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THE INTEGRABILITY PROBLEM FOR PSEUDOGROUP STRUCTURES

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CHAPTER I. GENERALITIES

1. Pseudogroup structures and integrability

The first section provides a brief introduction to the general theory of pseudogroup structures leading to a discussion of the integrability problem. Throughout the paper, manifolds and maps are assumed to be C^∞.

A *local diffeomorphism* of manifolds M and M' is a diffeomorphism defined on open subsets. When no confusion threatens we shall write $f\colon M \to M'$ even though the domain of f may be a proper subset of M.

A *pseudogroup* Γ on a manifold M is a collection of local diffeomorphisms of M satisfying five axioms:

1. *Composition.* If f and g belong to Γ, then $f \cdot g$ belongs to Γ whenever it is defined, i.e., whenever the domain of f equals the range of g.
2. *Inversion.* If $f \in \Gamma$, then $f^{-1} \in \Gamma$.
3. *Identity.* The identity map of M belongs to Γ.
4. *Restriction.* If $f \in \Gamma$, and U is any open subset of the domain of f, then the restriction $f|U \in \Gamma$.
5. *Local definition.* A local diffeomorphism f of M belongs to Γ, if each point in its domain admits a neighborhood U for which the restriction $f|U \in \Gamma$.

Γ is *transitive* if, given any two points x and y in M, there exists an $f \in \Gamma$ such that $f(x) = y$.

Pseudogroups properly belonging to the smooth category not only consist of smooth mappings, but their differential behavior varies smoothly from point to point. To formalize this, fix any point 0 in M to serve as *origin*; to a transitive pseudogroup Γ the choice of 0 is indifferent. For each positive integer k define $B^k(M)$ to be the collection of k-jets with source 0 of all maps in Γ defined at 0. Assigning to each jet its target provides a projection of $B^k(M)$ onto M. Denote by $G^k(M)$, or just G^k, the fiber over 0 in $B^k(M)$, the collection of k-jets of elements in Γ which fix the origin. G^k is a group under jet composition, acting on $B^k(M)$ to the right by composition. Moreover, the

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orbits of G^k are precisely the fibers of the projection $B^k(M) \to M$. Thus at least in a formal sense $B^k(M)$ is a principal G^k bundle over M, called the *k-th order structure bundle of* Γ. $B^k(M)$ is formally a subbundle of the smooth principal bundle $D^k(M, M)$ of all k-jets at 0 of local diffeomorphisms of M. We require that the differential behavior of Γ vary smoothly by requiring $B^k(M)$ to be a smooth subbundle of $D^k(M, M)$, and its structure group G^k to be a Lie subgroup of the structure group of $D^k(M, M)$.

We are interested in pseudogroups which are defined by a finite number of smooth differential conditions. Therefore we declare that a pseudogroup Γ on M is a *smooth pseudogroup* if its structure bundles are smooth as just explained, and if it satisfies the following differential form of the "local definition" axiom:

6. *Differential definition.* There exists an integer k such that a local diffeomorphism f of M belongs to Γ if at each point in its domain the k-jet of f equals the k-jet of a member of Γ. The smallest such k is called the order of Γ.

Pseudogroups Γ_M and Γ_N on manifolds M and N are *equivalent* if there exists a diffeomorphism $g: M \to N$ such that $g \cdot f \cdot g^{-1}$ belongs to Γ_N whenever f belongs to Γ_M, and g is called an *equivalence* of Γ_M and Γ_N. If u is an open subset of M, then the collection of all maps in Γ_M with domain and range in u constitutes a pseudogroup Γ_u on u, called the *restriction* of Γ_M to u. A local diffeomorphism of M onto N is called a *local equivalence* of Γ_M and Γ_N if it is an equivalence of the restriction pseudogroups on its domain and range. Two transitive pseudogroups are *locally equivalent* if there exists a local equivalence between them.

A Γ *structure* or Γ *atlas* on a manifold M' is an atlas of local diffeomorphisms of M into M' whose transition functions belong to Γ. Specifically it is a collection $\{f_i\}$ of local diffeomorphisms of M into M', which satisfy (a) $\bigcup \text{range}(f_i) = M'$ and (b) $f_i^{-1} \cdot f_j \in \Gamma$ for all i and j. We may assume the atlas to be complete in the sense that any local diffeomorphism of M into M' consistent with condition (b) actually belongs to the atlas.

M is referred to as the *model space* for the Γ structure on M'. Note that Γ itself defines a Γ structure on M, termed the *model* Γ *structure*.

Let M'' be another manifold possessing a Γ structure, and $g: M' \to M''$ a local diffeomorphism. Then g *preserves the* Γ *structures* if $g \cdot f_i$ belongs to the Γ atlas on M'' whenever f_i belongs to the Γ atlas on M' (and domain $(g) \supset$ range (f_i)). Every map belonging to the Γ atlas on M' is structure-preserving with respect to the model Γ structure on M.

For each k define the *k-th order structure bundle* $B^k(M')$ of the Γ structure on M' to be the set of k-jets at 0 of all charts in the atlas which are defined at 0. The target mapping makes $B^k(M')$ a principal G^k bundle over M', G^k acting to the right by composition. If $D^k(M, M')$ denotes the bundle over M' of all k-jets at 0 of local diffeomorphisms of M into M', then $B^k(M')$ is a reduction of $D^k(M, M')$ to the structure group G^k.

Any local diffeomorphism f of M into M' extends to a local diffeomorphism f^k of $D^k(M, M)$ into $D^k(M, M')$ called its *k-jet extension*, as follows: if $p \in D^k(M, M)$ has target m in the domain of f, then $f^k(p) = j^k_m(f) \cdot p$. f^k commutes with the right action of the structure group, so it is a morphism of bundles. We shall say that a local diffeomorphism f of M' into M'', two manifolds with Γ structures, *is k-th order structure-preserving* if its k-jet extension f^k takes $B^k(M')$ into $B^k(M'')$. If $f\colon M \to M'$ belongs to the almost structure on M', then it is k-th order structure-preserving for all k, with respect to the model structure on M. Conversely, Axiom 6 implies that if f is k-th order structure-preserving for some k at least as large as the order of Γ, then f belongs to the Γ structure on M'.

Suppose now that we are not given a Γ structure on M', but only a principal G^k subbundle $B^k(M')$ of $D^k(M, M')$ for some k at least as large as the order of Γ. We think of $B^k(M')$ as a k-th order specification of a Γ structure on M', and refer to it as a *k-th order almost Γ structure* on M'. (If $k = 1$ it is usually called a *G-structure* on M', where $G = G^1$ is the first order structure group of Γ.) We are concerned with the following general question: Is a given k-th order almost Γ structure actually the k-th order structure bundle of a (necessarily unique) Γ structure on M? As k is greater than or equal to the order of Γ, this is equivalent to asking whether every jet in $B^k(M')$ may be represented by a local diffeomorphism of M onto M' which is k-th order structure-preserving. If the answer is affirmative, the almost structure is said to be *integrable*.

This *integrability problem* is actually an infinite family of problems including many of deep and classical significance in differential geometry. Three of the best known examples are:

A. Take Γ to be the pseudogroup of local isometries of Euclidean space. A first order Γ structure on M' determines a Riemannian metric on M', and conversely every Riemannian metric determines a corresponding first order Γ structure. An almost structure is integrable if and only if M' is locally isometric to Euclidean space.

B. Let Γ be the pseudogroup of holomorphic local diffeomorphisms of C^n. Specifying a first order almost Γ structure on M' is equivalent to prescribing smoothly a complex vector space structure on the tangent space of M' at each point, an "almost complex structure" on M' in the standard sense. The almost Γ structure is integrable if and only if M' is a complex manifold (with the prescribed complex tangent bundle).

C. If Γ is the pseudogroup of all local diffeomorphisms of R^n respecting the leaves of the linear fibration $R^n \to R^m$ $(m < n)$, then first order almost Γ structures on M' correspond bijectively with smooth $(n - m)$-dimensional linear subbundles of the tangent bundle of M', also called differential distributions of rank $n - m$ on M'. The almost Γ structure is integrable if and only if the corresponding distribution is integrable in the sense of Frobenius, meaning that the distribution consists of the tangents to the leaves of a foliation of M'.

Criteria for the integrability of a k-th order almost Γ structure are of three classes. First is an obvious geometric consistency requirement. If $B^k(M')$ is integrable, then it will be only one of an infinite sequence of structure bundles on M'. For $l < k$ the l-th order structure bundle is entirely determined by $B^k(M')$; $B^l(M')$ must be the image of $B^k(M')$ under the natural projection $D^k(M, M') \to D^l(M, M')$. Integrability means that each jet p in $B^k(M')$ is represented by a local diffeomorphism f of M into M' which is l-th order structure-preserving for all l: $f^l: B^l(M) \to B^l(M')$. Of course the behavior of f^l is not determined by the single jet p, but some of it is. If $l < k$, then p does specify the $k - l$ jet of f^l on the fiber of $B^l(M)$ above the source 0 of p. So we may impose part of the condition that p be representable by a structure-preserving map, a condition extrinsic to p, by the intrinsic requirement that the image of $B^l(M)$ in $D^l(M, M')$ under the l-jet extension of (one and hence) any representative of p contact $B^l(M')$ to order $k - l$ along the fiber above the target of p. If p possesses this property for all $l < k$, it will be called a *structure-preserving k-jet.* For $B^k(M')$ to be integrable it is then necessary that it consist purely of structure-preserving jets. This is the consistency limitation mentioned earlier, which we now assume to pertain whenever we apply the term "almost structure".

Rather than asking immediately for a structure-preserving local diffeomorphism representing a jet p in $B^k(M')$ we may first attempt to find a structure preserving $(k + 1)$-jet representing p. In general there is an obstruction, a canonically defined two-form on $B^k(M')$; this obstruction vanishes at p if and only if p admits a structure-preserving $(k + 1)$-jet extension [1]. Thus the k-th order almost structure $B^k(M')$ may be extended to a (unique) $(k + 1)$-st order almost structure $B^{k+1}(M')$ precisely when the obstruction form vanishes identically. The same considerations now apply to $B^{k+1}(M')$; it is induced by an almost structure of order $k + 2$ if and only if its obstruction form vanishes. We shall say that a given almost structure $B^k(M')$ is *formally integrable* if it may be extended to an entire sequence of almost structure $\{B^l(M')\}$, which is uniquely determined by $B^k(M')$ as k is assumed at least as large as the order of Γ. So the second of the three aforementioned criteria for the integrability of an almost structure, its formal integrability, reduces to the vanishing of a sequence of obstruction forms. In fact, after a certain point depending only on the nature of the pseudogroup Γ, all of these obstructions automatically disappear. Therefore formal integrability depends only on a finite number of conditions.

In each of the three examples mentioned above, all obstructions save one are a priori null. In the Riemannian case, the substantial obstruction (the obstruction on $B^2(M')$ to finding three-jets of isometries) is of second order and is really nothing but the Riemannian curvature tensor carried from M' up to $B^2(M')$. Formal integrability of almost complex structures and of distributions (our other two examples) both depend on the first order obstruction. The vanishing of this tensor is equivalent to the classical conditions of the Newlander-Nirenberg and Frobenius theorems respectively.

Beginning with a k-th order almost Γ structure, we have filled in the entire sequence of structure bundles of the Γ structure which we seek. The final step in our quest is to determine whether a formally integrable almost Γ structure is actually integrable; can a Γ atlas on M' be fitted to the structure bundle sequence? Formal integrability tells us what the infinite jets, the Taylor expansions, of the elusive Γ structure must be. But are there true local diffeomorphisms of M into M' with acceptable infinite jets at all points? This is really a question about the solvability of certain partial differential equations, equations whose nature is determined by the pseudogroup Γ and to which the almost structure supplies inhomogeneous data. The formal integrability of the almost structure translates into the formal solvability of the corresponding equations; at every point, the equations are known to have solutions in terms of infinite jets of diffeomorphisms, or of formal power series.

In our three examples, celebrated theorems guarantee that formal integrability always implies C^∞ integrability. However, such is not always the case, for formal solvability of partial differential equations does not always imply the existence of smooth solutions. In [5] Guillemin and Sternberg exhibit a formally integrable first order almost structure which is not integrable.

It is the purpose of this paper to show that all formally integrable almost Γ structures are actually integrable whenever Γ is a *flat pseudogroup*—essentially any smooth pseudogroup on R^n which contains the translations. (Flat pseudogroups are defined in Chapter II. All of the standard pseudogroups on Euclidean space, including our three examples, are flat.) This integrability theorem was proved for analytic almost structures in 1909 by Elie Cartan, using the Cartan-Kähler theorem (which he created for this task) to solve the differential equations. The proof is accessible in Chapter III of [12]. In the C^∞ category the result has been obtained for various classes of flat pseudogroups; in particular the following solutions are important for us.

1. A pseudogroup is of *finite type* if $B^{k+1}(M) = B^k(M)$ under the natural projection map for some sufficiently large k. For such pseudogroups Victor Guillemin [1] proved the integrability theorem using geometric techniques.

2. Note that G^1, as one-jets of maps fixing $0 \in M$, is represented on the tangent space of M at 0. A pseudogroup for which this representation is irreducible is called an *irreducible preudogroup.* Algebraic analysis, presented in [12], shows that an irreducible flat pseudogroup is either of finite type or belongs to one of twelve classically studied species. As the integrability theorem is valid for each of these classical pseudogroups, it is established for all irreducible pseudogroups.

The technique used in this paper was also developed by Victor Guillemin. It consists of introducing a quotienting procedure locally into the category of almost structures by which a given almost structure may be resolved into successively simpler ones. Eventually one of these will be irreducible, and being formally integrable it will be integrable. Then an inductive argument shows how to extend a local diffeomorphism of one of these structures to a chart preserving the next structure in series, leading finally back to the original almost structure. The crucial constraints governing the process are linear partial differential equations whose coefficients are constant because Γ contains the translations. Such equations are vulnerable to a powerful local existence theorem of Malgrange and Ehrenpreis, which together with the Frobenius and Newlander-Nirenberg theorems constitutes our entire arsenal of partial differential equation weaponry. The essential panoply is completed by some algebraic results of Victor Guillemin.

