (1 + \eta\,|x'|^2)^{-q} \qquad \left(\eta > 0,\ q = 2p + \left[\frac{n}{2}\right] + 1\right). \tag{3.41}$$

The procedure is similar to that of deriving (3.15).

$$\begin{aligned}\int_{\Omega_j} \gamma\,|\nabla v_j|^2\,dx &= \int_{\Omega_j} \nabla(\gamma v_j)\cdot\nabla v_j\,dx - \int_{\Omega_j} v_j\cdot\nabla v_j \nabla\gamma\,dx \\ &= \int_{\Gamma_j'} \gamma v_j \frac{\partial v_j}{\partial\nu}\,dS - \int v_j \nabla v_j\cdot\nabla\gamma\,dx,\end{aligned} \tag{3.42}$$

from which it follows that

$$\begin{aligned}\int_{\Omega_j} \gamma\,|\nabla v_j|^2\,dx &\leqq c\eta \int_{\Omega_j} \gamma\,|v_j|^2\,dx + c\eta\int_{\Omega_j} \gamma\,|\nabla v_j|^2\,dx \\ &\quad + \delta\int_{\Gamma_j'} \gamma\,|v_j|^2\,dS + \frac{c}{\delta}\int_{\Gamma_j'} \gamma \left|\frac{\partial}{\partial\nu}(Lg_\lambda)\right|^2 dS.\end{aligned} \tag{3.43}$$

Applying Lemmas 1.1.5, 1.1.8 with $\varphi = \gamma^{1/2} v_j$ and taking η, δ to be sufficiently small, we get

$$(3.44)\qquad \int_{\Omega_j} \gamma\, |\nabla v_j|^2\, dx \leqq c \int_{\Gamma_j'} \gamma \left| \frac{\partial}{\partial \nu} (Lg_\lambda) \right|^2 dS.$$

By Lemma 1.1.5 (applied to $\varphi = \gamma^{1/2} v_j$) we also get

$$(3.45)\qquad \int_{\Omega_j} \gamma\, |v_j|^2\, dx \leqq c \int_{\Gamma_j'} \gamma \left| \frac{\partial}{\partial \nu} (Lg_\lambda) \right|^2 dS.$$

By (3.36), (3.41) it follows that the right hand side of (3.45) is bounded independently of j. Hence we can use a standard compactness argument to show that a subsequence of $\{v_j\}$ is weakly convergent to a solution $w_\lambda(x)$ (3.30)–(3.32), and

$$(3.46)\qquad \int_\Omega \gamma\, |w_\lambda|^2\, dx + \int_\Omega \gamma\, |\nabla w_\lambda|^2\, dx \leqq c \int_{\Gamma'} \gamma \left| \frac{\partial}{\partial \nu} (Lg_\lambda) \right|^2 dS.$$

Having constructed w_λ, it is now easily seen that $\chi_\lambda(x)$, given by (3.28) satisfies (3.24)–(3.26). This completes the proof of Theorem 3.1.

We now define the formal solution of the problem (3.2)–(3.5)

$$(3.47)\qquad \Psi(x, t) = \sum_\gamma \int_0^{1/\alpha} \Big[\sum_{j=0}^{s-1} \overline{(\chi_\lambda^{(\gamma)}, u_j^0)\mu_\lambda^j(t)} \Big] \chi_\lambda^{(\gamma)}(x)\, d\sigma^{(\gamma)}(\lambda)$$

where the μ_λ^j are defined by

$$(3.48)\qquad \left[p\left(\frac{d}{dt}\right) - \alpha + \frac{1}{\lambda} \right] \mu_\lambda^{j-1}(t) = 0,$$

$$(3.49)\qquad \frac{d^k}{dt^k} \mu_\lambda^{j-1}(0) = \delta_{kj} \qquad (0 \leqq k \leqq s - 1).$$

In analogy with the case where Ω is a bounded domain, we expect the $\chi_\lambda^{(\gamma)}$ to satisfy

$$(3.50)\qquad \frac{\partial}{\partial y} \chi_\lambda^{(\gamma)} + \alpha \zeta \chi_\lambda^{(\gamma)} = \frac{\zeta}{\lambda} \chi_\lambda^{(\gamma)} \quad \text{on} \quad \Gamma.$$

Hence Ψ satisfies (3.2)–(3.5) formally. In the following sections we shall interpret Ψ as a distribution and prove that it satisfies (3.2), (3.4) in the classical sense and (3.3), (3.5) in some "weak" sense. Even though the methods to be used apply to the case where p in (3.3) is any polynomial all of whose zeros have nonpositive real parts, we shall carry out the arguments only for the polynomial $p(z) = z^2$ that arises in water waves.

For simplicity we shall assume that throughout §§4–7 $H(e) = H$, *i.e.*, the sum $\sum_\gamma$ in (3.47) consists of one term. The general case will be treated in §8. Finally, it will be convenient to make the transformation

$$(3.51)\qquad \hat\lambda = \frac{1}{\lambda} - \alpha, \qquad \hat\chi_{\hat\lambda} = \chi_\lambda, \qquad d\hat\sigma(\hat\lambda) = d\sigma(\lambda).$$

Omitting, for simplicity, the "roofs," we can rewrite the formal solution in the form:

(3.52)

$$\Psi(x, t) = \int_0^\infty \overline{(\chi_\lambda, u^0)}[\cos (\lambda)^{1/2}t]\chi_\lambda(x)\, d\sigma(\lambda) + \int_0^\infty \overline{(\chi_\lambda, u_t^0)} \frac{\sin (\lambda)^{1/2}t}{(\lambda)^{1/2}} \chi_\lambda(x)\, d\sigma(\lambda).$$

4. Smoothness of the solution in the interior. For every $\varphi \in C_0^\infty(\Omega)$, define

$$(4.1)\qquad (\Psi, \varphi) = \int_0^\infty \overline{(\chi_\lambda, u^0)} \cos (\lambda)^{1/2}t(\chi_\lambda(x), \varphi(x))\, d\sigma(\lambda) + \int_0^\infty \overline{(\chi_\lambda, u_t^0)} \frac{\sin (\lambda)^{1/2}t}{(\lambda)^{1/2}} (\chi_\lambda(x), \varphi(x))\, d\sigma(\lambda),$$

where $(\chi_\lambda(x), \varphi(x)) = \int_{R^n} \chi_\lambda(x)\varphi(x)\, dx$.

Theorem 4.1. *Assume that u^0, u_t^0 belong to Φ and that the conditions (A_R), (3.10) hold. Then the functional Ψ defined by (4.1) is a distribution in Ω.*

Proof. It suffices to consider the case $u_t^0 = 0$, since the second integral in (4.1) can be treated in the same way that we shall treat the first integral. Writing

$$(4.2)\qquad (\Psi, \varphi) = \int_0^\infty \overline{(\chi_\lambda, u^0)} \cos (\lambda)^{1/2}t(Lg_\lambda(x), \varphi(x))\, d\sigma(\lambda) + \int_0^\infty \overline{(\chi_\lambda, u^0)} \cos (\lambda)^{1/2}t(w_\lambda(x), \varphi(x))\, d\sigma(\lambda) \equiv (\Psi_1, \varphi) + (\Psi_2, \varphi)$$

we shall prove that both integrals make sense and we shall estimate them.

In view of Theorem 3.1,

(4.3)

$$\int Lg_\lambda(x)\cdot\varphi(x)\, dx' = \int \left\{\int L[(1 + |x'|^2)^p\Gamma(x', y; \xi)] \frac{g_\lambda(\xi')}{(1 + |\xi'|^2)^p}\, d\xi \right\}\varphi(x)\, dx' = \int g_\lambda(\xi')\psi_y(\xi')\, d\xi'$$

where

$$(4.4)\qquad \psi_y(\xi') = (1 + |\xi'|^2)^{-p} \int L[(1 + |x'|^2)^p\Gamma(x', y; \xi)]\varphi(x', y)\, dx'.$$

Denote by M a bounded open set in Ω containing the support of φ, and such that $\bar{M} \cap \Gamma = \emptyset$. We can use (1.52) and obtain:

$$(4.5)\qquad |\psi_y(\xi')| \leqq c(1 + |\xi'|^2)^{-p}\, \|\varphi\|_y \qquad \left(\|\varphi\|_y = \left\{\int |\varphi(x', y)|^2\, dx'\right\}^{1/2}\right),$$

where c depends on M but not on φ.

For fixed y we now solve $Lf_y = \psi_y$ by $f_y = E * \psi_y$ (E defined in (2.18) with $m = n - 1$) and find, using (4.5), that

$$|(\nabla')^j f_\nu(\xi')| \leqq c(1 + |\xi'|^2)^{-p} \, ||\varphi||_\nu \qquad (0 \leqq j < \infty; c \text{ depends on } j).$$

Hence $\varphi \in \Phi$ and

(4.6) $$||f_\nu||_1 \leqq c \, ||\varphi||_\nu \qquad (||\cdot||_1 = \text{norm in } \Phi).$$

Consider now (Ψ_1, φ). From (4.3) we obtain

$$(Lg_\lambda(x), \varphi(x)) = \int g_\lambda(\xi')\Big(\int \psi_\nu(\xi') \, dy\Big) \, d\xi' = \int g_\lambda(\xi') Lf(\xi') \, d\xi'$$

where $f(\xi') = \int f_\nu(\xi') \, dy$. Since, clearly, f satisfies the conditions on φ in Lemma 2.3, we have, by (2.23),

(4.7) $$(Lg_\lambda(x), \varphi(x)) = (\chi_\lambda, f).$$

It follows that the left hand side of (4.7) is measurable with respect to $d\sigma(\lambda)$, and therefore the same is true of the integrand of (Ψ_1, φ). Furthermore, by Schwarz's inequality and (2.13),

(4.8) $$|(\Psi_1, \varphi)|^2 \leqq \int |(\chi_\lambda, u^0)|^2 \, d\sigma(\lambda) \int |(\chi_\lambda, f)|^2 \, d\sigma(\lambda)$$
$$= ||u^0||_1^2 \, ||f||_1^2 .$$

Using (4.6) we conclude that

(4.9) $$|(\Psi_1, \varphi)| \leqq c \, ||\varphi||_{L^2(M)} .$$

We turn to (Ψ_2, φ). First we prove:

Lemma 4.1. *$(w_\lambda(x), \varphi(x))$ is measurable with respect to $d\sigma(\lambda)$.*

Proof. Let Γ_{j1}, Γ_{j2} be subdomains of Γ_j' (defined following (3.36)) such that their boundaries are smooth and $\bar{\Gamma}_{j1} \subset \Gamma_{j2}$, $\bar{\Gamma}_{j2} \subset \Gamma_j'$. Suppose also that Γ_{j1} contains all the obstacles Γ_i and all the points of Γ_j' whose distance to the boundary of Γ_j' is $\geqq 1$. Let a, b be C^∞ functions on Γ_j' satisfying $a = 1$, $b = 0$ in Γ_{j2}, $a = 0$, $b = 1$ outside Γ_{j2}, and $0 \leqq a \leqq 1$, $0 \leqq b \leqq 1$, $a + b > 0$ throughout Γ_j'.

