The Instability and Vibration of Rotating Beams With Arbitrary Pretwist and an Elastically Restrained Root

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The governing differential equations and the boundary conditions for the coupled bending-bending-extensional vibration of a rotating nonuniform beam with arbitrary pretwist and an elastically restrained root are derived by Hamilton's principle. The semi-analytical solution procedure for an inextensional beam without taking account of the coriolis forces is derived. The coupled governing differential equations are transformed to be a vector characteristic governing equation. The frequency equation of the system is derived and expressed in terms of the transition matrix of the vector governing equation. A simple and efficient algorithm for determining the transition matrix of the general system with arbitrary pretwist is derived. The divergence in the Frobenius method does not exist in the proposed method. The frequency relations between different systems are revealed. The mechanism of instability is discovered. The influence of the rotatory inertia, the coupling effect of the rotating speed and the mass moment of inertia, the setting angle, the rotating speed and the spring constants on the natural frequencies, and the phenomenon of divergence instability are investigated. [DOI: 10.1115/1.1408615]

1 Introduction

Rotating beams, which have importance in many practical applications such as turbine blades, helicopter rotor blades, airplane propellers, and robot manipulators have been investigated for a long time. An interesting review of the subject can be found in the papers by Leissa [1], Ramamurti and Balasubramanian [2], Rosen [3] and Lin [4]. Much attention has been focused on the investigation of the unpretwisted beam. Most of research of the vibration problem of rotating pretwisted beam have been studied by using numerical method because of its complexity. No analytical solution for the vibration of a rotating pretwisted beam has been presented.

Considering the Bernoulli-Euler unpretwisted beam theory, the influence of tip mass, rotating speed, hub radius, setting angle, taper ratio, and elastic root restraints on the natural frequencies of transverse vibrations of a rotating beam were investigated by many investigators. Lee and Kuo [5] obtained the exact solution for the free vibrations of rotating unpretwisted beam with bending rigidity and mass density varying in arbitrary polynomial forms by taking the Frobenius method. Lee and Lin [6] studied the free vibration of unpretwisted Timoshenko beams. The two coupled characteristic differential equations governing the bending response uncouple into one complete fourth-order ordinary differential equation with variable coefficients in the angle of rotation. The four fundamental solutions of the uncoupled fourth-order ordinary differential equation were obtained using the Frobenius method. The frequency equation was expressed in terms of the four fundamental solutions. Similarly, one can decouple the two coupled governing characteristic differential equations of a rotating pretwisted beam into one complete eighth-order ordinary differential equation by taking many procedures of differentation. The variation of the coefficients of the uncoupled governing characteristic equation will be very large. Taking the Frobenius method, the corresponding eight fundamental solutions of the eighth-order ordinary differential equation can be expressed in power series. However, the fundamental solutions will be divergent because the region of convergence of a power series is usually limited.

Considering the Timoshenko unpretwisted beam theory, the influence of shear deformation and rotatory inertia on the bending vibrations of a rotating beam were investigated by numerous authors. Lee and Lin [6] studied numerically the coupling effect of the rotating speed and the mass moment of inertia on the natural frequencies and the phenomenon of divergence instability. Lin [4] obtained the generalized Green function of an nth-order ordinary differential equation. This Green function was used to obtain the closed-form solution for the forced vibration of a rotating Timoshenko beam. The prediction to the frequencies and the mechanism of divergence instability of a rotating beam have not been investigated.

For a nonrotating pretwisted beam, approximation methods are very useful tools to investigate the free vibrations of pretwisted beams where it is difficult to obtain exact solutions even for the simplest cases. These methods are the finite element method ([7]), the Rayleigh-Ritz method ([8]), the Reissner method ([9]) the Galerkin method ([10]), and the transfer matrix method ([11,12]). Lin [12] derived the exact field transfer matrix of a nonuniform nonrotating pretwisted beam with arbitrary pretwist and studied the performance of a beam with elastic boundary conditions. However, the exact field transfer matrix of a rotating pretwisted beam can not be derived in a similar way.

For a rotating pretwisted beam, Rao and Carnegie [13] used the Holzer-Myklestad approach to determine the natural frequencies and mode shapes of a cantilever pretwisted blade. Subrahmanyam and Kaza [14] studied the vibrations of a cantilever tapered pretwisted beam by using the Ritz method and the finite difference method. Subrahmanyam, Kulkarni, and Rao [15] used the Reissner method to study the vibration of a rotating pretwisted cantile-

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ver Timoshenko beam. Sisto and Chang [16] proposed a finite element method for a vibration analysis of rotating pretwisted beam. Young and Lin [17] studied the stability of a cantilever tapered pretwisted beam with varying speed by using the Galerkin method. Kar and Neogy [18] used the Ritz method to study the stability of a rotating pretwisted cantilever beam. Hernried [19] used the finite difference method to determine the natural frequencies of a rotating pretwisted nonuniform cantilever beam. Moreover, the author used the mode-superposition method to determine the forced vibration of the beam. Surace, Anghel, and Mares [20] derived the approximate method based on the use of structural influence function to determine the natural frequencies of a rotating cantilever pretwisted Bernoulli-Euler beam. As a result, no analytical solution has been given to the coupled bending-bending vibrations of a rotating nonuniform beam with arbitrary pretwist and an elastically restrained root.

In this paper, the governing differential equations for the coupled bending-bending-extensional vibration of a rotating nonuniform beam with arbitrary pretwist, an elastically restrained root, setting angle, hub radius, and rotating at a constant angular velocity, are derived by using Hamilton's principle. For an inextensional beam, without taking account of the coriolis force's effect, the three coupled governing differential equations are reduced to two coupled equations and the centrifugal force is obtained. The reduced coupled governing differential equations are transformed to a vector characteristic differential equation. The frequency equation of the system is derived and expressed in terms of the transition matrix of the vector governing equation. A simple and efficient algorithm for determining the semi-analytical transition matrix of the general system with nonuniform pretwist is derived. The frequency relation and the mechanism of instability of unpretwisted and pretwisted rotating beams are investigated. The influence of the rotatory inertia, the coupling effect of the rotating speed and the mass moment of inertia, the setting angle, the rotating speed and the spring constants on the natural frequencies, and the phenomenon of divergence instability are investigated.

2 Pretwisted Beam

2.1 Governing Equations and Boundary Conditions. Consider the coupled bending-bending-extensional vibration of a pretwisted and doubly symmetric nonuniform beam elastically restrained, mounted with setting angle θ on a hub with radius R, rotating with constant angular velocity Ω, as shown in Fig. 1. The displacement fields of the beam are

$$u=u_0(x,t)-z\frac{\partial w}{\partial x}-y\frac{\partial v}{\partial x},\quad v=v(x,t),\quad w=w(x,t),\quad (1)$$

where x, y, and z are the fixed frame coordinates. t is the time variable. The velocity vector of a point (x, y, z) in the beam is given by

$$\mathbf{V}=\left[\frac{\partial u}{\partial t}+\Omega\sin\theta(z+w)+\Omega\cos\theta(y+v)\right]\mathbf{i}+\left[\frac{\partial v}{\partial t}-\Omega\cos\theta(x+R+u)\right]\mathbf{j}+\left[\frac{\partial w}{\partial t}-\Omega\sin\theta(x+R+u)\right]\mathbf{k}. \quad (2)$$

The potential energy $\overline{U}$ and the kinetic energy $\overline{T}$ of the beam are

$$\overline{U}=\frac{1}{2}\int_0^L\int_A\sigma_{xx}\varepsilon_{xx}dAdx+\frac{1}{2}K_{z\theta}\left[\frac{\partial w(0,t)}{\partial x}\right]^2+\frac{1}{2}K_{zT}w^2(0,t)+\frac{1}{2}K_{y\theta}\left[\frac{\partial v(0,t)}{\partial x}\right]^2+\frac{1}{2}K_{yT}v^2(0,t), \quad (3)$$

