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## NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

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## LOADED WITH A DIELECTRIC PLUG

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## SUMMARY

A waveguide-fed rectangular aperture antenna, loaded with a dielectric plug, is considered as a variational problem. The formulation results in an aperture-admittance expression which contains the self and mutual admittances of the transverse electric waveguide modes $\mathrm{TE}_{01}$ and $\mathrm{TE}_{03}$. The theoretical results show that a gross distortion of the aperture field and a significant decrease in the aperture admittance occurs when the plug becomes a resonant cavity for the $\mathrm{TE}_{03}$ mode.

The input admittance is compared with microwave measurements for the condition in which the external half-space is air. The theory is generally in good agreement with experiment over the entire waveguide band, including the frequency interval around resonance. It is noted that the plug can be used to tune out the susceptance and improve the match of the antenna.

## INTRODUCTION

The waveguide-fed rectangular aperture is one of the few antennas for which the input admittance has been analyzed quantitatively as a function of the dielectric constant of the medium over the ground plane. The device is also relatively easy to construct and can be mounted flush to the surface of a flight vehicle. As such, the rectangular aperture antenna seems appealing for possible use in reentry plasma diagnostics.

An air-filled waveguide opening onto the vehicle skin is unsatisfactory because the flow field will be disrupted and hot gases will enter the vehicle through the open waveguide. Some means must therefore be used to seal the aperture internally from the environment. A nonlossy dielectric block, snugly inserted into the aperture with the face flush to the ground plane, is, for practical reasons, an excellent way to isolate the antenna. For example, if the unloaded section of the waveguide is long enough to attenuate modes of higher order, impedance measurements may easily be performed between the plug and the transition from the coaxial line to the waveguide. Also, since the feed radiates into an air-filled guide, problems inherent in matching impedance with dielectric
loaded transition from the coaxial line to the waveguide are avoided. However, the aperture may strongly excite standing waves of higher-order waveguide modes in the plug, which may significantly perturb the admittance. The effects of one of these modes, the $\mathrm{TE}_{03}$, is considered as a variational problem. (Examples of the variational approach are given in refs. 1 and 2.) In the formulation, the external medium is assumed to be an infinite half-space of lossy dielectric material; however, calculations are restricted to conditions for which the half-space is air. The results are compared with microwave measurements.

## SYMBOLS

A amplitude of $\mathrm{TE}_{03}$ mode in the unloaded portion of the waveguide
$\mathrm{A}_{03}$ defined in equation (8)
a short dimension of waveguide
b long dimension of waveguide

B susceptance
$\mathrm{C}_{l}\left(\mathrm{k}_{\mathrm{x}}\right), \mathrm{C}_{\mathrm{m}}\left(\mathrm{k}_{\mathrm{y}}\right) \quad$ modified Hodara functions defined in equation (A2) ( $l$ and m are integers)

D nondimensional amplitude of higher-order mode component of aperture electric field
$\mathrm{E}_{\mathrm{X}}, \mathrm{E}_{\mathrm{y}} \quad$ components of electric field intensity
$\mathrm{E}_{\mathrm{O}} \quad$ amplitude of incident wave
$\mathrm{E}_{\mathrm{X}}^{01} \quad$ amplitude of $\mathrm{TE}_{01}$ component of electric field at the aperture $\mathrm{E}_{\mathrm{X}}^{03} \quad$ amplitude of $\mathrm{TE}_{03}$ component of electric field at the aperture f frequency
$\mathrm{F}_{1}^{l \mathrm{~m}}\left(\mathrm{k}_{\mathrm{x}}, \mathrm{k}_{\mathrm{y}}\right), \mathrm{F}_{2}\left(\mathrm{k}_{\mathrm{x}}, \mathrm{k}_{\mathrm{y}}\right) \quad$ Fourier transforms defined in equations (A4)
$\mathrm{f}_{1}^{l \mathrm{~m}}, \mathrm{f}_{2} \quad$ inverse Fourier transforms defined in equations (A6)

| $\mathrm{f}\left(\mathrm{k}_{\mathrm{x}}, \mathrm{k}_{\mathrm{y}}\right), \mathrm{g}\left(\mathrm{k}_{\mathrm{x}}\right.$ | $\left.\mathrm{k}_{\mathrm{x}}, \mathrm{k}_{\mathrm{y}}\right) \quad$ Fourier transforms of the potentials $\psi$ and $\phi$, respectively |
| :---: | :---: |
| $\mathrm{g}(\mathrm{x}), \mathrm{h}(\mathrm{y})$ | integrals defined in equations (A9) |
| G | conductance |
| $\mathrm{H}_{\mathrm{X}}, \mathrm{H}_{\mathrm{y}}$ | components of magnetic field intensity |
| $\mathrm{H}_{\mathrm{y}}^{01}$ | amplitude of $\mathrm{TE}_{01}$ component of magnetic field at the aperture |
| $\mathrm{H}_{\mathrm{y}}^{03}$ | amplitude of $\mathrm{TE}_{03}$ component of magnetic field at the aperture |
| I | reaction integral defined in equation (10) |
| $j=\sqrt{-1}$ |  |
| $\mathrm{k}_{\mathrm{x}}, \mathrm{k}_{\mathrm{y}}, \mathrm{k}_{\mathrm{z}}$ | three principal continuous mode numbers in region III (fig. 1) |
| $\mathrm{k}_{\mathrm{z} 01}^{\prime}, \mathrm{k}_{\mathrm{z} 03}^{\prime}$ | propagation constants in the unloaded section of the waveguide defined in equations (2) |
| $\mathrm{k}_{\mathrm{z} 01}, \mathrm{k}_{\mathrm{z} 03}$ | the z-components of the propagation constants in the loaded section of the waveguide (defined in eqs. (4)) |
| $\mathrm{k}_{\mathrm{O}}$ | propagation constant of free space |
| $\mathrm{k}=\mathrm{k}_{\mathrm{r}}+\mathrm{j} \mathrm{k}_{\mathrm{i}}$ | propagation constant of the medium of region III (fig. 1) |
| $\mathrm{k}_{\mathrm{r}}, \mathrm{k}_{\mathrm{i}}$ | real and imaginary parts of $k$ |
| N | index of refraction |
| $\mathrm{R}, \theta, \mathrm{z}$ | cylindrical coordinates |
| t | time |
| $\mathrm{V}_{\mathrm{o}}$ | electric field amplitude of forward propagating $\mathrm{TE}_{01}$ mode in region II |
| Vro | electric field amplitude of backward propagating $\mathrm{TE}_{01}$ mode in region II |




## THEORY

The waveguide geometry considered is shown in figure 1. A rectangular waveguide opens onto a flat ground plane, which is covered with a half-space of lossy dielectric material. The waveguide is sealed from the external environment by a nonlossy dielectric prism of thickness $z_{0}$, which is inserted so that the face of the plug is flush with the ground plane. The width and height of the plug are assumed to be equal to the inner dimensions of the waveguide.

The dominant $\mathrm{TE}_{01}$ mode is incident upon the aperture, where several waveguide modes may be excited by the discontinuity at $z=0$. However, for conciseness, the aperture electric field is considered as a superposition of only two waveguide modes,
specifically, the $\mathrm{TE}_{01}$ and $\mathrm{TE}_{03}$. This simplification is equivalent to assuming that (see fig. 1): (1) only those modes which are symmetrical about the origin are excited; (2) the aperture discontinuity induces a negligible $\mathrm{E}_{\mathrm{y}}$ component; and (3) all induced modes with cutoff frequencies higher than the $\mathrm{TE}_{03}$ mode are negligible.

Under these assumptions, the input admittance, relative to a short circuit placed at $z=0$, is derived. This derivation is made by imposing variational techniques in connection with the boundary-value problem.