Consider the problem (3.37) with the condition on Γ_{bj} modified:

$$a \frac{\partial}{\partial \nu} v_j + b v_j = -a \frac{\partial}{\partial \nu} (Lg_\lambda) \quad \text{on} \quad \Gamma_{bj} .$$

Denote the solution of the modified problem by $w_{j\lambda}$. One can prove that the $w_{j\lambda}$ satisfy the inequality (3.39) and, consequently, a subsequence of $\{w_{j\lambda}\}$ is weakly convergent to w_λ. Since (by (3.46)) w_λ is uniquely determined, we see that actually the complete sequence $\{w_{j\lambda}\}$ is weakly convergent to w_λ. Thus, in particular,

$$\int w_{j\lambda}(x)\varphi(x) \, dx \to \int w_\lambda(x)\varphi(x) \, dx$$

for any $\varphi \in C_0^\infty(\Omega)$. Hence, if we prove that $\int w_{i\lambda}(x)\varphi(x)\,dx$ is measurable with respect to $d\sigma(\lambda)$, then the proof of the lemma will be completed.

Let $R_i(x;\,\xi)$ be the Robin function corresponding to the third boundary value problem for the $w_{i\lambda}$. We can write:

$$w_{i\lambda}(x) = -\int_{\Gamma_i'} R_i(x;z)a(z)\frac{\partial}{\partial\nu}Lg_\lambda(z)\,dS_z\,. \tag{4.10}$$

Substituting g_λ from (3.29) we find that if $M \subset \Omega_i$.

$$\int w_{i\lambda}(x)\varphi(x)\,dx = \int\left\{\int \varphi(x)\int\left\{R_i(x;z)b(z)\frac{\partial}{\partial\nu}[(1+|z'|^2)^p\Gamma(z;\xi')]\,dS_z\right\}dx\right\}$$

$$\cdot\frac{g_\lambda(\xi')}{(1+|\xi'|^2)^p}\,d\xi' = \int\frac{\psi_i(\xi')}{(1+|\xi'|^2)^p}\,g_\lambda(\xi')\,d\xi'.$$

Noting that $|z-x| \geqq$ const. > 0 and using (1.52), we conclude that $\psi_i(\xi')$ has bounded derivatives of any order. Setting

$$f_i(\xi') = E*\frac{\psi_i(\xi')}{(1+|\xi'|^2)^p}\,,$$

we find that f_i satisfies the conditions on φ in Lemma 2.3, and $Lf_i = \psi_i(1+|\xi'|^2)^{-p}$. Hence,

$$\int w_{i\lambda}(x)\varphi(x)\,dx = (\chi_\lambda\,,\,f_i),$$

from which it follows that the left hand side is measurable with respect to $d\sigma(\lambda)$.

Lemma 4.2. *$\int_M |w_\lambda(x)|^2\,dx$ and $\int_M |\nabla w_\lambda(x)|^2\,dx$ are measurable with respect to $d\sigma(\lambda)$.*

Proof. Consider the first integral. Since it is the limit of

$$I_{i\lambda} \equiv \int_M |w_{i\lambda}(x)|^2\,dx,$$

it suffices to prove that the latter is measurable. Using the representation (4.10) we find that

$$I_{i\lambda} = \iiint A(x,z)B(x,\zeta)$$

$$\cdot\left\{\int C(z,\xi')\frac{g_\lambda(\xi')}{(1+|\xi'|^2)^p}\,d\xi'\right\}\left\{D(\zeta,\xi')\frac{g_\lambda(\xi')}{(1+|\xi'|^2)^p}\,d\xi'\right\}dz\,d\zeta\,dx,$$

where A, B, C, D have bounded derivatives of any order. We can now proceed, as in the proof of Lemma 4.1, to show that each of the integrals in braces is measurable with respect to $d\sigma(\lambda)$. It follows that the same is true of $I_{i\lambda}$. The proof for $\int |\nabla w_\lambda|^2\,dx$ is similar.

Having proved Lemmas 4.1, 4.2 we proceed to estimate $(\Psi_2\,,\,\varphi)$. Clearly,

$$(4.11) \qquad |(\Psi_2, \varphi)|^2 \leqq \int |(\chi_\lambda, u^0)|^2 \, d\sigma(\lambda) \int |(w_\lambda(x), \varphi(x))|^2 \, d\sigma(\lambda)$$
$$\leqq c \, ||\varphi||^2_{L^2} \int \left\{ \int_M |w_\lambda(x)|^2 \, dx \right\} d\sigma(\lambda).$$

We now wish to use (3.46). We therefore need to estimate the right hand side of this inequality. Set

$$d_x(\xi') = \left\{ \frac{\partial}{\partial \nu} [L(1 + |x'|^2)^p \Gamma(x', y; \xi')]_{\Gamma'} \right\} (1 + |\xi'|^2)^{-p},$$
$$e_x(\xi') = E * d_x(\xi').$$

Then $Le_x = d_x$ and

$$(4.12) \qquad |(\nabla')^j e_x(\xi')| \leqq c(1 + |\xi'|^2)^{-p}(1 + |x|^2)^p$$
$$(j = 0, 1, 2, \cdots ; \quad c \text{ depends on } j)$$

if x is restricted Γ'.

It follows that

$$\int_{\Gamma'} \gamma \left| \frac{\partial}{\partial \nu} (Lg_\lambda) \right|^2 dS_x \leqq \int \gamma \left| \int Le_x(\xi') \cdot g_\lambda(\xi') \, d\xi' \right|^2 dS_x$$
$$= \int \gamma(x') \, |(\chi_\lambda, e_x)|^2 \, dS_x$$

and, integrating with respect to $d\sigma(\lambda)$ we obtain

$$(4.13) \qquad \int \left\{ \int_{\Gamma'} \gamma \left| \frac{\partial}{\partial \nu} (Lg_\lambda) \right|^2 dS \right\} d\sigma(\lambda) \leqq \int_{\Gamma'} \gamma(x') \, ||e_x||_1^2 \, dS_x \leqq c$$

by (3.41), (4.12). Hence, if we integrate (3.46) (with Ω replaced by M) with respect to $d\sigma(\lambda)$, we get

$$(4.14) \qquad \iint_M |w_\lambda(x)|^2 \, dx \, d\sigma(\lambda) + \iint_M |\nabla w_\lambda(x)|^2 \, dx \, d\lambda \leqq c$$

for any bounded domain M (c depends on M).

Using (4.14), we conclude from (4.11) that

$$(4.15) \qquad |(\Psi_2, \varphi)| \leqq c \, ||\varphi||^2_{L^2(M)} .$$

Combining this result with (4.9) we arrive at the following:

Lemma 4.3. *Let M be any bounded subdomain of Ω with $\bar{M} \cap \Gamma = \emptyset$. Then for any $\varphi \in C_0^\infty(M)$,*

$$(4.16) \qquad |(\Psi, \varphi)| \leqq c \, ||\varphi||_{L^2(M)} ,$$

where c is a constant (depending on M).

The assertion of the theorem is a consequence of the lemma.

Since $\Delta\chi_\lambda(x) = 0$, we have $(\chi_\lambda(x), \Delta\varphi(x)) = 0$ for any $\varphi \in C_0^\infty(\Omega)$. Hence, from (4.1), $(\Psi, \Delta\varphi) = 0$. This means that the distribution Ψ satisfies the Laplace equation in the sense of distributions. But, as is well known, Ψ is then equivalent to a harmonic function $u(x)$ in Ω, *i.e.*,

$$(\Psi, \varphi) = \int u(x)\varphi(x)\,dx \quad \text{for every} \quad \varphi \in C_0^\infty(\Omega). \tag{4.17}$$

We sum up:

Theorem 4.2. *Under the assumptions of Theorem* 4.1, *the distribution* Ψ *can be identified with a function u harmonic in* Ω.

5. Smoothness of the solution at the bottom. From (4.16), (4.17) we conclude that

$$\left|\int u(x)\varphi(x)\,dx\right| \leqq c\left\{\int |\varphi(x)|^2\,dx\right\}^{1/2} \tag{5.1}$$

for any $\varphi \in C_0^\infty(M)$ where M is any bounded subdomain of Ω with $\bar{M} \cap \Gamma = \emptyset$. Hence u defines a bounded linear functional in $L^2(M)$. It follows that $u \in L^2(M)$. Thus, $u \in L^2_{\text{loc}}(\Omega \cup \Gamma')$. (In general, if $u \in L^2(K)$ for compact subsets of a set A, we write $u \in L^2_{\text{loc}}(A)$.)

We next prove:

Lemma 5.1. ∇u *is in* $L^2_{\text{loc}}(\Omega \cup \Gamma')$.

Proof. It suffices to prove that for any M as before,

$$|(\Psi, D\varphi)| \leqq c\left\{\int |\varphi|^2\,dx\right\}^{1/2} \tag{5.2}$$

where φ is any function in $C_0^\infty(M)$. Indeed, since $u \in C^\infty(\Omega)$, it would then follow that

$$\left|\int Du\cdot\varphi\,dx\right| = \left|\int u\,D\varphi\,dx\right| = |(\Psi, D\varphi)| \leqq c\left\{\int |\varphi|^2\,dx\right\}^{1/2},$$

from which the assertion would follow.

Following the proof of (4.9) we find that

$$|(\Psi_1, D\varphi)| \leqq c\,||\varphi||_{L^2(M)}. \tag{5.3}$$

Indeed, the only change is that instead of L we now have to deal with DL.

Next, analogously to (4.11) we have

$$|(\Psi_2, D\varphi)|^2 \leqq c\int |(w_\lambda(x), D\varphi(x)|^2\,d\sigma(\lambda)$$

$$\leqq c\,||\varphi||^2_{L^2} \iint_M |Dw_\lambda(x)|^2\,dx\,d\sigma(\lambda).$$

Using (4.14) we get

$$|(\Psi_2\,,\,D\varphi)| \leqq c\,||\varphi||_{L^2}\,.$$

Combining this inequality with (5.3), (5.2) follows.

The method of proof of (5.3) gives also:

$$(5.4) \qquad |(\Psi_1\,,\,D^\beta\varphi)| \leqq c\,||\varphi||_{L^2} \qquad (\varphi \in C_0^\infty(M))$$

for any β. We now wish to derive also the inequality:

$$(5.5) \qquad |(\Psi_2\,,\,D^\beta\varphi)| \leqq c\,||\varphi||_{L^2}\,.$$

This is a consequence of

$$(5.6) \qquad \iint_M |D^\beta w_\lambda(x)|^2\,dx\,d\sigma(\lambda) \leqq c.$$

Lemma 5.2. *If Γ' is of class C^m then* (5.6) *holds for all* $|\beta| \leqq m$.

Proof. Note first that the inner integral in (5.6) is measurable with respect to $d\sigma(\lambda)$. Indeed, it suffices to prove it for M with $\bar{M} \subset \Omega$. Let M_t be a family of smooth domains which vary monotonically and smoothly with t $(0 \leqq t \leqq t_0)$ such that $M_0 = M \subset M_{t_0}$. Using the representation

$$w_{j\lambda} = c\int_{\partial M_t}\left(r^{2-n}\frac{\partial w_{j\lambda}}{\partial\nu} - w_{j\lambda}\frac{\partial}{\partial r}r^{2-n}\right)dS$$

and integrating with respect to t for $\epsilon \leqq t \leqq t_0$ for some $\epsilon > 0$, we find that, for any β, $D^\beta w_{j\lambda} \to D^\beta w_\lambda$ uniformly in M. Hence, if we prove that

$$I(\lambda) \equiv \int_M |D^\beta w_{j\lambda}(x)|^2\,dx$$

is measurable with respect to $d\sigma(\lambda)$, then the same holds for w_λ. Now, the measurability of $I(\lambda)$ can be proved similarly to proof, in Lemma 4.2, for the case $|\beta| = 1$.

