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THE INTEGRAL OF THE SCALAR CURVATURE OF COMPLETE MANIFOLDS WITHOUT CONJUGATE POINTS

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Abstract

We prove that the integral of the scalar curvature of a complete manifold M without conjugate points is nonpositive and vanishes only if M is flat, provided that the Ricci curvature on the unit tangent bundle SM has an integrable positive or negative part.

Introduction

A complete Riemannian manifold M is said to be without conjugate points if the geodesics of M contain no pair of conjugate points, equivalently, if any two distinct points of its universal covering, endowed with the induced metric, are joined by a unique geodesic. If the sectional curvature of M is nonpositive, then M has no conjugate points. However, there exist compact and complete noncompact manifolds without conjugate points and with sectional curvature of both signs (see [2] or [7] for examples).

The object of this paper is to prove the following result.

Theorem A. *Let M be a complete manifold without conjugate points. Suppose that the Ricci curvature on the unit tangent bundle SM has an integrable positive or negative part. Then the integral of the scalar curvature of M is nonpositive and vanishes only if M is flat.*

Theorem A generalizes results of several authors. The inequality is due to Cohn-Vossen [4] when M is two-dimensional and simply connected. The result was obtained by E. Hopf [8] for surfaces with finite volume and Gaussian curvature bounded from below. In [6] Green extended the result of E. Hopf for complete n-dimensional manifolds with finite volume and sectional curvature bounded from below. Finally, in [9] Innami proved the theorem for complete n-dimensional manifolds with the additional hypotheses that the integral of the Ricci curvature is finite, and the nonwandering set of SM decomposes into at most countably many invariant

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sets each of which has finite volume. In particular, he obtained the result for manifolds with finite volume and for simply connected manifolds.

Examples. In general, the integral of the Ricci curvature of a manifold without conjugate points may not exist. Let M be a compact manifold without conjugate points and sectional curvature of both signs, and let $\widetilde{M}$ be its universal covering with the induced metric. Then both positive and negative parts of the Ricci curvature of $\widetilde{M}$ are not integrable, because there are infinitely many disjoint fundamental domains in $\widetilde{M}$.

Gulliver [7, Theorem 3] has proved that if (M, g') has sectional curvature $K \leq 0$ and if $K = -c^2$, $c > 0$, on a normal ball B, centered in p, then there is a metric g without conjugate points on M such that $g = g'$ except on a compact subset of B, and the sectional curvatures of g are constant and positive on a neighborhood of p. Gulliver also observes that one can construct g by modifying the metric g' in a disjoint family of balls, each of which has constant negative sectional curvature. Taking (M, g') as the hyperbolic space we can obtain a manifold without conjugate points which does not cover any compact manifold and such that both positive and negative parts of the Ricci curvature are not integrable.

On the other hand, some examples may have infinite volume and finite total Ricci curvature (hence nonpositive by Theorem A). To see this, take a convex curve C on the xz-plane of a Euclidean 3-space R^3 such that C is the tractrix $z(x) = \int_0^{-\ln x} \sqrt{1 - e^{-2t}}\, dt$ for all $0 < x < \frac{1}{2}$ and $z(x)$ is constant for all $x \geq 1$. Let (S, g') be the surface of revolution obtained by rotation of C about the z-axis. Then every local perturbation of (S, g'), following Gulliver, has infinite volume and negative total Gaussian curvature.

We can easily see that the proof of Theorem A yields the following more general result.

Theorem B. *Let M be a complete manifold and let $Z \subset SM$ be a set invariant under the geodesic flow such that all geodesics γ_v, with $v \in Z$, contain no pair of conjugate points. Suppose that the Ricci curvature on Z has an integrable positive or negative part. Then*

$$\int_Z \mathrm{Ric}\, d\mu \leq 0,$$

where the equality holds only if the curvature tensor of M is identically zero on Z.