2. Quotients of pseudogroups and almost structures

Suppose that M is a fiber bundle over N. A local diffeomorphism f_M of M is said to *respect the fibration* if there exists a local diffeomorphism f_N on N, called the *quotient* of f_M, making the following square commute:

$$\begin{tikzcd} M \arrow[r, "f_M"] \arrow[d] & M \arrow[d] \\ N \arrow[r, "f_N"] & N \end{tikzcd}$$

If the origin in N is taken to be the image of the origin in M, the quotient concept induces a surjection from the bundle $\hat{D}^k(M,M)$ of all k jets at 0 of fibration respecting local diffeomorphisms on M, onto $D^k(N,N)$.

If Γ_M is a transitive smooth pseudogroup of fibration respecting maps on M, we would like to define a quotient pseudogruop $\Gamma_N$on N to consist of the quotients of elements of Γ_M. However it is not clear that this even makes sense, for the collection of quotient maps may not be closed under composition; two quotient maps may be composable while the elements of Γ_M inducing them have nonoverlapping domains and ranges. Unable to easily construct the desired object, we follow common mathematical procedure by formulating instead a definition. A smooth pseudogroup Γ_N on N will be called the *quotient pseudogroup* of Γ_M if for all k the projection $\hat{D}^k(M,M) \to D^k(N,N)$ maps $B^k(M)$ onto $B^k(N)$. If it exists, Γ_N is obviously unique and strictly minimal among smooth pseudogroups on N which contain the quotients of all elements of Γ_M. As Γ_M is transitive, Γ_N must also be. Γ_M will be said to be *fibrable* if a smooth quotient pseudogroup exists, and will be referred to as an *extension* of Γ_N. (We remark that the order of Γ_N may exceed the order of Γ_M.)

In the analytic category, Kuranishi and Rodrigues [7] have shown that every fibration respecting pseudogroup is locally fibrable. Furthermore, using the Cartan-Kähler theorem they proved that any element of Γ_N close enough to the identity in the C^k topology, for sufficiently large k, is locally a quotient of a map in Γ_M. We shall see that a fibration respecting flat pseudogroup is easily fibered, and will actually prove that every element of the quotient pseudogroup is locally the quotient of a map belonging to Γ_M.

The collection Γ_0 of all maps in Γ_M which induce the identity on N, forms a pseudogroup termed the *kernel pseudogroup* of the quotient $\Gamma_M \to \Gamma_N$. To a large extent, the kernel pseudogroup holds the secret of the extension; this vague statement will become quite clear through the remainder of the paper. If Γ_0 contains only the identity (plus its local restrictions), Γ_M is called a *prolongation* of Γ_N, and is essentially isomorphic to it.

Let K denote the fiber through 0 in M. The restrictions of all maps in Γ_M which carry the single fiber K into itself constitute a pseudogroup on K, although it is not obvious that it is a smooth pseudogroup. There is no real difficulty with the smoothness of its structure bundles, but Axiom 6 is not evident. Again we introduce a definition. A smooth pseudogroup Γ_K will be called the *fiber pseudogroup* of Γ_M if the restriction map of $\hat{D}^k(M, M)|K$ onto $D^k(K, K)$ maps $B^k(M)$ onto $B^k(K)$ for all k. (Of course we take the origin of M to be the origin of K as well.) When it exists, Γ_K is surely unique, and minimal among smooth pseudogroups containing the restrictions of maps in Γ_M which preserve K. Again, for flat pseudogroups there will be no difficulty establishing the existence of a smooth fiber pseudogroup.

If Γ_M is transitive and possesses a fiber pseudogroup Γ_K, then it induces, by restriction of its maps to K, a Γ_K structure on every fiber of $M \to N$. We shall say that a fibration respecting local diffeomorphism of M *preserves fiber structures* if its restriction to any fiber, considered as a map into the image fiber, preserves the Γ_K structures. In particular, all elements of Γ_M preserve fiber structures. If Γ_M is also fibrable, we shall call it a Γ_K *extension* of its quotient pseudogroup.

Γ_K and Γ_N tell us quite a bit about the transitive smooth pseudogroup Γ_M: it consists of certain local diffeomorphisms of M which preserve fiber structures and induce maps belonging to Γ_N. But it is crucial to recognize that the quotient and fiber pseudogroups do not completely specify Γ_M; to qualify for membership, a map on M must not only behave properly along each fiber and permute fibers in an acceptable manner, but its action on the fibers must vary consistently with certain constraints which we have referred to as the "secret" of the extension. Information about these constraints is provided by the kernel pseudogroup Γ_0. Thus Γ_K extensions of Γ_N may range from the one extreme where $\Gamma_0 = \{\text{identity}\}$ and Γ_M is a prolongation of Γ_N, to the opposite extreme where Γ_0 contains all maps preserving the fiber structures and inducing the identity on N. In the latter case one can easily show (from Axiom 6) that Γ_M contains all lifts of elements in Γ_N which preserve the fiber structures; we refer to this pseudogroup as the *trivial Γ_K extension* of Γ_N.

Similar considerations apply to almost structures. If $M' \rightarrow N'$ is another bundle, define $\hat{D}^k(M, M')$ to be the bundle of all k-jets at 0 of fibration respecting local diffeomorphisms of M into M'. Suppose Γ_M has a quotient pseudogroup Γ_N, and $B^k(M')$ is a k-th order almost Γ_M structure on a manifold M'. $B^k(M')$ is a *fibrable almost structure* if there is a fibration $M' \rightarrow N'$ such that $B^k(M') \subset \hat{D}^k(M, M')$, and furthermore if there exists a k-th order almost Γ_N structure $B^k(N')$ such that the natural quotient projection $\hat{D}^k(M, M') \rightarrow D^k(N, N')$ carries $B^k(M')$ onto $B^k(N')$. The almost Γ_M structure on M' is *locally fibrable* if every point in M' has a neighborhood upon which the induced almost Γ_M structure is fibrable. The following result will be of use.

Proposition 2.1. *If the transitive smooth pseudogroup Γ_M is fibrable, then every formally integrable almost Γ_M structure is locally fibrable, and the local quotients are formally integrable almost Γ_N structures.*

The proof of this theorem, to be found in [11], is a straightforward application of a metamathematical generality central to this subject: since any formally integrable almost Γ_M structure is identical with the model up to infinite order at all points, any naturally defined jet condition which holds true on the model must hold true on all formally integrable almost Γ_M structures. For example, to fiber the underlying manifold in Proposition 2.1 one simply invokes the Frobenius theorem, which asserts that local fibrability is a two-jet condition. A similar instance of this principle is the next theorem, whose proof is also in [11].

Proposition 2.2. *Suppose the fibrable smooth pseudogroup Γ_M admits a smooth fiber pseudogroup. Then every fibrable almost Γ_M structure $B^k(M')$ naturally restricts to an almost Γ_K structure on each fiber of M', which is formally integrable if $B^k(M')$ is.*

Now suppose that Γ_M is a fibrable pseudogroup, and $B^k(M')$ a fibrable almost Γ_M structure. Suppose further that there is a local diffeomorphism $f_N : N \rightarrow N'$ preserving the k-th order quotient Γ_N structures. Can one locally find a map $f_M : M \rightarrow M'$ around any point in M', which preserves k-th order structures and completes the following commutative square?

$$\begin{tikzcd}
M \arrow[r, dashed, "f_M"] \arrow[d] & M' \arrow[d] \\
N \arrow[r, "f_N"] & N'
\end{tikzcd}$$

We shall refer to this as the *lifting problem*. Its affirmative answer for all fibrable almost Γ_M structures and all such structure preserving maps f_N will be called the *lifting theorem* for the quotient $\Gamma_M \rightarrow \Gamma_N$.

We are primarily interested in studying the *integrability theorem* for various smooth pseudogroups Γ_M, which avers that all formally integrable almost Γ_M structures are actually integrable. Recall that this is a local question; $B^k(M')$

is integrable if and only if every point in M' is in the range of a structure preserving local diffeomorphism from M to M'. In view of Proposition 2.1 we may recognize the fundamental formula which will allow us to pursue the integrability theorem by inductive techniques:

$$\text{Integrability theorem for } \Gamma_N + \text{Lifting theorem for } \Gamma_M \to \Gamma_N \Rightarrow \text{Integrability theorem for } \Gamma_M.$$

A special case of the lifting theorem is easily proven, and will be of technical value in more subtle instances. The proof is in [11].

Proposition 2.3. *Suppose that* Γ_M *is the trivial* Γ_K *extension of* Γ_N, *and that the integrability theorem is true for* Γ_K. *Then the lifting theorem is true for* $\Gamma_M \to \Gamma_N$.

The technical application we have in mind for this is the following. Suppose Γ_M is any Γ_K extension, and $B^k(M')$ a fibrable formally integrable almost Γ_M structure. Γ_M is contained in the pseudogroup $\tilde{\Gamma}_M$ of all local diffeomorphisms of M which preserve fiber structures, the trivial Γ_K extension of the pseudogroup of all local diffeomorphisms of N. Of course, the integrability theorem for this huge pseudogroup on N is true, and so Proposition 2.3 together with the fundamental formula implies the validity of the integrability theorem for $\tilde{\Gamma}_M$. By enlarging its structure group, we may embed $B^k(M')$ in a formally integrable almost $\tilde{\Gamma}_M$ structure $\tilde{B}^k(M')$. Then the integrability theorem for $\tilde{B}^k(M')$ gives us in particular the following fact:

Proposition 2.4. *Suppose that* Γ_M *is a* Γ_K *extension, and that the integrability theorem is true for* Γ_K. *Then any jet in a fibrable, formally integrable almost* Γ_M *structure* $B^k(M')$ *may be represented by a fibration respecting local diffeomorphism of* M *into* M' *which preserves fiber structures.*

3. The Lie algebra of a pseudogroup

As the local properties of a finite dimensional Lie group are characterized by a more tractable algebraic entity, its Lie algebra, so too for smooth pseudogroups. In fact, one may consider a Lie group to be a pseudogroup of local transformations on a manifold, namely, left translations acting on the underlying space of the group itself. Viewed thus, the Lie algebra consists of global vector fields who local one-parameter groups belong to the pseudogroup. For general pseudogroups we do not wish to entangle ourselves in global questions, so we sheafify the definition of Lie algebras. The *Lie algebra sheaf* $\mathscr{L}$ of a smooth pseudogroup Γ on M is the sheaf of smooth vector fields on M whose local one-parameter transformation groups belong to Γ. Vector fields belonging to $\mathscr{L}$ will be called Γ *vector fields.*

If f is a local diffeomorphism of M, then its differential defines a transformation f_* of vector fields X on its domain into vector fields f_*X on its range, and $\exp_t(f_*X) = f\cdot(\exp_t X)\cdot f^{-1}$. In particular, for f belonging to Γ, X is a Γ

vector field if and only if f_*X is. Thus the Lie algebra sheaf of Γ is invariant under the action of Γ.

The nature of $\mathscr{L}$ may be further explored through the concept of k-jet extensions of vector fields. If X is a smooth vector field defined (in an open set) on M, and $\exp_t X$ its local one-parameter group of transformations on M, then the k-jet extensions $(\exp_t X)^k$ define a one-parameter group on $D^k = D^k(M, M)$. We define the *k-jet extension* X^k of X to be the smooth vector field generating $(\exp_t X)^k$. So by definition: $\exp_t (X^k) = (\exp_t' X)^k$.

Because Γ is a smooth pseudogroup, a local diffeomorphism of M belongs to Γ if and only if its k-jet extension map preserves the structure bundles $B^k = B^k(M)$ for all k. Since $(\exp_t X)^k = \exp_t (X^k)$, we observe

Proposition 3.1. *A vector field X on M is a Γ vector field if and only if its k-jet prolongation X^k is tangent to B^k for all k.*

From the readily verified relations $(X + Y)^k = X^k + Y^k$ and $[X, Y]^k = [X^k, Y^k]$, Proposition 3.1 implies that $\mathscr{L}$ is in fact a sheaf of Lie algebras.

If p is any point of D^k with target m, the assignment $X \rightarrow X^k(p)$ defines a linear map from the germs of vector fields at m to the tangent space T_pD^k. If X vanishes to order $k + 1$ at m, then $X^k(p) = 0$; so by passage to the quotient, the map $X \rightarrow X^k(p)$ induces a linear map from the vector space $J^k_m(TM)$ of k-jets of vector fields at m to the tangent space T_pD^k. It is well known [4] that $J^k_m(TM) \rightarrow T_pD^k$ is an isomorphism.

In particular, if p and q in D^k both have the same target, then the tangent spaces of D^k at p and q are both isomorphic to $J^k_m(TM)$ and hence isomorphic to each other. In fact, the isomorphism $T_pD^k \rightarrow T_qD^k$ is just the differential of the global transformation of D^k defined by the right action of the element $p^{-1}q$ in the structure group of D^k. This follows from the observation that each extension vector field X^k on D^k is invariant under the right action of the structure group; for $\exp_t (X^k)$ corresponds to the left action of $(\exp_t X)^k$ and therefore commutes with the right action of the structure group. Hence right multiplication by $p^{-1}q$ carries $X^k(p)$ into $X^k(q)$.

The space of *infinite jets of vector fields* is defined as the inverse limit $J^\infty_m(TM) = \varprojlim J^k_m(TM)$. If each $J^k_m(TM)$ is topologized discretely, $J^\infty_m(TM)$ is a complete infinite dimensional topological vector space. The k-jet maps $X \rightarrow j^k_m(X)$ from the vector space of germs of vector fields at m into $J^k_m(TM)$ are consistent with the projections $J^k_m(TM) \rightarrow J^{k-1}_m(TM)$, and hence by the universal property of $J^\infty_m(TM)$ they induce an *infinite jet* map $X \rightarrow j^\infty_m(X) \in J^\infty_m(TM)$. Of course, in the C^∞ category this map is surjective.

If X and Y are two vector fields at m, the k-jet of $[X, Y]$ at m is completely determined by the $(k + 1)$-jets of X and Y. Therefore the Lie bracket defines a bilinear map $[\ ,\ ]$: $J^{k+1}_m(TM) \times J^{k+1}_m(TM) \rightarrow J^k_m(TM)$. Passing to the inverse limit, the bracket operation makes $J^\infty_m(TM)$ into a complete topological Lie algebra.