We shall prove (5.6) in case $\bar{M}$ intersects Γ', since if $\bar{M} \cap \Gamma' = \emptyset$ then we already know that u is in $C^\infty(M)$ and there is no point in deriving (5.6). For definiteness let us assume that ∂M is composed of two parts: N_1 lying on Γ_b and N_2 lying in Ω. Let N_0 be a subset of N_1 whose distance to N_2 is positive, and let $\eta \in C^\infty(\bar{M})$, $\eta = 1$ in a neighborhood M_0 of N_0, $\eta = 0$ in a neighborhood of N_1. We further impose the condition that $\partial\eta/\partial\nu = 0$ on N_1.

Consider the function $v = \eta w_\lambda$ in M. It satisfies:

$$\Delta v = -\nabla\eta\cdot\nabla w_\lambda - w_\lambda\Delta\eta \quad \text{in} \quad M,$$

$$\frac{\partial v}{\partial\nu} = 0 \quad \text{on} \quad N_2\,,$$

$$\frac{\partial v}{\partial\nu} = -\eta\frac{\partial}{\partial\nu}(Lg_\lambda) \quad \text{on} \quad N_1\,.$$

Using a known L^2 *a priori* inequality for elliptic equations (see, for instance, [1]), we get

$$(5.7)\qquad ||D^2v||_{L^2(M)} \leqq c\,||\nabla\eta\cdot\nabla u + u\Delta\eta||_{L^2(M)} + c\left|\left|\eta\frac{\partial}{\partial\nu}(Lg_\lambda)\right|\right|_{H^1(N_1)}.$$

Note that

$$(5.8)\qquad \left|\left|\eta\frac{\partial}{\partial\nu}(Lg)_\lambda\right|\right|_{H^1(N_1)} \leqq c\left|\left|\frac{\partial}{\partial\nu}(Lg_\lambda)\right|\right|_{L^2(N^1)} + c\sum_{j=1}^{n-1}\left|\left|\frac{\partial}{\partial s_j}\left(\frac{\partial}{\partial\nu}(Lg_\lambda)\right)\right|\right|_{L^2(N^1)}$$

where $\partial/\partial s_j$ are tangential derivatives along N^1. Now, the first term on the right hand side of (5.8) can be estimated by an inequality similar to (4.13). The same is true of each of the terms in the sum. (Indeed, we can write

$$\int_{N_1}\left|\frac{\partial}{\partial s_j}\left(\frac{\partial}{\partial\nu}(Lg_\lambda)\right)\right|^2 dS \leqq \int_{N_1}\left|\int Le_x^j(\xi')\cdot g_\lambda(\xi')\,d\xi'\right|^2 dS$$

where $e_x^j = E * d_x^j$, $d_x^j = \partial d_x/\partial s_j$, and d_x is the function defined following (4.11).) We conclude that

$$\int\left|\left|\eta\frac{\partial}{\partial\nu}(Lg_\lambda)\right|\right|^2_{H^1(N_1)} d\sigma(\lambda) \leqq c < \infty.$$

Combining this result with (4.14), it follows from (5.7) that

$$\iint_M |D^2(\eta w_\lambda)|^2\,dx\,d\sigma(\lambda) \leqq c < \infty.$$

Hence

$$\iint_{M_0} |D^2 w_\lambda|^2\,dx\,d\sigma(\lambda) \leqq c.$$

Since M, M_0 were quite arbitrary, the proof of (5.6) is complete for $|\beta| = 2$, in case $\bar{M}$ intersects Γ_b. If M intersects some of the Γ_i, then the proof is similar.

Finally, the proof for any β follows step-by-step, analogously to the proof for $|\beta| = 2$.

From (5.5), (5.4) and (4.6) we conclude:

Theorem 5.1. *Let the assumptions of Theorem* 4.1 *be satisfied. If* Γ' *is of class* C^m *then* u *is in* $H^m_{\text{loc}}(\Omega \cup \Gamma')$.

Corollary. *If* Γ' *is of class* C^m *with* $m \geqq [(n+2)/2] + m_0$, *then* u *belongs to* $C^{m_0}(\Omega \cap \Gamma')$.

The corollary follows by using the Sobolev inequality.

6. Validity of the boundary condition at the bottom. We assume throughout this section that the assumptions of Theorem 4.1 are valid and that Γ' is of class C^m with some $m \geqq [(n+2)/2] + 1$ so that u is continuously differentiable in $\Omega \cup \Gamma'$. We shall prove that $\partial u/\partial\nu = 0$ on Γ'.

We first prove two lemmas. The first will be used in Sec. 7 while the second will be used in the present section.

Lemma 6.1. *Let* $A_0 = \{(x', y_0);\ x' \in A\}$ *where* A *is a bounded domain in* R^{n-1}, *and assume that* $\bar{A}_0 \subset \Omega$. *Then for any* $\varphi(x') \in C_0^\infty(A)$,

$$\text{(6.1)}\quad \int_A u(x', y_0)\varphi(x')\,dx' = \int_0^\infty \overline{(\chi_\lambda, u^0)} \cos(\lambda)^{1/2}t\left[\int_A \chi_\lambda(x', y_0)\varphi(x')\,dx'\right] d\sigma(\lambda)$$
$$+ \int_0^\infty \overline{(\chi_\lambda, u_t^0)} \frac{\sin(\lambda)^{1/2}t}{(\lambda)^{1/2}} \left[\int_A \chi_\lambda(x', y_0)\varphi(x')\,dx'\right] d\sigma(\lambda).$$

Proof. It is enough to consider the case where $u_t^0 \equiv 0$. Let $\zeta_m(y)$ be a C^∞ function satisfying: $\zeta_m(y) = 0$ if $|y - y_0| > 1/m$, $\zeta_m(y) \geqq 0$ and $\int \zeta_m(y)\,dy = 1$. Let m_0 be such that if $m \geqq m_0$ then the closure of the set $A_m = \{(x', y);\ x \in A,\ |y - y_0| < 1/m\}$ lies in Ω. Then, for $m \geqq m_0$ we have

$$\text{(6.2)}\quad \iint u(x)\varphi(x')\zeta_m(y)\,dx'\,dy$$
$$= \int \overline{(\chi_\lambda, u^0)} \cos(\lambda)^{1/2}t\left[\iint \chi_\lambda(x)\varphi(x')\zeta_m(y)\,dx'\,dy\right] d\sigma(\lambda).$$

As $m \to \infty$ the left hand side of (6.2) converges to the left hand side of (6.1). Hence it remains to prove the same for the right hand sides of (6.2), (6.1). Their difference is equal to

$$\int \overline{(\chi_\lambda, u^0)} \cos(\lambda)^{1/2}t\left\{\iint L[g_\lambda(x', y_0) - g_\lambda(x', y)]\varphi(x')\zeta_m(y)\,dx'\,dy\right\} d\sigma(\lambda)$$
$$+ \int \overline{(\chi_\lambda, u^0)} \cos(\lambda)^{1/2}t\left\{\iint [w_\lambda(x', y_0) - w_\lambda(x', y)]\varphi(x')\zeta_m(y)\,dx'\,dy\right\} d\sigma(\lambda)$$
$$\equiv I_m + J_m\,.$$

Thus, if $I_m \to 0$, $J_m \to 0$ as $m \to \infty$, then the lemma follows.

Writing $g_\lambda(x', y_0) - g_\lambda(x', y)$ in its explicit form (by (3.29)), we find that

$$|I_m|^2 \leqq c \int \left|\int L\varphi_m(\xi')\cdot g(\xi')\,d\xi'\right|^2 d\sigma(\lambda) = c\int |(\chi_\lambda, \varphi_m)|^2\,d\sigma(\lambda) = c\,||\varphi_m||_1^2\,,$$

where $\varphi_m = E * \psi_m$ (so that $L\varphi_m = \psi_m$) and

$$\text{(6.3)}\quad \psi_m(\xi') = (1 + |\xi'|^2)^{-p} \iint (1 + |x'|^2)^p$$
$$\cdot[\Gamma(x', y; \xi') - \Gamma(x', y_0; \xi')]\varphi(x')\zeta_m(y)\,dx'\,dy.$$

From (1.52) it follows that each derivative of ψ_m (and hence also of φ_m) is bounded by a constant times $(1 + |\xi'|^2)^{-p}/m$. Hence,

$$|I_m| \leqq \frac{c}{m} \to 0 \quad \text{as} \quad m \to \infty\,.$$

To prove that $J_m \to 0$ it suffices to prove that

$$K_m \equiv \int \left\{ \iint |w_\lambda(x', y_0) - w_\lambda(x', y)|^2 \zeta_m(y) \, dx' \, dy \right\} d\sigma(\lambda) \to 0 \quad \text{as} \quad m \to \infty .$$

By the mean value theorem,

$$|w_\lambda(x', y_0) - w_\lambda(x', y)|^2 \leqq |y_0 - y| \int_{y_1}^{y_2} \left| \frac{\partial}{\partial \eta} w_\lambda(x', \eta) \right|^2 d\eta$$

where $y_1 = y_0 - 1/m_0$, $y_2 = y_0 + 1/m_0$. Since $|y - y_0| \leqq 1/m$, we obtain

$$K_m \leqq \frac{1}{m} \int \left\{ \int_{A_{m_0}} |\nabla w_\lambda(x)|^2 \, dx \right\} d\sigma(\lambda) \leqq \frac{c}{m}$$

where (4.14) has been used. This completes the proof of the lemma.

Lemma 6.2. *Let A be a bounded domain in R^{n-1} and let $B = \{(x', y); x' \in A, y_0 < y < y_1\}$ be a domain with $\bar{B} \subset \Omega \cup \Gamma'$. Let $\varphi(x) \in C^\infty(B)$ such that $\varphi(x', y) = 0$ in a neighborhood of $\partial_0 B = \{(x', y); x' \in \partial A, y_0 \leqq y \leqq y_1\}$. Then*

$$(6.4) \qquad \int_B u(x)\varphi(x) \, dx = \int_0^\infty \overline{(\chi_\lambda , u^0)} \cos (\lambda)^{1/2} t \left[\int_B \chi_\lambda(x)\varphi(x) \, dx \right] d\sigma(\lambda) + \int_0^\infty (\chi_\lambda , u_t^0) \frac{\sin (\lambda)^{1/2} t}{(\lambda)^{1/2}} \left[\int_B \chi_\lambda(x)\varphi(x) \, dx \right] d\sigma(\lambda).$$

Proof. The proof is similar to the proof of Lemma 6.1. We define $\eta_m(y)$ to be C^∞ functions satisfying: $\eta_m(y) = 1$ if $y_0 + 1/m < y < y_1 - 1/m$, $\eta_m(y) = 0$ if $y < y_0 + 1/2m$ or if $y > y_1 - 1/2m$, and $0 \leqq \eta_m(y) \leqq 1$ elsewhere. Then we apply (4.1) to $\eta_m(y)\varphi(x', y)$ and we prove that the corresponding sides of the equality converge to the corresponding sides of (6.3). Instead of the ψ_m given in (6.3) we now have

$$\psi_m(\xi') = (1 + |\xi'|^2)^{-p} \iint (1 + |x'|^2)^p \Gamma(x', y; \xi')\varphi(x)[1 - \eta_m(y)] \, dx' \, dy,$$

and each derivative of ψ_m is bounded by a constant times $(1 + |\xi'|^2)^{-p}/m$.