The Boundary-Value Problem at $z=-z_{0}$
The waveguide solutions in region $I$ (fig. 1) consist of an incident $\mathrm{TE}_{01}$ mode and reflected $\mathrm{TE}_{01}$ and $\mathrm{TE}_{03}$ modes. As such, the tangential field components may be
 written as

$$
\left.\begin{array}{l}
E_{X}^{I}=E_{o}\left(e^{-j k_{z 01}^{\prime}}+\Gamma e^{j k_{z 01}^{\prime}}\right) \cos \frac{\pi y}{b}+A e^{j k_{z 03}^{\prime}} \cos \frac{3 \pi y}{b} \\
E_{y}^{I}=0  \tag{1}\\
H_{x}^{I}=0 \\
H_{y}^{I}=Y_{01}^{\prime} E_{o}\left(e^{-j k_{z 01}^{\prime} z}-\Gamma e^{j k_{z 01}^{\prime}}\right) \cos \frac{\pi y}{b}-Y_{03}^{\prime} A e^{j k_{z 03}^{\prime}} \cos \frac{3 \pi y}{b}
\end{array}\right\}
$$

where a harmonic time dependence of $e^{j \omega t}$ is implicit, and

$$
\left.\begin{array}{l}
\mathrm{k}_{\mathrm{z} 01}^{\prime}=\mathrm{k}_{\mathrm{o}} \sqrt{1-\left(\frac{\pi}{\mathrm{k}_{\mathrm{o}} \mathrm{~b}}\right)^{2}}=\mathrm{k}_{\mathrm{o}} \frac{\mathrm{Y}_{01}^{\prime}}{\mathrm{Y}_{\mathrm{O}}}  \tag{2}\\
\mathrm{k}_{\mathrm{z} 03}^{\prime}=-j \mathrm{k}_{\mathrm{o}} \sqrt{\left(\frac{3 \pi}{\mathrm{k}_{\mathrm{o}} \mathrm{~b}}\right)^{2}-1}=\mathrm{k}_{\mathrm{o}} \frac{\mathrm{Y}_{03}^{\prime}}{\mathrm{Y}_{\mathrm{o}}}
\end{array}\right\}
$$

The quantity $\Gamma$ is the reflection coefficient, and $A$ is the unknown amplitude coefficient of the $\mathrm{TE}_{03}$ mode.

In the dielectric plug ( $-z_{0} \leqq \mathrm{z} \leqq 0$ ), a superposition of modal standing waves exists, and the tangential fields are therefore given by

$$
\begin{align*}
& E_{X}^{I I}=\left(V_{o} e^{-j k_{z 01} z}+V_{r o} e^{j k_{z 01}}{ }^{z}\right) \cos \frac{\pi y}{b}+\left(V_{3} e^{-j k_{z 03^{z}}^{Z}}+V_{4} e^{j k_{z 03} z}\right) \cos \frac{3 \pi y}{b} \\
& \mathrm{E}_{\mathrm{y}}^{\mathrm{II}}=0  \tag{3}\\
& \mathrm{H}_{\mathrm{x}}^{\mathrm{II}}=0 \\
& \left.H_{y}^{I I}=Y_{01}\left(V_{o} e^{-j k_{z 01} Z}-V_{r o} e^{j k_{z 01} z}\right) \cos \frac{\pi y}{b}+Y_{03}\left(V_{3} e^{-j k_{z 03} z}-V_{4} e^{j k_{z 03}}{ }^{z}\right) \cos \frac{3 \pi y}{b}\right)
\end{align*}
$$

where

$$
\begin{align*}
& \mathrm{k}_{\mathrm{z} 01}=\mathrm{k}_{\mathrm{o}} \sqrt{\mathrm{~N}^{2}-\left(\frac{\pi}{\mathrm{k}_{\mathrm{o}} \mathrm{~b}}\right)^{2}}=\mathrm{k}_{\mathrm{O}} \frac{\mathrm{Y}_{01}}{\mathrm{Y}_{\mathrm{O}}} \\
& \mathrm{k}_{\mathrm{z} 03}=\mathrm{k}_{\mathrm{o}} \sqrt{\mathrm{~N}^{2}-\left(\frac{3 \pi}{\mathrm{k}_{\mathrm{o}} \mathrm{~b}}\right)^{2}}=\mathrm{k}_{\mathrm{o}} \frac{\mathrm{Y}_{03}}{\mathrm{Y}_{\mathrm{O}}} \quad\left(\mathrm{~N}^{2}>\left(\frac{3 \pi}{\mathrm{k}_{\mathrm{o}} \mathrm{~b}}\right)^{2}\right)  \tag{4}\\
& \mathrm{k}_{\mathrm{z} 03}=-\mathrm{j} \mathrm{k}_{\mathrm{o}} \sqrt{\left(\frac{3 \pi}{\mathrm{k}_{\mathrm{o}} \mathrm{~b}}\right)^{2}-\mathrm{N}^{2}}=\mathrm{k}_{\mathrm{O}} \frac{\mathrm{Y}_{03}}{\mathrm{Y}_{\mathrm{O}}} \quad\left(\mathrm{~N}^{2}<\left(\frac{3 \pi}{\mathrm{k}_{\mathrm{o}} \mathrm{~b}}\right)^{2}\right)
\end{align*}
$$

The orthogonality of waveguide modes requires that the corresponding modal values of $\mathrm{E}_{\mathrm{X}}$ and $\mathrm{H}_{\mathrm{y}}$ be continuous at $\mathrm{z}=-\mathrm{z}_{\mathrm{o}}$; therefore, equations (1) and (3) immediately lead to the relations

$$
\begin{gather*}
Y_{01}^{\prime}\left(e^{j k_{z 01}^{\prime} z_{o}}-\Gamma e^{-j k_{z 01}^{\prime} z_{o}}\right)  \tag{5a}\\
Y_{01}\left(e^{j k_{z 01}^{\prime} z_{o}}+\Gamma e^{-j k_{z 01}^{\prime} z_{o}}\right) \tag{5b}
\end{gather*} \frac{\left(V_{o} e^{j k} z 01^{z_{o}}-V_{r o} e^{-j k_{z 01} z_{o}}\right)}{\left(V_{o} e^{j k_{z 01} z_{0}}+V_{r o} e^{-j k_{z 01} z_{o}}\right)}
$$

Equation (5a) expresses a relationship between the reflection coefficient and the normalized aperture admittance, $\mathrm{y}_{\mathrm{ap}}=\frac{\mathrm{V}_{\mathrm{o}}-\mathrm{V}_{\mathrm{ro}}}{\mathrm{V}_{\mathrm{o}}+\mathrm{V}_{\mathrm{ro}}}$, which is to be calculated.

Equation (5b) can be manipulated to give

$$
\begin{align*}
\mathrm{V}_{3}+\mathrm{V}_{4} & =\frac{2 \mathrm{~V}_{3}}{\mathrm{Y}_{03}-\mathrm{Y}_{03}^{\prime}} e^{j \mathrm{k}_{\mathrm{z} 03^{\mathrm{z}_{\mathrm{o}}}}^{\cos \mathrm{k}_{\mathrm{z} 03^{\mathrm{z}_{\mathrm{O}}}}\left(\mathrm{Y}_{03}+\mathrm{jY}_{03}^{\prime} \tan \mathrm{k}_{\left.\mathrm{z} 03^{\mathrm{z}_{\mathrm{o}}}\right)}\right.}} \begin{aligned}
& \equiv \mathrm{D}\left(\mathrm{~V}_{\mathrm{o}}+\mathrm{V}_{\mathrm{ro}}\right)
\end{aligned}
\end{align*}
$$

where $D$ is an unknown coefficient, and

$$
\begin{align*}
\mathrm{V}_{3}-\mathrm{V}_{4} & =-\frac{2 \mathrm{~V}_{3}}{\mathrm{Y}_{03}-\mathrm{Y}_{03}^{\prime}} e^{j \mathrm{k}_{\mathrm{z} 03} \mathrm{z}_{\mathrm{O}}} \cos \mathrm{k}_{\mathrm{z} 03^{\mathrm{z}_{\mathrm{O}}}\left(\mathrm{Y}_{03}^{\prime}+j Y_{03} \tan \mathrm{k}_{\mathrm{z} 03^{\mathrm{z}_{\mathrm{o}}}}\right)} \\
& \equiv-\mathrm{A}_{03} \mathrm{D}\left(\mathrm{~V}_{\mathrm{o}}+\mathrm{V}_{\mathrm{ro}}\right) \tag{7}
\end{align*}
$$

The factor $A_{03}$ in equation (7) is defined as

$$
\begin{equation*}
A_{03}=\frac{Y_{03}^{\prime}+j Y_{03} \tan k_{z 03} z_{o}}{Y_{03}+j Y_{03}^{\prime} \tan k_{z 03} z_{o}} \tag{8}
\end{equation*}
$$