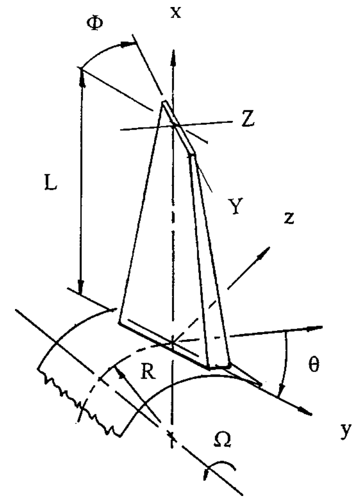

Fig. 1 Geometry and coordinate system of a rotating pretwisted beam

$$\overline{T}=\frac{1}{2}\int_0^L\int_A\left[\left(\frac{\partial u}{\partial t}\right)^2+\left(\frac{\partial v}{\partial t}\right)^2+\left(\frac{\partial w}{\partial t}\right)^2\right]\rho dAdx, \quad (4)$$

where A is the cross-sectional area of the beam. K_{yT}, $K_{y\theta}$, K_{zT}, and $K_{z\theta}$ are the translational and rotational spring constants at the left end of the beam in the y and z-directions, respectively. L is the length of the beam. ρ is the mass density per unit volume. $\sigma_{xx}, \varepsilon_{xx}$ are the normal stress and strain in the x-direction, respectively. Application of Hamilton's principle yields the following governing differential equations:

$$-\frac{\partial^2}{\partial x^2}\left(EI_{yy}\frac{\partial^2 w}{\partial x^2}+EI_{yz}\frac{\partial^2 v}{\partial x^2}\right)+\frac{\partial}{\partial x}\left(N\frac{\partial w}{\partial x}\right)+\frac{\partial}{\partial x}\left(J_y\frac{\partial^3 w}{\partial t^2\partial x}+J_x\frac{\partial^3 v}{\partial t^2\partial x}\right)-\Omega^2\frac{\partial}{\partial x}\left(J_y\frac{\partial w}{\partial x}+J_x\frac{\partial v}{\partial x}\right)-\rho A\left(\frac{\partial^2 w}{\partial t^2}-\Omega\sin\theta\frac{\partial u_0}{\partial t}\right)+\rho A\Omega\sin\theta\left(\frac{\partial u_0}{\partial t}+\Omega\sin\theta w+\Omega\cos\theta v\right)=0, \quad (5)$$

$$-\frac{\partial^2}{\partial x^2}\left(EI_{yz}\frac{\partial^2 w}{\partial x^2}+EI_{zz}\frac{\partial^2 v}{\partial x^2}\right)+\frac{\partial}{\partial x}\left(N\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial x}\left(J_x\frac{\partial^3 w}{\partial t^2\partial x}+J_z\frac{\partial^3 v}{\partial t^2\partial x}\right)-\Omega^2\frac{\partial}{\partial x}\left(J_x\frac{\partial w}{\partial x}+J_z\frac{\partial v}{\partial x}\right)-\rho A\left(\frac{\partial^2 v}{\partial t^2}-\Omega\cos\theta\frac{\partial u_0}{\partial t}\right)+\rho A\Omega\cos\theta\left(\frac{\partial u_0}{\partial t}+\Omega\sin\theta w+\Omega\cos\theta v\right)=0, \quad (6)$$

$$\frac{\partial N}{\partial x}-\rho A\frac{\partial^2 u_0}{\partial t^2}-2\rho A\Omega\left(\sin\theta\frac{\partial w}{\partial t}+\cos\theta\frac{\partial v}{\partial t}\right)+\rho A\Omega^2(x+u_0)=0, \quad (7)$$

and the associated boundary conditions:
at $x=0$:

$$u_0=0, \quad (8)$$

$$EI_{yy}\frac{\partial^2 w}{\partial x^2}+EI_{yz}\frac{\partial^2 v}{\partial x^2}-K_{z\theta}\frac{\partial w}{\partial x}=0, \quad (9)$$

$$\frac{\partial}{\partial x}\left(EI_{yy}\frac{\partial^2 w}{\partial x^2}+EI_{yz}\frac{\partial^2 v}{\partial x^2}\right)-N\frac{\partial w}{\partial x}-J_y\frac{\partial^3 w}{\partial x\partial t^2}-J_x\frac{\partial^3 v}{\partial x\partial t^2}+\Omega^2\left(J_y\frac{\partial w}{\partial x}+J_x\frac{\partial v}{\partial x}\right)+K_{zT}w=0, \quad (10)$$

$$EI_{yz}\frac{\partial^2 w}{\partial x^2}+EI_{zz}\frac{\partial^2 v}{\partial x^2}-K_{y\theta}\frac{\partial v}{\partial x}=0, \quad (11)$$

$$\frac{\partial}{\partial x}\left(EI_{yz}\frac{\partial^2 w}{\partial x^2}+EI_{zz}\frac{\partial^2 v}{\partial x^2}\right)-N\frac{\partial v}{\partial x}-J_x\frac{\partial^3 w}{\partial x\partial t^2}-J_z\frac{\partial^3 v}{\partial x\partial t^2}+\Omega^2\left(J_x\frac{\partial w}{\partial x}+J_z\frac{\partial v}{\partial x}\right)+K_{yT}w=0, \quad (12)$$

at $x=L$:

$$N(L)=0, \quad (13)$$

$$EI_{yy}\frac{\partial^2 w}{\partial x^2}+EI_{yz}\frac{\partial^2 v}{\partial x^2}=0, \quad (14)$$

$$\frac{\partial}{\partial x}\left(EI_{yy}\frac{\partial^2 w}{\partial x^2}+EI_{yz}\frac{\partial^2 v}{\partial x^2}\right)-N\frac{\partial w}{\partial x}-J_y\frac{\partial^3 w}{\partial x\partial t^2}-J_x\frac{\partial^3 v}{\partial x\partial t^2}+\Omega^2\left(J_y\frac{\partial w}{\partial x}+J_x\frac{\partial v}{\partial x}\right)=0, \quad (15)$$

$$EI_{yz}\frac{\partial^2 w}{\partial x^2}+EI_{zz}\frac{\partial^2 v}{\partial x^2}=0, \quad (16)$$

$$\frac{\partial}{\partial x}\left(EI_{yz}\frac{\partial^2 w}{\partial x^2}+EI_{zz}\frac{\partial^2 v}{\partial x^2}\right)-N\frac{\partial v}{\partial x}-J_x\frac{\partial^3 w}{\partial x\partial t^2}-J_z\frac{\partial^3 v}{\partial x\partial t^2}+\Omega^2\left(J_x\frac{\partial w}{\partial x}+J_z\frac{\partial v}{\partial x}\right)=0, \quad (17)$$

where E is Young's modulus. I is the area moment inertia of the beam. J_x, J_y, and J_z are mass moment of inertia per unit length about the x, y and z-axes, respectively. N is the centrifugal force. It is observed that in Eq. (7) the centrifugal force N is related to u_0, v, and w. The centrifugal force N is the parameter of Eqs. (5) and (6) in terms of u_0, v, and w. Thus the system is nonlinear. It is hard to obtain the solution of the system. But if an inextensional beam without the Coriolis force effect is considered, the system becomes linear and the corresponding solution can be easily obtained. Moreover, the lateral vibration of a blade subjected to low rotational speed is dominant and the effect of the Coriolis force may be neglected [14].