For all k, designate with excusable ambiguity the k-jet of the identity map of M at 0 by i. The isomorphisms $J_0^k(TM) \to T_iD^k$ transform the bracket into a bilinear operation $[\ ,\ ]\colon T_iD^{k+1} \times T_iD^{k+1} \to T_iD^k$ characterized by the equality $[X^{k+1}(i), Y^{k+1}(i)] = [X^k, Y^k](i)$ for all vector fields X, Y defined about 0 on M. Also note the isomorphism $J_0^\infty(TM) \to \varprojlim T_iD^k$.

Return to the smooth pseudogroup Γ on M, with structure bundles B^k. The bracket operation takes $T_iB^{k+1} \times T_iB^{k+1} \to T_iB^k$. For if X^{k+1} and Y^{k+1} are tangent to B^{k+1} at i, then X^k, Y^k, and therefore $[X^k, Y^k]$ are tangent to B^k at i. Thus the closed subspace $L = \varprojlim T_iB^k$ of $J_0^\infty(TM)$ is a Lie subalgebra. L is called the *formal Lie algebra* of Γ. As all infinite jets of vector fields at 0 are actually representable by smooth vector fields (i.e., j_0^∞ is surjective), observe that $L = \{j_0^\infty(X)\colon X^k \text{ is tangent to } B^k \text{ along the fiber } G^k \text{ above } 0, \forall k\}$.

If f is any local diffeomorphism of M, and X a vector field on its domain, then $(f_*X)^k = (f^k)_*X^k$. In particular, suppose f belongs to Γ and fixes the origin; then $(f_*X)^k$ is tangent to B^k along the fiber G^k if and only if X is. Therefore L is invariant under the induced isomorphism $f_*\colon J_0^\infty(TX) \to J_0^\infty(TX)$.

Define $L^{(0)}$ to be the subalgebra of infinite jets of vector fields belonging to L which are zero at 0. If $X(0) = 0$, then $X^k(i)$ is tangent to the fiber of $D^k \to M$, and conversely. Therefore the natural projection $L \to T_iB^k$ takes $L^{(0)}$ onto the tangent space T_iG^k to the fiber. T_iG^k may be identified with the Lie algebra of right invariant vector fields on G^k; in fact, if $j_0^\infty(X) \in L^{(0)}$, then the restriction of X^k to G^k is a right invariant vector field on G^k. Thus L completely determines the Lie algebras of all of the structure groups of Γ.

The infinite jet map of germs of vector fields at 0 into $J_0^\infty(TM)$ carries the stalk $\mathscr{L}_0$ of $\mathscr{L}$ at 0 into L. For any reasonable smooth pseudogroup, $j_0^\infty(\mathscr{L}_0)$ is dense in L. In particular, this is always true in the analytic category, and at least for flat pseudogroups in the C^∞ category.

We have now defined four fundamental objects in the theory of smooth pseudogroups: pseudogroups themselves, structure bundles, Lie algebra sheaves, and formal Lie algebras. The interrelations of these concepts are obviously intimate, although not as precisely delineated as the interplay between a Lie group and its Lie algebra. Our approach focuses on the formal Lie algebra, for it is most amenable to algebraic analysis. First we shall interpret $J_m^\infty(TM)$ from a more convenient algebraic viewpoint.

If V^* is the dual of a finite dimension real vector space V, then denote by $F^k(V^*)$ the space of homogeneous polynomials of degree k on V and the k-fold symmetric product of V^*. Thus $F(V^*) = \prod_{k=0}^\infty F^k(V^*)$ is the ring of formal power series on V. Giving each F^k the discrete topology, F is a complete infinite dimensional real commutative algebra. If V is taken to be the tangent space V_m of M at m, F may be naturally identified with the ring $J_m^\infty(\boldsymbol{R})$ of infinite jets of real valued functions at m. (If $\{x_1, \cdots, x_n\}$ are local coordinates, and $\{X_i = dx_i(m)\}$ the corresponding basis for V_m^*, then $F(V_m^*)$ is just the ring of formal power series in $X_1, \cdots, X_n$. The infinite jet of a function corresponds to its formal power series expansion in terms of these coordinates.)

Define $\operatorname{Der} F$ to be the Lie algebra of continuous derivations of F. With the topology of pointwise convergence ($X_n \to X$ in $\operatorname{Der} F$ if and only if $X_n f \to Xf$ in F for all $f \in F$), $\operatorname{Der} F$ is a complete topological Lie algebra. The grading of F induces a grading $\operatorname{Der} F = \prod_{l=-1}^{\infty} \operatorname{Der}^l F$, where $\operatorname{Der}^l F = \{X \in \operatorname{Der} F\colon XF^k \subset F^{k+l} \text{ for all } k\}$. The grading is consistent with the Lie algebra structure: $[\operatorname{Der}^l F, \operatorname{Der}^k F] \subset \operatorname{Der}^{l+k} F$. As a subspace of $\operatorname{Der} F$ each $\operatorname{Der}^l F$ is discrete, and the topology on $\operatorname{Der} F$ is just the product topology. Introduce the notation $\operatorname{Der}^{(k)} F$ for the subalgebra $\prod_{l \geq k} \operatorname{Der}^l F$. For each $k > 0$, $\operatorname{Der}^{(k)} F$ is an ideal of $\operatorname{Der}^{(0)} F$.

Every $v \in V$ defines a linear functional on V^* and thus a linear map of $F^1 = V^*$ into $F^0 = \boldsymbol{R}$. Since F^1 and F^0 generate F, v extends uniquely to a continuous derivation of F with degree -1, which we continue to denote by v. Moreover, every continuous derivation of F is a sum of derivations of the form fv with $f \in F$ and $v \in V$. So there is a natural isomorphism $\operatorname{Der} F = F \otimes V$ identifying $\operatorname{Der}^l F = F^{l+1} \otimes V$.

For a more concerete description, let $\{\partial/\partial X_1, \cdots, \partial/\partial X_n\}$ be the basis dual to a basis $\{X_1, \cdots, X_n\}$ for V^*. Each $\partial/\partial X_i \in V$ acts on F as the formal derivative of power series with respect to X_i. $\operatorname{Der} F$ is the Lie algebra of formal power series vector fields $f_1\partial/\partial X_1 + \cdots + f_n\partial/\partial X_n$, the coefficients f_i belonging to F. $\operatorname{Der}^l F$ is the set of formal fields in which each f_i is a homogeneous polynomial of degree $l + 1$.

If V is taken to be $V_m = T_m M$, $\operatorname{Der} F$ may be naturally identified with $J^\infty_m(TM)$. Any vector field Y at m determines a derivation on the ring of real valued functions at 0, thereby inducing a derivation of $J^\infty_m(\boldsymbol{R}) = F$. In terms of coordinates $\{x_1, \cdots, x_n\}$ on M, $Y = y_1\partial/\partial x_1 + \cdots + y_n\partial/\partial x_n$ where each y_i is a function. The corresponding element of $\operatorname{Der} F$ is the analogous formal power series vector field in which each y_i is replaced by its infinite jet, or formal power series expansion, at m.

The kernel of the surjection $\operatorname{Der} F(V_m^*) \cong J^\infty_m(TM) \to J^k_m(TM)$ is $\operatorname{Der}^{(k)} F$, so we may identify $J^k_m(TM)$ with $\operatorname{Der} F(V_m^*)/\operatorname{Der}^{(k)} F(V_m^*)$. $\operatorname{Der}^{(0)} F(V_m^*)$ (consistent with the notation introduced earlier) corresponds to the subalgebra of $J^\infty_0(TM)$ consisting of infinite jets of vector fields which vanish at the origin. Thus at the point $m = 0$, $\operatorname{Der}^{(0)} F/\operatorname{Der}^{(k)} F$ may be identified with the Lie algebra of the structure group of D^k.

If g is any local diffeomorphism of M at m, composition with g defines an isomorphism of the ring of germs of functions at $g(m)$ into the ring of germs of functions at m. Passing to infinite jets, this yields an isomorphism $F(V_{g(m)}) \to F(V_m)$. Furthermore, the collection of all isomorphisms of $F(V_{g(m)}) \to F(V_m)$ may be identified by this procedure with the set of infinite jets of diffeomorphisms having source m and target $g(m)$. Any such isomorphism α defines an isomorphism $\alpha_*\colon \operatorname{Der} F(V_m) \to \operatorname{Der} F(V_{g(m)})$ by $\alpha_*(X) = \alpha^{-1}X\alpha$. When α is identified with $j^\infty_m(g)$, α_* is just the isomorphism $g_*\colon J^\infty_m(TM) \to J^\infty_{g(m)}(TM)$ induced by the map g_* on the germs of vector fields.

CHAPTER II. FLAT PSEUDOGROUPS

1. Connceted flat pseudogroups

A flat pseudogroup Γ is a smooth pseudogroup on a real vector space M, which contains the translations and has a graded formal Lie algebra. By definition, a subalgebra L of $\operatorname{Der} F(V^*)$ is *graded* if $L = \prod_{k=-1}^{\infty} L \cap \operatorname{Der}^k F$. Define $L^k = L \cap \operatorname{Der}^k F$ and $L^{(k)} = \prod_{l \geq k} L^l$. As Γ contains the translations, L^{-1} is all of $V = T_0 M$, and of course Γ is transitive. In general, we shall refer to a graded subalgebra of $\operatorname{Der} F(V^*)$ which contains V as a *flat subalgebra*.

Every flat subalgebra L of $\operatorname{Der} F(V^*)$ may be realized as the formal Lie algebra of a flat pseudogroup. In fact, one may easily recognize a maximal flat pseudogroup on M, whose formal Lie algebra is L. At each point $m \in M$ define the subalgebra L_m of $\operatorname{Der} F(V_m^*)$ to be $(\tau_m)_* L$, where $V_m = T_m M$ and τ_m is the diffeomorphism "translation by m". We observed in the last section that if Γ is any pseudogroup with Lie algebra L and f an element of Γ such that $f(0) = 0$, then $f_* L = L$. It follows that if Γ contains the translations, and g is any element of Γ whatsoever, then $g_* L_m = L_{g(m)}$ for all m in the domain of g. (Consider $f = \tau_{g(m)}^{-1} \cdot g \cdot \tau_m$.) Therefore the pseudogroup consisting of all local diffeomorphisms g of M satisfying this property contains every flat pseudogroup with formal Lie algebra L.

More significant is the existence of a minimal flat pseudogroup whose formal Lie algebra is L. A pseudogroup Γ will be said to be *connected* if its structure groups G^k are all connected. We shall demonstrate that any flat subalgebra L of $\operatorname{Der} F(V^*)$ has a unique connected flat pseudogroup Γ on M. Note that if $\tilde{\Gamma}$ is any other flat pseudogroup on M corresponding to L, then the structure group $\tilde{G}^k$ of $\tilde{\Gamma}$ and the structure group G^k of Γ have the same Lie algebra, namely, the image of $L^{(0)}$ in $T_i D^k$. Therefore G^k must be contained in $\tilde{G}^k$ as its identity component. As both Γ an $\tilde{\Gamma}$ contain the translations, it follows that the structure bundle B^k of Γ is contained in the structure bundle $\tilde{B}^k$ of $\tilde{\Gamma}$; for the fiber of B^k over $m \in M$ is $\tau_m^k G^k$, and that of $\tilde{B}^k$ is $\tau_m^k \tilde{G}^k$. Any morphism of D^k preserving a subbundle of $\tilde{B}^k$ must preserve $\tilde{B}^k$ itself, so the k-jet extension of every element of Γ preserves $\tilde{B}^k$. Because this holds for all k, and $\tilde{\Gamma}$ is a smooth pseudogroup, $\Gamma \subset \tilde{\Gamma}$. Thus every flat pseudogroup on M with formal Lie algebra L contains the connected pseudogroup Γ. The importance of this is

Proposition 1.1. *The integrability theorem is true for all flat pseudogroups if it is true for all connected flat pseudogroups.*

Proof. This presumes, of course, the claimed existence of connected flat pseudogroups for every flat algebra L. Let Γ and $\tilde{\Gamma}$ be as above, and suppose $\tilde{B}^k(M')$ is a formally integrable almost $\tilde{\Gamma}$ structure. Given any point $m \in M'$, shrink M' sufficiently around m so that $\tilde{B}^k(M')$ may be trivialized. Let $B^k(M')$

be any connected component of $\tilde{B}^k(M')$. $B^k(M')$ is a formally integrable almost Γ structure on M', so if the integrability theorem holds for Γ, then there exists a local diffeomorphism $f\colon M \to M'$ hitting m such that $f^k\colon B^k(M) \to B^k(M')$. As f^k is a local morphism of $D^k(M, M) \to D^k(M, M')$, we conclude $f^k\colon \tilde{B}^k(M) \to \tilde{B}^k(M')$ as well. q.e.d.

The construction of the connected pseudogroup Γ is self evident. The structure group G^k must be the connected subgroup of the structure group of D^k, whose Lie algebra is the image of $L^{(0)}$ in $T_i D^k$. The bundle B^k must be the unique translation invariant subbundle of D^k with group G^k; that is, the fiber of B^k over $m \in M$ is $\tau^k_m G^k$. Finally, we must define Γ to be the pseudogroup of all local diffeomorphisms of M whose k-jet extensions preserve B^k for all k. Γ is obviously a pseudogroup containing the translations. The fact that Γ satisfies Axiom 6—that it has finite order—results from an algebraic finiteness property of the Lie algebra L (derived ultimately from Hilbert's basis theorem); the proof is in [11].

An immediate consequence of the definitions, plus Proposition I.3.1, is the following characterization of the Lie algebra sheaf $\mathscr{L}$ of Γ.

Proposition 1.2. *A vector field X on M is a Γ vector field if and only if $j^\infty_m(X) \in L_m$ at every point m in the domain of X.*

Of great significance is the fact that $j^\infty_0(\mathscr{L}_0)$ is dense in L. The difficulty of establishing this in general is an annoying complication in the approach to arbitrary smooth pseudogroups through their infinitesimal transformations, and conversely the ease with which it may be proved in the flat case is one of the major reasons for limiting our attention to graded algebras. Note that L has plenty of formal vector fields which are actually convergent and therefore define analytic vector fields about $0 \in M$. In fact, since L is graded, its polynomial vector fields $\bigoplus_{k=-1}^{\infty} L^k$ are dense in L. Consequently the following lemma suffices to prove the denseness of $j^\infty_0(\mathscr{L}_0)$.