It therefore follows that the tangential fields at $\mathrm{z}=0$ (eqs. (3)) are given by

$$
\left.\begin{array}{l}
\mathrm{E}_{\mathrm{x}}^{\mathrm{II}}(0)=\left(\mathrm{V}_{\mathrm{o}}+\mathrm{V}_{\mathrm{ro}}\right)\left(\cos \frac{\pi \mathrm{y}}{\mathrm{~b}}+\mathrm{D} \cos \frac{3 \pi \mathrm{y}}{\mathrm{~b}}\right) \\
\mathrm{E}_{\mathrm{y}}^{\mathrm{II}}(0)=\mathrm{H}_{\mathrm{x}}^{\mathrm{II}}(0)=0  \tag{9}\\
\mathrm{H}_{\mathrm{y}}^{\mathrm{II}}(0)=\mathrm{Y}_{01}\left(\mathrm{~V}_{\mathrm{o}}-\mathrm{V}_{\mathrm{ro}}\right) \cos \frac{\pi \mathrm{y}}{\mathrm{~b}}-\mathrm{Y}_{03} \mathrm{~A}_{03} \mathrm{D}\left(\mathrm{~V}_{\mathrm{o}}+\mathrm{V}_{\mathrm{ro}}\right) \cos \frac{3 \pi \mathrm{y}}{\mathrm{~b}}
\end{array}\right\}
$$

The unknown coefficient D and the aperture admittance may now be solved by variational techniques.

The Variational Expression for the Admittance
In order to develop a useful admittance expression, the following reaction integral is considered

$$
\begin{equation*}
I=\int_{-a / 2}^{a / 2} \int_{-b / 2}^{b / 2} E_{x}(x, y, 0) H_{y}(x, y, 0) d x d y \tag{10}
\end{equation*}
$$

This integral has the dimensions of power, but does not contain a complex conjugate of either the electric field intensity or magnetic field intensity in the integrand, which is characteristic of stationary admittance expressions (ref. 3). Substituting equations (9) into equation (10) gives an expression for the integral $I$ in terms of the quantities in region II $\left(-z_{0} \leqq z \leqq 0\right)$. The result is

$$
\begin{equation*}
I=Y_{01}\left(V_{o}+V_{r o}\right)^{2} \frac{a b}{2}\left(\frac{V_{o}-V_{r o}}{V_{o}+V_{r o}}-\frac{Y_{03}}{Y_{01}} A_{03} D^{2}\right) \tag{11}
\end{equation*}
$$

The expression $\frac{V_{0}-V_{\text {ro }}}{V_{0}+V_{\text {ro }}}$ is the normalized aperture admittance $y_{\text {ap }}$. Therefore, equation (11) may be rearranged to give

$$
\begin{equation*}
\mathrm{y}_{\mathrm{ap}}=\frac{2}{\mathrm{ab}} \frac{\mathrm{I}}{\mathrm{Y}_{01}\left(\mathrm{~V}_{\mathrm{o}}+\mathrm{V}_{\mathrm{ro}}\right)^{2}}+\frac{\mathrm{Y}_{03}}{\mathrm{Y}_{01}} \mathrm{~A}_{03} \mathrm{D}^{2} \tag{12}
\end{equation*}
$$

It is assumed that reaction is conserved across the aperture; therefore
$I=\int_{-a / 2}^{a / 2} \int_{-b / 2}^{b / 2} E_{x}^{I I}(x, y, 0) H_{y}^{I I}(x, y, 0) d x d y=\int_{-a / 2}^{a / 2} \int_{-b / 2}^{b / 2} E_{x}^{I I I}(x, y, 0) H_{y}^{I I I}(x, y, 0) d x d y$

Since the fields in region III consist of continuous modes, it is more convenient to work with the Fourier transforms of $\mathrm{E}_{\mathrm{x}}$ and $\mathrm{H}_{\mathrm{y}}$ in which case Parseval's theorem (ref. 3, p. 182) may be used to re-express the last integral in equation (13) as follows:

$$
\begin{align*}
& \mathrm{I}=\int_{-\mathrm{a} / 2}^{\mathrm{a} / 2} \int_{-\mathrm{b} / 2}^{\mathrm{b} / 2} \mathrm{E}_{\mathrm{x}}^{\mathrm{III}}(\mathrm{x}, \mathrm{y}, 0) \mathrm{H}_{\mathrm{y}}^{\mathrm{III}}(\mathrm{x}, \mathrm{y}, 0) \mathrm{dx} d \mathrm{~d} \\
& \equiv \frac{1}{(2 \pi)^{2}} \int_{-\infty}^{\infty} \int_{\mathrm{x}}^{\infty} \overline{\mathrm{E}}_{\mathrm{x}}^{\mathrm{II}}  \tag{14}\\
&\left(\mathrm{k}_{\mathrm{x}}, \mathrm{k}_{\mathrm{y}}, 0\right) \overline{\bar{H}}_{\mathrm{y}}^{\mathrm{III}}\left(\mathrm{k}_{\mathrm{x}}, \mathrm{k}_{\mathrm{y}}, 0\right) \mathrm{dk}_{\mathrm{x}} d \mathrm{k}_{\mathrm{y}}
\end{align*}
$$

where $\overline{\overline{\mathrm{E}}}_{\mathrm{x}}^{\mathrm{III}}\left(\mathrm{k}_{\mathrm{x}}, \mathrm{k}_{\mathrm{y}}, 0\right)$ and $\overline{\overline{\mathrm{H}}}_{\mathrm{y}}^{\mathrm{III}}\left(\mathrm{k}_{\mathrm{x}}, \mathrm{k}_{\mathrm{y}}, 0\right)$ are the double Fourier transforms of the field components; that is

$$
\begin{equation*}
\left.\overline{\bar{E}}_{x}\left(k_{x}, k_{y}, 0\right)=\int_{-\infty}^{\infty} \int_{x} E_{x}(x, y, 0) e^{j\left(k_{x} x+k_{y} y\right.}\right) d x d y \tag{15}
\end{equation*}
$$

The substitution of equation (14) into equation (12) gives

$$
\begin{equation*}
\mathrm{y}_{\mathrm{ap}}=\frac{2}{\mathrm{abY}} 01\left(\mathrm{~V}_{\mathrm{o}}+\mathrm{V}_{\mathrm{ro}}\right)^{2} \frac{1}{(2 \pi)^{2}} \int_{-\infty}^{\infty} \int_{\mathrm{x}} \overline{\overline{\mathrm{E}}}_{\mathrm{x}}^{\mathrm{III}}(0) \overline{\bar{H}}_{\mathrm{y}}^{\mathrm{III}}(0) \mathrm{dk}_{\mathrm{x}} \mathrm{dk} \mathrm{k}_{\mathrm{y}}+\frac{\mathrm{Y}_{03}}{\mathrm{Y}_{01}} \mathrm{~A}_{03} \mathrm{D}^{2} \tag{16}
\end{equation*}
$$

The exterior problem ( $z>0$ ) must now be solved in order to express $\overline{\bar{E}}_{\mathrm{X}}^{\mathrm{III}}(0)$ and $\overline{\mathrm{H}}_{\mathrm{y}}^{\mathrm{II}}(0)$ in terms of the unknown coefficient D . When this solution is completed, the stationary character of $y_{\mathrm{ap}}$ in equation (16), for which a proof is included in reference 1, may be incorporated to solve for $D$. This solution is demonstrated in the next section.