For an inextensional beam, without taking account of the Coriolis force effect, the centrifugal force in Eqs. (7) and (13) can be expressed as

$$N(x)=\Omega^2\int_x^L \rho A(R+x)\,dx. \quad (18)$$

In terms of the following dimensionless quantities,

$$B_{ij}=E(x)I_{ij}(x)/[E(0)I_{yy}(0)], \quad i,j=x,y,z \quad g_i=J_i(x)/J_y(0),$$
$$m=\rho(x)A(x)/[\rho(0)A(0)], \quad n(\xi)=\alpha^2\int_\xi^1 m(\chi)(r+\chi)\,d\chi,$$
$$r=R/L, \quad V=v/L,$$
$$W=w/L, \quad \alpha^2=\rho(0)A(0)\Omega^2L^4/[E(0)I_{yy}(0)],$$
$$\beta_1=K_{z\theta}L/[E(0)I_{yy}(0)], \quad \beta_2=K_{zT}L/[E(0)I_{yy}(0)],$$
$$\beta_3=K_{y\theta}L/[E(0)I_{yy}(0)], \quad \beta_4=K_{yT}L/[E(0)I_{yy}(0)],$$
$$\eta=J_y(0)/[\rho(0)A(0)L^2], \quad \Lambda^2=\rho(0)A(0)\omega^2L^4/[E(0)I_{yy}(0)],$$
$$\xi=x/L, \quad \tau=t/L^2\sqrt{E(0)I_{yy}(0)/\rho(0)A(0)},$$
$$\gamma_{i1}=\frac{\beta_i}{1+\beta_i}, \quad \gamma_{i2}=\frac{1}{1+\beta_i}, \quad (19)$$

the dimensionless governing characteristic differential equations of motion for harmonic vibration with circular frequency ω are written as

$$-\frac{d^2}{d\xi^2}\left(B_{yy}\frac{d^2W}{d\xi^2}+B_{yz}\frac{d^2V}{d\xi^2}\right)+\frac{d}{d\xi}\left(n\frac{dW}{d\xi}\right)-\eta(\alpha^2+\Lambda^2)\frac{d}{d\xi}\left(g_y\frac{dW}{d\xi}+g_x\frac{dV}{d\xi}\right)+m(\alpha^2\sin^2\theta+\Lambda^2)W+m\alpha^2\sin\theta\cos\theta V=0, \quad (20)$$

$$-\frac{d^2}{d\xi^2}\left(B_{yz}\frac{d^2W}{d\xi^2}+B_{zz}\frac{d^2V}{d\xi^2}\right)+\frac{d}{d\xi}\left(n\frac{dV}{d\xi}\right)-\eta(\alpha^2+\Lambda^2)\frac{d}{d\xi}\left(g_x\frac{dW}{d\xi}+g_z\frac{dV}{d\xi}\right)+m(\alpha^2\cos^2\theta+\Lambda^2)V+m\alpha^2\sin\theta\cos\theta W=0, \quad \xi\in(0,1) \quad (21)$$

and the associated dimensionless elastic boundary conditions are given as follows:
at $\xi=0$:

$$\gamma_{12}\left(B_{yy}\frac{d^2W}{d\xi^2}+B_{yz}\frac{d^2V}{d\xi^2}\right)-\gamma_{11}\frac{dW}{d\xi}=0, \quad (22)$$

$$\gamma_{22}\left\{\frac{d}{d\xi}\left(B_{yy}\frac{d^2W}{d\xi^2}+B_{yz}\frac{d^2V}{d\xi^2}\right)-n\frac{dW}{d\xi}+\eta(\alpha^2+\Lambda^2)\times\left(g_y\frac{dW}{d\xi}+g_x\frac{dV}{d\xi}\right)\right\}+\gamma_{21}W=0, \quad (23)$$

$$\gamma_{32}\left(B_{yz}\frac{d^2W}{d\xi^2}+B_{zz}\frac{d^2V}{d\xi^2}\right)-\gamma_{31}\frac{dV}{d\xi}=0, \quad (24)$$

$$\gamma_{42}\left\{\frac{d}{d\xi}\left(B_{yz}\frac{d^2W}{d\xi^2}+B_{zz}\frac{d^2V}{d\xi^2}\right)-n\frac{dV}{d\xi}+\eta(\alpha^2+\Lambda^2)\times\left(g_x\frac{dW}{d\xi}+g_z\frac{dV}{d\xi}\right)\right\}+\gamma_{41}V=0 \quad (25)$$

at $\xi=1$:

$$B_{yy}\frac{d^2W}{d\xi^2}+B_{yz}\frac{d^2V}{d\xi^2}=0, \quad (26)$$

$$\frac{d}{d\xi}\left(B_{yy}\frac{d^2W}{d\xi^2}+B_{yz}\frac{d^2V}{d\xi^2}\right)+\eta(\alpha^2+\Lambda^2)\left(g_y\frac{dW}{d\xi}+g_x\frac{dV}{d\xi}\right)=0, \quad (27)$$

$$B_{yz}\frac{d^2W}{d\xi^2}+B_{zz}\frac{d^2V}{d\xi^2}=0, \quad (28)$$

$$\frac{d}{d\xi}\left(B_{yz}\frac{d^2W}{d\xi^2}+B_{zz}\frac{d^2V}{d\xi^2}\right)+\eta(\alpha^2+\Lambda^2)\left(g_x\frac{dW}{d\xi}+g_z\frac{dV}{d\xi}\right)=0. \tag{29}$$

It should be noted that considering the velocity field (2) results in the coupling effect of the rotating speed and the mass moment of inertia, $\eta(\alpha^2+\Lambda^2)d(g_y dW/d\xi+g_x dV/d\xi)/d\xi$ and $\eta(\alpha^2+\Lambda^2)d(g_x dW/d\xi+g_z dV/d\xi)/d\xi$. However, if the displacements of any point in the cross section of the beam is the same as that of the center of the cross section, i.e., $u=u_0(x,t)$, $v=v(x,t)$, and $w=w(x,t)$, these effects cannot be considered ([17]). Furthermore, when the setting angle is zero, the governing equations become the same as those given by Rosen [3].

It can be observed that the second terms in Eqs. (20) and (21) present the effect of the centrifugal force n to increase the stiffness of the beam. Because the second and third terms in Eqs. (20) and (21) are positive and negative, respectively, the coupling effect of the rotating speed and the mass moment of inertia, i.e., the third terms in Eqs. (20) and (21), presents an axial compressive force to decrease the stiffness of the beam. Moreover, because the coupling effect includes the product of the rotating speed α and the rotatory inertia η, the coupling effect on the frequencies is great for the system with large parameters α and η.

2.2 Solution Method.

2.2.1 Transformed Vector Governing Equation. Defining the state variables as

$$x_1=W,\quad x_2=\frac{dW}{d\xi},\quad x_3=\frac{d^2W}{d\xi^2},\quad x_4=\frac{d^3W}{d\xi^3},$$

$$x_5=V,\quad x_6=\frac{dV}{d\xi},\quad x_7=\frac{d^2V}{d\xi^2},\quad x_8=\frac{d^3V}{d\xi^3}, \tag{30}$$

Eqs. (20) and (21) can be written as, respectively,

$$a_1\frac{dx_4}{d\xi}+a_2\frac{dx_3}{d\xi}+a_3\frac{dx_2}{d\xi}+a_4\frac{dx_1}{d\xi}+a_5x_1+a_6\frac{dx_8}{d\xi}+a_7\frac{dx_7}{d\xi}$$
$$+a_8\frac{dx_6}{d\xi}+a_9\frac{dx_5}{d\xi}+a_{10}x_5=0, \tag{31}$$

$$a_{11}\frac{dx_4}{d\xi}+a_{12}\frac{dx_3}{d\xi}+a_{13}\frac{dx_2}{d\xi}+a_{14}\frac{dx_1}{d\xi}+a_{15}x_1+a_{16}\frac{dx_8}{d\xi}+a_{17}\frac{dx_7}{d\xi}$$
$$+a_{18}\frac{dx_6}{d\xi}+a_{19}\frac{dx_5}{d\xi}+a_{20}x_5=0, \tag{32}$$

where

$$a_1=B_{yy},\quad a_2=2\,\frac{dB_{yy}}{d\xi},$$

$$a_3=\frac{d^2B_{yy}}{d\xi^2}-n+\eta(\alpha^2+\Lambda^2)g_y,\quad a_4=-\frac{dn}{d\xi}+\eta(\alpha^2+\Lambda^2)\frac{dg_y}{d\xi},$$

$$a_5=-m(\alpha^2\sin^2\theta+\Lambda^2),\quad a_6=a_{11}=B_{yz},$$

$$a_7=a_{12}=2\,\frac{dB_{yz}}{d\xi},\quad a_8=a_{13}=\frac{d^2B_{yz}}{d\xi^2}+\eta(\alpha^2+\Lambda^2)g_x,$$

$$a_9=a_{14}=\eta(\alpha^2+\Lambda^2)\frac{dg_x}{d\xi},\quad a_{10}=a_{15}=-m\alpha^2\sin\theta\cos\theta,$$

$$a_{16}=B_{zz},\quad a_{17}=2\,\frac{dB_{zz}}{d\xi},\quad a_{18}=\frac{d^2B_{zz}}{d\xi^2}-n+\eta(\alpha^2+\Lambda^2)g_z,$$

$$a_{19}=-\frac{dn}{d\xi}+\eta(\alpha^2+\Lambda^2)\frac{dg_z}{d\xi},\quad a_{20}=-m(\alpha^2\cos^2\theta+\Lambda^2). \tag{33}$$