Proposition 1.3. *If X is an analytic vector field on an open ball in M whose infinite jet $j^\infty_0(X)$ belongs to L, then X is a Γ vector field.*

Proof. Apply the Campbell-Hausdorff formula (see [11]).

From the proposition one can also show that every jet in G^k is in fact representable by a diffeomorphism belonging to Γ. It follows that the bundle B^k is truly the structure bundle of Γ (see [11]).

We conclude this section by observing the following link between the pseudogroup Γ and the algebraic structure of L:

Proposition 1.6. *Every closed ideal I of L is invariant under the action of Γ: $f_* I = I$ for all $f \in \Gamma$ which fix the origin.*

Proof. Since I is closed and $\{L^{(k)}\}$ is a neighborhood basis at $0 \in L$, it suffices to show that $f_* I = I$ modulo $L^{(k)}$ for all k. But $j^{k+1}_0(f) \in G^{k+1}$ implies that there are polynomial fields $X_1, \cdots, X_l \in L$ such that f_* agrees with $(\exp X_1)_* \bullet \cdots \bullet (\exp X_l)_*$ modulo $L^{(k)}$. As I is a closed ideal, it is invariant under each $(\exp X_a)_* = \exp(\operatorname{ad} X_a)$.

2. Fibrations of connected flat pseudogroups

Suppose that $M \to N$ is a fibration of the vector space M respected by the connected flat pseudogroup Γ_M, and that the fiber K through $0 \in M$ is connected. Since Γ_M contains the translations, K is a subgroup and therefore a vector subspace of M. Furthermore, $N = M/K$, and the fibration is just the canonical projection. Let L_M be the formal Lie algebra of Γ_M.

The tangent space W of K at 0 is invariant under the differentials of all maps in Γ_M which fix 0. Equivalently, W is a subspace of V invariant under the *linear isotropy group* of Γ_M, the first structure group $G^1(M)$ considered naturally as a subgroup of $GL(V)$. Thus W is also invariant under the *linear isotropy algebra* $L_M^0 \hookrightarrow gl(V)$, the Lie algebra of $G^1(M)$. Conversely, any L_M^0 invariant subspace of V defines a linear fibration of M respected by Γ_M. (Note that the representation of L_M^0 on V induced by the representation $G^1(M) \to GL(V)$ is just the bracket operation of L_M^0 and $L_M^{-1} = V$.)

We wish to manufacture flat pseudogroups Γ_N on N and Γ_K on K (necessarily connected) which serve as quotient and fiber pseudogroups of Γ_M. To do so is easy; we simply look for the corresponding formal flat algebras.

The tangent space to N at 0 is $U = V/W$, and U^* may be identified with the annihilator of W in V^*. It is evident that the representation of L_M^0 on $V^* = F^1(V^*)$ is dual to its representation on $V = L^{-1}$ by Lie bracket. Thus W is an L_M^0 invariant subspace of V if and only if U^* is an L_M^0 invariant subspace of V^*. The following is easily proven:

Lemma. *U^* is an L_M^0 invariant subspace of V^* if and only if $F(U^*)$ is an L_M invariant subring of $F(V^*)$.*

Thus, if U^* is an L_M^0 invariant subspace of V^*, then restriction to $F(U^*)$ defines a representation $L_M \to \operatorname{Der} F(U^*)$. The image L_N of L_M, called the *quotient algebra* of L_M, is clearly a flat subalgebra of $\operatorname{Der} F(U^*)$. Γ_N is defined to be the corresponding connected flat pseudogroup on N.

If X_M is a vector field on M, whose one-parameter group respects the fibration $M \to N$, then there is a quotient vector field X_N on N related to it by the differential of the fibration map. In fact, X_N is the infinitesimal generator of the one-parameter group on N consisting of the quotients of the transformations in the one-parameter group of X_M. The graded homomorphism $L_M \to \operatorname{Der} F(U^*)$ which we have constructed algebraically is just the infinite jet completion of the natural quotient map of germs of Γ_M vector fields at 0 into germs of vector fields at 0 in N.

The fact that Γ_N is really the quotient pseudogroup of Γ_M is an easy consequence of the naturality of its definition. The requisite property is that $B^k(N)$ be the image of $B^k(M)$ under the quotient map $\hat{D}^k(M, M) \to D^k(N, N)$, for which it suffices to show that the homomorphism of structure groups takes $G^k(M)$ onto $G^k(N)$. As the latter are both connected Lie groups, this is automatic from the commutative diagram:

$$\begin{tikzcd}
L_M^{(0)} \arrow[r] \arrow[d] & T_iG^k(M) \arrow[r, hook] & T_i\hat{D}^k(M,M) \arrow[d] \\
L_N^{(0)} \arrow[r] & T_iG^k(N) \arrow[r, hook] & T_iD^k(N,N) \ .
\end{tikzcd}$$

When we have proved the lifting theorem for the quotient $\Gamma_M \to \Gamma_N$, we will obtain as a special case the satisfying fact that Γ_N really consists only of the quotients of the maps in Γ_M, at least locally, that is, at every point in its domain any $f_N \in \Gamma_N$ is locally the quotient of a map $f_M \in \Gamma_M$.

The definition of the fiber pseudogroup Γ_K is similar. If X_M is a vector field on M, whose one-parameter group maps K into itself, then X_M is tangent to K and restricts to a vector field X_K on K. The one-parameter group of X_K is, of course, just the restriction to K of the one-parameter group of X_M. To find the appropriate flat algebra L_K on K, simply mimic as follows the restriction process on the level of infinite jets of vector fields.

The ring of germs at 0 of real functions on K is naturally isomorphic to the ring of germs of functions on M modulo the subring of germs which vanish along K. Passing to infinite jets, there is an isomorphism of $F(W^*)$ with the quotient of $F(V^*)$ by the ideal $U^*F(V^*)$. The subalgebra of L_M preserving $U^*F(V^*)$ is $W \oplus L_M^{(0)}$; W preserves $U^*F(V^*)$ because it kills U^*, while $L_M^{(0)}$ does because it maps U^* into $F^{(1)}(U^*)$. So there is a representation $W \oplus L_M^{(0)} \to \operatorname{Der} F(W^*)$, which is the algebraic version of the restriction procedure on infinite jets of vector fields. The *fiber algebra* L_K is taken to be the image of $W + L_M^{(0)}$; it is obviously a flat subalgebra of $\operatorname{Der} F(W^*)$. That the corresponding connected flat pseudogroup Γ_K is actually the fiber pseudogroup of Γ_M follows from the natural commutative diagram:

$$\begin{tikzcd}
L_M^{(0)} \arrow[r] \arrow[d] & T_iG^k(M) \arrow[r, hook] & \hat{D}^k(M,M)\,|\,K \arrow[d] \\
L_K^{(0)} \arrow[r] & T_iG^k(K) \arrow[r, hook] & D^k(K,K) \ .
\end{tikzcd}$$

Of particular interest are linear quotients $M \to N$ which are minimal among those respected by Γ_M, equivalently those for which W is a minimal L_M^0 invariant subspace of V. As the representation of L_K^0 on W is just the restriction to W of the action of L_M^0 on V, minimality is further equivalent to the assumption that W is irreducible under L_K^0. In general, a flat algebra L such that L^{-1} is irreducible under L^0 is called an *irreducible algebra*, and corresponding flat pseudogroups are *irreducible pseudogroups*. Such algebras are quite well understood.

Proposition 2.1. *To any irreducible flat albegra L one of the following four alternatives must pertain*:

1. $L^k = 0$ *for all* $k > 0$.
2. $L^1 \neq 0$ *but* $L^k = 0$ *for* $k > 1$, *and* L *is simple.*

3. *L is infinite dimensional and simple.*

4. *L is infinite dimensional, $[L, L]$ is simple, and the codimension of $[L, L]$ is at most two.*

Furthermore, the infinite dimensional cases are completely characterized. There are precisely twelve classes of infinite dimensional irreducible flat Lie algebras, and the corresponding pseudogroups are all classical ones [10], [12]. A reference for the finite dimensional algebras is [6]. As mentioned in the first chapter, the integrability theorem is known for all irreducible flat pseudogroups.

Irreducible algebras of the first type are said to be *affine*; those of the remaining three types are *primitive*. Irreducible flat pseudogroups are also said to be affine or primitive, depending on their formal Lie algebras. When $M \to N$ is a minimal linear fibration, Γ_M and L_M will be called *affine extensions* or *primitive extensions* of Γ_N and L_N according as the classification of Γ_K or L_K.

The inetgrability theorem for arbitrary flat pseudogroups depends upon three inputs to our "fundamental formula": the lifting theorems for affine and primitive extensions, and the known integrability theorem for irreducible flat pseudogroups. The primitive lifting theorem is indeed that, primitive rather than subtle; it can be established with little difficulty but a bit of tedium, using the techniques of Singer-Sternberg [12] and a theorem on minimal ideals in infinite dimensional Lie algebras [2]. The proof of the affine lifting theorem, which requires deep results from partial differential equation theory, will comprise the remainder of this paper.

CHAPTER III. THE AFFINE LIFTING PROBLEM

1. The kernel algebra

If the secret of a flat pseudogroup extension $\Gamma_M \to \Gamma_N$ resides in the kernel pseudogroup, the secret of the kernel pseudogroup resides in turn in the kernel algebra, then there is an L ideal in A, which contains all other L ideals in $L_M \to L_N$. In this chapter we adhere to the notation used heretofore, with the additional assumption that Γ_M is an affine extension of Γ_N.

A simple algebraic observation will be useful. If A is any subspace of a Lie algebra, then there is an L ideal in A, which contains all other L ideals in A. Specifically, it consists of the set of elements $a \in A$ for which $\operatorname{ad} x_1 \cdot \operatorname{ad} x_2 \cdot \cdots \cdot \operatorname{ad} x_t(a) \in A$ no matter what the finite set $\{x_1, \cdots, x_t\} \subset L$ might be. If A is a graded subpsace of a graded algebra L, then this ideal is also graded.

The fiber algebra L_K is assumed in this chapter to be affine. Therefore $L_K^{(1)} = 0$, so $L_K = W + g$, where $g = L_K^0$ is the *fiber isotropy algebra,* a Lie subalgebra of $gl(W)$. As the action of g on W is irreducible, standard results in the theory of linear groups imply that g is either semisimple or the direct sum of a semisimple algebra with a center. Moreover the center consists either of all real multiples of the identity map in $gl(W)$, or else W has a complex vector space structure for which the center consists of all complex multiples of the identity.

The kernel I of $L_M \rightarrow L_N$ is a graded ideal of L_M with $I^{-1} = W$. In particular, I is an ideal in $W + L_M^{(0)}$, so its image in L_K must be of the form $W + h$, where h is an ideal of g, the *kernel fiber isotropy algebra*. Being an ideal of g, h also admits a decomposition $h = h_0 \oplus h_1 \oplus \cdots \oplus h_r$, where $h_1, \cdots, h_r$ are simple ideals, and the center h_0 is either zero, $\boldsymbol{R}\cdot$identity, or $\boldsymbol{C}\cdot$identity.

As W is an ideal in $W + h$, its preimage A in I is an ideal of I. However A need not be an ideal of L, so we define I_A to be the largest L ideal in A. Note that I_A is abelian. For W is abelian, hence $[I_A, I_A]$ is in the kernel of $W + L_M^{(0)} \rightarrow W + g$. But because $L_M \supset V$, $L_M^{(0)}$ can obviously contain no nontrivial ideal of L; therefore $[I_A, I_A] = 0$. It is also easy to check that I_A contains W, by direct application of the specific description of I_A given at the beginning of the section.

Define $\bar{L}_M = L_M/I_A$, and $\bar{I} = I/I_A$. The map $I \rightarrow W + h$ induces a Lie algebra homomorphism $\bar{I} \rightarrow h = h_0 \oplus h_1 \oplus \cdots \oplus h_r$. For each $j = 0, 1, \cdots, r$ define $\bar{I}_j$ to be the largest $\bar{L}_M$ ideal inside the preimage of h_j in $\bar{I}$. Victor Guillemin has shown [3] that the graded ideals $\bar{I}_j$ mimic the decomposition of h by the ideals h_j. That is,

Proposition 1.1. 1. $\bar{I} = \bar{I}_0 \oplus \bar{I}_1 \oplus \cdots \oplus \bar{I}_r$.

2. *The image of $\bar{I}_j$ in h is precisely h_j; in fact, the degree zero map $\bar{I}_j^0 \rightarrow h_j$ is an isomorphism.*

3. *For $j > 0$ there are no nontrivial proper closed subideals of $\bar{L}_M$ in $\bar{I}_j$.*

4. *$\bar{I}_0$ is abelian.*

Recall the following form of Shur's Lemma. Suppose k is a real simple Lie algebra, and Δ the collection of all linear maps of k, which commute with ad x for every $x \in k$. Then either Δ is isomorphic to $\mathbf{R}$, in which case k is said to be of *real type*, or Δ is isomorphic to $\boldsymbol{C}$, and k is said to be of *complex type*. Δ is an artifice for intrinsically discovering the largest field of scalars over which k may be considered to be defined. Thus k is of complex type precisely when it has a complex Lie algebra structure extending the real structure.

The adjoint representation of $\bar{L}_M$ induces a representation $L_M \rightarrow \mathrm{Der}\,(\bar{I}_j)$ for any j. Thus $\bar{I}_j$ is a module over the Abelian Lie subalgebra $V \subset L_M$. Now assume h_j to be of real type, and define V_j to be the commutator of $\bar{I}_j$ in V: $V_j = \{v \in V: v\bar{I}_j = 0\}$. Define $U_j^* \subset V^*$ to be the annihilator of V_j. Since $V_j \supset W$, $U_j^* \subset U^*$, the annihilator of W. In [2] Guillemin proved the following characterization of the ideal $\bar{I}_j$.