## The Boundary-Value Problem at $\mathrm{z}=0$

The fields in region III ( $z>0$ ) are a superposition of continuous modes, and may be derived from TE and TM potential functions, $\psi$ and $\phi$, respectively, such that

$$
\left.\begin{array}{l}
\mathrm{E}_{\mathrm{x}}^{\mathrm{III}}=-\frac{\partial \psi}{\partial \mathrm{y}}+\frac{1}{j \omega \epsilon} \frac{\partial^{2} \phi}{\partial \mathrm{x} \partial \mathrm{z}} \\
\mathrm{E}_{\mathrm{y}}^{\mathrm{III}}=\frac{\partial \psi}{\partial \mathrm{x}}+\frac{1}{j \omega \epsilon} \frac{\partial^{2} \phi}{\partial \mathrm{y} \partial \mathrm{z}} \\
\mathrm{H}_{\mathrm{x}}^{\mathrm{III}}=\frac{1}{j \omega \mu_{\mathrm{O}}} \frac{\partial^{2} \psi}{\partial \mathrm{x} \partial \mathrm{z}}+\frac{\partial \phi}{\partial \mathrm{y}}  \tag{17}\\
\mathrm{H}_{\mathrm{y}}^{\mathrm{III}}=\frac{1}{j \omega \mu_{\mathrm{O}}} \frac{\partial^{2} \psi}{\partial \mathrm{y} \partial \mathrm{z}}-\frac{\partial \phi}{\partial \mathrm{x}}
\end{array}\right\}
$$

where the solutions to $\psi$ and $\phi$ are given by

$$
\left.\begin{array}{l}
\psi=\frac{1}{(2 \pi)^{2}} \int_{-\infty}^{\infty} \int_{-\infty}^{-j\left(k_{x} x+k_{y} y+k_{z} z\right)} d k_{x} d k_{y} \\
\phi=\frac{1}{(2 \pi)^{2}} \iint_{-\infty}^{\infty} g e^{-j\left(k_{x} x+k_{y} y+k_{z} z\right)} d k_{x} d k_{y} \tag{18}
\end{array}\right\}
$$

The mode number $\mathrm{k}_{\mathrm{Z}}$ is defined by the relationship

$$
\mathrm{k}_{\mathrm{z}}^{2}=\mathrm{k}^{2}-\mathrm{k}_{\mathrm{x}}^{2}-\mathrm{k}_{\mathrm{y}}^{2}
$$

where k is the complex wave number of the medium of region III. A choice of the branch of $\mathrm{k}_{\mathrm{Z}}$ must be made so that

$$
\left.\begin{array}{l}
\operatorname{Re} \mathrm{k}_{\mathrm{Z}}>0 \\
\operatorname{Im} \mathrm{k}_{\mathrm{Z}}<0 \tag{19}
\end{array}\right\}
$$

Now, let $\overline{\overline{\mathrm{E}}}_{\mathrm{x}}^{\mathrm{III}}(\mathrm{z}), \quad \overline{\overline{\mathrm{E}}}_{\mathrm{y}}^{\mathrm{III}}(\mathrm{z}), \overline{\overline{\mathrm{H}}}_{\mathrm{x}}^{\mathrm{III}}(\mathrm{z})$, and $\overline{\overline{\mathrm{H}}}_{\mathrm{y}}^{\mathrm{III}}(\mathrm{z})$ be the double Fourier transforms of their respective components in physical space; namely,

The substitution of equations (20) and (18) into equations (17) results in the following equations:

$$
\left.\begin{array}{l}
\overline{\bar{E}}_{\mathrm{x}}^{\mathrm{III}}(\mathrm{z})=j\left(\mathrm{k}_{\mathrm{y}} \mathrm{f}+\frac{\mathrm{k}_{\mathrm{x}} \mathrm{k}_{\mathrm{z}}}{\omega \epsilon} \mathrm{~g}\right) \mathrm{e}^{-j \mathrm{k}_{\mathrm{z}} \mathrm{z}} \\
\overline{\overline{\mathbf{E}}}_{\mathrm{y}}^{\mathrm{III}}(\mathrm{z})=-j\left(\mathrm{k}_{\mathrm{x}} \mathrm{f}-\frac{\mathrm{k}_{\mathrm{y}} \mathrm{k}_{\mathrm{z}}}{\omega \epsilon} \mathrm{~g}\right) \mathrm{e}^{-j k_{z_{z}}} \tag{21}
\end{array}\right\}
$$

At $z=0$, the tangential components of $\overrightarrow{\mathrm{E}}$ are continuous across the aperture; that is, $\overline{\bar{E}}_{\mathrm{x}}^{\mathrm{II}}(0)=\overline{\overline{\mathrm{E}}}_{\mathrm{x}}^{\mathrm{III}}(0)$ and $\overline{\overline{\mathrm{E}}}_{\mathrm{y}}^{\mathrm{II}}(0)=\overline{\overline{\mathrm{E}}}_{\mathrm{y}}^{\mathrm{III}}(0)$. The modal coefficients f and g may therefore be expressed in terms of the transforms of the tangential components of $\overrightarrow{\mathrm{E}}{ }^{\mathrm{II}}$. The result is

$$
\left.\begin{array}{l}
\mathrm{f}=\mathrm{j} \frac{\left(\mathrm{k}_{\mathrm{x}} \overline{\overline{\mathrm{E}}}_{\mathrm{y}}^{\mathrm{II}}(0)-\mathrm{k}_{\mathrm{y}} \overline{\overline{\mathrm{E}}}_{\mathrm{x}}^{\mathrm{I}}(0)\right)}{\mathrm{k}_{\mathrm{x}}^{2}+\mathrm{k}_{\mathrm{y}}{ }^{2}} \\
\mathrm{~g}=-\mathrm{j} \frac{\left(\mathrm{k}_{\mathrm{x}} \overline{\overline{\mathrm{E}}}_{\mathrm{x}}^{\mathrm{II}}(0)+\mathrm{k}_{\mathrm{y}} \overline{\overline{\mathrm{E}}}_{\mathrm{y}}^{\mathrm{II}}(0)\right) \omega \epsilon}{\mathrm{k}_{\mathrm{z}}\left(\mathrm{k}_{\mathrm{x}}^{2}+\mathrm{k}_{\mathrm{y}}^{2}\right)} \tag{22}
\end{array}\right\}
$$

From equations (17), (20), and (22), it also follows that

$$
\begin{equation*}
\overline{\bar{H}}_{\mathrm{y}}^{\mathrm{III}}(0)=\frac{\left(\mathrm{k}^{2}-\mathrm{k}_{\mathrm{y}}^{2}\right) \overline{\overline{\mathrm{E}}}_{\mathrm{x}}^{\mathrm{II}}(0)+\mathrm{k}_{\mathrm{y}} \mathrm{k}_{\mathrm{x}} \overline{\overline{\mathrm{E}}}_{\mathrm{y}}^{\mathrm{II}}(0)}{\omega \mu_{\mathrm{o}} \mathrm{k}_{\mathrm{z}}} \tag{23}
\end{equation*}
$$

Remaining to be solved are the transforms, $\overline{\mathrm{E}}_{\mathrm{X}}^{\mathrm{II}}(0)$ and $\overline{\overline{\mathrm{E}}}_{\mathrm{y}}^{\mathrm{II}}(0) \quad$ (eqs. (9)), with substitution of the quantity

$$
\begin{equation*}
\overline{\overline{\mathrm{E}}}_{\mathrm{x}}^{\mathrm{III}}(0) \overline{\overline{\mathrm{H}}}_{\mathrm{y}}^{\mathrm{III}}(0) \equiv \overline{\overline{\mathrm{E}}}_{\mathrm{x}}^{\mathrm{II}}(0) \overline{\overline{\mathrm{H}}}_{\mathrm{y}}^{\mathrm{III}}(0) \tag{24}
\end{equation*}
$$

into the aperture admittance expression.
The transforms $\overline{\bar{E}}_{\mathrm{x}}^{\mathrm{II}}(0)$ and $\overline{\overline{\mathrm{E}}}_{\mathrm{y}}^{\mathrm{II}}(0)$ are given by

$$
\begin{align*}
& \overline{\overline{\mathrm{E}}} \mathrm{y} \\
& \mathrm{II}(0)
\end{aligned}=0 \quad \begin{aligned}
\overline{\bar{E}}_{\mathrm{x}}^{\mathrm{III}}(0) & =\overline{\overline{\mathrm{E}}}_{\mathrm{x}}^{\mathrm{II}}(0)=\left(\mathrm{V}_{\mathrm{O}}+\mathrm{V}_{\mathrm{ro}}\right) \int_{-\mathrm{a} / 2}^{a / 2} \int_{-\mathrm{b} / 2}^{\mathrm{b} / 2}\left(\cos \frac{\pi y}{\mathrm{~b}}+\mathrm{D} \cos \frac{3 \pi y}{\mathrm{~b}}\right) e^{j\left(\mathrm{k}_{\mathrm{x}} \mathrm{x}+\mathrm{k}_{\mathrm{y}} \mathrm{y}\right)} d x d y  \tag{25}\\
& \equiv\left(\mathrm{~V}_{\mathrm{O}}+\mathrm{V}_{\mathrm{ro}}\right) \mathrm{C}_{0}\left(\mathrm{k}_{\mathrm{x}}\right)\left[\mathrm{C}_{1}\left(\mathrm{k}_{\mathrm{y}}\right)+D C_{3}\left(\mathrm{k}_{\mathrm{y}}\right)\right]
\end{align*}
$$

where the $C_{l}$ 's and $C_{m}$ 's are functions which are similar to those defined by Hodara in reference 4 and are explicitly defined as

$$
\left.\begin{array}{l}
C_{0}\left(k_{x}\right)=\int_{-a / 2}^{a / 2} e^{j k_{x} x} d x=\frac{a \sin \left(k_{x} a / 2\right)}{k_{x} a / 2} \\
C_{1}\left(k_{y}\right)=\int_{-b / 2}^{b / 2} \cos \frac{\pi y}{b} e^{j k_{y} y} d y=\frac{2 \pi b \cos \left(k_{y} b / 2\right)}{\pi^{2}-\left(k_{y} b\right)^{2}}  \tag{26}\\
C_{3}\left(k_{y}\right)=\int_{-b / 2}^{b / 2} \cos \frac{3 \pi y}{b} e^{j k_{y} y} d y=\frac{-6 \pi b \cos \left(k_{y} b / 2\right)}{(3 \pi)^{2}-\left(k_{y} b\right)^{2}}
\end{array}\right\}
$$