Multiplying Eq. (31) by a_{16} and Eq. (32) by a_6 and subtracting the latter from the former, one obtains

$$\frac{dx_4}{d\xi}=\sum_{j=1}^{8}c_jx_j, \tag{34}$$

where

$$c_1=-(a_5a_{16}-a_{15}a_6)/s,\quad c_2=-(a_4a_{16}-a_{14}a_6)/s,$$
$$c_3=-(a_3a_{16}-a_{13}a_6)/s,\quad c_4=-(a_2a_{16}-a_{12}a_6)/s,$$
$$c_5=-(a_{10}a_{16}-a_{20}a_6)/s,\quad c_6=-(a_9a_{16}-a_{19}a_6)/s,$$
$$c_7=-(a_8a_{16}-a_{18}a_6)/s,\quad c_8=-(a_7a_{16}-a_{17}a_6)/s, \tag{35}$$

in which $s=a_1a_{16}-a_{11}a_6$. Similarly, multiplying Eq. (31) by a_{11} and Eq. (32) by a_1 and subtracting the latter from the former, one obtains

$$\frac{dx_8}{d\xi}=\sum_{j=1}^{8}\overline{c_j}x_j, \tag{36}$$

where

$$\overline{c_1}=-(a_5a_{11}-a_{15}a_1)/\overline{s},\quad \overline{c_2}=-(a_4a_{11}-a_{14}a_1)/\overline{s},$$
$$\overline{c_3}=-(a_3a_{11}-a_{13}a_1)/\overline{s},\quad \overline{c_4}=-(a_2a_{11}-a_{12}a_1)/\overline{s},$$
$$\overline{c_5}=-(a_{10}a_{11}-a_{20}a_1)/\overline{s},\quad \overline{c_6}=-(a_9a_{11}-a_{19}a_1)/\overline{s},$$
$$\overline{c_7}=-(a_8a_{11}-a_{18}a_1)/\overline{s},\quad \overline{c_8}=-(a_7a_{11}-a_{17}a_1)/\overline{s}, \tag{37}$$

in which $\overline{s}=a_6a_{11}-a_{16}a_1$. Based on the relations (30), (34), and (36), the transformed vector characteristic governing equation can be obtained as follows:

$$\frac{d\mathbf{X}}{d\xi}=\mathbf{A}(\xi)\mathbf{X}(\xi), \tag{38}$$

where

$$X(\xi)=[x_1\ \ x_2\ \ x_3\ \ x_4\ \ x_5\ \ x_6\ \ x_7\ \ x_8]^T,$$

$$A(\xi)=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8\\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\ \overline{c_1} & \overline{c_2} & \overline{c_3} & \overline{c_4} & \overline{c_5} & \overline{c_6} & \overline{c_7} & \overline{c_8} \end{bmatrix}, \tag{39}$$

in which the superscript "T" is the symbol of transpose of a matrix.

2.2.2 Frequency Equation. The solution of Eq. (38) can be expressed as

$$\mathbf{X}(\xi)=\mathbf{T}(\xi,0)\mathbf{X}(0), \tag{40}$$

where $\mathbf{T}(\xi,0)$ is the transition matrix from 0 to ξ, to be determined. Moreover, the state variables at $\xi=1$ can be written as

$$x_i(1)=\sum_{j=1}^{8}T_{ij}(1,0)x_j(0),\quad i=1,2,\ldots,8 \tag{41}$$

where T_{ij} is the elements of the transition matrix from 0 to 1. Expressing the boundary conditions (22)–(25) in terms of the state variables $\{x_1(0),x_2(0),\ldots,x_8(0)\}$ and substituting Eq. (41) into the boundary conditions (26)–(29), the frequency equation of the system is obtained,

$$|\chi_{ij}|_{8\times8}=0, \tag{42}$$

where

$$\chi_{11}=\chi_{14}=\chi_{15}=\chi_{16}=\chi_{18}=0,\quad \chi_{12}=-\gamma_{11},$$
$$\chi_{13}=\gamma_{12}B_{yy}(0),\quad \chi_{17}=\gamma_{12}B_{yz}(0);$$
$$\chi_{21}=\gamma_{21},\quad \chi_{22}=\gamma_{22}[\eta(\alpha^2+\Lambda^2)g_y(0)-n(0)],$$
$$\chi_{23}=\gamma_{22}B'_{yy}(0),\quad \chi_{24}=\gamma_{22}B_{yy}(0),$$
$$\chi_{25}=0,\quad \chi_{26}=\gamma_{22}\eta(\alpha^2+\Lambda^2)g_x(0),$$
$$\chi_{27}=\gamma_{22}B'_{yz}(0),\quad \chi_{28}=\gamma_{22}B_{yz}(0);$$
$$\chi_{31}=\chi_{32}=\chi_{34}=\chi_{35}=\chi_{38}=0,\quad \chi_{33}=\gamma_{32}B_{yz}(0),$$
$$\chi_{36}=-\gamma_{31},\quad \chi_{37}=\gamma_{32}B_{zz}(0);$$
$$\chi_{41}=0,\quad \chi_{42}=\gamma_{42}\eta(\alpha^2+\Lambda^2)g_x(0),$$
$$\chi_{43}=\gamma_{42}B'_{yz}(0),\quad \chi_{44}=\gamma_{42}B_{yz}(0),$$
$$\chi_{45}=\gamma_{41},\quad \chi_{46}=\gamma_{42}[\eta(\alpha^2+\Lambda^2)g_z(0)-n(0)],$$
$$\chi_{47}=\gamma_{42}B'_{zz}(0),\quad \chi_{48}=\gamma_{42}B_{zz}(0);$$
$$\chi_{5j}=B_{yy}(1)T_{3j}(1,0)+B_{yz}(1)T_{7j}(1,0),\quad j=1,2,\ldots,8;$$
$$\begin{aligned}\chi_{6j}=\;&\eta(\alpha^2+\Lambda^2)g_y(1)T_{2j}(1,0)+B'_{yy}(1)T_{3j}(1,0)\\&+B_{yy}(1)T_{4j}(1,0)+\eta(\alpha^2+\Lambda^2)g_x(1)T_{6j}(1,0)\\&+B'_{yz}(1)T_{7j}(1,0)+B_{yz}(1)T_{8j}(1,0),\quad j=1,2,\ldots,8;\end{aligned}$$
$$\chi_{7j}=B_{yz}(1)T_{3j}(1,0)+B_{zz}(1)T_{7j}(1,0),\quad j=1,2,\ldots,8;$$
$$\begin{aligned}\chi_{8j}=\;&\eta(\alpha^2+\Lambda^2)g_x(1)T_{2j}(1,0)+B'_{yz}(1)T_{3j}(1,0)\\&+B_{yz}(1)T_{4j}(1,0)+\eta(\alpha^2+\Lambda^2)g_z(1)T_{6j}(1,0)\\&+B'_{zz}(1)T_{7j}(1,0)+B_{zz}(1)T_{8j}(1,0),j=1,2,\ldots,8.\end{aligned}\tag{43}$$

Letting $\gamma_{11}=\gamma_{21}=\gamma_{31}=\gamma_{41}=1$ and $\gamma_{12}=\gamma_{22}=\gamma_{32}=\gamma_{42}=0$, the frequency equation for a cantilever beam can be obtained.

2.2.3 Semi-analytical Transition Matrix. It is well known ([21]) that the following Peano-Baker series is the closed-form transition matrix of Eq. (38)

$$\begin{aligned}\mathbf{T}(\xi,\xi_{i-1})=\;&\mathbf{I}+\int_{\xi_{i-1}}^{\xi}\mathbf{A}(\chi_1)d\chi_1+\int_{\xi_{i-1}}^{\xi}\mathbf{A}(\chi_1)\int_{\xi_{i-1}}^{\chi_1}\mathbf{A}(\chi_2)d\chi_2d\chi_1\\&+\int_{\xi_{i-1}}^{\xi}\mathbf{A}(\chi_1)\int_{\xi_{i-1}}^{\chi_1}\mathbf{A}(\chi_2)\int_{\xi_{i-1}}^{\chi_2}\mathbf{A}(\chi_3)d\chi_3d\chi_2d\chi_1\\&+\cdots.\end{aligned}\tag{44}$$

However, it is impossible to determine the multiple integrals of the series analytically or numerically. Hence, an approximate transition matrix is required. In this paper, a simple and efficient algorithm is developed to find the approximate transition matrix.