This paper is organized as follows. In §1 we state the maximal ergodic theorem and describe the so-called E. Hopf's decomposition of SM. These are the basic tools of our proof of Theorem A. In §2 we introduce a measure on the space of orbits on SM. It will allow us to decompose the Liouville measure of SM on its dissipative and conservative parts. §3 contains a description of the selfadjoint Riccati tensor along geodesics with no conjugate points, constructed by Green. In particular, we obtain simple proofs of two results of Ambrose [1] and of Chicone and Ehrlich [3] about the integral of the Ricci curvature along such geodesics. In §4 we prove Theorem A.

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1. Preliminaries

Let M be a complete Riemannian manifold with dimension n, and let SM be its unit tangent bundle. Denote by $T_t\colon SM \to SM$, $t \in \mathbf{R}$, the geodesic flow of M, that is, $T_t(v) = \dot{\gamma}_v(t)$, where γ_v is the geodesic determined by v. Let μ be the Liouville measure defined on the Borel sets of SM. It is known that the geodesic flow preserves the measure μ which is σ-finite.

In what follows we refer to [10] as a basic reference.

1.1. Maximal Ergodic Theorem. *Let f be an integrable function on SM and let $D \subset SM$ be a T_t-invariant Borel set. Define*

$$E(f) = \left\{ v \in D;\ \sup_{s>0} \int_0^s f(T_t(v))\, dt > 0 \right\}.$$

Then $\int_{E(f)} f\, d\mu \geq 0$. Also, if we set $T = T_1$, and if A is a T-invariant Borel set and

$$E[f] = \left\{ v \in A;\ \sup_{n\geq 0} \sum_{j=0}^{n} f(T^j(v)) > 0 \right\},$$

then

$$\int_{E[f]} f\, d\mu \geq 0.$$

1.2. The decomposition of E. Hopf of SM. Let f_0 be an integrable function on SM. Then the Borel sets

$$D^+ = \left\{ v \in SM;\ \int_0^\infty f_0(T_t(v))\, dt < \infty \right\}, \qquad C^+ = SM \backslash D^+$$

are T_t-invariant; they are independent of f_0 in the following sense: if

$f_1 \geq 0$ is another integrable function, then the set

$$E = \left\{ v \in SM ; \int_0^\infty f_1(T_t(v))\,dt = \infty \right\} \cap D^+$$

has measure zero. To see this, we note that E is a T_t-invariant set. By the Maximal Ergodic Theorem 1.1, we conclude that $\int_E (af_1 - f_0)\,d\mu \geq 0$ for every $a > 0$. Letting $a \to 0$, we get $\int_E f_0\,d\mu = 0$. Since $f_0 > 0$, $\mu(E) = 0$, as we wished to show. Now, if $f_1 > 0$, interchanging f_0 and f_1, we obtain the independence of D^+, up to a subset of measure zero.

The decomposition $SM = D^+ \cup C^+$ is called E. Hopf's decomposition of SM associated to the geodesic flow $(T_t)_{t\in\mathbf{R}}$. The components D^+ and C^+ are called, respectively, the dissipative and the conservative parts of the decomposition. Denote by $SM = D^- \cup C^-$ the decomposition of E. Hopf of SM associated to the inverse flow $(T_{-t})_{t\in\mathbf{R}}$. Then

$$D^- = \left\{ v \in SM ; \int_{-\infty}^0 f_0(T_t(v))\,dt < \infty \right\}.$$

From what we have seen above, it follows that if $D = D^+ \cap D^-$, then for every integrable function f on SM, the Lebesgue integral

$$\int_{-\infty}^\infty f(T_t(v))\,dt \tag{1.2.1}$$

exists for almost all $v \in D$. Also, if $g_0 > 0$ is integrable, then the T-invariant sets

$$D(T) = \left\{ v \in SM ; \sum_{-\infty}^{\infty} g_0(T^j(v)) < \infty \right\}, \qquad C(T) = SM \backslash D(T)$$

are independent of g_0, up to a subset of measure zero. Moreover, if $g \geq 0$ is integrable, then

$$\sum_{-\infty}^{\infty} g(T^j(v)) < \infty \quad \text{for almost all } v \in D(T). \tag{1.2.2}$$

The following lemma establishes a connection between the sets D and $D(T)$.