Substituting equations (25) and (23) into equation (16) yields the aperture admittance expression in the form

$$
\begin{align*}
\mathrm{y}_{\mathrm{ap}}= & \frac{1}{\mathrm{Y}_{01}} \frac{2}{\omega \mu_{\mathrm{o}} \mathrm{ab}} \frac{1}{(2 \pi)^{2}} \iint_{-\infty}^{\infty} \frac{\mathrm{k}^{2}-\mathrm{k}_{\mathrm{y}}^{2}}{\mathrm{k}_{\mathrm{z}}} \mathrm{C}_{0}\left(\mathrm{k}_{\mathrm{x}}\right) \mathrm{C}_{0}\left(\mathrm{k}_{\mathrm{x}}\right)\left[\mathrm{C}_{1}\left(\mathrm{k}_{\mathrm{y}}\right) \mathrm{C}_{1}\left(\mathrm{k}_{\mathrm{y}}\right)+2 \mathrm{DC} \mathrm{C}_{1}\left(\mathrm{k}_{\mathrm{y}}\right) \mathrm{C}_{3}\left(\mathrm{k}_{\mathrm{y}}\right)\right. \\
& \left.+\mathrm{D}^{2} \mathrm{C}_{3}\left(\mathrm{k}_{\mathrm{y}}\right) \mathrm{C}_{3}\left(\mathrm{k}_{\mathrm{y}}\right)\right] d \mathrm{k}_{\mathrm{x}} d \mathrm{k}_{\mathrm{y}}+\frac{\mathrm{Y}_{03}}{\mathrm{Y}_{01}} \mathrm{~A}_{03} \mathrm{D}^{2} \tag{27}
\end{align*}
$$

If the following relations are defined

$$
\begin{array}{r}
\mathrm{Y}_{l \mathrm{~m}}=\mathrm{Y}_{\mathrm{m} l}=\frac{2}{\omega \mu_{\mathrm{o}} \mathrm{ab}} \frac{1}{(2 \pi)^{2}} \iint_{-\infty}^{\infty} \frac{\mathrm{k}^{2}-\mathrm{k}_{\mathrm{y}}^{2}}{\mathrm{k}_{\mathrm{z}}} \mathrm{C}_{0}\left(\mathrm{k}_{\mathrm{x}}\right) \mathrm{C}_{0}\left(\mathrm{k}_{\mathrm{x}}\right) \mathrm{C}_{l}\left(\mathrm{k}_{\mathrm{y}}\right) \mathrm{C}_{\mathrm{m}}\left(\mathrm{k}_{\mathrm{y}}\right) \mathrm{dk}_{\mathrm{x}} \mathrm{dk}_{\mathrm{y}} \\
(l=1,3 ; \mathrm{m}=1,3) \tag{28}
\end{array}
$$

equation (27) finally reduces to

$$
\begin{equation*}
\mathrm{y}_{\mathrm{ap}}=\frac{\mathrm{Y}_{11}}{\mathrm{Y}_{01}}+2 \mathrm{D} \frac{\mathrm{Y}_{13}}{\mathrm{Y}_{01}}+\mathrm{D}^{2}\left(\frac{\mathrm{Y}_{33}}{\mathrm{Y}_{01}}+\frac{\mathrm{Y}_{03}}{\mathrm{Y}_{01}} \mathrm{~A}_{03}\right) \tag{29}
\end{equation*}
$$

Since $y_{a p}$ is stationary with respect to assumed aperture field distributions, it follows that

$$
\begin{equation*}
\frac{\delta \mathrm{y}_{\mathrm{ap}}}{\delta \mathrm{D}}=0 \tag{30}
\end{equation*}
$$

Therefore,

$$
\begin{equation*}
D=-\frac{Y_{13} / Y_{01}}{\frac{Y_{33}}{Y_{01}}+\frac{Y_{03}}{Y_{01}} A_{03}} \tag{31}
\end{equation*}
$$

At this point, the problem is basically solved. Equation (31) can be substituted into equation (29) to give

$$
\begin{equation*}
\mathrm{y}_{\mathrm{ap}}=\frac{\mathrm{Y}_{11}}{\mathrm{Y}_{01}}-\frac{\left(\mathrm{Y}_{13} / \mathrm{Y}_{01}\right)^{2}}{\frac{\mathrm{Y}_{33}}{\mathrm{Y}_{01}}+\frac{\mathrm{Y}_{03}}{\mathrm{Y}_{01}} A_{03}} \tag{32}
\end{equation*}
$$

and the electric field distribution at the aperture is computed by substituting equation (31) into equations (9). This substitution results in

$$
\begin{equation*}
\mathrm{E}_{\mathrm{x}}^{\mathrm{II}}(0)=\left(\mathrm{V}_{\mathrm{O}}+\mathrm{V}_{\mathrm{ro}}\right)\left[\cos \frac{\pi \mathrm{y}}{\mathrm{~b}}-\left(\frac{\mathrm{Y}_{13} / \mathrm{Y}_{01}}{\frac{\mathrm{Y}_{33}}{\mathrm{Y}_{01}}+\frac{\mathrm{Y}_{03}}{\mathrm{Y}_{01}} \mathrm{~A}_{03}}\right) \cos \frac{3 \pi \mathrm{y}}{\mathrm{~b}}\right] \tag{33}
\end{equation*}
$$

If the techniques described by Compton (ref. 1) are used, the integrals $\mathrm{Y}_{l \mathrm{~m}}$ may be manipulated into forms which are convenient for numerical evaluation. These expressions are given in the appendix.

The aperture admittance $\mathrm{y}_{\mathrm{ap}}$ must now be related to a measurable quantity; namely, the reflection coefficient.

## Relationship Between Aperture Admittance and Input Admittance

The input admittance of the antenna relative to a short circuit at the reference plane $\mathrm{z}=0$ is given by

$$
\begin{equation*}
\mathrm{y}_{\mathrm{in}}=\frac{1-\bar{\Gamma}}{1+\bar{\Gamma}} \tag{34}
\end{equation*}
$$

with $\bar{\Gamma}$ defined as

$$
\begin{equation*}
\bar{\Gamma}=\Gamma e^{-\mathrm{j}\left(\phi_{\mathrm{S}} \pm \pi\right)} \tag{35}
\end{equation*}
$$

The exponential factor in equation (35) is required in order to account for the phase shift introduced by the dielectric plug when the aperture is short-circuited.