By approximating the coefficient matrix $\mathbf{A}(\xi)$ by n piecewise constant coefficient matrices $\mathbf{A}(s_i)$, $i=1,2,\ldots,n$, one obtains a characteristic governing equation with constant coefficient matrix. Here s_i can be any value between $[\xi_{i-1},\xi_i]$ and ξ_i denotes the coordinate position at the end of the ith subsection. Consequently, the transition matrix of the ith subsection from ξ_{i-1} to ξ can be obtained:

$$\mathbf{T}(\xi,\xi_{i-1})=e^{\mathbf{A}(s_i)(\xi-\xi_{i-1})},\quad s_i,\quad \xi\in(\xi_{i-1},\xi_i).\tag{45}$$

After applying the composition property of the transition matrix, i.e., $\mathbf{T}(\xi_{i+1},\xi_{i-1})=\mathbf{T}(\xi_{i+1},\xi_i)\mathbf{T}(\xi_i,\xi_{i-1})$, the overall transition matrix is obtained:

$$\mathbf{T}(\xi,0)=\prod_{i=j}^{1}\mathbf{T}(s_i),\quad \xi\in(\xi_{j-1},\xi_j),\quad s_i\in(\xi_{i-1},\xi_i).\tag{46}$$

It can be observed that if the coefficient matrix $\mathbf{A}$ in the Peano-Baker series is constant, the transition matrix (44) is the same as Eq. (45). Hence, when the number of subsections approaches infinity, the approximate transition matrix becomes the exact one. It can be obtained at any desired level of accuracy by taking a suitable number of subsections. It should be noted here that while numerically determining the natural frequencies, the rate and tendency of the convergence of the solutions will be different according to the coordinate position s_i in the piecewise constant matrix selected as different values between $[\xi_{i-1},\xi_i]$. In this paper, for a better rate of convergence, s_i is taken to be $(\xi_{i-1}+\xi_i)/2$.

3 Frequency Relations

3.1 Frequency Relations of Pretwisted Beams. The relations among the setting angle θ, the rotatory inertia η, and the frequency Λ of rotating beams are studied. Meanwhile, one expects to predict the parameters of some system according to the parameters of another system. Two systems denoted as "a" and "b" have the same parameters except θ, η, and Λ. It is observed from the governing Eqs. (20) and (21) that if there exist the following relations, the two systems are similar:

$$\eta_a(\alpha^2+\Lambda_{a,i}^2)=\eta_b(\alpha^2+\Lambda_{b,i}^2),\tag{47}$$
$$\alpha^2\sin^2\theta_a+\Lambda_{a,i}^2=\alpha^2\sin^2\theta_b+\Lambda_{b,i}^2,\tag{48}$$
$$\alpha^2\cos^2\theta_a+\Lambda_{a,i}^2=\alpha^2\cos^2\theta_b+\Lambda_{b,i}^2,\tag{49}$$
$$\sin 2\theta_a=\sin 2\theta_b,\tag{50}$$

where i denotes the ith frequency. Assume that all the parameters of the system a are given and the parameters $\{\eta_b,\theta_b,\Lambda_{b,i}\}$ are unknown. It is obvious that Eqs. (48) and (49) do not be satisfied simultaneously unless $\theta_a=\theta_b$ and $\Lambda_{a,i}=\Lambda_{b,i}$. Substituting $\Lambda_{a,i}=\Lambda_{b,i}$ into Eq. (47), $\eta_a=\eta_b$. It is trivial that the two systems are the same as each other. In other words, one can't predict exactly the parameters of the system b from the parameters of the system a via the relations (47)–(50).

However, if the relation (49) is approximated and its effect is very small, the parameters of the system b can be accurately predicted from the parameters of the system a via the relations (47), (48), and (50). It is observed from Eq. (21) that the integrated parameter of the relation (49) is the coefficient for the deflection v. When the pretwisted angle is small, the deflection w is dominant and the effect of the relation (49) is very small. Moreover, the stiffer the system is, the larger the frequencies are. For a stiffer system its frequency Λ is greatly larger than the rotating speed α and the relation (49) can be approximated.

3.2.1 Frequency Relations and Mechanism of Instability of Unpretwisted Beams. Letting $V=\Phi=0$ in Eqs. (20)–(27), the governing equations and the boundary conditions of an unpretwisted Rayleigh beam can be obtained, respectively, as follows:

$$\begin{aligned}&\frac{d^2}{d\xi^2}\left(B_{yy}\frac{d^2W}{d\xi^2}\right)-\frac{d}{d\xi}\left\{[n-\eta(\alpha^2+\Lambda^2)g_y]\frac{dW}{d\xi}\right\}\\&\quad-m(\alpha^2\sin^2\theta+\Lambda^2)W=0,\end{aligned}\tag{51}$$

at $\xi=0$:

$$\gamma_{12}B_{yy}\frac{d^2W}{d\xi^2}-\gamma_{11}\frac{dW}{d\xi}=0,\tag{52}$$

$$\gamma_{22}\left\{\frac{d}{dx}\left(B_{yy}\frac{d^2W}{d\xi^2}\right)-[n-\eta(\alpha^2+\Lambda^2)g_y]\frac{dW}{d\xi}\right\}+\gamma_{21}W=0,\tag{53}$$

at $\xi=1$:

$$\frac{d^2W}{d\xi^2}=0,\tag{54}$$

$$\frac{d}{dx}\left(B_{yy}\frac{d^2W}{d\xi^2}\right)-[n-\eta(\alpha^2+\Lambda^2)g_y]\frac{dW}{d\xi}=0. \tag{55}$$

One can obtain from Eqs. (51)–(55) the relations (47) and (48) which can be satisfied simultaneously. Given all the parameters of the system *a* and the setting angle θ_b for the system *b*, the corresponding frequencies $\Lambda_{b,i}$ and the rotatory inertia η_b of the system *b* can be predicted exactly by using the relations (48) and (47), respectively.

It should be noted that the critical state of instability is $\Lambda_1^2=0$. If $\Lambda_1^2<0$, the natural frequency is imaginery and the divergent instability ([6]) occurs. Substituting $\Lambda_{a,1}$ and the associated parameters into the relations (47) and (48), one can predict whether the divergent instability will happen to the system *b*. Letting $\Lambda_{b,1}^2=0$, the critical rotatory inertia and the critical setting angle are obtained from Eqs. (47) and (48), respectively,

$$(\eta_b)_{\text{critical}}=\eta_a(1+\Lambda_{a,1}^2/\alpha^2), \tag{56}$$

$$(\theta_b)_{\text{critical}}=\sin^{-1}\sqrt{\sin^2\theta_a+\Lambda_{a,1}^2/\alpha^2} \tag{57}$$

under the following necessary condition of the divergent instability

$$\sin^2\theta_a+\Lambda_{a,1}^2/\alpha^2<1. \tag{58}$$

Because the effects of the rotatory inertia η and the setting angle θ are to decrease the frequencies of the system, under the condition (58) the domain of instability is $\{(\theta,\eta)|\sin^{-1}\sqrt{(\sin^2\theta_a+\Lambda_{a,1}^2/\alpha^2)}\leqslant\theta\leqslant\pi/2,\ \eta\geqslant\eta_a(1+\Lambda_{a,1}^2/\alpha^2)\}$.

For Bernoulli-Euler beams without taking account of the effect of the rotatory inertia η, only the relation (48) exists. The following frequency relation can be obtained by substracting the relation in the $(i+j)$th mode by that in the jth mode,

$$\Lambda_{a,i+j}^2-\Lambda_{a,i}^2=\Lambda_{b,i+j}^2-\Lambda_{b,i}^2. \tag{59}$$

It should be noted that for Bernoulli-Euler beams the condition of instability (58) is sufficient.