The analog of J_m is now bounded by the square root of

$$c \left\{ \iint_B |w_\lambda(x)|^2 \, dx \, d\sigma(\lambda) \right\} \int_B |\varphi(x)[1 - \eta_m(y)]|^2 \, dx$$

and therefore it converges to 0 as $m \to \infty$. The proof is thereby completed.

Theorem 6.1. *Let the assumptions of Theorem 4.1 be satisfied and let $\Gamma' \in C^m$ for some $m \geqq [(n + 2)/2] + 1$. Then $\partial u/\partial \nu = 0$ on Γ_b.*

Proof. It suffices to consider the case where $u_t^0 = 0$. For any $r > R$, set $\Gamma_{b,r} = \Gamma_b \cap \{(x', y); |x'| < r\}$. Let $\triangle_r$ be a smooth manifold lying in Ω, with $\partial \triangle_r = \partial \Gamma_{b,r}$ and such that the domain Ω_r bounded by $\overline{\triangle}_r \cup \Gamma_{b,r}$ has a smooth boundary also at $\partial \triangle_r$. Similarly we construct $\triangle_{r_m}$, Ω_{r_m} for Γ_{b,r_m} where $r_m =$

$r - 1/m$. We can choose the $\triangle_{r_m}$ such that

$$\bar{\Omega}_{r_m} \subset (\text{int. } \Omega_{r_{m+1}}) \cup \Gamma_{b,r_{m+1}} .$$

Let φ be any smooth function in $\bar{\Omega}_r$, satisfying

$$\triangle\varphi = 0 \quad \text{in} \quad \Omega_r , \qquad \frac{\partial\varphi}{\partial\nu} = 0 \quad \text{on} \quad \Gamma_{b,r} . \tag{6.5}$$

For fixed m let $\eta \in C^\infty(\bar{\Omega}_r)$ be such that $\eta = 1$ in Ω_{r_m}, $\eta = 0$ outside $\Omega_{r_{m+1}}$, $0 \leqq \eta \leqq 1$ in Ω_r and $\partial\eta/\partial\nu = 0$ on $\Gamma_{b,r}$. Consider the function $\psi = \eta\varphi$. It satisfies:

$$\triangle\psi = 0 \text{ in } \Omega_{r_m} \text{ and outside } \Omega_{r_{m+1}} , \tag{6.6}$$

$$\frac{\partial\psi}{\partial\nu} = 0 \quad \text{on} \quad \Gamma_{b,r} . \tag{6.7}$$

By the assumption (A_R), $h(x') \equiv -\lambda$ if $|x'| > R$. Denote by M the support of $\triangle\psi$ and introduce the sets

$$M_1 = \{(x', y); (x', y) \in M, -\lambda + \epsilon < y < 0\},$$
$$M_2 = \{(x', y); (x', y) \in M, -\lambda < y < -\lambda + 2\epsilon\}.$$

If ϵ is sufficiently small then M_1, M_2 form an open covering of M and (by an appropriate construction of the Ω_{r_m}) we may assume that M_2 is contained in a set B as in Lemma 6.2.

Let $\alpha_j \in C^\infty(M)$ $(j = 1, 2)$ be such that $0 \leqq \alpha_j$, $\alpha_1 + \alpha_2 \equiv 1$, $\alpha_1(x', y) = 0$ if $y < -\lambda + \epsilon$, $\alpha_2(x', y) = 0$ if $y > -\lambda + 2\epsilon$. We claim that

$$\begin{aligned} &\int_{\Omega_r} u(x)\alpha_j(x)\triangle\psi(x)\,dx \\ &\qquad = \int \overline{(\chi_\lambda , u^0)} \cos (\lambda)^{1/2} t \left[\int_{\Omega_r} \chi_\lambda(x)\alpha_j(x)\triangle\psi(x)\,dx\right] d\sigma(\lambda). \end{aligned} \tag{6.8}$$

Indeed, for $j = 1$ this follows from (4.1) and (4.17), whereas for $j = 2$ this follows from Lemma 6.2. Now,

$$\int_{\Omega_r} \chi_\lambda \triangle\psi\,dx = \int_{\partial\Omega_r} \left(\chi_\lambda \frac{\partial\psi}{\partial\nu} - \psi \frac{\partial\chi_\lambda}{\partial\nu}\right) dS = 0 \tag{6.9}$$

since $\partial\chi_\lambda/\partial\nu = 0$ and $\partial\psi/\partial\nu = 0$ on $\Gamma_{b,r}$ whereas $\psi = 0$ in a neighborhood of $\triangle_r$. Summing over j in (6.8), and using (6.9), we get

$$\int_{\Omega_r} u(x)\triangle\psi(x)\,dx = 0.$$

But

$$\int_{\Omega_r} u\triangle\psi\,dx = \int_{\Gamma_{b,r}} \left(\frac{\partial\psi}{\partial\nu} u - \psi \frac{\partial u}{\partial\nu}\right) dS = -\int_{\Gamma_{b,r}} \eta\varphi \frac{\partial u}{\partial\nu}\,dS.$$

Hence, $\int_{\Gamma_{b,r}} \eta\varphi(\partial u/\partial\nu)\, dS = 0$. Using this relation for $\eta = \eta_m$ where $m \to \infty$, we get

$$\int_{\Gamma_{b,r}} \varphi \frac{\partial u}{\partial \nu}\, dS = 0 \tag{6.10}$$

for any φ satisfying (6.5).

The proof of the theorem is completed by using the following lemma.

Lemma 6.3. *Let G be a bounded domain whose boundary ∂G is smooth and consists of a finite number of smooth connected manifolds. Let Γ_1 be a subdomain of ∂G such that $\partial\Gamma_1$ is smooth. Set $\Gamma_2 = \partial G \backslash \bar{\Gamma}_1$. Let f be any continuous function on $\bar{\Gamma}_1$ such that*

$$\int_{\Gamma_1} f\varphi\, dS = 0 \tag{6.11}$$

for any smooth φ in $\bar{G}$ satisfying

$$\triangle\varphi = 0 \quad \text{in} \quad G, \qquad \frac{\partial\varphi}{\partial\nu} = 0 \quad \text{on} \quad \Gamma_1 . \tag{6.12}$$

Then $f \equiv 0$.

Proof. Let $N(x;\ \xi)$ be the Neumann function for the Laplace equation in G and let $c_0 = \partial N(x;\xi)/\partial\nu$ (c_0 is a constant). Introduce $N_0(x;\xi) = N(x,\xi) - c_0$. The function

$$\varphi(x) = \int_{\Gamma_2} N_0(x;\xi)\mu(\xi)\, dS$$

satisfies (6.12) for any smooth function $\mu(\xi)$ on Γ_2. Substituting it in (6.11) and changing the order of integration, we get

$$\int_{\Gamma_2} \left\{ \int_{\Gamma_1} N_0(x;\xi) f(x)\, dS \right\} \mu(\xi)\, dS = 0.$$

Hence

$$g(\xi) \equiv \int_{\Gamma_1} N_0(x;\xi) f(x)\, dS$$

satisfies $g = 0$ on Γ_2 (since $N_0(x;\ \xi) = N_0(\xi,\ x)$). But we also have $\triangle g = 0$ in G, $\partial g/\partial\nu = 0$ on Γ_2. By a well known theorem on continuation of harmonic functions, $g \equiv 0$ in G as well as outside G. Using the jump relations of $\partial g/\partial\nu$ at Γ_1, we then conclude that $f \equiv 0$.

7. Validity of the boundary condition at $y = 0$. For simplicity we shall first consider the case where $\zeta(x') \equiv 1$ and $u_t^0 = 0$. We shall say that $\varphi(x')$ belongs to C_*^∞ if $\varphi \in C^\infty(R^{n-1})$ and if each derivative of φ decreases faster than any power of $|x'|^{-1}$ as $x' \to \infty$. We denote by δ a fixed positive number such that the strip $-\delta \leqq y < 0$ lies in G.

Lemma 7.1. Let (A_R) be satisfied. If $f \varepsilon C_*^\infty$ then the solution φ of (3.7)–(3.9) (with $\zeta \equiv 1$) satisfies: For each y, $-\delta \leqq y \leqq 0$, $\varphi(x', y)$, $\varphi_y(x', y)$ belong to C_*^∞ and for any r, α ($r = 0, 1, 2, \cdots, 0 \leqq |\alpha| < \infty$),

$$|D_{x'}^\alpha \varphi(x', y)| + |D_{x'}^\alpha \varphi_y(x', y)| \leqq C(1 + |x'|)^{-r}, \tag{7.1}$$

$$|D_{x'}^\alpha \varphi(x', y) - D_{x'}^\alpha \varphi(x', 0)| \leqq c(y)(1 + |x'|)^{-r}, \tag{7.2}$$

$$|D_{x'}^\alpha \varphi_y(x', y) - D_{x'}^\alpha \varphi_y(x', 0)| \leqq c(y)(1 + |x'|)^{-r}, \tag{7.3}$$

where C is a constant independent of y and $c(y) \to 0$ as $y \to 0$ (C and $c(y)$ depend on r, α).

Proof. From the proof of Lemma 3.1 we have

$$\int_{R^{n-1}} M(x') \, |D_{x'}^\alpha \varphi(x', y)|^2 \, dx' \leqq C \qquad (-\delta \leqq y \leqq 0) \tag{7.4}$$

for any α and for any $M(x')$ of the form $(1 + \sigma |x'|^2)^r$, where σ is a sufficiently small positive integer. By the Sobolev inequality we then get

$$|D_{x'}^\alpha \varphi(x', y)| \leqq C(1 + |x'|)^{-r},$$

for any r, α.

To complete the proof of (7.1) it suffices to show that

$$\int_{R^{n-1}} M(x') \, |D_{x'}^\alpha \varphi_y(x', y)|^2 \, dx' \leqq C, \tag{7.5}$$

for then we can use the Sobolev inequality as before. It is clearly sufficient to prove (7.5) when the integration is restricted to $|x'| > R^*$ (R^* was introduced in the proof of Lemma 3.1). Setting $w = D_y \varphi$ we have (compare the proof of Lemma 3.1),

$$\triangle w = 0 \quad \text{in} \quad \Omega^*,$$

$$w_y + \alpha w = F \quad \text{on} \quad \Gamma^* \qquad \left(F = -\sum_{i=1}^{n-1} \frac{\partial^2 \varphi}{\partial x_i^2} + \alpha f - \alpha \varphi\right),$$

$$w_\nu + \alpha w = F_0 \quad \text{on} \quad \Gamma_b^* \qquad \left(F_0 = -\sum_{i=1}^{n-1} \frac{\partial^2 \varphi}{\partial x_i^2} \text{ since } w = 0 \quad \text{on} \quad \Gamma_b^*\right),$$

$$w = D_y \varphi \quad \text{on} \quad \Gamma_0^* .$$

Even though in (3.17) $w_\nu = 0$ on Γ_b^*, the present system is not essentially different from (3.17) since F_0 belongs to C_*^∞. By a treatment similar to that of (3.17) we can then derive (7.5).

(7.2) is a consequence of (7.1) and the mean value theorem.