Using the relation $y_{a p}=\frac{V_{o}-V_{\text {ro }}}{V_{o}+V_{r o}}$ in equation (5a) yields

$$
\begin{equation*}
\Gamma e^{-2 j k_{z 01}^{\prime} z_{o}}=\frac{1-y}{1+y} \tag{36}
\end{equation*}
$$

where

$$
\begin{equation*}
y=\frac{Y_{01}}{Y_{01}^{\prime}} \frac{\left(y_{a p}+j \tan k_{z 01} z_{o}\right)}{\left(1+j y_{a p} \tan k_{z 01}^{z_{o}}\right)} \tag{37}
\end{equation*}
$$

When a short circuit is placed over the aperture, $E$ goes to zero at $z=0$; therefore, equations (3) demand that

$$
\begin{equation*}
\mathrm{V}_{\mathrm{ro}, \mathrm{~s}}=-\mathrm{V}_{\mathrm{o}, \mathrm{~s}} \tag{38}
\end{equation*}
$$

where the subscript $s$ denotes the short-circuit values of $V_{\text {ro }}$ and $V_{0}$.
Equation (5a) therefore gives

$$
\begin{equation*}
\Gamma_{S} e^{-2 j k_{z 01}^{\prime} z_{0}}=e^{j \phi_{S}} e^{-2 j k_{z 01}^{\prime} z_{0}}=\frac{j \tan k_{z 01} z_{o}-\frac{Y_{01}}{Y_{01}^{\prime}}}{j \tan k_{z 01} z_{o}+\frac{Y_{01}}{Y_{01}^{\prime}}} \tag{39}
\end{equation*}
$$

The substitution of equations (36) and (39) into equation (34) results in

$$
\begin{equation*}
y_{i n}=\frac{\tan k_{z 01} z_{o}+j \frac{Y_{01}}{Y_{01}^{\prime}} y}{y \tan k_{z 01} z_{o}+j \frac{Y_{01}}{Y_{01}^{\prime}}} \equiv g_{i n}+j b_{i n} \tag{40}
\end{equation*}
$$

where $y$ is defined in equation (37).

Interpretation of the Theoretical Results
The expression for the aperture admittance has been derived in the form (eq. (32)):

$$
\mathrm{y}_{\mathrm{ap}}=\frac{\mathrm{Y}_{11}}{\mathrm{Y}_{01}}-\frac{\left(\mathrm{Y}_{13} / \mathrm{Y}_{01}\right)^{2}}{\frac{\mathrm{Y}_{33}}{\mathrm{Y}_{01}}+\frac{\mathrm{Y}_{03}}{\mathrm{Y}_{01}} A_{03}}
$$

where $Y_{11}, Y_{13}$, and $Y_{33}$ are double integrals which must be solved numerically. In order to attach physical significance to these integrals, consider equations (9) as expressed in the following form

$$
\left.\begin{array}{l}
\mathrm{E}_{\mathrm{x}}^{\mathrm{II}}(0)=\mathrm{E}_{\mathrm{x}}^{01} \cos \frac{\pi \mathrm{y}}{\mathrm{~b}}+\mathrm{E}_{\mathrm{x}}^{03} \cos \frac{3 \pi \mathrm{y}}{\mathrm{~b}}  \tag{41}\\
\mathrm{H}_{\mathrm{y}}^{\mathrm{II}}(0)=\mathrm{H}_{\mathrm{y}}^{01} \cos \frac{\pi \mathrm{y}}{\mathrm{~b}}+\mathrm{H}_{\mathrm{y}}^{03} \cos \frac{3 \pi \mathrm{y}}{\mathrm{~b}}
\end{array}\right\}
$$

where

$$
\left.\begin{array}{l}
\mathrm{E}_{\mathrm{x}}^{01}=\mathrm{V}_{\mathrm{o}}+\mathrm{V}_{\mathrm{ro}} \\
\mathrm{E}_{\mathrm{x}}^{03}=\mathrm{D}\left(\mathrm{~V}_{\mathrm{o}}+\mathrm{V}_{\mathrm{ro}}\right) \\
\mathrm{H}_{\mathrm{y}}^{01}=\mathrm{Y}_{01}\left(\mathrm{~V}_{\mathrm{o}}-\mathrm{V}_{\mathrm{ro}}\right)  \tag{42}\\
\mathrm{H}_{\mathrm{y}}^{03}=-\mathrm{Y}_{03} \mathrm{~A}_{03} \mathrm{D}\left(\mathrm{~V}_{\mathrm{o}}+\mathrm{V}_{\mathrm{ro}}\right)
\end{array}\right\}
$$

Since $y_{a p}=\frac{V_{o}-V_{r o}}{V_{o}+V_{r o}}$, it follows from equations (42) that equation (32) can be expressed in the form

$$
\begin{equation*}
\mathrm{Y}_{01} \mathrm{y}_{\mathrm{ap}}=\frac{\mathrm{H}_{\mathrm{y}}^{01}}{\mathrm{E}_{\mathrm{X}}^{01}}=\mathrm{Y}_{11}-\frac{\mathrm{Y}_{13}^{2}}{\mathrm{Y}_{33}-\frac{\mathrm{H}_{\mathrm{y}}^{03}}{\mathrm{E}_{\mathrm{x}}^{03}}} \tag{43}
\end{equation*}
$$

Equations (31) and (42) also give the relationship

$$
\begin{equation*}
\frac{\mathrm{E}_{\mathrm{x}}^{03}}{\mathrm{E}_{\mathrm{x}}^{01}}=-\frac{\mathrm{Y}_{13}}{\mathrm{Y}_{33}-\frac{\mathrm{H}_{\mathrm{y}}^{03}}{\mathrm{E}_{\mathrm{x}}^{03}}} \tag{44}
\end{equation*}
$$

The substitution of equation (44) into equation (43) results in

$$
\begin{equation*}
\mathrm{H}_{\mathrm{y}}^{01}=\mathrm{E}_{\mathrm{x}}^{01} \mathrm{Y}_{11}+\mathrm{Y}_{13} \mathrm{E}_{\mathrm{x}}^{03} \tag{45}
\end{equation*}
$$

and equation (44) can be rewritten as

$$
\begin{equation*}
\mathrm{H}_{\mathrm{y}}^{03}=\mathrm{E}_{\mathrm{x}}^{01} \mathrm{Y}_{13}+\mathrm{Y}_{33} \mathrm{E}_{\mathrm{x}}^{03} \tag{46}
\end{equation*}
$$

In other words, $Y_{11}$ and $Y_{33}$ may be regarded as the self-admittances of the $T E_{01}$ mode and $\mathrm{TE}_{03}$ mode, respectively. The integral $\mathrm{Y}_{13}$ may be interpreted as the mutual coupling between the modes.

Inspection of equation (32) reveals that the aperture will strongly excite a $\mathrm{TE}_{03}$ mode if the mutual admittance is large and if the aperture admittance of the $\mathrm{TE}_{03}$ mode is nearly matched to the $\mathrm{TE}_{03}$ admittance of the waveguide; that is, if $\mathrm{Y}_{13}^{2}$ is not small and $\mathrm{Y}_{33} \approx-\mathrm{Y}_{03} \mathrm{~A}_{03}$. The numerical results will show that these conditions are satisfied at some point in the waveguide band.

## DISCUSSION AND RESULTS

The explicit forms of the admittance integrals $\mathrm{Y}_{11} / \mathrm{Y}_{\mathrm{O}}, \quad \mathrm{Y}_{13} / \mathrm{Y}_{\mathrm{O}}$, and $\mathrm{Y}_{33} / \mathrm{Y}_{\mathrm{O}}$, as given in the appendix, were numerically solved in a manner discussed by Compton (ref. 1). As a partial check of the programing, values of the integral $Y_{11} / Y_{O}$ were compared with the one-mode lossy half-space results of R. C. Rudduck. ${ }^{1}$ The numbers differed only in the fourth significant figure, which was considered to be satisfactory agreement.

To the authors' knowledge, the other two integrals have not been computed by previous investigators; therefore, an experimental check appeared to be in order. A commercial RG-49/U C-band waveguide of inner dimensions 4.755 by 2.215 centimeters was selected as an aperture feed; and a slightly undersized rectangular quartz prism, arbitrarily cut to a thickness of $0.75 \mathrm{in} .(1.9 \mathrm{~cm})$ was cemented into the guide flush with the aperture to within $-0.005 \mathrm{in} .(0.13 \mathrm{~mm}$ ). A standard $3.62-\mathrm{in}$. ( 9.2 cm ) diameter circular flange was used as a ground plane. It was experimentally ascertained that the flange was large enough to appear as an infinite ground plane, as far as the admittance is concerned.