4 Instability of Rotating Beams

4.1 Beam With Infinite Translational Root Spring Constant.

4.1.1 Pretwisted Beam. Consider the free vibration of pretwisted Bernoulli-Euler beams with infinite translational root spring constant and without the rotational root spring, i.e., $\gamma_{12}=\gamma_{21}=\gamma_{32}=\gamma_{41}=1$ and $\gamma_{11}=\gamma_{22}=\gamma_{31}=\gamma_{42}=0$. It is assumed that there exists the free vibration of rigid-body motion and its mode shape is

$$W=w_0\xi \quad \text{and} \quad V=v_0\xi, \tag{60}$$

where w_0 and v_0 are constants. Equation (60) satisfies the boundary conditions (22)–(29). Substituting Eq. (60) into the governing Eqs. (20) and (21), one can obtain

$$-w_0\alpha^2m(r+\xi)+m(\alpha^2\sin^2\theta+\Lambda^2)\xi w_0+m\alpha^2\sin\theta\cos\theta v_0\xi=0,$$

$$-v_0\alpha^2m(r+\xi)+m\alpha^2\sin\theta\cos\theta w_0\xi+m(\Lambda^2+\alpha^2\cos^2\theta)v_0\xi=0, \tag{61a}$$

where the first terms of Eq. (61*a*) are derived from the second terms of Eqs. (20) and (21), respectively. Letting $r=0$, the following conditions are obtained:

$$(\Lambda^2-\alpha^2\cos^2\theta)w_0+\alpha^2\sin\theta\cos\theta v_0=0,$$

$$\alpha^2\sin\theta\cos\theta w_0+(\Lambda^2-\alpha^2\sin^2\theta)v_0=0. \tag{61b}$$

The first two eigenvalues are $\Lambda_1^2=0$ and $\Lambda_2^2=\alpha^2$. Because the square of fundamental frequency is zero, the instability will happen to a pretwisted Bernoulli-Euler beam. Because the effect of rotatory inertia is to decrease the frequencies, the instability will also happen to a pretwisted Rayleigh beam. However, for a pretwisted Bernoulli-Euler beam with $r>0$, $\gamma_{11}>0$, and $\gamma_{31}>0$ its fundamental frequency will be greater than the value of zero. The reason is that when the hub radius *r* and the rotational root spring constants γ_{11} and γ_{31} are increased from zero, the fundamental frequency of a pretwisted Bernoulli-Euler beam is increased from zero. This means that the instability will not happen to a pretwisted Bernoulli-Euler beam with $r>0$, $\gamma_{21}=\gamma_{41}=1$, $\gamma_{11}>0$, and $\gamma_{31}>0$.

4.1.2 Unpretwisted Beam. Letting $\theta=v_0=r=0$ in Eq. (61*b*), the corresponding fundamental frequency and the mode shape of an unpretwisted Bernoulli-Euler beam $\beta_2\to\infty$ are obtained, respectively,

$$\Lambda_1^2=\alpha^2, \quad W=w_0\xi. \tag{62}$$

One can predict via Eq. (48) that when $\theta_b=\pi/2$, the associated fundamental frequency $\Lambda_{b,1}=0$. When the hub radius *r* or the rotational root spring constant β_1 is increased, the fundamental frequency of the beam with $\theta_b=\pi/2$ is increased to be larger than zero and the instability will not happen. It is well known that when the setting angle is decreased, the frequencies are increased. Thus the instability will not happen also for the beam with $\beta_2\to\infty$, $r>0$, $\beta_1>0$ and $\theta<\pi/2$. It is concluded that in spite of the setting angle θ and the rotating speed α the instability does not happen to a Bernoulli-Euler beam with $\beta_2\to\infty$, $\beta_1>0$, and $r>0$. On the other hand, it can be observed that when $r>0$ or $\beta_1>0$, the fundamental frequency Λ_1 of the beam with $\beta_2\to\infty$ and $\theta=0$ is increased to be larger than α and the condition of instability (58) is not satisfied. This predicts also the above conclusion.

Because the frequencies of Rayleigh and Timoshenko beams taking account of the rotatory inertia η are smaller than those of Bernoulli-Euler beams under the same conditions, the fundamental frequencies of the unpretwisted Rayleigh and Timoshenko beams with $\eta>0$, $\theta=r=\beta_1=0$, and $\beta_2\to\infty$ will be less than α and the necessary condition of instability (58) is satisfied. The fundamental frequency is smaller than α and the necessary condition instability (58) is satisfied until *r* and β_1 are increased to be large enough. In other words, the instability will happen to the unpretwisted Rayleigh and Timoshenko beams with $\beta_2\to\infty$, $\beta_1>0$, $r>0$, and $\eta>\eta_{\text{critical}}$.

4.2 Beam With Infinite Rotational Root Spring Constants.

4.2.1 Pretwisted Beam. Consider the free vibration of a beam with infinite rotational root spring constant and without translational root spring, i.e., $\gamma_{11}=\gamma_{22}=\gamma_{31}=\gamma_{42}=1$ and $\gamma_{12}=\gamma_{21}=\gamma_{32}=\gamma_{41}=0$. It is assumed that there exists a rigid-body free-vibration motion and its mode shape is

$$W=w_0 \quad \text{and} \quad V=v_0, \tag{63}$$

where w_0 and v_0 are constants. Eq. (63) satisfies the boundary conditions (22)–(29). Substituting Eq. (63) into the governing Eqs. (20) and (21), the following conditions are obtained:

$$(\alpha^2\sin^2\theta+\Lambda^2)w_0+\alpha^2\sin\theta\cos\theta v_0=0,$$

$$\alpha^2\sin\theta\cos\theta w_0+(\alpha^2\cos^2\theta+\Lambda^2)v_0=0. \tag{64}$$

Equation (64) results in that the eigenvalue and the mode shape are $\Lambda^2=-\alpha^2$ and $v_0=\cot\theta w_0$. This means that the rigid-body free-vibration motion is unstable. Moreover, when the translational root spring constant is increased to a critical value from zero, the eigenvalue Λ^2 is increased to zero from the value of $-\alpha^2$. It is concluded that when the translational root spring constant is smaller than the critical value, the instability will occur.

4.2.2 Unpretwisted Beam. Consider the free vibration of rigid-body motion of a unpretwisted beam with infinite rotational

Table 1 Convergence pattern of dimensionless frequencies of a rotating pretwisted cantilever doubly tapered beam $[B_{yy}=(1-0.1\xi)^4\cos^2\xi\pi/6+4(1-0.1\xi)^4\sin^2\xi\pi/6,\ B_{zz}=4(1-0.1\xi)^4\cos^2\xi\pi/6+(1-0.1\xi)^4\sin^2\xi\pi/6,\ B_{yz}=1.5(1-0.1\xi)^4\sin\xi\pi/3,\ \alpha=3.0,\ \eta=0]$

Number of Subsections	Λ_1	Λ_2	Λ_3	Λ_4	Λ_5
5	5.100	7.791	23.747	44.036	63.632
10	5.119	7.775	23.802	43.897	63.790
20	5.124	7.771	23.817	43.861	63.836
30	5.124	7.770	23.820	43.854	63.844
40	5.125	7.770	23.821	43.851	63.848
50	5.125	7.769	23.821	43.850	63.849
60	5.125	7.769	23.822	43.850	63.850
70	5.125	7.769	23.822	43.850	63.850

root spring constant and without the translational root spring. Substituting the mode shape $W=w_0$ which satisfies the boundary conditions (52)–(55) into Eq. (51), the following condition is obtained:

$$\alpha^2\sin^2\theta+\Lambda^2=0, \tag{65}$$

which satisfies the condition of instability (58). Letting $\theta=0$, one obtains from Eq. (64) that the fundamental frequency Λ is zero. When the translational spring constant is increased, the fundamental frequency is increased from zero. The condition of instability (58) is satisfied until the translational root spring constant is larger than a critical value. This means that when the translational root spring constant is smaller than a critical value, the condition (58) will be satisfied and the instability will occur. Because for Rayleigh and Timoshenko beams which the effect of rotatory inertia is considered their fundamendal frequencies are smaller than that of a Bernoulli-Euler beam under the same parameters, the condition (58) for Rayleigh and Timoshenko beams is satisfied as soon as the condition for a Bernoulli-Euler beam is satisfied.