If we prove that

$$|D_{x'}^\alpha D_y^2 \varphi(x', y)| \leqq C(1 + |x'|)^r \qquad (-\delta \leqq y \leqq 0) \tag{7.6}$$

for any r, α, then (7.3) follows also by using the mean value theorem. But, since $\triangle \varphi = 0$, (7.6) is a consequence of (7.1).

Lemma 7.2. *Let (A_R) hold and let $u^0 \in \Phi$. Then there exists an $r = r(n)$ and a function $w(x')$ in $L^2(R^{n-1})$ such that for $-\delta \leqq y \leqq 0$,*

$$\frac{|u(x', y, t)|}{(1 + |x'|^2)^r} \leqq w(x'), \tag{7.7}$$

and for any $\varphi \in C_*^\infty$,

$$\int u(x', y, t)\varphi(x')\, dx' = \int (\chi_\lambda, u^0) \cos (\lambda)^{1/2}t\left[\int \chi_\lambda(x', y)\varphi(x')\, dx'\right] d\sigma(\lambda). \tag{7.8}$$

Proof. For $\varphi \in C_0^\infty$, (7.8) is the assertion of Lemma 6.1. A review of the estimates for the right hand side of (7.8), given in §4, shows that this integral exists for all $\varphi \in C_*^\infty$ and it is bounded by

$$c \int (1 + |x'|^2)^r\, |\varphi(x')|^2\, dx' \qquad (r = r(n)) \tag{7.9}$$

where c is a constant independent of y. Using this bound in (7.8) for $\varphi \in C_0^\infty$, we find that (7.7) holds. For any $\varphi \in C_0^\infty$, (7.8) now follows by using (7.8) with φ replaced by $\alpha_\nu\varphi$, for a suitable sequence of $\alpha_\nu \in C_0^\infty$, and taking $\nu \to \infty$. Indeed, using (7.7) we obtain $\int u\alpha_\nu\varphi \to \int u\varphi$, whereas by slightly modifying estimates derived in §4 (compare (7.9)) we get

$$\left|\int (\chi_\lambda, u^0) \cos (\lambda)^{1/2}t\left\{\int \chi_\lambda(x', y)[\alpha_\nu(x')\varphi(x') - \varphi(x')]\, dx'\right\} d\sigma(\lambda)\right|^2$$
$$\leqq c \int (1 + |x'|^2)^r\, |1 - \alpha_\nu(x')|^2\, |\varphi(x')|^2\, dx' \to 0 \quad \text{as} \quad \nu \to \infty.$$

Lemma 7.3. *Let (A_R) hold and let $\varphi(x', y)$ be a C^∞ function for $x' \in R^n$, $-\delta \leqq y \leqq 0$ having the following properties: For each y, $\varphi(x', y) \in C_*^\infty$ and for any $r \geqq 0$, $|\alpha| \geqq 0$,*

$$\int_{R^{n-1}} (1 + |x'|^2)^r\, |D_x^\alpha\varphi(x, y) - D_x^\alpha\varphi(x, 0)|^2\, dx' \leqq c(y), \qquad c(y) \to 0 \quad \text{if} \quad y \to 0.$$

Then

$$\lim_{y \to 0} \int (\chi_\lambda, u^0) \cos (\lambda)^{1/2}t\left[\int \chi_\lambda(x', y)\varphi(x', y)\, dx'\right] d\sigma(\lambda) \tag{7.10}$$
$$= \int (\chi_\lambda, u^0) \cos (\lambda)^{1/2}t(\chi_\lambda, \varphi(\cdot, 0))\, d\sigma(\lambda).$$

Proof. Consider first the case where φ is independent of y; *i.e.*, $\varphi(x', y) = \varphi(x')$. Using (3.28), we see that (7.10) would follow if we could prove that

$$\int \left|\int [w_\lambda(x', y) - w_\lambda(x', 0)]\varphi(x')\, dx'\right|^2 d\sigma(\lambda) \to 0 \quad \text{as} \quad y \to 0 \tag{7.11}$$

and that

(7.12)
$$\int \left| \int g_\lambda(\xi')\psi_y(\xi')\, d\xi' \right|^2 d\sigma(\lambda) \to 0 \quad \text{as} \quad y \to 0$$

where

(7.13)
$$\psi_y(\xi') = (1 + |\xi'|^2)^{-p} \int_{R^{n-1}} [\Gamma(x', y, \xi) - \Gamma(x', 0, \xi')](1 + |x'|^2)^p L\varphi(x')\, dx'.$$

Note that the last integral is taken in the sense of the principal value.

By known estimates on the derivatives of integrals of the form $\int \Gamma\mu\, dx'$ with μ sufficiently smooth, it follows that

$$|D^\alpha \psi_y(\xi')| \leqq c(y)(1 + |\xi'|^2)^{-p} \qquad (c(y) \to 0 \text{ if } y \to 0).$$

From this we can easily deduce (7.12).

To prove (7.11) we use the mean value theorem and (3.46).

Consider now the general case and write

$$\varphi(x', y) = \varphi(x', 0) + [\varphi(x', y) - \varphi(x', 0)].$$

Since (7.10) holds with φ replaced by $\varphi(x', 0)$, it remains to prove that

$$\int (\chi_\lambda, u^0) \cos (\lambda)^{1/2} t \left\{ \int \chi_\lambda(x', y)[\varphi(x', y) - \varphi(x', 0)]\, dx' \right\} d\sigma(\lambda) \to 0 \quad \text{as} \quad y \to 0.$$

This is proved again by using the representation (3.28).

Lemma 7.4. *Let (A_R) hold and let $u^0 \varepsilon \Phi$. Then, for any $\varphi \varepsilon C_*^\infty$, $-\delta < y < 0$,*

(7.14)
$$\int u_y(x', y, t)\varphi(x')\, dx' = \int (\chi_\lambda, u^0) \cos (\lambda)^{1/2} t \left[\int \frac{\partial}{\partial y} \chi_\lambda(x', y)\varphi(x')\, dx' \right] d\sigma(\lambda).$$

Note that the outer integrand on the right hand side is measurable with respect to $d\sigma(\lambda)$ and that the integral is finite. Indeed, this follows by slightly modifying the proofs for the case where $\partial\chi_\lambda/\partial y$ is replaced by χ_λ.

Proof. From (7.7) and the fact that u is harmonic, it follows that

(7.15)
$$|u_y(x', y, t)| \leqq b(y)(1 + |x'|^2)^r w_0(x') \qquad \left(\int |w_0(x')|^2\, dx' < \infty \right)$$

where $b(y)$ is a bounded function in any closed subinterval of $(-\delta, 0)$. Hence the y-derivative of the left hand side of (7.8) is equal to the left hand side of (7.14). To prove the same assertion for the right hand side, we take a finite difference with respect to y. If we can show that

(7.16)
$$\int \left| \int \left[\frac{\chi_\lambda(x', y + h) - \chi_\lambda(x', y)}{h} - \frac{\partial \chi_\lambda(x', y)}{\partial y} \right] \varphi(x')\, dx' \right|^2 d\sigma(\lambda) \to 0 \quad \text{as} \quad h \to 0,$$

then (7.14) is completely proved.

Using (3.28), the representation (3.29) and the bound

$$\iint (1 + |x'|^2)^r \, |D^\beta w_\lambda(x)|^2 \, dx \, d\sigma(\lambda) < \infty \tag{7.17}$$

for $|\beta| \leqq 2$ and some $r > 0$, the proof of (7.16) follows without difficulty. Note that (7.17) for $|\beta| \leqq 1$ was proved in §3 (see (3.46)). For $|\beta| = 2$, we have only proved (5.6), but that proof, when slightly modified, yields (7.17).

By Green's formula we have:

$$\int \frac{\partial}{\partial y} \chi_\lambda(x', y) \cdot \varphi(x', y) \, dx' = -\int \chi_\lambda(x', y) \varphi_y(x', y) \, dx' \tag{7.18}$$

for any φ as in Lemma 7.1. Writing (7.14) for $\varphi = \varphi(x', y)$ and using (7.18), we obtain

$$\int u_y(x', y, t)\varphi(x', y) = \int (\chi_\lambda, u^0) \cos (\lambda)^{1/2} t \left[\int \chi_\lambda(x', y)\varphi_y(x', y) \, dx' \right] d\sigma(\lambda).$$

Taking $y \to 0$ and using Lemmas 7.1, 7.3, we get:

$$\lim_{y \to 0} \int u_y(x', y, t)\varphi(x', y) = \int (\chi_\lambda, u^0) \cos (\lambda)^{1/2} t (\chi_\lambda, \varphi_y(\cdot, 0)) \, d\sigma(\lambda). \tag{7.19}$$

We shall now need the following condition:

(U^0) There exists a function U^0 satisfying

$$\triangle U^0 = 0 \quad \text{in} \quad \Omega,$$

$$\frac{\partial U^0}{\partial y} + \alpha U^0 = f^0 \quad \text{on} \quad \Gamma,$$

$$\frac{\partial}{\partial \nu} U^0 = 0 \quad \text{on} \quad \Gamma',$$

such that $U^0 = u^0$ on Γ and u^0, f^0 belong to Φ.

Similarly we define the condition (U_t^0) for u_t^0.

Observe that $u^0 = Tf^0$.

At this point we recall that the χ_λ of §2 satisfy (2.7) so that after having performed the transformation (3.51) (and omitting the "roofs"), we have:

$$(\chi_\lambda, \varphi) = (\lambda + \alpha)(\chi_\lambda, T\varphi) \quad \text{if} \quad \varphi \in \Phi. \tag{7.20}$$

Therefore, if u^0 satisfies the condition (U^0),

$$\lambda(\chi_\lambda, u^0) = \lambda(\chi_\lambda, Tf^0) = \frac{\lambda}{\lambda + \alpha} (\chi_\lambda, f^0).$$

Consequently,

$$\int |\lambda(\chi_\lambda, u^0)|^2 \, d\sigma(\lambda) < \infty. \tag{7.21}$$

Assume (U^0) and consider (7.8) with $\varphi(x')$ replaced by a function $\varphi(x', y)$ as in Lemma 7.1. If we differentiate formally twice with respect to t, we obtain

$$\frac{\partial^2}{\partial t^2} \int u(x', y, t)\varphi(x', y) = -\int (\chi_\lambda, u^0)\lambda \cos (\lambda)^{1/2} t \left[\int \chi_\lambda(x', y)\varphi(x', y)\, dx' \right] d\sigma(\lambda). \tag{7.22}$$

To prove (7.22) rigorously, we take finite differences, and then go to the limit using the Lebesgue convergence theorem and (7.21).