Proposition 1.2. *If h_j is of real type, then $\bar{I}_j = h_j \otimes F(U_j^*)$, where $U_j^* \subset U^*$ is the annihilator of the commutator of $\bar{I}_j$ in V.*

Of course $h_j \otimes F(U_j^*)$ is naturally a Lie algebra, being the tensor product of a Lie algebra and a commutative associative algebra. For the complex case, we first quote [2]:

Proposition 1.3. *If h_j is of complex type, then $\bar{I}_j$ has a unique complex Lie algebra structure consistent with the complex structure of h_j and making each derivation* ad $\bar{x}$, *for $\bar{x} \in \bar{L}_M$, complex linear.*

So the representation $L_M \to \operatorname{Der}(\bar{I}_j)$ defined by $x \to \operatorname{ad} \bar{x}$ extends to a complex linear representation $L_M \otimes \boldsymbol{C} \to \operatorname{Der}(\bar{I}_j)$. In particular, $\bar{I}_j$ is a $V \otimes \boldsymbol{C}$ module. (Throughout the paper, tensor products are taken over $\boldsymbol{R}$ unless otherwise annotated.) Define V_j to be the commutator of $\bar{I}_j$ in $V \otimes \boldsymbol{C}$, $V_j = \{z \in V \otimes \boldsymbol{C}\colon z\bar{I}_j = 0\}$, and define $U_j^* \subset (V \otimes \boldsymbol{C})^* = V^* \otimes \boldsymbol{C}$ to be the annihilator of V_j. Since $V_j \supset W \otimes \boldsymbol{C}$, U_j^* is a complex subspace of $U^* \otimes \boldsymbol{C}$. The complex analogue of Proposition 1.2 is [2]:

Proposition 1.4. *If h_j is of complex type, then $\bar{I}_j = h_j \otimes_{\boldsymbol{C}} F(U_j^*)$, where the complex subspace U_j^* of $U^* \otimes \boldsymbol{C}$ is the annihilator of the commutator of $\bar{I}_j$ in $V \otimes \boldsymbol{C}$.*

2. Description of affine pseudogroup extensions

The intent of this section is to analyze more specifically the behavior of the pseudogroup Γ_M along fibers, especially the behavior of the kernel pseudogroup. We begin by discussing the fiber pseudogroup Γ_K.

The first structure group $G^1(K)$ of Γ_K may be considered naturally embedded in $Gl(W)$ as the connected subgroup with Lie algebra $g \subset gl(W)$. As $L_K^l = 0$ for $l > 0$, each of the higher structure groups $G^l(K)$ is isomorphic to $G^1(K)$ under the natural projection. It is very easily seen that Γ_K itself consists precisely of the global affine transformations of K belonging to the group $A = G + T$, plus the restrictions of these transformations to open subsets of K, where T is the group of translations of K, and G is the group of linear maps of K whose differentials at 0 belong to $G^1(K)$. (Since K is a vector space, one may identify T with K and W, whereby G is identified with $G^1(K)$. However, hopefully for notational and conceptual clarity, we will continue to use the symbols separately in their appropriate contexts.) Note that the flat pseudogroup Γ_K is of finite type, so the integrability theorem is known for formally integrable almost Γ_K structures.

Fix forever a linear section of the projection $M \to N$, or equivalently a decomposition $M = N \times K$. Then any map $f_M \in \Gamma_M$, since it preserves fiber structures, must have the simple form $f_M(n, z) = (f_N(n), a(n)z)$, where $f_N \in \Gamma_N$ is the quotient map and $a\colon N \to A$ is a smooth map. The mystery is the nature of the function a, the manner in which the action of f_M along fibers is permitted to vary from one fiber to the next. To some extent the function a may be rigidly controlled by f_N; for example, if Γ_M is a prolongation of Γ_N, a is uniquely specified by f_N. However in general there is some flexibility in the function a, and it is this flexibility which is characterized by the kernel pseudogroup Γ_0. In fact, it is clear that all maps in Γ_M which cover f_N are given by $(n, z) \to (f_N(n), a(n)b(n)z)$, where $b\colon N \to A$ ranges over all functions for which $(n, z) \to (n, b(n)z)$ belongs to Γ_0.

Thus presupposing the existence of at least one lift $f_M \in \Gamma_M$ of an element $f_N \in \Gamma_N$ we see that the kernel pseudogroup parametrizes the collection of all lifts. In order to extend this simple observation to almost structures, we wish to examine the k-jet representation of Γ_0. Obviously if $g_M \in \Gamma_0$ fixes the origin, then its k-jet at 0 belongs to the kernel of the homomorphism $G^k(M) \to G^k(N)$. In fact, such k jets arising from Γ_0 completely exhaust the kernel:

Proposition 2.1. *Every jet in the kernel of the homomorphism $G^k(M) \to G^k(N)$ has a representative in the kernel pseudogroup.*

Proof. See [11].

How does the kernel pseudogroup act on fibers? Note in general that if $f_M(n, z) = (f_N(n), a(n)z)$ and $f_M(0) = 0$, then $a(0) \in A$ belongs to its linear subgroup G and is just the image of $j_0^1(f_M)$ under the fiber restriction homomorphism $G^1(M) \to G^1(K) \simeq G$. Define H as the image in G of the kernel C^1 of $G^1(M) \to G^1(N)$. Then it is clear that every element of the kernel pseudogroup is of the form $(n, z) \to (n, b(n)z)$ where $b\colon N \to H + T \subset A$. As the Lie algebra of C^1 is l^0, the Lie algebra of H is the image of l^0 in g, namely, the kernel fiber isotropy algebra h.

The structure of H is rather rigidly prescribed. As noted earlier, because g is irreducibly represented on K it must be of the form $g = g_0 \oplus g_1 \oplus \cdots \oplus g_s$, where each g_i is simple for $i > 0$, and the center g_0 is either $0, \boldsymbol{R}\cdot$(identity), or $\boldsymbol{C}\cdot$(identity) for some complex structure on K. Correspondingly the group G is a product $G = G_0 \times G_1 \times \cdots \times G_s$, where each G_i is simple for $i > 0$ and G_0 is the center. Since G is connected, each of these subgroup factors must also be connected. In particular G_0 is either the identity, all positive real multiples of the identity, or all nonzero complex multiples of the identity for some complex structure on K. Now $h = h_0 \oplus h_1 \oplus \cdots \oplus h_r$ is a Lie subalgebra of g. For $j > 0$ each h_j must be one of the simple components g_i of g; hence there is a unique Lie subgroup H_j of G with Lie algebra h_j, namely, the corresponding connected simple subgroup G_i. The center H_0 of H is a subgroup of G_0 with Lie algebra h_0; it need not be connected in instances when G_0 is complex. In sum, $H = H_0 \times H_1 \times \cdots \times H_r$ where each H_j is a simple connected normal subgroup if $j > 0$ and H_0 is some Lie subgroup of the nonzero complex numbers.

3. Constructing the ladder

We shall approach the lifting theorem for affine pseudogroup extensions by performing a preliminary prolongation of the problem to a bundle of "fiber 1-jets". A whole ladder of intermediate structures will be naturally defined, and any given structure preserving map of quotient spaces will be carried up the ladder rung by rung. So suppose that M' is fibered over N', $B^k(M')$ a fibrable, formally integrable almost Γ_M structure on M' with quotient B (N'), and $f_N\colon N \to N'$ a structure preserving map of the quotient structures. Around any given point in M' we seek a structure preserving local diffeomorphism f_M completing a commutative square:

$$\begin{tikzcd} M \arrow[r, "f_M"] \arrow[d] & M' \arrow[d] \\ N \arrow[r, "f_N"] & N' . \end{tikzcd}$$

The integrability of the induced almost structures on the fibers of M' provides local charts of the affine space K into each fiber of M'. As a convenience, we shall assume that each fiber is actually globally diffeomorphic to K via one of these charts. This entails no loss of generality, for the lifting theorem which we seek to prove is local, and the given almost Γ_M structure on M' is at least locally equivalent to an almost structure on a space whose fibers admit global affine structure preserving diffeomorphisms onto K (see [11]).

Consider heuristically the important special case in which $M' = M$ and $N' = N$. Then the lifting problem reduces to showing that any $f_N \in \Gamma_N$ is locally a quotient of an element $f_M \in \Gamma_M$. Such an f_M is specifically represented, via the fixed decomposition $M = N \times K$, as $f_M(n, z) = (f_N(n), a(n)z), a(n) \in A$. The question of constructing f_M is a matter of manufacturing a suitable function $a\colon N \to A$. If one such lift exists, then all other possibilities are $(n, z) \to (f_N(n), a(n)b(n)z)$ where the map $(n, z) \to (n, b(n)z)$ belongs to the kernel pseudogroup. In particular $b\colon N \to H + T$. So for each n, the image of $a(n)$ in $A/(H + T) = G/H$ is uniquely determined by f_N. As for the remainder of the function a, its "component" in each factor of $H_0 \times H_1 \times \cdots \times H_r + T$ is arbitrary within the constrains imposed by the kernel pseudogroup. Therefore we are led to study the possibility of finding an acceptable function a by seeking its various pieces within the realm of the kernel pseudogroup. The general situation is quite analogous except that some inhomogeneous data intervenes.

The formally integrable almost Γ_M structure on M' determines a formally integrable almost Γ_K structure on any fiber K'_n of M', where $n \in N'$ is its image below. Define a bundle $P' \to N'$ by taking for the fiber over $n \in N'$ the set of all global structure preserving charts from K to K'_n. P' is a principal A bundle over N', which is smooth because the almost Γ_K structures vary smoothly from fiber to fiber. P' is also a principal G bundle over M'; the bundle projection assigns to the chart $p \in P'$ of K onto K'_n the point $p(0) \in M'$. (In fact, P' may be identified with the bundle $\bigcup_{N'} B^1(K'_n)$ over $\bigcup_{N'} K'_n = M'$.) On the model space the corresponding bundle is denoted by P.

Suppose $f_M\colon M \to M'$ is a lift of f_N which at least preserves fiber almost structures. (In particular, any lift which actually preserves the almost Γ_M structures must preserve fiber structures.) Then f_M defines by composition a local diffeomorphism $f_P\colon P \to P'$ which is just the one-jet extension of f_M along each fiber; f_P will be refered to as the *fiber extension* of f_M. f_P commutes with the right action of A, i.e., it is an A morphism, and it lifts f_N:

$$\begin{tikzcd}
P \arrow[r, "f_P"] \arrow[d] & P' \arrow[d] \\
M \arrow[r, "f_M"] \arrow[d] & M' \arrow[d] \\
N \arrow[r, "f_N"] & N' .
\end{tikzcd}$$

Conversely any A morphism $f_P\colon P \to P'$ is induced by some map $f_M\colon M \to M'$ which preserves fiber structures; specifically, $f_M(n, z) = f_P(\tau_n)(z)$ where $\tau_n\colon M \to M$ is "translation by n". To find an f_M which preserves not only fiber structures but actually the almost Γ_M structure, we prolong $B^k(M')$ to P' and search for a morphism f_P which preserves the prolonged structure.

Let $q \in B^k(M')$ be any jet in the almost Γ_M structure on M'. According to Proposition I.2.4 the jet q may be represented by the local diffeomorphism $f_M\colon M \to M'$ which preserves fiber structures, and thus induces a morphism $f_P\colon P \to P'$. Providing P with an origin O_P, the identity map of K, the k-jet of f_P at O_P is entirely determined by q; we shall call it the *fiber extension* of q. The fiber extensions of all jets in $B^k(M')$ constitute a bundle $B^k(P') \to P'$, and in particular on the model space a bundle $B^k(P) \to P$.

Define $D_c^k(P, P')$ to be the bundle of k-jets of local morphisms of P into P' with source O_P. All of its elements are jets respecting the fibrations $P \to M$ and $P' \to M'$, so our usual quotienting procedure defines a projection $D_c^k(P, P') \to D^k(M, M')$. Restricted to $B^k(P') \subset D_c^k(P, P')$, the projection has image $B^k(M')$ and in fact is inverse to the fiber extension map used to construct $B^k(P')$.

Suppose now that $f_P\colon P \to P'$ is the fiber extension of $f_M\colon M \to M'$. (Implicit is the assumption that f_M preserve fiber structures.) Then the following diagram of k-jet extensions commutes:

$$\begin{tikzcd}
D_c^k(P, P) \arrow[r, "f_P^k"] \arrow[d] & D_c^k(P, P') \arrow[d] \\
D^k(M, M) \arrow[r, "f_M^k"] & D^k(M, M') .
\end{tikzcd}$$

Since $B^k(P') \to B^k(M')$ is bijective, we note that $f_P^k\colon B^k(P) \to B^k(P')$ if and only if $f_M^k\colon B^k(M) \to B^k(M')$. As it is clear that any local diffeomorphism f_P

of P into P' for which $f_P^k \colon B^k(P) \to B^k(P')$ must be a local morphism and therefore a fiber extension of some such map f_M, we conclude that the problem of integrating $B^k(M')$ is equivalent to that of integrating the prolongation $B^k(P')$.

The use of the word "prolongation" is consistent with earlier use. In fact, $B^k(P)$ is the k-th structure bundle of the pseudogroup Γ_P on P consisting of all fiber extensions of elements of Γ_M, and Γ_P is a prolongation of Γ_M. The argument immediately preceeding shows, for the special case where $M' = M$, that any local diffeomorphism of P preserving $B^k(P)$ is actually the fiber extension of an element of Γ_M; consequently Γ_P does satisfy axiom 6, so is a smooth pseudogroup on P. For an arbitrary M', the bundle $B^k(P')$ is a k-th order almost Γ_P structure on P'. As $B^k(M')$ is formally integrable, so is $B^k(P')$; its higher order bundles are the fiber extensions of the higher order bundles on M'.

Via the composite projection $B^k(P) \to B^k(M) \to B^k(N)$, the structure group $G^k(P)$ maps into but not onto $G^k(N)$. Define $\tilde{G}^k(P)$ to be the entire preimage in $B^k(P)$ of $G^k(N)$. Thus $\tilde{G}^k(P)$ consists of k-jets at O_P of elements of Γ_P which take the fiber over 0 in $P \to N$ into itself. But we may assume that any local morphism of P is defined on an A invariant domain, and obviously the k-jet of such a morphism at any point of the fiber through O_P is determined by its k-jet at O_P. Therefore composition of local morphisms preserving the fiber through O_P defines a group structure on $\tilde{G}^k(P)$ making $\tilde{G}^k(P) \to G^k(N)$ a homomorphism.