It is concluded that for both pretwisted and unpretwisted beams with infinite rotational root spring constants the instability will occur when the translational root spring constant is smaller than a critical value.

5 Numerical Results and Discussion

To demonstrate the efficiency and convergence of the proposed method, the first five frequencies are determined for a rotating pretwisted cantilever doubly tapered beam. In Table 1, the convergence pattern of dimensionless frequencies of the beam is shown. It shows that the natural frequencies determined by the proposed method converge very rapidly. Even when the number of subsections is only five, the differences between these solutions and the converged solutions are less than 0.5 percent.

For comparison, a uniformly pretwisted cantilever beam with constant cross section is considered. The natural frequencies obtained by the proposed method as well as those given by Subrahmanyam and Kaza [14] and Lin [12] are tabulated in Table 2.

Table 2 Effect of inertia constant η on the dimensionless frequencies of a rotating pretwisted cantilever beam $[B_{yy}=\cos^2\xi\Phi+4\sin^2\xi\Phi,\ B_{zz}=4\cos^2\xi\Phi+\sin^2\xi\Phi,\ B_{yz}=1.5\sin 2\xi\Phi,\ \alpha^*=\alpha/3.51602]$

Φ	α^*	Mode Number	$\eta=0$ #	$\eta=0$ ##	$\eta=0$ present	$\eta=0.0001$ present	$\eta=0.001$ present
30 deg	0	1	3.5245	3.5245	3.5245	3.5235	3.5149
		2	6.9585	6.9585	6.9586	6.9526	6.8994
		3	22.339	22.338	22.339	22.298	21.945
		4	42.896	42.896	42.898	42.649	40.576
		5	63.423	63.419	63.423	63.138	60.758
	1	1	5.1824	-	5.1824	5.1804	5.1632
		2	7.1461	-	7.1462	7.1386	7.0705
		3	24.055	-	24.055	24.010	23.618
		4	43.735	-	43.737	43.479	41.335
		5	65.103	-	65.104	64.811	62.367
	3	1	8.2156	-	8.2156	8.1990	8.0502
		2	11.749	-	11.748	11.743	11.694
		3	34.834	-	34.834	34.763	34.136
		4	49.804	-	49.805	49.488	46.845
		5	77.191	-	77.193	76.843	73.907
90 deg	0	1	3.5900	3.5899	3.5900	3.5882	3.5716
		2	6.4847	6.4849	6.4850	6.4815	6.4500
		3	24.531	24.530	24.530	24.457	23.833
		4	37.457	37.459	37.460	37.317	36.096
		5	72.973	72.962	72.965	72.470	68.460
	1	1	5.1120	-	5.1121	5.1086	5.0780
		2	6.8250	-	6.8253	6.8202	6.7753
		3	26.041	-	26.039	25.960	25.281
		4	38.533	-	38.536	38.385	37.102
		5	74.400	-	74.392	73.887	69.798
	3	1	7.9774	-	7.9776	7.9688	7.8908
		2	11.804	-	11.804	11.792	11.688
		3	35.872	-	35.871	35.755	34.741
		4	46.044	-	46.047	45.841	44.101
		5	84.936	-	84.929	84.353	79.672

#: given by Subrahmanyam and Kaza [14]; ##: given by Lin [12]

Table 3 The frequency relations between rotating unpretwisted Bernoulli-Euler beams $[\eta=0,\ r=0,\ B_{yy}=(1-0.2\xi)^3,\ m=(1-0.2\xi)]$

	α	θ	Λ_1^2	Λ_2^2	Λ_3^2	$\Lambda_2^2-\Lambda_1^2$	$\Lambda_3^2-\Lambda_2^2$
		0 deg	17.8707	450.0783	3223.7197	432.2076	2773.6414
	2	45 deg	15.8707	448.0783	3221.7197	432.2076	2773.6414
$\beta_{TL}\to\infty$		90 deg	13.8707	446.0783	3219.7197	432.2076	2773.6414
$\beta_{\theta L}\to\infty$		0 deg	55.5119	648.9802	3750.9751	593.4683	3101.9949
	6	45 deg	37.5119	630.9802	3732.9751	593.4683	3101.9949
		90 deg	19.5119	612.9802	3714.9751	593.4683	3101.9949
		0 deg	16.8900	421.6602	3023.3622	404.7702	2601.7020
	2	45 deg	14.8900	419.6602	3021.3622	404.7702	2601.7020
$\beta_{TL}\to\infty$		90 deg	12.8900	417.6602	3019.3622	404.7702	2601.7020
$\beta_{\theta L}=50$		0 deg	53.6052	616.3647	3540.1596	562.7595	2923.7949
	6	45 deg	35.6052	598.3647	3522.1596	562.7595	2923.7949
		90 deg	17.6052	580.3647	3504.1596	562.7595	2923.7949

Subrahmanyam and Kaza [14] studied the vibration of a rotating pretwisted cantilever beam by using the finite difference method and the Ritz method. Lin [12] studied the vibration of a nonrotating nonuniform pretwisted beam by using the modified transfer matrix method. Subrahmanyam and Kaza [14] did not consider the effect of the rotatory inertia and the coupling effect of the rotating speed and the mass moment of inertia. Without considering these effects, i.e., $\eta=0$, excellent agreement is obtained between the previous numerical results and those by the proposed method. Moreover, the effect of the inertia constant η will decrease greatly the natural frequencies. The effect of the inertia constant η on the natural frequencies of higher modes is relatively greater than that on the natural frequencies of lower modes. As the rotating speed α increases, the effect of the inertia constant η on the natural frequencies increases. The reason is that the coupling effect includes the product of the rotating speed α and the rotatory inertia η.

The frequency relations (48) and (59) between rotating unpretwisted Bernoulli-Euler beams is proved numerically in Table 3. The frequency relations (47)–(50) among rotating pretwisted beams are proved numerically in Table 4. A pretwisted cantilever beam with a small pretwisted angle is considered in Table 4. It is shown that the prediction of frequency via the relations (47), (48), and (50) is very accurate.

Figure 2 verifies the facts revealed in Sections 4.1.1 and 4.2.1 that the instability will happen to a pretwisted Rayleigh beam with infinite translational root spring constants, but not to a pretwisted Bernoulli-Euler beam with $r>0$, $\theta<\pi/4$ and $\gamma_{21}=\gamma_{41}=1$. Moreover, the instability will happen to the pretwisted Rayleigh and Bernoulli-Euler beams with infinite rotational root spring constants.

The influence of the rotating speed on the first three natural frequencies of doubly tapered beams with nonuniform pretwists is shown in Fig. 3. It is observed that the effect of the rotating speed on the first two frequencies are almost the same for all three systems. However, the effect on the higher mode frequencies are greatly different.

Figure 4 shows the influence of the total pretwist angle Φ on the first four frequencies of cantilever beams with different ratio of area moment inertia in the z and y-directions $I_{ZZ}(0)/I_{YY}(0)$. If the cross section of the beam is almost square, e.g., $I_{ZZ}(0)/I_{YY}(0)=2$, the influence of the total pretwist angle Φ on the frequencies is small. However, when $I_{ZZ}(0)/I_{YY}(0)=100$, the influence of the total pretwist angle Φ on the frequencies is great. The influence on the frequencies of higher modes is greater than that on the frequencies of lower modes.