The admittance integrals were then computed in 200 MHz intervals over the 4.0 GHz to 6.0 GHz band, with air chosen as the external medium. The resultant real and imaginary parts of $Y_{13} / Y_{O}$ and $Y_{33} / Y_{O}$ are plotted in figure 2. For reference,

[^1]the quantity $-\frac{\mathrm{Y}_{03} \mathrm{~A}_{03}}{\mathrm{Y}_{\mathrm{O}}}$ (eq. (8)) is plotted with $\mathrm{Y}_{33} / \mathrm{Y}_{\mathrm{O}}$. The interesting feature of figure 2(b) is that $-\frac{\mathrm{Y}_{03} \mathrm{~A}_{03}}{\mathrm{Y}_{\mathrm{O}}}$, which is purely capacitative above cutoff of the $\mathrm{TE}_{03}$ mode, cancels the inductive susceptance of $\mathrm{Y}_{33} / \mathrm{Y}_{\mathrm{O}}$ at a frequency of about 5.44 GHz . Because the real part of $Y_{33} / Y_{0}$ is small over the band, the second term of the aperture admittance expression (eq. (32))
$$
\mathrm{y}_{\mathrm{ap}}=\frac{\mathrm{Y}_{11}}{\mathrm{Y}_{01}}-\frac{\left(\mathrm{Y}_{13} / \mathrm{Y}_{01}\right)^{2}}{\frac{\mathrm{Y}_{33}}{\mathrm{Y}_{01}}+\frac{\mathrm{Y}_{03}}{\mathrm{Y}_{01}} \mathrm{~A}_{03}}
$$
becomes large and affects the aperture admittance at frequencies near 5.44 GHz . The frequency interval of the computations was therefore reduced near $f=5.44 \mathrm{GHz}$ in order to observe the detailed behavior. The results are given in figure 3, where a sharp drop in both the aperture conductance $G_{a p} / Y_{O}$ and the aperture susceptance $\mathrm{B}_{\mathrm{ap}} / \mathrm{Y}_{\mathrm{O}}$ is noted near 5.44 GHz . The decrease in $\mathrm{G}_{\mathrm{ap}} / \mathrm{Y}_{\mathrm{O}}$ indicates that power is being coupled out of the $\mathrm{TE}_{01}$ mode and appearing as stored energy in the $\mathrm{TE}_{03}$ mode. Physically, the plug becomes a resonant cavity for the $\mathrm{TE}_{03}$ mode, with a Q of about 200. The integral $\mathrm{Y}_{11} / \mathrm{Y}_{\mathrm{O}}$ would be the aperture admittance if the $\mathrm{TE}_{03}$ mode were not included in the theory, and is plotted for reference in figure 3. From inspection, it is obvious that a one-mode description is an inadequate representation of the aperture admittance near resonance.

The aperture field distribution $\left|\frac{\mathrm{E}_{\mathrm{x}}^{\mathrm{II}}(0)}{\overline{\mathrm{V}_{\mathrm{o}}+\mathrm{Vrol}}}\right|^{2}$ (eq. (33)) is plotted in figure 4. It is vividly shown in figures 4 (a) and 4(b) that substantial changes occur in the electric field distribution as the frequency is swept over a small band about resonance. It is of further interest to note that the $\mathrm{TE}_{01}$ and $\mathrm{TE}_{03}$ modes add in phase at resonance. If the frequency is significantly above or below resonance, the aperture field distribution does not markedly differ from the $\cos ^{2} \frac{\pi y}{b} \mathrm{TE}_{01}$ mode variation, as shown in figure 4(c).

The input admittance was calculated from equation (40), and the results are shown in figure 5. The experimental data (circles) generally agree with the theory over the entire waveguide band. In particular, the resonant dip predicted by the theory is unmistakably confirmed experimentally. The assumptions made in the theory therefore appear to be justified.

In this particular experiment, the plug introduces a large amount of input capacitance at midband, which diminishes the resonance effects noted in figure 3. This effect may not be obtained in general because the magnitude and frequency location of the perturbation depends upon the index of refraction and thickness of the plug, the waveguide dimensions, and the dielectric constant of the external medium.

(b) Plot of $\frac{Y_{33}}{Y_{0}}=\frac{G_{33}}{Y_{0}}+\frac{j B_{33}}{Y_{0}}$ and $-\frac{Y_{03} A_{03}}{Y_{0}}$.

Figure 2.- Plot of the higher-order admittance integrals as a function of frequency.


Figure 3.- Plot of the aperture admittance and first-order admittance integral as a function of frequency.


Figure 4.- Plots of the aperture electric field distribution across the aperture.

(a) Plot of $\mathrm{gin}_{\mathrm{in}}$ as a function of frequency.

(b) Plot of $\mathrm{b}_{\mathrm{in}}$ as a function of frequency.

Figure 5.- Experimental and theoretical values of the input admittance as a function of frequency.

If the waveguide is not loaded, the susceptance of this antenna is capacitative over the entire band, and the standing wave ratio at $f=4.18 \mathrm{GHz}$ is 1.6 . However, when the $0.75-\mathrm{in}$. ( 1.9 cm ) quartz plug is inserted into the waveguide, the susceptance is tuned out at $\mathrm{f}=4.18 \mathrm{GHz}$; and the resultant standing wave ratio is 1.25 . Since the tuning point can be changed by choosing an appropriate plug thickness, the free-space match of the antenna can easily be improved at a given frequency in the waveguide band.

Calculations and experiments have been performed at X -band, and similar results were obtained.

## CONCLUDING REMARKS

The problem of a dielectric-plug loaded waveguide opening onto a flat conducting ground plane was investigated by variational techniques. The aperture field was assumed to consist of a superposition of the incident $\mathrm{TE}_{01}$ mode and an induced $T E_{03}$ mode. Theoretical and experimental results have shown that the dielectric plug can become a resonant cavity for the $\mathrm{TE}_{03}$ mode at a discrete frequency in the waveguide band. The coupling of stored energy into the cavity causes distortions in the electric field distribution at the aperture and perturbs the admittance near the resonant frequency. The results show that the susceptance of the antenna can be tuned out to improve the match by adjusting the thickness of the plug.

Langley Research Center,
National Aeronautics and Space Administration, Langley Station, Hampton, Va., October 10, 1967, 129-01-03-03-23.

## APPENDIX

## REDUCTION OF THE ADMITTANCE INTEGRALS

If methods described by Compton (ref. 1) are used, the admittance integrals $Y_{l m}$ may be manipulated into forms which are suitable to numerical evaluation. It was shown in equation (28) that

$$
\begin{equation*}
\mathrm{Y}_{l \mathrm{~m}}=\mathrm{Y}_{\mathrm{m} l}=\frac{2}{\omega \mu_{\mathrm{o}} \mathrm{ab}} \frac{1}{(2 \pi)^{2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\mathrm{k}^{2}-\mathrm{k}_{\mathrm{y}}^{2}} \frac{\mathrm{k}_{\mathrm{z}}}{} \mathrm{C}_{0}\left(\mathrm{k}_{\mathrm{x}}\right) \mathrm{C}_{0}\left(\mathrm{k}_{\mathrm{x}}\right) \mathrm{C}_{l}\left(\mathrm{k}_{\mathrm{y}}\right) \mathrm{C}_{\mathrm{m}}\left(\mathrm{k}_{\mathrm{y}}\right) \mathrm{dk}_{\mathrm{x}} \mathrm{dk}_{\mathrm{y}} \tag{A1}
\end{equation*}
$$

where, from a generalization of equations (26), the $C_{l}$ 's are given by

$$
\begin{equation*}
\mathrm{C}_{l}\left(\mathrm{k}_{\mathrm{y}}\right)=-\frac{2 l \pi \mathrm{bj}^{l+1} \cos \left(\mathrm{k}_{\mathrm{y}} \frac{\mathrm{~b}}{2}\right)}{(l \pi)^{2}-\left(\mathrm{k}_{\mathrm{y}} \mathrm{~b}\right)^{2}} \tag{A2}
\end{equation*}
$$

When the explicit forms of the $C_{l}$ 's are substituted into equation (A1), the integral $\mathrm{Y}_{l \mathrm{~m}}$ may be arranged in the following form

$$
\begin{equation*}
Y_{l \mathrm{~m}}=\frac{1}{(2 \pi)^{2}} \int_{-\infty}^{\infty} \int_{1} \mathrm{~F}_{1}^{l \mathrm{~m}}\left(\mathrm{k}_{\mathrm{x}}, \mathrm{k}_{\mathrm{y}}\right) \mathrm{F}_{2}\left(\mathrm{k}_{\mathrm{x}}, \mathrm{k}_{\mathrm{y}}\right) \mathrm{dk}_{\mathrm{x}} \mathrm{dk}_{\mathrm{y}} \tag{A3}
\end{equation*}
$$

where

$$
\left.\left.\begin{array}{l}
\mathrm{F}_{1}^{l \mathrm{~m}}\left(\mathrm{k}_{\mathrm{x}}, \mathrm{k}_{\mathrm{y}}\right)=-\mathrm{j}^{l+\mathrm{m}_{\mathrm{m} l}} \quad 2 \mathrm{ab}\left(\mathrm{k}^{2}-\mathrm{k}_{\mathrm{y}}^{2}\right) \sin ^{2}\left(\mathrm{k}_{\mathrm{x}} \frac{\mathrm{a}}{2}\right) \cos ^{2}\left(\mathrm{k}_{\mathrm{y}} \frac{\mathrm{~b}}{2}\right)
\end{array}\right\} \mu_{\mathrm{o}}\left(\mathrm{k}_{\mathrm{x}} \frac{\mathrm{a}}{2}\right)^{2}\left[\left(\mathrm{k}_{\mathrm{y}} \mathrm{~b}\right)^{2}-(2 \pi)^{2}\right]\left[\left(\mathrm{k}_{\mathrm{y}} \mathrm{~b}\right)^{2}-(\mathrm{m} \pi)^{2}\right]\right\}
$$