The decomposition $M = N \times K$ induces a trivialization $P = N \times A$, where $n \to \tau_n$ is the global section corresponding to $(n, i) \subset N \times A$, with i the identity of A. If $f_M \in \Gamma_M$ has the specific representation $f_M(n, z) = (f_N(n), b(n)z)$, then its fiber extension on $N \times A$ is $f_P(n, a) = (f_N(n), b(n)a)$. Any two jets in $\tilde{G}^k(P)$ are k-jets at $O_P = (0, i)$ of fiber extensions f_P and g_P, where $f_N(0) = 0 = g_N(0)$, and the group product $j^k_{0_P}(f_P) \cdot j^k_{0_P}(g_P)$ in $\tilde{G}^k(P)$ is the k-jet at O_P of the morphism $f_P \cdot g_P$.

Any local morphism of P into P' may also be considered to have an A invariant domain, so $\tilde{G}^k(P)$ as well as $G^k(P)$ acts on $B^k(P')$ to the right. In fact, $B^k(P')$ is a principal $G^k(P)$ bundle over P' and a principal $\tilde{G}^k(P)$ bundle over N'.

We decompose the bundle $P' \to N'$ by the right action of A. First define Q' to be the quotient of P' by the right action of the normal subgroup T of A, so the points of Q' are the orbits of T in P'. $Q' \to N'$ is a principal $A/T = G$ bundle, and $P' \to Q'$ a principal T bundle.

R'_0 is then defined as the quotient of Q' by the normal subgroup H_0 of G. $R'_0 \to N'$ is a principal G/H_0 bundle, and $Q' \to R'_0$ a principal H_0 bundle. Continuing, define R'_j for $j = 1, \cdots, r$ to be the quotient of R'_{j-1} by H_j. $R'_j \to N'$ is a principal $G/H_0 \times H_1 \times \cdots \times H_j$ bundle, and $R'_{j-1} \to R'_j$ a principal H_j bundle.

For the model structure the corresponding bundles are denoted Q, R , etc. It we use the letter S to denote any of these models, the pseudogroup Γ_P on P respects the fibration $P \to S$, and the collection of quotient maps defines a pseudogroup Γ_S on S. We are not yet in a position to affirm that Γ_S satisfies Axiom 6 except, as already observed, for $S = P$. This will emerge in the course of the argument; however, we shall see that its validity for all S is equivalent to the lifting theorem for the special case of the model almost structure.

Each bundle S has an origin O_S, the image of the origin of P. The jets in $D^k_c(P, P')$ all respect the fibration $P \to S$, so the general quotient procedure yields a projection $D^k_c(P, P') \to D^k_c(S, S')$. Define the bundle $B^k(S') \to S'$ to be the image of $B^k(P')$ under quotienting. In particular, $B^k(S)$ is the k-th order structure bundle of Γ_S. The groups $G^k(S)$ and $\tilde{G}^k(S)$ are the images of the analogous groups on P; $\tilde{G}^k(S)$ is the preimage of $G^k(N)$ with respect to the quotient projection $B^k(S) \to B^k(N)$. Note that, except for P', the spaces S' are not bundles over M' although they do fiber over N'. $B^k(S')$ is a principal $G^k(S)$ bundle over S' and a principal $\tilde{G}^k(S)$ bundle over N'.

Denote by $\Gamma^\sharp_S$ the pseudogroup of all local morphisms of S whose k-jet extensions preserve $B^k(S)$. (Note that any local map of S whose k-jet extension preserves $B^k(S)$ must in fact be a local morphism.) We known $\Gamma^\sharp_P = \Gamma_P$, but as mentioned above the other equalities $\Gamma^\sharp_S = \Gamma_S$ are deeper. $\Gamma^\sharp_S$ is a smooth pseudogroup on S which may be called the formal completion of Γ_S. $B^k(S)$ is the k-th order structure bundle of both Γ_S and $\Gamma^\sharp_S$. In fact, since $\Gamma_P = \Gamma^\sharp_P$ it is clear that all of the structure bundles of Γ_S and $\Gamma^\sharp_S$ are identical. $B^k(S')$ is a formally integrable almost Γ_S or $\Gamma^\sharp_S$ structure on S'. (Its higher order almost structures are of course those induced by the higher order almost structures corresponding to $B^k(M')$.)

Our strategy should now be evident. Beginning with a prescribed structure preserving map f_N, we attempt successively to find morphisms

$$\begin{tikzcd}
S \arrow[r, dashed, "f_S"] \arrow[d] & S' \arrow[d] \\
N \arrow[r, "f_N"] & N'
\end{tikzcd}$$

such that $f^k_S\colon B^k(S) \to B^k(S')$. Once we reach the bundle $S' = P'$, we will have established the lifting theorem for all affine extensions. Pictographically, we shall ascend the following ladder (for reference, the structure groups of each level as a principal bundle over the level immediately below have been included):

$$\begin{tikzcd}
P \arrow[r, dashed] \arrow[d] & P' \arrow[d] & {} \arrow[d, phantom, "\rbrace\, T"] \\
Q \arrow[r, dashed] \arrow[d] & Q' \arrow[d] & {} \arrow[d, phantom, "\rbrace\, H_0"] \\
R_0 \arrow[r, dashed] \arrow[d] & R'_0 \arrow[d] & {} \\
\vdots \arrow[d] & \vdots \arrow[d] & {} \\
R_{j-1} \arrow[r, dashed] \arrow[d] & R'_{j-1} \arrow[d] & {} \arrow[d, phantom, "\rbrace\, H_j"] \\
R_j \arrow[r, dashed] \arrow[d] & R'_j \arrow[d] & {} \\
\vdots \arrow[d] & \vdots \arrow[d] & {} \\
R_r \arrow[r, dashed] \arrow[d] & R'_r \arrow[d] & {} \arrow[d, phantom, "\rbrace\, G/H"] \\
N \arrow[r, "f_N"'] & N' . & {}
\end{tikzcd}$$

4. One small step . . .

The first step up the ladder is effortless, for the pseudogroup on R_r is a prolongation of the pseudogroup on N. In this section, the only ladder bundle to be discussed is R_r, so we shall delete the subscript and simply refer to it as R. Every element of the pseudogroup Γ_R is induced by the fiber extension of an element of Γ_M. If the quotient map in Γ_N is the identity, then the action along each fiber of M belongs to $H + T$. As $R = N \times A/(H + T)$, the action on R is also trivial, so indeed Γ_R is a prolongation of Γ_N.

On the jet level this implies that the homomorphism $\tilde{G}^k(R) \to G^k(N)$ is bijective. It is surely surjective, for the homomorphisms in the commutative triangle

$$\begin{tikzcd}
\tilde{G}^k(P) \arrow[rr] \arrow[dr] & & \tilde{G}(R) \arrow[dl] \\
& G^k(N) &
\end{tikzcd}$$

are surjective. Any element $p_R \in \tilde{G}^k(R)$ is induced by the fiber extension of a jet $p_M \in B^k(M)$ whose target belongs to K. We may compose p_M with a translation to bring its target to the origin and still not alter the induced jet on R; thus p_M may be taken in $G^k(M)$. The image of p_R in $G^k(N)$ is the identity if and only if the same is true of p_M. However, according to Proposition 2.1, if p_M belongs to the kernel of $G^k(M) \to G^k(N)$, then it is the k-jet at 0 of an element $f_M \in \Gamma_0$. As observed just above, the quotiented fiber extension f_R of f_M on R is then the identity, so as p_R is by definition the k-jet of f_R, $p_R =$ identity. Thus $\tilde{G}^k(R) \to G^k(N)$ is injective as well as surjective.

Consequently, we note that the projections $B^k(R) \to B^k(N)$ and $B^k(R') \to B^k(N')$ are bijective. Now suppose that $f_N \colon N \to N'$ is a structure preserving map. Then its k-jet extension f_N^k is a $G^k(N)$ local morphism of $B^k(N) \to B^k(N')$. Define f_R^k to be the unique map lifting f_N^k:

$$\begin{tikzcd}
B^k(R) \arrow[r, "f_R^k"] \arrow[d] & B^k(R') \arrow[d] \\
B^k(N) \arrow[r, "f_N^k"] & B^k(N') .
\end{tikzcd}$$

Perforce, f_R^k is a $\tilde{G}^k(R)$ morphism, which we need only to demonstrate to be the k-jet extension of its quotient map $f_R \colon R \to R'$.

The reason f_R^k is indeed the k-jet extension of f_R is that this condition is a functorial jet criterion. Let $f_R^{k-1} \colon B^{k-1}(R) \to B^{k-1}(R')$ be the quotient map induced by f_R^k, and p_R^k any jet in $B^k(R')$. That the jet p_R^k preserves lower order bundles (i.e., that $B^k(R')$ is admissible as an almost structure) follows immediately from the analogous property of any jet $p_M^k \in B^k(M')$ which induces p_R^k. Therefore p_R^k defines a linear isomorphism $(p_R^k)_*$ of the tangent space of $B^{k-1}(R)$ at i, the $(k-1)$-jet of the identity map of R, onto the tangent space of $B^{k-1}(R')$ at the projection p_R^{k-1} of p_R^k. The same is true, of course, for jets $q_R^k \in B^k(R)$. It is a standard fact (see [4]) that f_R^k is the k-jet prolongation of f_R if and only if the following triangle commutes no matter which point q_R^k is selected from $B^k(R)$:

$$\begin{tikzcd}
 & T_i B^{k-1}(R) \arrow[dl, "(q_R^k)_*"'] \arrow[dr, "(f_R^k q_R^k)_*"] & \\
T_{q_R^{k-1}} B^{k-1}(R) \arrow[rr, "df_R^{k-1}"'] & & T_{f_R^{k-1} q_R^{k-1}} B^{k-1}(R') .
\end{tikzcd}$$

By pushing this critical triangle down to the level of $N \xrightarrow{f_N} N'$ via the bijections $B^k(R) \to B^k(N), B^k(R') \to B^k(N')$, we obtain the triangle

$$\begin{tikzcd}
& T_iB^{k-1}(N) \arrow[dl, "(q_N^k)_*"'] \arrow[dr, "(f_N^k q_N^k)_*"] & \\
T_{q_N^{k-1}}B^{k-1}(N) \arrow[rr, "df_N^{k-1}"'] & & T_{f_N^{k-1}q_N^{k-1}}B^{k-1}(N') \, .
\end{tikzcd}$$

Here commutativity holds because f_N^k is in fact the k-jet extension of f_N. It follows that commutativity holds above, so that $f_R\colon R \to R'$ in a local morphism whose k-jet extension is the prescribed map f_R^k. As $f_R^k\colon B^k(R) \to B^k(R'), f_R$ is the desired lift of f_N.

5. The simple quotients of real type

This section begins with some relevant generalities. Suppose S is any of the model bundles in the ladder, and O_S its origin. There is a natural surjection of L_M onto the tangent space of S at O_S defined by the usual procedure of prolonging vector fields. If X is a Γ_M vector field, and $\exp_t X \in \Gamma_M$ is its local one-parameter group of transformations, then the vector field X_S is defined to be the infinitesimal generator of the quotiented fiber extension group of transformations $(\exp_t X)_S$ on S. Passing to jets of vector fields, this prolongation process defines in particular a surjection $L_M \to T_{O_S}(S)$. (Surjectivity is obvious, for the map is nothing but the surjective extension map $L_M \to T_iB^1(M)$ followed by the derivative of the projection $B^1(M) \to \bigcup_N B^1(K_n) = P \to S$.)

Every one-jet $p_S \in G^1(S)$ may be identified with a linear isomorphism $(p_S)_*$ of $T_{O_S}(S)$. The jet p_S is represented by the map $f_S \in \Gamma_S$ induced by an element $f_M \in \Gamma_M$, and by definition $(p_S)_* = df_S$. Necessarily $f_M(0) = 0$, and from the equality $((f_M)_*X)_S = (f_S)_*X_S$ is derived the commutativity of the square:

$$\begin{tikzcd}
L_M \arrow[r, "(f_M)_*"] \arrow[d] & L_M \arrow[d] \\
T_{O_S}(S) \arrow[r, "(p_S)_*"] & T_{O_S}(S) \, .
\end{tikzcd}$$

If E is a closed ideal of L_M, then its image $(E_S)_{O_S}$ in $T_{O_S}(S)$ is invariant under the action of $G^1(S)$. For by Proposition II.1.6, if $f_M \in \Gamma_M$ fixes the origin, then $(f_M)_*E = E$. Consequently the subspace $(E_S)_{O_S}$ extends uniquely to a distribution E_S on S by the action of $B^1(S)$: if $s \in S$, then $(E_S)_s = (p_S)_*(E_S)_{O_S}$ for any one-jet $p_S \in B^1(S)$ with target s. Since any other jet in $B^1(S)$ with the same target differs from p_S by an element of $G^1(S)$, $(E_S)_s$ is well defined. In the same way, the almost structure $B^1(S')$ defines a distribution $E_{S'}$ on S'.

At each point $m \in M$ the formal Lie algebra L_m of infinite jets of vector fields at m was defined to be the translate of L_M. The ideal E is carried to an ideal E_m of L_m, and we shall speak of a vector field X on M as an E field if $j^\infty_m(X) \in E_m$ for all m. (In particular, L_M fields are just Γ_M vector fields.) By construction, if X is an E field on M, then its extension X_S on S is a section of the tangent subbundle E_S; furthermore, with the Campbell-Hausdorff formula one can easily show that the extensions of E fields on S span E_S at every point. This yields several bits of information. First, since the fields X_S are all invariant under the right action of the structure group of $S \to N$, the distribution E_S is right invariant. Second, the distribution E_S is integrable in the sense of Frobenius. For if X and Y are E fields, then so is $[X, Y]$. As $[X_S, Y_S] = [X, Y]_S$, the Frobenius condition is satisfied by a spanning set of vector fields everywhere, which suffices to guarantee the integrability of E_S.

One may conclude from the integrability of E_S that the distribution $E_{S'}$ is also integrable, and moreover that every jet in the almost structure $B^k(S')$ is representable by a morphism from S into S' which preserves the distributions. (Complete the pseudogroup Γ_S to the Frobenius pseudogroup of all maps preserving E_S, enlarge $B^k(S')$ to a formally integrable almost Frobenius structure, and apply the Frobenius theorem.) Thus E_S and $E_{S'}$ determine foliations on S and S' respectively, and the almost structure on S' consists of certain jets of foliation-respecting morphisms.