1.2.3. Lemma. *If $g \geq 0$ is integrable on SM, then $\sum_{-\infty}^{\infty} g(T^j(v)) < \infty$ for almost all $v \in D$.*

Proof. Set $g_0(x) = \int_0^1 f_0(T_t(x))\,dt$, $x \in SM$. Then $g_0 > 0$ and

$$\int_{SM} g_0\,d\mu = \int_{SM} \int_0^1 f_0(T_t(x))\,dt\,d\mu = \int_{SM} f_0\,d\mu.$$

Thus g_0 is integrable and by (1.2.1) we have

$$\sum_{-\infty}^{\infty} g_0(T^j(v)) = \int_{-\infty}^{\infty} f_0(T_t(v))\,dt < \infty$$

for every $v \in D$. If we choose this g_0 in the definition of $D(T)$, then $D \subset D(T)$. In view of (1.2.2), this completes the proof.

2. A measure on the space of orbits

Let $\sim$ be the equivalence relation in SM defined by: $v \sim w$ if and only if there exists $j \in \mathbf{Z}$ such that $T^j(v) = w$. Let $SM/\sim$ be the space of orbits, and denote by $\pi: SM \to SM/\sim$ the natural projection $\pi(v) = [v]$. Consider in $SM/\sim$ the σ-algebra $\widetilde{\beta}$ induced by π. Then $\pi(E)$ is measurable for every Borel set E, because $\pi^{-1}(\pi(E)) = \bigcup_{-\infty}^{\infty} T^j(E)$. A Borel set E is called a wandering set if $T^j(E) \cap E = \phi$ for every $j \geq 1$. For each $\widetilde{E} \in \widetilde{\beta}$, let

$$\tilde{\mu}(\widetilde{E}) = \sup\{\mu(E);\, E \subset \pi^{-1}(\widetilde{E}),\ E \text{ a wandering set}\}.$$

2.1. Lemma. *$\tilde{\mu}$ is a measure on $SM/\sim$ with the property that $\tilde{\mu}(\pi(E)) = \mu(E)$ for every wandering set E.*

Proof. Let $\widetilde{E}_n$, $n \in \mathbf{N}$, be measurable and disjoint sets in $\widetilde{\beta}$. We will first show that

$$\tilde{\mu}\left(\bigcup_1^{\infty} \widetilde{E}_n\right) = \sum_1^{\infty} \tilde{\mu}(\widetilde{E}_n). \tag{2.2}$$

Let $F \subset \pi^{-1}\left(\bigcup_1^{\infty} \widetilde{E}_n\right)$ be a wandering set. Then the sets $F \cap \pi^{-1}(\widetilde{E}_n)$, $n \in \mathbf{N}$, are disjoint wandering sets, so $\tilde{\mu}(\widetilde{E}_n) \geq \mu(F \cap \pi^{-1}(\widetilde{E}_n))$ for every $n \geq 1$. Therefore

$$\mu(F) = \mu\left(F \cap \bigcup_1^{\infty} \pi^{-1}(\widetilde{E}_n)\right) = \mu\left(\bigcup_1^{\infty} F \cap \pi^{-1}(\widetilde{E}_n)\right)$$
$$= \sum_1^{\infty} \mu(F \cap \pi^{-1}(\widetilde{E}_n)) \leq \sum_1^{\infty} \tilde{\mu}(\widetilde{E}_n).$$

This implies that $\tilde{\mu}\left(\bigcup_1^{\infty} \widetilde{E}_n\right) \leq \sum_1^{\infty} \tilde{\mu}(\widetilde{E}_n)$.