6 Conclusion

A solution procedure for the bending-bending vibration of a rotating nonuniform beam with arbitrary pretwist and an elastically restrained root is derived. A simple and efficient algorithm for deriving the semianalytical transition matrix of the general system with nonuniform pretwist is proposed. The algorithm can be applied to linear control systems. The divergence in the Frobenius method does not exist in the proposed method. The frequency

Table 4 The prediction of the fundamental frequency Λ_b of pretwisted cantilever beams $[\alpha=0.1,\ \eta_a=0.001,\ r=1,\ B_{yy}=(1-0.1\xi)\cos^2\Phi\xi+1000(1-0.1\xi)^3\sin^2\Phi\xi,\ \ B_{zz}=1000(1-0.1\xi)^3\cos^2\Phi\xi+(1-0.1\xi)\sin^2\Phi\xi,\ \ B_{yz}=(5000(1-0.1\xi)^3-0.5(1-0.1\xi))\sin 2\Phi\xi]$

Φ	θ_a	Λ_a	θ_b	η_b	Λ_b	$\overline{\Lambda}_b$
	0 deg	3.62623	90 deg	0.00100076	3.62485	3.62485
0.1 deg	20 deg	3.62607	70 deg	0.00100058	3.62501	3.62501
	40 deg	3.62566	50 deg	0.00100013	3.62543	3.62543
	60 deg	3.62520	30 deg	0.00099942	3.62589	3.62589
	0 deg	3.61501	90 deg	0.00100077	3.61363	3.61415
5 deg	20 deg	3.61493	70 deg	0.00100059	3.61387	3.61426
	40 deg	3.61468	50 deg	0.00100013	3.61444	3.61453
	60 deg	3.61439	30 deg	0.00099962	3.61509	3.61483
	0 deg	3.53790	90 deg	0.00100080	3.53648	3.54079
15 deg	20 deg	3.53829	70 deg	0.00100061	3.53721	3.54051
	40 deg	3.53920	50 deg	0.00100014	3.53895	3.53970
	60 deg	3.54019	30 deg	0.00099960	3.54090	3.53870
	0 deg	3.44536	90 deg	0.00100084	3.44391	3.45405
25 deg	20 deg	3.44647	70 deg	0.00100064	3.44536	3.45312
	40 deg	3.44914	50 deg	0.00100015	3.44889	3.45065
	60 deg	3.45212	30 deg	0.00099958	3.45285	3.44778

Λ_b: determined by using Eq. (48)
$\overline{\Lambda}_b$: substituting Λ_b into Eq. (47), η_b is obtained. Further, determine $\overline{\Lambda}_b$ by using the proposed method for the general system.

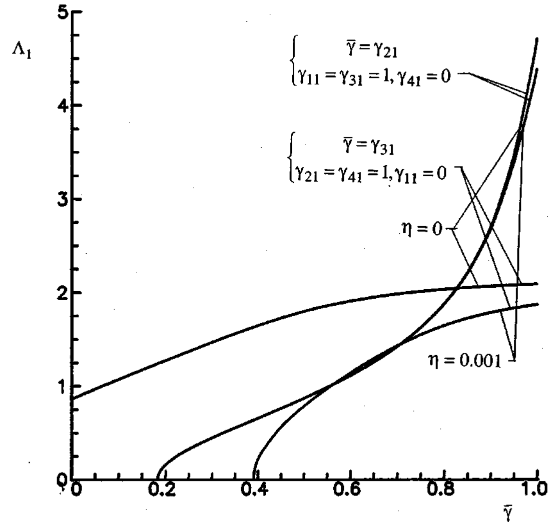

Fig. 2 The influence of the root spring constants on the instability of a pretwisted tapered beam $[B_{yy}=(1-0.1\xi)\cos^2\pi\xi/4+100(1-0.1\xi)^3\sin^2\pi\xi/4,\ B_{zz}=100(1-0.1\xi)^3\cos^2\pi\xi/4+(1-0.1\xi)\sin^2\pi\xi/4,\ B_{yz}=[50(1-0.1\xi)^3-0.5(1-0.1\xi)]\sin\pi\xi/2,\ \alpha=2,\ \theta=30\ \mathrm{deg},\ r=0.1]$

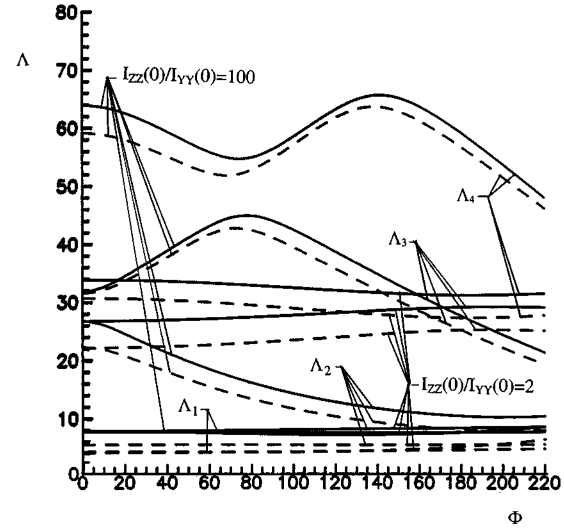

Fig. 4 The influence of the total pretwist angle Φ on the first four natural frequencies of cantilever doubly tapered beams $[B_{yy}=(1-0.1\xi)^4\cos^2\xi\Phi+I_{ZZ}(0)/I_{YY}(0)(1-0.1\xi)^4\sin^2\xi\Phi,\ B_{ZZ}=I_{ZZ}(0)/I_{YY}(0)(1-0.1\xi)^4\cos^2\xi\Phi+(1-0.1\xi)^4\sin^2\xi\Phi,\ B_{yz}=I_{ZZ}(0)/(2I_{YY}(0))(1-0.1\xi)^4\sin^2\xi\Phi,\ \eta=0.001,\ \theta=0,\ r=1;$ ——: $\alpha=4;$ – – –: $\alpha=1]$

relations among different systems are revealed. The mechanisms of instability is discovered. The effects of several parameters on the instability of rotating beams is investigated. It is shown that

1 due to the coupling effect of the rotational speed and the rotatory inertia, when the rotating speed α increases, the effect of the inertia constant η on the natural frequencies increases.

2 the effect of the rotatory inertia on the natural frequencies of higher modes is relatively greater than that on the natural frequencies of lower modes.

3 the instability does not happen to a unpretwisted Bernoulli-Euler beam with infinite translational root spring constant and $r>0$. However, if the rotational root spring constant is smaller than a critical value, the instability will happen to the Rayleigh and Timoshenko unpretwisted beams with infinite translational spring root constant, $r>0$ and $\theta>0$.

4 if the translational root spring constant is smaller than a critical value, the instability will happen to Bernoulli-Euler, Rayleigh, and Timoshenko unpretwisted and pretwisted beams with infinite rotational root spring constant.

5 the instability will not happen to a pretwisted Bernoulli-Euler beam with $r>0$, $\gamma_{21}=\gamma_{41}=1$, $\gamma_{11}>0$, and $\gamma_{31}>0$. The instability will happen to pretwisted Rayleigh and Timoshenko pretwisted beams.

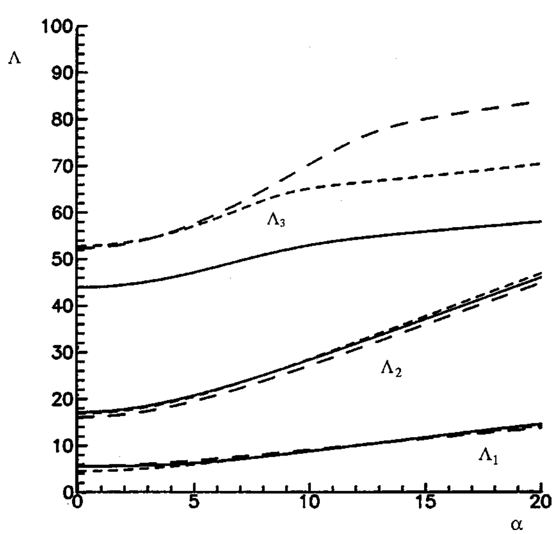

Fig. 3 The influence of the rotating speed α on the first three natural frequencies of cantilever doubly tapered beams with uniform and nonuniform pretwists $[B_{yy}=(1-0.1\xi)^4\cos^2\varphi+100(1-0.1\xi)^4\sin^2\varphi,\ B_{zz}=100(1-0.1\xi)^4\cos^2\varphi+(1-0.1\xi)^4\sin^2\varphi,\ B_{yz}=49.5(1-0.1\xi)^4\sin^2\varphi,\ \eta=0.001,\ \theta=\pi/3,\ r=0.2;$ ——: $\varphi=\pi\xi^2/2;$ – – –: $\varphi=\pi/2\sin(\xi\pi/2);$ – – –: $\varphi=\pi\xi/2]$

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