Several trivialities might be noted. The whole discussion above is entirely natural. Therefore, if $\bar{S}$ is a bundle lower in the ladder than S, then the triangle

$$\begin{tikzcd}
 & T_{O_S}(S) \arrow[dd] \\
L_M \arrow[ur] \arrow[dr] & \\
 & T_{O_{\bar{S}}}(\bar{S})
\end{tikzcd}$$

commutes. For the bundle Q (and hence all lower bundles), the map $L_M \to T_{O_Q}(Q)$ kills the abelian ideal I_A and thereby induces a surjection $\bar{L}_M \to T_{O_Q}(Q)$. For the polynomial vector fields are dense in I_A, and the one-parameter group of any polynomial field in I_A acts on each fiber of M by translation; since $Q = P/T$, the induced one-parameter group on Q is the identity. Similarly, for each bundle R_j the map $\bar{L}_M \to T_{O_j}(R_j)$ kills the ideal $\bar{I}_0 \oplus \bar{I}_1 \oplus \cdots \oplus \bar{I}_j$.

We now turn our attention to the problem of finding a lifting map

$$\begin{tikzcd}
R_{j-1} \arrow[r, dashed, "f_{j-1}"] \arrow[d] & R'_{j-1} \arrow[d] \\
R_j \arrow[r, "f_j"] & R'
\end{tikzcd}$$

where f_j is an extant structure preserving morphism, and f_{j-1} is required to be a structure preserving morphism as well. In this section the simple algebra h_j is assumed to be of real type, and the index j is fixed throughout.

As $\bar{I}_j \rightarrow T_{O_j}(R_j)$ is zero, the image of $\bar{I}_j$ in $T_{O_{j-1}}(R_{j-1})$ is vertical, tangent to the fiber of $R_{j-1} \rightarrow R_j$. This fiber is just the group H_j, and the map of $\bar{I}_j$ into the Lie algebra of H_j is just the surjection $\bar{I}_j \rightarrow h_j$ defined in § 1.

Part of a horizontal complement to the vertical tangent space allows invariant definition. The adjoint representation of $L_M \rightarrow \mathrm{Der}\,(L_M)$ induces a representation $L_M \rightarrow \mathrm{Der}\,(\bar{I}_j)$. Define the ideal E of L_M to be the kernel of this representation. We claim that the distribution $E_{R_{j-1}}$ on R_{j-1} is horizontal. For the image $\bar{E}$ of E in $\bar{L}_M$ is just the commutator of $\bar{I}_j$, $\bar{E} = \{x \in \bar{L}_M\colon [x, \bar{I}_j] = 0\}$, and therefore $\bar{E}$ is disjoint from $\bar{I}_j$. (As $\bar{I}_j^0 = h_j$ is simple, $\bar{E} \cap \bar{I}_j^0 = 0$. Then $\bar{E} \cap \bar{I}_j$ is a proper closed subideal in $\bar{I}_j$ which must be zero according to Proposition 1.1.) Therefore the image of $\bar{E} \rightarrow T_{O_{j-1}}(R_{j-1})$ does not intersect the vertical component, so at least at O_{j-1} the distribution $E_{R_{j-1}}$ is horizontal. Because both the vertical distribution and $E_{R_{j-1}}$ are preserved by the action of $B^1(R_{j-1})$, they are everywhere disjoint. Likewise the distribution $E_{R'_{j-1}}$ on R'_{j-1} is horizontal.

The distributions $E_{R_{j-1}}$ and $E_{R'_{j-1}}$ on R_{j-1} and R'_{j-1} are integrable by preceeding general considerations. Any structure-preserving lift $f_{j-1}\colon R_{j-1} \rightarrow R'_{j-1}$ of f_j must respect the foliations defined by these distributions. We now show that this is the only constraint governing the lift. The key item is the use of the algebraic structure theorem for $\bar{I}_j$ to characterize the kernel pseudogroup of the quotient $\Gamma^{\#}_{R_{j-1}} \rightarrow \Gamma^{\#}_{R_j}$. The kernel is shown to be the entire collection of morphisms of R_{j-1} which respect the foliation and induce the identity on R_j.

Let U_j^* be the subspace of the dual U^* to the tangent space U of N at 0 defined by Proposition 1.2, so that $\bar{I}_j = h_j \otimes F(U_j^*)$. The annihilator of U_j^* is a subspace of U, which may be considered as the tangent space of the leaf through 0 in a linear foliation of N. (In fact, it is easy to check that under the projection $R_{j-1} \rightarrow N$, this foliation of N is induced by the foliation by $E_{R_{j-1}}$ on R_{j-1}.) From the fact that $\bar{I}_j = h_j \otimes F(U_j^*)$, we can easily deduce that the pseudogroup $\Gamma^{\#}_{R_{j-1}}$ contains every morphism of the bundle $R_{j-1} = N \times G/(H_0 \times \cdots \times H_{j-1})$ of the form $(n, a) \rightarrow (n, h(n)a)$ where $h\colon N \rightarrow H_j$ is any function being constant on the leaves of the foliation of N. In fact, since H_j is connected we can locally write $h(n)$ as a product $(\exp X_1(n)) \cdot \cdots \cdot (\exp X_t(n))$ around any point N, which we may for convenience take to be the origin, where each X_i is a function from N to h_j which is constant on leaves of the foliation of N. Since M is fibered over N, the functions X_i may be defined on M by requiring that they be constant on the fibers of $M \to N$. Then for each $i = 1, \cdots, t$ the infinite jet $j_0^\infty(X_i)$ belongs to $h_j \otimes F(U_j^*) = \bar{I}_j$. Let $Y_i \in L_M$ be any preimage of $j_0^\infty(X_i)$ under the surjection $L_M \to \bar{L}_M$, and for any specific index l let Z_i be the unique polynomial vector field on M of degree at most l, which equals Y_i modulo $L_M^{(l)}$. Then $f_M = \exp Z_1 \cdot \cdots \cdot \exp Z_t$ belongs to Γ_M and has the form $(n, z) \to (n, b(n)z)$, where $b\colon N \to H_j$ is a function constant on the leaves in N, and $j_0^l(b) = j_0^l(h)$. Thus the l-jet of the given morphism of R_{j-1} at points above $0 \in N$ equals the l-jet of the induced morphism $f_{R_{j-1}} \in \Gamma_{R_{j-1}}$. By translating to other points in N, we conclude that the given morphism may be approximated everywhere by elements of the pseudogroup $\Gamma_{R_{j-1}}$ up to arbitrary order, so that the morphism itself must belong to $\Gamma^\#_{R_{j-1}}$.

As every morphism θ of R_{j-1} inducing the identity on R_j is of the form $\theta(n, a) = (n, h(n)a)$, where $h\colon N \to H_j$, to complete our characterization of the kernel pseudogroup of $\Gamma^\#_{R_{j-1}} \to \Gamma^\#_{R_j}$ we need only to demonstrate that if θ respects the foliation on R_{j-1}, then in fact h must be constant on the leaves of the foliation of N. Let $u \in U$ be any vector annihilated by U_j^*. Since $M = N \times K$, we may extend u to a constant vector field on M. Then u is an E field, so its extension $u_{R_{j-1}}$ to a vector field on R_{j-1} is a section of the distribution $E_{R_{j-1}}$. By naturality, $u_{R_{j-1}}$ projects under the map $R_{j-1} \to R_j$ to the vector field u_{R_j} on R_j. Since $\theta\colon R_{j-1} \to R_{j-1}$ induces the identity on R_j, $\theta_* u_{R_{j-1}}$ also projects to u_{R_j}. Thus $(\theta_* u_{R_{j-1}}) - u_{R_{j-1}}$ is vertical with respect to $R_{j-1} \to R_j$. But the distribution $E_{R_{j-1}}$ is horizontal, so if θ preserves $E_{R_{j-1}}$ we must have $\theta_* u_{R_{j-1}} = u_{R_{j-1}}$. This equation is equivalent, in terms of the specific representation of θ on the trivialized bundle R_{j-1}, to the relation $uh = 0$, where the constant vector field u on N operates on the function h in the usual way for vector fields. But $uh = 0$ for all $u \in U$ annihilated by U_j^* if and only if h is constant along the leaves of the foliation in N.

Now we are ready to construct the lift $f_{j-1}\colon R_{j-1} \to R'_{j-1}$. Just choose it to be any morphism covering f_j and mapping leaves of the $E_{R_{j-1}}$ foliation in R_{j-1} into leaves of the $E_{R'_{j-1}}$ foliation in R'_{j-1}. Such an f_{j-1} may always be found at least locally. For at any point r_{j-1} in R_{j-1} above a point r_j in R_j, the horizontal space $E_{R_{j-1}}$ maps bijectively onto the space E_{R_j} at r_j via the projection $R_{j-1} \to R_j$. Therefore this projection locally restricts to a diffeomorphism of leaves of the $E_{R_{j-1}}$ foliation in R_{j-1} onto leaves of the E_j foliation in R_j. A similar assertion holds for $R'_{j-1} \to R'_j$. As the morphism $f_j\colon R_j \to R'_j$ preserves structure, it must map leaves of the E_{R_j} foliation into leaves of the $E_{R'_j}$ foliation. Therefore a lift f_{j-1} may be locally found mapping leaves of the $E_{R_{j-1}}$ foliation into leaves of the $E_{R'_{j-1}}$ foliation. Because $E_{R_{j-1}}$ and $E_{R'_{j-1}}$ are both invariant under the right action of the structure group of $R_{j-1} \to N$, f_{j-1} may also be required to be a morphism.

To prove f_{j-1} preserves the k-th order almost structures we must show that if $p_{j-1} \in B^k(R_{j-1})$, then $p'_{j-1} = f^k_{j-1}(p_{j-1})$ belongs to $B^k(R'_{j-1})$. Let p_j be the image of p_{j-1} under the projection $B^k(R_{j-1}) \to B^k(R_j)$. Then $p'_j = f^k_j(p_j)$ belongs to $B^k(R'_j)$ because f_j preserves structure. Let q'_{j-1} be any point above p'_j in the fibration $B^k(R'_{j-1}) \to B^k(R'_j)$. Then $(q'_{j-1})^{-1}p'_{j-1}$ is the k-jet of a morphism of R_{j-1}, which induces the identity on R_j and which respects the foliation of R_{j-1}. (For both p'_{j-1} and q'_{j-1} are foliation respecting jets.) Therefore, by the characterization of the kernel pseudogroup, $(q'_{j-1})^{-1}p'_{j-1}$ belongs to $\tilde{G}(R_{j-1})$. Hence $p'_{j-1} = q'_{j-1}\cdot((q'_{j-1})^{-1}p'_{j-1})$ belongs to $B^k(R'_{j-1})$ as desired.

6. The simple quotients of complex type

When the group H_j is of complex type, the lift

$$\begin{tikzcd}
R_{j-1} \arrow[r, dashed, "f_{j-1}"] \arrow[d] & R'_{j-1} \arrow[d] \\
R_j \arrow[r, "f_j"] & R'_j
\end{tikzcd}$$

is accomplished by a straightforward complexification of the procedure developed in the last section for the real type quotients. The distributions now are subbundles of the complexified tangent bundle, and must be tamed by a complexification of the Frobenius theorem, an amalgam of the real Frobenius theorem with the Newlander-Nirenberg theorem. First we complexify the general discussion of the previous section.

Again let S be any bundle in our ladder, and consider the complex linear map $L_M \otimes C \to T_{0_S}(S) \otimes C$. Any ideal E of the complex Lie algebra $L_M \otimes C$ induces right invariant distributions E_S and $E_{S'}$ on S and S' just as before, E_S and $E_{S'}$ now being complex subbundles of $T(S) \otimes C$ and $T(S') \otimes C$ respectively.

The standard Frobenius theorem says that a real linear subbundle E_S of $T(S)$ is locally equivalent to a translation invariant tangent subbundle on a Euclidean space, via local diffeomorphisms of the Euclidean space into S, if and only if the vector field sections of E_S are closed under the Lie bracket operation. The complex analogue of this, proved in [8] by reduction to the two special cases of classical Frobenius theorem and the Newlander-Nirenberg theorem, is the following.

Theorem 6.1. *A complex linear subbundle E_S of $T(S) \otimes C$ is locally equivalent to a translation invariant subbundle of the complexified tangent bundle of a Euclidean space, via local diffeomorphisms of the Euclidean space into S, if and only if the sections of E_S and of the associated bundle $E_S + \bar{E}_S$ are both closed under Lie bracket.*

With analytic input from this complex Frobenius theorem and algebraic input from Proposition 1.4, the lift is now established by mimicry of the real case. The essential fact is that the kernel pseudogroup of $\Gamma^\#_{R_{j-1}} \to \Gamma^\#_{R_j}$ is again uncomplicated; here it consists of all morphisms $(n, a) \to (n, h(n)a)$ where

$h(n) = h(x, y, z)$ is constant in certain variables y, holomorphic in some complex variables z, and arbitrarily smooth in other variables x. The details are carefully worked out in [11].

7. The center

The problem of finding a structure preserving morphism f_Q lifting a given structure preserving morphism f_{R_0}

$$\begin{tikzcd}
Q \arrow[r, dashed, "f_Q"] \arrow[d] & Q' \arrow[d] \\
R^0 \arrow[r, "f_{R_0}"] & R_0'
\end{tikzcd}$$

is of a more substantial nature than the earlier lifting questions. Choose any morphism f_Q lifting f_{R_0}. (This may always be done locally, which suffices.) Then all other lifts are of the form $f_Q \cdot s$, where s is a morphism of Q inducing the identity on R_0. By means of the trivializations $Q = N \times G$ and $R_0 = N \times G/H_0$, the morphisms of Q inducing the identity on R_0 may simply be considered as functions $s\colon N \to H_0$. (Thus given $s\colon N \to H_0$ the corresponding morphism of Q is $(n, a) \to (n, s(n)a)$.) It will be shown that $f_Q \cdot s$ preserves structure precisely when the function s satisfies a certain formally solvable inhomogeneous constant coefficient linear partial differential equation, whereupon a theorem of Malgrange and Ehrenpreis will conjure the solution.

$B^k(Q)$ and $B^k(Q')$ are principal $\tilde{G}^k(Q)$ bundles over N and N' respectively. Since the k-jet prolongation of any morphism on Q commutes with the right action of $\tilde{G}^k(Q)$, one needs only to demonstrate that $(f_Q \cdot s)^k$ carries one point in each fiber of $B^k(Q) \to N$ into $B^k(Q')$ in order to conclude that $(f_Q \cdot s)^k\colon B^k(Q) \to B^k(Q')$. The easiest point to examine in the fiber over $n \in N$ is obviously the k-jet at O_S of the morphism induced by "translation by n" on M; denote by i_n this jet whose target is $(n, i) \in Q$, i being the indentity of G.