Taking $y \to 0$ in (7.22) and using Lemma 7.3 (or, rather, the proof of Lemma 7.3), we obtain

$$\lim_{y\to 0} \frac{\partial^2}{\partial t^2} \int u(x', y, t)\varphi(x', y)\, dx' = -\int (\chi_\lambda, u^0)\lambda \cos (\lambda)^{1/2} t(\chi_\lambda, \varphi(\cdot, 0))\, d\sigma(\lambda). \tag{7.23}$$

We claim that the right hand sides of (7.23) and of (7.19) are equal except for the sign. It suffices to prove that

$$(\chi_\lambda, \varphi_y(\cdot, 0)) = \lambda(\chi_\lambda, \varphi(\cdot, 0)). \tag{7.24}$$

Now,

$$(\chi_\lambda, f) = (\chi_\lambda, \varphi_y + \alpha\varphi) = \alpha(\chi_\lambda, \varphi) + (\chi_\lambda, \varphi_y).$$

Employing (7.20) we then get

$$(\chi_\lambda, \varphi_y) = (\lambda + \alpha)(\chi_\lambda, Tf) - \alpha(\chi_\lambda, \varphi) = \lambda(\chi_\lambda, \varphi).$$

We have thus proved that

$$\lim_{y\to 0} \int \left(\frac{\partial^2}{\partial t^2} + \frac{\partial}{\partial y}\right) u(x', y, t)\cdot\varphi(x', y)\, dx' = 0. \tag{7.25}$$

The derivative $\partial^2/\partial t^2$ has to be taken in the sense of distributions, *i.e.*, $\int (\partial^2 u/\partial t^2)\varphi = (\partial^2/\partial t^2) \int u\varphi$. If $u_t^0 \not\equiv 0$ but u_t^0 satisfies (U_t^0), then the same result is valid with obvious modifications in the proof. We can therefore state:

Theorem 7.1. *Let the assumption (A_R) hold and let u^0, u_t^0 satisfy the conditions (U^0) and (U_t^0) respectively. Then the solution $u(x, t)$ satisfies* (7.25), *for any $\varphi(x)$ as in Lemma* 7.1.

It remains to consider the initial condition on Γ. Suppose that first $u_t^0 \equiv 0$. From Lemmas 7.2, 7.3 we get

$$\lim_{y\to 0} \int u(x', y, t)\varphi(x')\, dx' = \int (\chi_\lambda, u^0) \cos (\lambda)^{1/2} t(\chi_\lambda, \varphi)\, d\sigma(\lambda).$$

It is easily proved that the right hand side is continuous in t. Hence, if we take $t \to 0$ and use (2.8) we get

$$\lim_{t\to 0}\lim_{y\to 0}\int u(x', y, t)\varphi(x')\,dx' = \int u^0(x')\varphi(x')\,dx'. \tag{7.26}$$

Similarly we get

$$\lim_{y\to 0}\lim_{t\to 0}\int u(x', y, t)\varphi(x')\,dx' = \int u^0(x')\varphi(x')\,dx'. \tag{7.27}$$

If u^0 satisfies (U^0) then we can similarly prove that

$$\lim_{t\to 0}\lim_{y\to 0}\int \frac{\partial}{\partial t}u(x', y, t)\varphi(x')\,dx' = \int u_t^0(x')\varphi(x')\,dx', \tag{7.28}$$

$$\lim_{y\to 0}\lim_{t\to 0}\int \frac{\partial}{\partial t}u(x', y, t)\varphi(x')\,dx' = \int u_t^0(x')\varphi(x')\,dx', \tag{7.29}$$

where, of course, $u_t^0 \equiv 0$. If $u_t^0 \not\equiv 0$ and u_t^0 is in Φ (but does not necessarily satisfy (U_t^0)), then (7.26)–(7.29) still hold. We sum up:

Theorem 7.2. *Let the assumption* (A_R) *hold and let* u^0 *satisfy* (U^0) *and* $u_t^0 \varepsilon \Phi$. *Then the solution* $u(x, t)$ *satisfies* (7.26)–(7.29).

Remark 1. If in Theorem 7.1 we replace the assumptions (U^0), (U_t^0) by the assumption that u^0, u_t^0 belong to Φ, then we can prove (7.25) in an integrated form, *i.e.*,

$$\lim_{y\to 0}\left\{\int u(x', y, t)\varphi(x', y)\,dx' + \int_0^t dt\int_0^t dt\int u_y(x', y, t)\varphi(x', y)\,dx'\right\} = \int u^0(x')\varphi(x', 0)\,dx' + t\int u_t^0(x')\varphi(x', 0)\,dx'. \tag{7.30}$$

(7.28), (7.29) are also satisfied in an integrated form.

Remark 2. The relations (7.25), (7.30) extend to the case where $\zeta(x') \not\equiv 1$. Then, instead of (7.25) we have:

$$\lim_{y\to 0}\int \left(\zeta(x')\frac{\partial^2}{\partial t^2} + \frac{\partial}{\partial y}\right)u(x', y, t)\cdot\varphi(x', y)\,dx' = 0. \tag{7.31}$$

Here we have to assume that $\zeta \varepsilon C_*^\infty$.

Remark 3. The space C_*^∞ can be replaced throughout this section by a space $C_*^{m,r}$ of functions $\varphi(x')$ satisfying

$$|D^\alpha\varphi(x')| \leqq c(1 + |x'|)^{-r} \qquad (0 \leqq |\alpha| \leqq m).$$

Here m and r are some constants depending only on n. Remark 2 is also valid if $\zeta \varepsilon C_*^{m,r}$ (instead of $\zeta \varepsilon C_*^\infty$).

Remark 4. In Lemma 7.2 we have proved (7.7) provided $u^0 \varepsilon \Phi$, $u_t^0 = 0$. If $u_t^0 \not\equiv 0$ then we can derive:

(7.32) $$|u(x', y, t)| \leqq (1 + |x'|^2)^r w(x')(1 + t), \qquad \int |w(x')|^2 \, dx' < \infty$$

for all $-\delta < y < 0$.

Remark 5. If $u(x, t)$ is known to be smooth in $G \cup \Gamma$ and if u_{tt}, u_y belong to $L^2(\Gamma)$, then (7.25) reduces to

(7.33) $$u_{tt} + u_y = 0 \quad \text{on} \quad \Gamma.$$

Indeed, in that case we get

(7.34) $$\int g_t(x')\varphi(x', 0) \, dx' = 0 \qquad (g_t(x') \equiv [u_{tt} + u_y]_\Gamma)$$

for any φ as in Lemma 7.1. But (7.34) means that $(g_t, Tf) = 0$, *i.e.*, $(Tg_t, f) = 0$. Since f is an arbitrary function in C_*^∞, $Tg_t = 0$. Consequently $g_t = 0$.

The assertion (7.33) should hold even if u_{tt}, u_y are not in $L^2(\Gamma)$ but have polynomial growth instead. In that case the previous argument does not apply, but a proof analogous to the proof of Lemma 6.3 can probably be given.

8. Final form of the existence theorem. In the previous sections we have imposed two restrictions. The first one is the condition (A_R) on Γ_b and the second is the assumption that T has a simple spectrum, *i.e.*, the sum in (3.47) consists of just one term. In this section we shall eliminate these restrictions by employing a sequence of truncations. For the sake of clarity, we shall first still maintain the assumption that the spectrum of T is simple.

We assume that

(8.1) $$\nabla' h(x') = 0(1) \quad \text{as} \quad |x'| \to \infty.$$

Choose any monotone sequence of positive numbers R_k, $R_k \to \infty$. For each k, we modify $h(x')$ so that the new function $h_k(x')$ satisfies:

$$|\nabla' h_k(x')| \leqq C \qquad (C \text{ independent of } k),$$

$h_k(x') = h(x')$ for $|x'| \leqq R_k$, and the condition (A_R) is satisfied for h_k with some $R = R(k) > 0$.

Denote by Ω_k the domain obtained from Ω by this modification, and let $u_k(x, t)$ be the solution of Theorem 4.1 corresponding to Ω_k. Reviewing the proof of Theorem 5.1 we see (here we use Lemmas 1.1.8, 1.1.9) that the local H^m bounds on u_k hold with constants independent of k. Hence, by employing a compactness argument (as in the proof of Theorem 1.5.1) we conclude that the sequence

$$v_k(x, t) \equiv \int_0^t u_k(x, t) \, dt$$

has a subsequence which converges uniformly with its first x-derivatives in every compact subset of $(\Omega \cup \Gamma') \times [0, \infty)$. Denoting the limit by $v(x, t)$ we have $\triangle v = 0$ in Ω, $\partial v/\partial \nu = 0$ on Γ_b.

In order to prove that $\partial v/\partial \nu = 0$ on the obstacles Γ_j ($j = 1, \cdots, j_0$) we have to truncate them at the same time that we truncate Γ_b. We require a portion of each of the truncated domains Γ_j to be flat and horizontal. Therefore we make the following definition.

Definition. We say that Γ_j has the property (B) if Γ_j is of class C^p for some $p \geqq [(n + 2)/2] + 1$ and if Γ_j is a disjoint finite union of sets $L_{j\mu}$, $M_{j\mu}$, $N_{j\nu}$ having the following properties: $M_{j\mu}$ lies on a hyperplane $y = \text{const.} = \lambda_\mu$, it is topologically an $(n - 1)$-dimensional shell with an inner smooth boundary $\partial_0 M_{j\mu}$ and an outer smooth boundary $\partial_1 M_{j\mu}$; $L_{j\mu}$ is a C^p manifold with boundary, and its boundary coincides with $\partial_0 M_{j\mu}$; the manifold $L_{j\mu} \cup M_{j\mu}$ is C^p also at $\partial_0 M_{j\mu}$. Finally, the $N_{j\nu}$ are C^p portions on Γ_j (they "connect" the $\partial_1 M_{j\mu}$).

By modifying the proof of Theorem 6.1 we find that if Γ_j satisfies the condition (B), then $\partial u/\partial \nu = 0$ on $\bigcup_\mu L_{j\mu}$.

We need the following condition:

(B_∞) There exists an increasing sequence of open submanifolds Γ_{jq} of Γ_j with the following properties: For each q there exists a manifold $\hat{\Gamma}_{jq}$ satisfying (B) with $\bigcup_\mu L_{j\mu}$ equal to Γ_{jq}; furthermore, for some $p \geqq [(n + 2)/2] + 1$, the C^p norm of the Γ_{jq} is bounded by a constant independent of q.

If (B_∞) holds for each Γ_j, then we set

$$\Gamma'' = \Gamma_b \cup \left(\bigcup_{j=1}^{j_0} \Gamma_j^* \right) \quad \text{where} \quad \Gamma_j^* = \bigcup_{q=1}^{\infty} \Gamma_{jq} .$$

For general classes of smooth domain (for instance for convex smooth domains) the boundary, say Γ_j, satisfies the condition (B_∞) with Γ_j^* being all of Γ_j except for some smooth $(n - 2)$-dimensional submanifold.