From Parseval's theorem, $Y_{l m}$ may be converted to an integration over physical space such that

$$
\begin{align*}
Y_{l \mathrm{~m}} & =\frac{1}{(2 \pi)^{2}} \int_{-\infty}^{\infty} \int_{1} \mathrm{~F}_{1}^{l \mathrm{~m}^{*}}\left(\mathrm{k}_{\mathrm{x}}, \mathrm{k}_{\mathrm{y}}\right) \mathrm{F}_{2}\left(\mathrm{k}_{\mathrm{x}}, \mathrm{k}_{\mathrm{y}}\right) \mathrm{dk}_{\mathrm{x}} d \mathrm{k}_{\mathrm{y}} \\
& =\int_{-\infty}^{\infty} \int_{\mathrm{f}_{1}}^{l \mathrm{~m}^{*}}(\mathrm{x}, \mathrm{y}) \mathrm{f}_{2}(\mathrm{x}, \mathrm{y}) \mathrm{dx} d \mathrm{dy} \tag{A5}
\end{align*}
$$

## APPENDIX

if $f_{1}$ and $f_{2}$ are the Fourier transforms of $F_{1}$ and $F_{2}$, respectively; that is

$$
\left.\begin{array}{l}
f_{1}^{l m}(x, y)=\frac{1}{(2 \pi)^{2}} \iint_{-\infty}^{\infty} F_{1}^{l m}\left(k_{x}, k_{y}\right) e^{-j\left(k_{x} x+k_{y} y\right)} d k_{x} d k_{y}  \tag{A6}\\
f_{2}(x, y)=\frac{1}{(2 \pi)^{2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_{2}\left(k_{x}, k_{y}\right) e^{-j\left(k_{x} x+k_{y} y\right)} d k_{x} d k_{y}
\end{array}\right\}
$$

From Harrington (ref. 3), the Fourier transform of $\mathrm{F}_{2}\left(\mathrm{k}_{\mathrm{x}}, \mathrm{k}_{\mathrm{y}}\right)$ is

$$
\begin{equation*}
f_{2}(x, y)=\int_{-\infty}^{\infty} \int^{e^{-j\left(k_{x} x+k_{y} y\right)} d k_{x} d k_{y}}=\frac{2 \pi j e^{-j k \sqrt{x^{2}+y^{2}}}}{\sqrt{x^{2}+y^{2}}} \tag{A7}
\end{equation*}
$$

The Fourier transform of $F_{1}\left(\mathrm{k}_{\mathrm{x}}, \mathrm{k}_{\mathrm{y}}\right)$ may be separated into the form

$$
\begin{align*}
f_{1}^{l m}(x, y) & =\frac{2 b}{\pi^{2} a \omega \mu_{0}} \int_{-\infty}^{\infty} \frac{\sin ^{2} \frac{k_{x} a}{2}}{k_{x}^{2}} e^{-j k_{x} x} d k_{x} \int_{-\infty}^{\infty} d k_{y}\left\{\frac{-j^{l+m} m l\left(k^{2}-k_{y}^{2}\right) \cos ^{2} \frac{k_{y} b}{2} e^{-j k_{y} y}}{\left[\left(k_{y} b\right)^{2}-(l \pi)^{2}\right]\left[\left(k_{y} b\right)^{2}-(m \pi)^{2}\right]}\right\} \\
& \equiv \frac{2 b}{\pi^{2} a \omega \mu_{o}} \mathrm{~g}(\mathrm{x}) \mathrm{h}(\mathrm{y})^{l m} \tag{A8}
\end{align*}
$$

These integrals may be evaluated by the calculus of residues. The result is

$$
\begin{align*}
& \left.\begin{array}{ll}
g(x)=\frac{\pi}{2}(a-|x|) & (|x|<a) \\
g(x)=0 & (|x|>a) \\
h(y)^{l m}=-j^{l+m} \frac{1}{2 \pi^{2} b}\left\{\frac{\left[k^{2}-\left(\frac{m \pi}{b}\right)^{2}\right] l \sin \left(\frac{m \pi|y|}{b}\right)-\left[k^{2}-\left(\frac{l \pi}{b}\right)^{2}\right] m \sin \left(\frac{l \pi|y|}{b}\right)}{l^{2}-m^{2}}\right.
\end{array}\right\}, ~ l l  \tag{A9}\\
& (|\mathrm{y}|<\mathrm{b} ; \quad l \neq \mathrm{m} ; \quad l, \mathrm{~m} \text { odd }) \\
& \mathrm{h}(\mathrm{y})^{l \mathrm{~m}}=\frac{1}{4 \pi \mathrm{bl}}\left\{\frac{1}{\pi}\left[\mathrm{k}^{2}+\left(\frac{2 \pi}{\mathrm{~b}}\right)^{2}\right] \sin \left(\frac{2 \pi|\mathrm{y}|}{\mathrm{b}}\right)+\frac{l}{\mathrm{~b}}\left[\mathrm{k}^{2}-\left(\frac{2 \pi}{\mathrm{~b}}\right)^{2}\right](\mathrm{b}-|\mathrm{y}|) \cos \left(\frac{l \pi|\mathrm{y}|}{\mathrm{b}}\right)\right\} \\
& \text { ( }|\mathrm{y}|<\mathrm{b} ; \quad l=\mathrm{m} \neq 0 ; \quad l, \mathrm{~m} \text { odd }) \\
& h(y)^{l m}=0 \\
& (|y|>b \text { for all } l, m)
\end{align*}
$$

## APPENDIX

Therefore, the admittance integrals are given by

$$
\begin{equation*}
Y_{l m}=j \frac{16 b}{\pi a \omega \mu_{o}} \int_{0}^{a} \int_{0}^{b} \frac{g(x) h(y)^{l m} e^{-j k} \sqrt{x^{2}+y^{2}}}{\sqrt{x^{2}+y^{2}}} d x d y \tag{A10}
\end{equation*}
$$

where

$$
\begin{equation*}
k=k_{r}-j k_{i}=k_{o} \sqrt{\epsilon_{r}-j \epsilon_{i}} \tag{A11}
\end{equation*}
$$

The singularity at $\mathrm{x}=\mathrm{y}=0$ may be removed by converting to normalized polar coordinates, with $k_{0} x=R \cos \theta$ and $k_{0} y=R \sin \theta$. This requires splitting up the integral defined in equation (A10) into the form

$$
\begin{equation*}
\mathrm{Y}_{l \mathrm{~m}}=\int_{\theta=0}^{\theta_{\mathrm{O}}} \int_{\mathrm{R}=0}^{\mathrm{k}_{\mathrm{o}} \mathrm{a} / \cos \theta} \cdots+\int_{\theta=\theta_{\mathrm{O}}}^{\pi / 2} \int_{\mathrm{R}=0}^{\mathrm{k}_{\mathrm{o}} \mathrm{~b} / \sin \theta} \ldots \tag{A12}
\end{equation*}
$$

where $\tan \theta_{\mathrm{O}}=\frac{\mathrm{b}}{\mathrm{a}}$.

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Details on the availability of these publications may be obtained from:
SCIENTIFIC AND TECHNICAL INFORMATION DIVISION


[^0]:    For sale by the Clearinghouse for Federal Scientific and Technical Information
    Springfield, Virginia 22151 - CFSTI price $\$ 3.00$

[^1]:    ${ }^{1}$ Unpublished calculations furnished under NASA grant NsG-448, Ohio State University, 1967.