The image $\bar{i}_n$ of i_n under $B^k(Q) \to B^k(R_0)$ is the k-jet of the morphism induced by "translation by n" on R_0, with target $(n, \bar{i}) \in R_0$, where $\bar{i}$ is the identity of G/H_0. Because f_{R_0} preserves structure, we conclude that $f_{R_0}^k(\bar{i}_n) \in B^k(R_0')$. Pick a jet p_n with target $f_Q(n, i) \in Q'$ which belongs to $B^k(Q')$ and covers $f_{R_0}^k(\bar{i}_n)$ via $B^k(Q') \to B^k(R_0')$; the existence of p_n results from the surjectivity of $G^k(Q) \to G^k(R_0)$. Then f_Q preserves the k-th order structures if and only if for all $n \in N$, $f_Q^k(i_n)$ belongs to the fiber of $B^k(Q')$ over the point $f_Q(i, n) \in Q'$, or equivalently $f_Q^k(i_n) = p_n \cdot q_n$ for some element $q_n \in G^k(Q)$. As $f_Q^k(i_n)$ and p_n both project to $f_{R_0}^k(\bar{i}_n) \in B^k(R_0')$, the element q_n must belong to the kernel $G_0^k(Q)$ of $G^k(Q) \to G^k(R_0)$. The choice of p_n, which may be made smoothly in n (at least locally), provides the inhomogeneous data for the differential equation $f_Q^k(i_n) = p_n \cdot q_n$, which must be solved for some smooth assignment $n \to q_n$ if f_Q^k is to preserve the k-th order structure.

Having chosen p_n to be data for f_Q, data for the other candidates $f_Q \cdot s$ may be generated naturally. For note that the morphism of Q corresponding to a constant function $N \rightarrow c \in H_0$ belongs to Γ_Q. (For by definition any $c \in H$ is the restriction to K of a linear transformation C on M belonging to Γ_M and inducing the identity on N. If $(n, z) \in N \times K = M$, then $C(n, z) = C(n, 0) + C(0, z) = (n, x_n) + (0, c(z)) = (n, c(z) + x_n)$. So the action of C on the fiber $K_n = n \times K$ belongs to $c + T \subset A$, and hence the extension of C to a morphism in Γ_Q is just the map corresponding to $N \rightarrow c$.) Therefore the k-jet of the constant morphism corresponding to an element $c \in H_0$ defines an element $c^k \in \tilde{G}^k(Q)$, which acts on $B^k(Q')$ to the right. In particular, for any $s\colon N \rightarrow H_0$ and any point $n \in N$, the element $s(n)^k \in \tilde{G}^k(Q)$ acts on $B^k(Q')$. The target of $p_n \cdot s(n)^k$ is $f_Q(n, s(n)) = (f_Q \cdot s)(n, i)$.

So the morphism $f_Q \cdot s$ of Q into Q' preserves k-structure precisely when the function $s\colon N \rightarrow H_0$ solves the partial differential equation $(f_Q \cdot s)^k(i_n) \in p_n \cdot s(n)^k \cdot G_0^k(Q)$, or $(s(n)^{-1})^k \cdot p_n^{-1} \cdot f_Q^k \cdot s^k(i_n) \in G_0^k(Q)$. Let us isolate the unknown function s from the inhomogeneous data. Note that for fixed $n \in N$, $p_n^{-1} \cdot j_{(n,i)}^k(f_Q)$ is the k-jet of a morphism of Q with source (n, i) and target $(0, i) = O_Q$. Its projection in R_0 is, by choice of p_n, the jet $f_{R_0}^k(i_n)^{-1} \cdot f_{R_0}^k(i_0) = i_n^{-1}$. Therefore $p_n^{-1} \cdot j_{(n,i)}^k(f_Q)$ is the k-jet at $(n, i) \in Q$ of a morphism of the form $(x, a) \rightarrow (x - n, \rho_n(x - a)a)$, where $(x, a) \in N \times G = Q$ and the function $\rho_n\colon N \rightarrow H_0$ is any function whose k-jet at $0 \in N$ is a specific value. (Note that $\rho_n(0) = 1$.) Substituting the fact that i_n is the jet of $(x, a) \rightarrow (x + n, a)$ at $(0, i) \in Q$, we see that $(s(n)^{-1})^k \cdot p_n^{-1} \cdot f_Q^k \cdot s^k(i_n)$ is the k-jet at $(0, i) \in Q$ of the morphism $(x, a) \rightarrow (x, s(n)^{-1}\rho_n(x) \cdot s(x + n)a)$. In accordance with our identification of morphisms of Q which induce the identity on R_0, with the corresponding H_0 valued functions on N, we may consider the group $G_0^k(Q)$ to be a subgroup of the commutative group $J_0^k(H_0)$ of k-jets of H_0 valued functions on N with source $0 \in N$ and target $1 \in H_0$. Considered thus, the condition on s is that the k-jet at $0 \in N$ of the function $x \rightarrow s(n)^{-1}\rho_n(x)s(x + n)$ belong to $G_0^k(Q)$. The k-jet of $\rho_n\colon N \rightarrow H_0$ is all that is specifically determined about ρ_n; call it r_n. If $c \in H_0$ is any element, let the same symbol denote the k-jet at $0 \in N$ of the constant function $N \rightarrow c \in H_0$. As usual, let τ_n denote "translation by n". Then we may recapitulate the differential condition on s in the following explicit form.

Summary. A local morphism $f_Q \cdot s$ of Q into Q' preserves the k-th order almost structure if and only if, for all n, the function $s\colon N \rightarrow H_0$ solves the k-jet equation

$$(*) \qquad j_0^k(s \cdot \tau_n) \cdot s(n)^{-1} \cdot r_n \in G_0^k(Q) \subset J_0^k(H_0)\ .$$

To understand the equation $(*)$, note that by the definition of $G_0^k(Q)$, the morphism of Q corresponding to a function $s\colon N \rightarrow H_0$ belongs to $\Gamma_Q^\sharp$ if and only if $j_0^k(s \cdot \tau_n) \cdot s(n)^{-1} \in G_0^k(Q)$ for all n. Thus $(*)$ is nothing but the structure equation for the kernel pseudogroup of $\Gamma_Q^\sharp \rightarrow \Gamma_{R_0}^\sharp$ with inhomogeneous data r_n.

Because $\rho_n(0) = 1$, each ρ_n maps N into the identity component of H_0. Also, if s satisfies $(*)$ so will s multiplied by any constant function; therefore there exists a solution $s\colon N \to H_0$ if and only if there is a solution mapping N into the identity component of H_0. In short, we may assume H_0 to be connected, in which it consists either of $\{1\}$, the positive real numbers, or the nonzero complex numbers. (If $H_0 = \{1\}$, $Q = R_0$ and there is nothing to prove.) If s satisfies $(*)$, so will s multiplied by any function $N \to H_0$ which solves the corresponding homogeneous equation (i.e., whose morphism of Q preserves structure). Therefore there exists a solution to $(*)$ if and only if there exists a solution for which the jet $j_0^k(s\cdot\tau_n)\cdot s(n)^{-1}\cdot r_n$ belongs to the identity component of $G_0^k(Q)$ at (one and hence) every n. Consequently we may also assume $G_0^k(Q)$ to be connected.

We now transform $(*)$ into linear form by applying the logarithm. If H_0 is the positive reals, then we may use the real log: $H_0 \to \boldsymbol{R}$. This transforms functions from N to H_0 into functions from N to $\boldsymbol{R}$, and the commutative multiplicative group $J_0^k(H_0)$ into the vector space $J_0^k(\boldsymbol{R})$ of k-jets at 0 of real valued functions on N. As $\log G_0^k(Q)$ is a connected Lie subgroup of $J_0^k(\boldsymbol{R})$, it is a vector subspace. Then we see that $(*)$ has a solution $s\colon N \to H_0$ if and only if there is a solution $t = \log(s)\colon N \to \boldsymbol{R}$ of the linear equation $j_0^k(t\cdot\tau_n) - t(n) + \log r_n \in \log G_0^k(Q)$. This may be made to look more familiar if we let D be a linear map of $J_0^k(\boldsymbol{R})$ onto some $\boldsymbol{R}^l$ whose kernel is $\log G_0^k(Q)$. Defining for any $t\colon N \to \boldsymbol{R}$ the function $\tilde{D}t\colon N \to \boldsymbol{R}^l$ by $(\tilde{D}t)(n) = D(j_0^k(t\cdot\tau_n) - t(n))$, the operator $\tilde{D}$ is nothing but a real constant coefficient linear partial differential operator on N. Denote by $u(n)$ the value $-D(r_n)$. Then for a real center H_0 we may state our

Conclusion. *There exists everywhere local morphisms of Q into Q' preserving the k-th order structures if and only if there exists locally real valued functions t on N satisfying the constant coefficient linear partial differential equation $\tilde{D}t = u$.*

When H_0 is complex, there is no more difficulty. Since all of the r_n and all of the elements of $G_0^k(Q)$ are k-jets of complex functions on N with source $0 \in N$ and target $1 \in H_0$, we may use the standard branch of the logarithm defined in any neighborhood of 1 to transform these jets. Then the local existence of a solution s for $(*)$ is equivalent to the local existence of a complex valued function t on N satisfying $j_0^k(t\cdot\tau_n) - t(n) + \log r_n \in \log G_0^k(Q)$. (For if any solution t exists near the point $n \in N$, then there exists a solution with $t(n) = 0$. Since $\exp = (\log)^{-1}$ in a neighborhood of $0 \in \boldsymbol{C}$, if $t(n) = 0$ then $s = \exp(t)$ solves $(*)$ near n. Conversely, if $(*)$ has a solution near n, then it has a solution for which $s(n) = 1$, and therefore $t = \log(s)$ solves the linear equation.) So the conclusion above remains valid with the understanding that $\boldsymbol{R}$ must be replaced by $\boldsymbol{C}$.

We are finally prepared to invoke [9]:

Theorem 7.1. *Let $\tilde{D}$ be a constant coefficient partial differential operator* *defined near the point n in the vector space N, and let u bev any smooth l-tuple of functions also defined near n. Then the equation* $\tilde{D}t = u$ *has a smooth solution t defined around n if and only if the equation is formally solvable in a neighborhood of n. That is, for all x in some neighborhood of n and all positive integers i, the equation* $j_x^i(\tilde{D}t) = j_x^i(u)$ *is solvable.*

The theorem, due to Malgrange and Ehrenpreis, is valid in either the real or complex category. One may trace through the derivation of our equation $\tilde{D}t = u$ to show that its formal solvability is equivalent to formal integrability of $B^k(Q')$. For, if for each i there exists a solution to $j_x^i(\tilde{D}t) = j_x^i(u)$ at x, then the k-jet extension of the morphisms $f_Q \cdot (\exp t)$ of Q into Q' takes $B^k(Q)$ to image manifolds in $D_c^k(Q, Q')$ which contact $B^k(Q')$ at points above $x \in N$ to arbitrarily high order. Conversely, the existence of a sequence of morphisms satisfying the latter condition and lifting f_{R_0} implies the existence of the requisite sequence of functions t. Thus the known formal integrability of $B^k(Q')$ allows us to apply Theorem 7.1 and to find the lift f_Q.

8. Final step: The Abelian quotient

The construction of the final lift

$$\begin{tikzcd} P \arrow[r, dashed, "f_P"] \arrow[d] & P' \arrow[d] \\ Q \arrow[r, "f_Q"] & Q' \end{tikzcd}$$

is nearly identical with the construction of f_Q in the last section. Once more the only condition is an inhomogeneous form of the constant coefficient partial differential equation characterizing the kernel pseudogroup of $\Gamma_P^\# \to \Gamma_Q^\#$. In fact, this lift is somewhat easier than the last to accomplish, since the equation is linear at inception; there is no need to transform logarithmically.

Again begin by choosing any morphism f_P lifting f_Q. The morphisms of P inducing the identity on Q are then identified with smooth functions on N with values in the translation group T of K, and all morphisms of P into P' lifting f_Q are given by $f_P \cdot s$ for some $s\colon N \to T$. Now repeat the argument of § 7, changing the referents Q, R_0, and H_0 to P, Q, and T respectively, to conclude that the morphism $f_P \cdot s$ preserves the almost structure if and only if s satisfies the k-jet equation $j_0^k(s\cdot\tau_n)\cdot s(n)^{-1}\cdot r_n \in G_0^k(P)$ for all n. Here r_n is a k-jet at 0 of a T valued function on N with source $0 \in N$ and target the identity of T, and the kernel $G_0^k(P)$ of $G^k(P) \to G^k(Q)$ is considered to be a subgroup of the group $J_0^k(T)$ of k-jets of T-valued functions with source $0 \in N$. (All elements of $G_0^k(P)$ must have the identity in T as target.)

The multiplicatively written group T is canonically isomorphic to the additive vector group K, so we may transform the above equation into the linear equatin of k-jets of K valued functions

$$(*) \qquad j_0^k(s \cdot \tau_n) - s(n) + r_n \in G_0^k(P) \subset J_0^k(K) \ .$$

Because we may modify any solution s of $(*)$ by any solution of the corresponding homogeneous equation (where $r_n = 0$), $(*)$ is solvable if and only if it remains solvable when $G_0^k(P)$ is replaced by its identity component. Therefore we may assume $G_0^k(P)$ to be connected and hence a vector subspace of $J_0^k(K)$. Choose a linear map D of $J_0^k(K)$ onto some $\boldsymbol{R}^l$ with kernel $G_0^k(P)$, set $u(n) = -D(r_n)$, and let $\tilde{D}$ be the constant coefficient partial differential operator on K valued functions defined by $(\tilde{D}s)(n) = D(j_0^k(s \cdot \tau_n) - s(n))$. Then there exist local morphisms $P \to P'$ everywhere lifting f_Q and preserving k-th order structure if and only if the equation $\tilde{D}s = u$ is everywhere locally solvable on N. As before, the formal solvability of the equation is equivalent to the known formal integrability of $B^k(P')$. Application of Theorem 7.1 (interpreted in the category of K valued functions) completes the proof.

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