Assuming that (B_∞) holds for all the Γ_j, we can now modify $h(x')$ into $h_k(x')$ and the Γ_j into $\hat{\Gamma}_{jk}$. Denoting the solution corresponding to the modified domain by u_k, we find as before that a subsequence of $\{\int^t u_k(x, t)\, dt\}$ is convergent to a function $v(x, t)$ satisfying

$$\frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \Gamma''. \tag{8.2}$$

We now turn the case where the spectrum of T is not simple, so that the sum in (3.47) consists of more than one term. If the sum is finite and the number of its terms is bounded independently of the above truncations, then, of course, all the previous results hold with trivial changes in the arguments. Let us then suppose that the sum is infinite. We then have to truncate the initial data. Thus we replace u^0, u_t^0 by their projections u^{0k}, u_t^{0k} into the space $\sum_{\gamma=1}^{k} H(e^\gamma)$. We truncate also the boundary Γ' as before, and denote the corresponding solution by u_k. It satisfies

$$\triangle u_k = 0 \quad \text{in} \quad \Omega_k ,$$

$$\frac{\partial u_k}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_k'$$

where Ω_k is the truncated domain and Γ_k' is an appropriate part of the boundary of $\partial\Omega_k\backslash\Gamma$.

u_k satisfies the conditions on $y = 0$ in the sense of Theorem 7.1, 7.2 with u^0, u_t^0 replaced by u^{0k}, u_t^{0k}. Setting $v_k(x, t) = \int_0^t u_k(x, t)\,dt$, we wish to prove that a subsequence of $\{v_k\}$ is uniformly convergent in compact subsets of $(\Omega \times \Gamma'') \times [0, \infty)$. To do so we have to prove that all the local H^m bounds that we have obtained for u_k are independent of k. But this follows by reviewing the method of their derivation in conjunction with the inequality

$$\left|\sum_\gamma \int (\chi_\lambda^{(\gamma)}, u^{0k})\left[\int \chi_\lambda^{(\gamma)}\varphi\,dx\right] d\sigma^{(\gamma)}(\lambda)\right|^2 \leqq \sum_\gamma \int |(\chi_\lambda^{(\gamma)}, u^{0k})|^2\,d\sigma^{(\gamma)}(\lambda) \sum_\gamma \int \left|\int \chi_\lambda^{(\gamma)}\varphi\,dx\right|^2 d\sigma^{(\gamma)}(\lambda).$$

Summary. Let u^0, u_t^0 satisfy (U^0) and (U_t^0) respectively. Let $h \in C^p$ for some $p \geqq [(n + 2)/2] + 1$ and let (8.1) hold. Finally, let (B_∞) hold for all the Γ_j. Then there exists an harmonic function $v(x, t)$, for $t \geqq 0$, satisfying (8.2), and satisfying

$$v = u^0, \qquad v_t = u_t^0 \quad \text{on} \quad \Gamma \quad \text{for} \quad t = 0,$$

$$v_{tt} + v_y = 0 \quad \text{on} \quad \Gamma \quad \text{for} \quad t \geqq 0$$

in the following sense: There exists a sequence of functions $\{v_k(x, t)\}$ converging to $v(x, t)$ uniformly on compact subsets of Ω, and there exist sequences $\{u^{0k}\}$, $\{u_t^{0k}\}$ converging in $L^2(\Gamma)$ to u^0 and u_t^0, respectively, such that $v_k(x, t) = \int_0^t u_k(x, t)\,dt$ and each $u_k(x, t)$ satisfies (7.26)–(7.28) with u^0, u_t^0 replaced by u^{0k} and u_t^{0k}, respectively.

If u^0, u_t^0 are only in Φ, then the conditions at $y = 0$ are satisfied in weaker form (compare Remark 1, §7).

Remark 1. If $\zeta \not\equiv 1$ then analogous results hold provided ζ is as in Remark 3 of §7. So far ζ was always a function bounded from above and below by positive constants. If $\zeta(x')$ approaches 0 as $x' \to \infty$, then all the previous results still remain true. Furthermore, if $|\zeta(x')| \leqq c(1 + |x'|)^{-r}$ for some r sufficiently large (depending only on n) then $g_\lambda(x')$ is in $L^2(R^{n-1})$. The extension $g_\lambda(x', y)$ can then be constructed in the form $G_0 * g_\lambda$ where G_0 is Green's function G_0 for the Laplacian in $y < 0$. Thus there is no need to employ the results of §1.

Remark 2. All the results of this chapter remain true if Γ_b is not everywhere given in the form $y = -h(x')$. Thus, Γ_b may be any smooth manifold in some cylinder $|x'| < R_0$ whereas in the cylinder $|x'| > R_0$ it is given by $y = -h(x')$ with $h(x')$ is smooth and satisfies (8.1).

Chapter 4. Uniqueness

In this chapter we prove a uniqueness theorem for solutions with exponential growth at infinity. The solutions are assumed to satisfy the boundary condition on $y = 0$ in the classical sense. Thus, the uniqueness theorem will not imply that the solutions of Chapters 1–3 are unique. This leaves a gap between the existence theorems of Chapters 1–3 and the uniqueness theorem proved here.

In this chapter we assume, as in Chapter 3, that Ω is a domain in R^n whose boundary consists of three parts. The first part is the hyperplane $y = 0$, which we denote by Γ. The second part is the "bottom" Γ_b, a manifold defined by $y = -h(x')$, where $h(x')$ is defined on all R^{n-1} and satisfies

$$0 < h_0 \leqq h(x') \leqq h_1 < \infty. \tag{1.1}$$

The third part consists of a finite number of compact manifolds Γ_i lying between Γ and Γ_b. We assume that Γ_b and the Γ_i are in $C^{2+\rho}$ for some $0 < \rho < 1$. We write $\Gamma' = \Gamma \cup (\bigcup_i \Gamma_i)$. Finally, we assume that

$$|\nabla' h(x')| = 0(1) \quad \text{as} \quad |x'| \to \infty. \tag{1.2}$$

We shall consider the problem

$$\Delta u = 0 \quad \text{in} \quad \Omega, \tag{1.3}$$

$$u_{tt} + u_y = 0 \quad \text{on} \quad \Gamma, \tag{1.4}$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \Gamma', \tag{1.5}$$

$$u|_{t=0} = 0 \quad \text{on} \quad \Gamma, \qquad u_t|_{t=0} = 0 \quad \text{on} \quad \Gamma. \tag{1.6}$$

We shall say that u is a *classical solution* of (1.3)–(1.6) if u, ∇u, u_t, u_{tt} are continuous in $\bar{\Omega} \times [0, \infty)$ and if (1.3)–(1.6) are satisfied.

The main result of the present chapter is the following:

Theorem 1.1. *Let the foregoing assumptions on Ω be satisfied and let $u(x, t)$ be a classical solution of* (1.3)–(1.6) *satisfying*

$$|u(x, t)| \leqq Ce^{\epsilon|x'| + \alpha t}, \tag{1.7}$$

$$|\nabla u(x, t)| + |u_t(x, t)| \leqq C_0(x)e^{\alpha t} \tag{1.8}$$

for all $x \in \bar{\Omega}$, $t \geqq 0$, where C, α, ϵ are positive constants and $C_0(x)$ is any locally bounded function. If ϵ is sufficiently small (depending only on Γ') then $u(x, t) \equiv 0$.

Proof. The Laplace transform of $u(x, t)$,

$$\hat{u}(x, s) = \int_0^\infty e^{-st} u(x, t)\, dt,$$

is defined and analytic for Re $s > \alpha$. Moreover, under the hypotheses stated, $\hat{u}$ satisfies

$$\Delta \hat{u} = 0 \quad \text{in} \quad \Omega, \tag{1.9}$$

$$\hat{u}_y + s^2 \hat{u} = 0 \quad \text{on} \quad \Gamma, \tag{1.10}$$

(1.11) $$\hat{u}_\nu = 0 \quad \text{on} \quad \Gamma',$$

(1.12) $$|\hat{u}(x, s)| \leqq Ce^{\epsilon|x'|}.$$

We shall show that (1.9)–(1.12) implies that $\hat{u} \equiv 0$. The theorem will follow from this.

Lemma 1.1. *There exists a function $\psi \in C^2(\bar{\Omega})$ satisfying: $\psi \in H^1(\Omega)$, $\psi \geqq 0$ in Ω, and*

(1.13) $$\triangle\psi = 0 \quad \textit{in} \quad \Omega,$$

(1.14) $$\psi_\nu = 1 \quad \textit{on the } \Gamma_j\ ,$$

(1.15) $$\psi_\nu = 0 \quad \textit{on} \quad \Gamma_b\ ,$$

(1.16) $$\psi_\nu + \alpha\psi = 0 \quad \textit{on} \quad \Gamma \qquad (\alpha \textit{ positive constant}).$$

Proof. The construction of a function ψ in $H^1(\Omega)$ which satisfies (1.13)–(1.16) is obtained by slightly modifying the construction of a solution for the system (3.3.7)–(3.3.9). Indeed, the only difference occurs in deriving the *a priori* bounds on the sequence of solutions which approximate ψ. In §3.3 we have denoted these solutions by φ_i (φ_i is defined in Ω_i). For the present system, let us denote the corresponding solutions by ψ_i . Then we have to estimate the additional terms

$$I_{ij} \equiv \int_{\Gamma_j} \psi_i \frac{\partial\psi_i}{\partial\nu}\, dS.$$

Since $\partial\psi_i/\partial\nu = 1$ on Γ_j , we get

$$|I_{ij}| \leqq \delta \int_{\Gamma_j} |\psi_i|^2\, dS + \frac{c}{\delta} \qquad (c \text{ constant})$$

for any $\delta > 0$. If we now use Lemma 1.1.8 and choose δ sufficiently small, then we obtain the bounds

$$||\nabla\psi_i||_{L^2(\Omega_i)} + ||\psi_i||_{L^2(\Omega_i)} + ||\psi_i||_{L^2(\Gamma_i^0)} \leqq c \qquad (c \text{ constant}).$$

Hence, we can choose a subsequence of $\{\psi_i\}$ which converges to a function ψ satisfying (1.13)–(1.16). ψ is also in $H^1(\Omega)$.

Using arguments similar to those of the proof of the first part of Theorem 1.2.2, we find that ψ is also in $C^2(\bar{\Omega})$. It remains to show that $\psi \geqq 0$.

Analogously to (1.2.17), we represent ψ locally in terms of a fundamental solution; near the boundaries Γ_b and Γ we have to use a local Neumann's function and a local Robin's function, respectively. Because of the assumption (1.2), the first derivatives of the Neumann function are bounded independently of the location where ψ is represented. The same is true, of course, of the local Robin function.

From this integral representation and from the fact that $\psi \in H^1(\Omega)$, we conclude that $\psi(x) \to 0$ as $|x| \to \infty$. Applying now the maximum principle we find, in view of the conditions (1.14)–(1.16), that ψ cannot take negative values in Ω. This completes the proof of the lemma.

Let $k = 1 + \sup_{x'} |\nabla' h(x')|$, $H = h_1 + 1$, and define $\delta = 1/4H$,

$$\eta = \min\left(\frac{1}{4Hk}, \frac{1}{4hH^2}\right), \tag{1.17}$$

$$\chi(x) = (H - y - \delta y^2) \exp[\eta(1 + |x'|^2)^{1/2}] + \beta\psi(x) \tag{1.18}$$

where β is a positive constant.

Lemma 1.2. *Let s be positive, $s^2 > \alpha$, $s^2 > 1/H$. If β is sufficiently large (independently of s), then*

$$\Delta\chi \leqq 0 \quad \text{in} \quad \Omega, \tag{1.19}$$

$$\chi_\nu \geqq 0 \quad \text{on} \quad \Gamma', \tag{1.20}$$

$$\chi_\nu + s^2\chi > 0 \quad \text{on} \quad \Gamma. \tag{1.21}$$

Proof. (1.19) and (1.20) on Γ_b can be verified by a simple calculation. On Γ_i we have

$$\chi_\nu = \frac{\partial}{\partial\nu}\{(H - y - \delta y^2) \exp[\eta(1 + |x'|^2)^{1/2}]\} + \beta.$$

Since Γ_i is bounded, (1.20) on Γ_i now follows by taking β to be sufficiently large.

Finally,

$$[\chi_\nu + s^2\chi]_{y=0} = (Hs^2 - 1) \exp[\eta(1 + |x'|^2)^{1/2}] + \beta(s^2 - \alpha)\psi(x', 0) > 0,$$

since, as we saw in Lemma 1.1, $\psi \geqq 0$.

We return to the proof of Theorem 1.1. We shall prove that $\hat{u}(x, s) \equiv 0$ if $\epsilon < \eta$ (η is defined by (1.17)). Denote by Ω_R the domain $\Omega \cap \{(x', y); |x'| < R\}$. Given any $\gamma > 0$, we have, by (1.12), (1.18),

$$|\hat{u}(x, s)| \leqq Ce^{\epsilon R} \leqq \gamma\chi(x) \quad \text{if} \quad |x'| = R, \tag{1.22}$$

provided R is sufficiently large. Consider the function

$$v(x, s) \equiv \gamma\chi(x) \pm \hat{u}(x, s).$$

In view of (1.9), (1.19), $\Delta v \leqq 0$, so that v does not assume a negative minimum in the interior of Ω_R. Since, by (1.20), (1.21) and (1.10), (1.11),

$$v_\nu \geqq 0 \quad \text{on} \quad \Gamma', \qquad v_\nu + s^2 v > 0 \quad \text{on} \quad \Gamma,$$

v cannot assume such a negative minimum on $\Gamma \cup \Gamma'$. Since, finally, $v \geqq 0$ on $|x'| = R$ (by (1.22)), it follows that $v \geqq 0$ in Ω_R. Hence,

$$|\hat{u}(x, s)| \leqq \gamma\chi(x).$$

For (x, s) fixed, we use this inequality with $\gamma = \gamma_m \to 0$. We conclude that $\hat{u}(x, s) = 0$. This completes the proof of the theorem.

Appendix. A Related Problem

1. Introduction. In this appendix, we shall be concerned with a problem different from that of water waves. It is related, however, in that a parameter t appears only in the "free surface" boundary condition. We shall describe briefly two methods for solving this new problem; we emphasize that these methods *fail* to solve the problem of water waves.

To describe the problem, let Ω be a domain of the sort we considered already in Chapters 2–4. For simplicity only, we assume there are no obstacles present. Thus, we assume we have a function $h(x')$ defined on R^{n-1}, belonging to $C^{2+\rho}$ for some $0 < \rho < 1$, and satisfying

$$0 < h_0 \leqq h(x') \leqq h_1 < \infty .$$

Then, Ω will be the set of all points $(x', y) \in R^n$ such that

$$-h(x') < y < 0.$$

Define Γ and Γ_b as in Chapter 3.

We shall consider the following problem:

$$\Delta u = 0 \quad \text{in} \quad \Omega, \tag{1.1}$$

$$u_\nu = 0 \quad \text{on} \quad \Gamma_b , \tag{1.2}$$

$$u_y + u_t = 0 \quad \text{on} \quad \Gamma, \tag{1.3}$$

$$u|_{t=0} = u^0. \tag{1.4}$$

This problem mainly differs from the problem of water waves in that only one derivative with respect to t appears in (1.3). Accordingly, only u — and not u_t — is assumed given initially.

2. An energy inequality. We shall derive an energy-type inequality for the solution u of (1.1)–(1.4). While an energy inequality follows easily when $u \in H^1(\Omega)$, the point of this section is to show that there is an inequality of energy-type even if $u^0(x')$ grows exponentially as $|x'| \to \infty$. The method of deriving the inequality fails in the case of water waves.

We assume that Ω, Γ, Γ_b are as in §1 and denote by Ω_R , Γ_R the intersections Ω, Γ with $|x'| < R$. We introduce the notation:

$$((f, g))_R = \int_{\Omega_R} f\bar{g}\, dx, \qquad ||f||_R^2 = \int_{\Omega_R} |f|^2\, dx, \qquad ||f||^2 = \int_\Omega |f|^2\, dx.$$

Let $\zeta(x') = \exp[\delta(1 + |x'|^2)^{1/2}]$ where δ is a sufficiently small positive number, depending only on Γ_b . Clearly

$$\frac{|\nabla\zeta(x')|}{\zeta(x')} \leqq \delta.$$

Let $\{R_m\}$ be an increasing sequence of positive numbers, $R_m \to \infty$, and let $0 < \delta_0 < \delta$. Let $u(x, t)$ be a smooth solution of (1.1)–(1.4) and suppose that

$$\int_\Gamma \zeta(x') \, |u^0(x')|^2 \, dx' < \infty, \tag{2.1}$$

$$|u(x', y, t)| + |\nabla' u(x', y, t)| \leqq Ce^{\delta_0 |x'|}(|x'| = R_m, \, m = 1, 2, \cdots). \tag{2.2}$$

Then

$$((\zeta u, \triangle u))_R + ((\nabla(\zeta u) \cdot \nabla u))_R$$
$$= \int_{\Gamma_R} \zeta u \bar{u}_\nu \, dx' + \int_{|x'|=R} \zeta u \bar{u}_\nu \, dS = -\frac{1}{2} \frac{d}{dt} \int_{\Gamma_R} \zeta \, |u|^2 \, dx' + \int_{|x'|=R} \zeta u \bar{u}_\nu \, dS.$$

Since

$$||u \nabla \zeta \cdot \nabla u||_R \leqq 2\delta \, ||\zeta^{1/2} u||^2 + 2\delta \, ||\zeta^{1/2} u||^2,$$

we get

$$||\zeta^{1/2} \nabla u||_R^2 + \frac{1}{2} \frac{d}{dt} \int_{\Gamma_R} \zeta \, |u|^2 \leqq 2\delta \, ||\zeta^{1/2} u||_R^2 + 2\delta \, ||\zeta^{1/2} \nabla u||^2 + \int_{|x'|=R} |\zeta u u_\nu| \, dS. \tag{2.3}$$

Estimate $||\zeta^{1/2} u||_R^2$ by Lemma 1.1.7. Taking δ positive sufficiently small, integrate with respect to t. We find, upon letting $R = R_m \to \infty$ and using (2.2),

$$\int_0^t ||\zeta^{1/2} \nabla u||^2 \, dt + \frac{1}{2} \int_\Gamma \zeta \, |u|^2 \, dx' \leqq c\delta \int_0^t \int_\Gamma \zeta \, |u|^2 \, dx' \, dt + \frac{1}{2} \int_\Gamma \zeta \, |u^0|^2 \, dx'.$$

Hence,

$$\int_0^t ||\zeta^{1/2} \nabla u||^2 \, dt + \frac{1}{2} \int_0^t \zeta \, |u|^2 \, dx' \leqq Ce^{ct} \int_\Gamma \zeta \, |u^0|^2 \, dx', \tag{2.4}$$

where C, c are constants.

This inequality gives a uniqueness theorem for solutions of (1.1)–(1.4). Its analog for bounded domains can be used to prove an existence theorem for (1.1)–(1.4) when u^0 satisfies (2.1). Indeed, we can proceed by the method of Chap. 1, but instead of using the energy inequalities (1.5.2), we now use inequalities of the form (2.4) in Ω_j, for solutions u_j of an appropriate truncated problem. (The derivation of these inequalities for the u_j is similar to the derivation of (2.4).)

3. The method of Laplace transforms. We give another method of proving the existence of a solution of (1.1)–(1.4) for data u^0 which may increase exponentially. Set $p(x'; \eta) = \exp [\eta(1 + |x'|^2)^{1/2}]$ where η is the positive constant occurring in (4.1.18). We shall assume that

$$|u^0(x')| \leqq Cp(x'; \eta_0) \qquad (\eta_0 < \eta). \tag{3.1}$$

If $u = ve^{\alpha t}$ $(\alpha > 0)$ is a solution of (1.1)–(1.4) and if

$$|v(x, t)| + |\nabla v(x, t)| \leqq cp(x'; \eta_0)e^{\lambda t} \qquad (\lambda > 0),$$

then the Laplace transform $\hat{v}(x, s)$ is defined and analytic for Re $s > \lambda$, and

$$\Delta\hat{v} = 0 \quad \text{in} \quad \Omega, \tag{3.2}$$

$$\hat{v}_\nu = 0 \quad \text{on} \quad \Gamma_b\,, \tag{3.3}$$

$$\hat{v}_y + \alpha\hat{v} + s\hat{v} = u^0 \quad \text{on} \quad \Gamma, \tag{3.4}$$

$$|\hat{v}(x, s)| \leqq c_0 p(x'; \eta_0) \qquad (c_0 \text{ constant}). \tag{3.5}$$

We now consider the system (3.2)–(3.4) for s real, $s > \lambda$.

If u^0 is smooth (say, $u^0 \in C^{1+\rho}$ for some $0 < \rho < 1$) then one can prove that there exists a classical solution of (3.2)–(3.4). Indeed, one first considers truncated problems (with Ω replaced by any one of a sequence of bounded domains). The solutions v_j of the truncated problems exist, by Theorem 1.2.1. Now one uses a compactness argument based upon an *a priori* energy inequality. This inequality is obtained from Green's formula applied to the pair v_j, $p(x'; -\eta_2)v_j$, for any $\eta_2 > \eta_0$.

Taking finite differences with respect to s, one can prove that $\partial\hat{v}/\partial s$ exists and satisfies the equations obtained from (3.2)–(3.4) by differentiating once with respect to s. Similarly one can show that all the derivatives of $\hat{v}$ with respect to s exist and satisfy the expected equations. The crucial step now is the following lemma:

Lemma 3.1. *Let $\chi(x)$ be the function in* (4.1.18) *with $\beta = 0$. There exists a constant B such that*

$$\left|\frac{\partial^m}{\partial s^m}\hat{v}(x, s)\right| \leqq \frac{B}{s^{m+1}}\chi(x)m! \qquad (m = 0, 1, 2, \cdots). \tag{3.6}$$

From the Post-Widder formula [9] we immediately conclude the first part of the following theorem:

Theorem 3.1. *$\hat{v}(x, s)$ is the Laplace transform of a function $v(x, t)$ satisfying: $|v(x, t)| \leqq B\chi(x)$. $u = ve^{\alpha t}$ is a solution of* (1.1)–(1.4) *in the following sense: u is a classical solution of* (1.1), (1.2), *while the Laplace transform of v satisfies* (3.4).

The second part of the theorem follows from (3.2), (3.3) and the uniqueness theorem for the Laplace transform.

It remains to prove Lemma 3.1.

To prove (3.6) for $m = 1$ we compare $\hat{v}$ with $\gamma\chi(x)/s$ where γ is some positive constant. Since the η_0 in (3.1) is less than η, we can slightly modify the proof of the uniqueness theorem in Chapter 4 and conclude that

$$|\hat{v}(x, s)| \leqq \frac{\gamma}{s}\chi(x).$$

We proceed to establish (3.6) by induction. To pass from m to $m + 1$, we note that the function $w \equiv \partial^{m+1}\hat{u}/\partial s^{m+1}$ satisfies:

$$\Delta w = 0 \quad \text{in} \quad \Omega,$$

$$w_\nu = 0 \quad \text{on} \quad \Gamma_b\,,$$

$$w_\nu + \alpha w + s w = -(m+1)\,\frac{\partial^m \hat{u}}{\partial s^m} \quad \text{on} \quad \Gamma.$$

We now compare w with $B\chi(x)(m+1)!/s^{m+1}$. Using the inductive assumption we find that

$$B\chi(x)(m+1)!/s^{m+1} \pm w \geqq 0.$$

This gives (3.6) for $m + 1$.

The estimate (3.6) is fairly delicate. We cannot solve the water waves problem by the present method because we are unable to prove an estimate analogous to (3.6) for that problem.

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