To show the other inequality, we may suppose that $\tilde{\mu}(\widetilde{E}_n) < \infty$ for all n. Now let $\varepsilon > 0$. Then for every $n \in \mathbf{N}$ there exists a wandering set $F_n \subset \pi^{-1}(\widetilde{E}_n)$ such that

$$\tilde{\mu}(\widetilde{E}_n) < \mu(F_n) + \varepsilon/2^n.$$

Thus $\bigcup_1^\infty F_n \subset \pi^{-1}\left(\bigcup_1^\infty \widetilde{E}_n\right)$ is a wandering set and

$$\sum_1^\infty \tilde{\mu}(\widetilde{E}_n) \le \sum_1^\infty \mu(F_n) + \varepsilon = \mu\left(\bigcup_1^\infty F_n\right) + \varepsilon \le \tilde{\mu}\left(\bigcup_1^\infty \widetilde{E}_n\right) + \varepsilon.$$

Letting $\varepsilon \to 0$ we obtain (2.2).

Now let E be a wandering set. It is clear that $\mu(E) \le \tilde{\mu}(\pi(E))$. If $B \subset \pi^{-1}(\pi(E))$ is a wandering set, then

$$\begin{aligned}\mu(B) &= \mu\left(B \cap \bigcup_{-\infty}^{\infty} T^j(E)\right) = \mu\left(\bigcup_{-\infty}^{\infty} B \cap T^j(E)\right)\\ &= \sum_{-\infty}^{\infty} \mu(B \cap T^j(E)) = \sum_{-\infty}^{\infty} \mu(T^{-j}(B) \cap E)\\ &= \mu\left(\bigcup_{-\infty}^{\infty} T^{-j}(B) \cap E\right) \le \mu(E).\end{aligned}$$

This shows that $\tilde{\mu}(\pi(E)) = \mu(E)$, as we wished to prove. q.e.d.

Given a function f integrable on SM, then by (1.2.1) $x \mapsto \int_{-\infty}^{\infty} f(T_t(x))\,dt$, $x \in D$, defines a measurable function on the Borel sets of D. Furthermore, it is constant in each orbit on D. Thus $[x] \mapsto \int_{-\infty}^{\infty} f(T_t(x))\,dt$, $[x] \in \pi(D)$, defines a $\tilde{\mu}$-measurable function on $\pi(D)$.

2.3. Proposition. *If f is an integrable function on SM, then*

$$\int_D f\,d\mu = \int_{\pi(D)} \int_{-\infty}^{\infty} f(T_t(v))\,dt\,d\tilde{\mu}.$$

Proof. Taking $\overline{g}(x) = \int_0^1 f(T_t(x))\,dt$ and using (1.2.1) we obtain $\sum_{-\infty}^{\infty} \overline{g}(T^j(v)) = \int_{-\infty}^{\infty} f(T_t(v))\,dt$ for almost all $v \in D$. Since $\int_D \overline{g}\,d\mu = \int_D \int_0^1 f(T_t(x))\,dt\,d\mu = \int_D f\,d\mu$, it suffices to show that

$$\int_D g\,d\mu = \int_{\pi(D)} \sum_{-\infty}^{\infty} g(T^j(v))\,d\tilde{\mu}, \tag{2.4}$$

for every integrable function g on SM. For this, it suffices to show (2.4) for $g = \chi_E$, where $E \subset D$ and $\mu(E) < \infty$. The general case follows by linearity and by applying the dominated convergence theorem.

If E is a wandering set, then $\sum_{-\infty}^{\infty} \chi_E(T^j(x)) = 1$ or 0 according as $\pi(x) \in \pi(E)$ or $\pi(x) \notin \pi(E)$, respectively. Therefore

$$\sum_{-\infty}^{\infty} \chi_E(T^j(x)) = \chi_{\pi(E)}(\pi(x))$$

for every $x \in SM$. Since $E \subset D$,

$$\int_{\pi(D)} \sum_{-\infty}^{\infty} \chi_E(T^j(x))\, d\tilde{\mu} = \tilde{\mu}(\pi(E)) = \mu(E) = \int_D \chi_E\, d\mu.$$

Now, if $E \subset D$ is an arbitrary Borel set with $\mu(E) < \infty$, then by Lemma 1.2.3, $\sum_{-\infty}^{\infty} \chi_E(T^j(x)) < \infty$ for almost all $x \in D$. Therefore, if E_0 is the set of the elements $x \in E$ such that $T^j(x) \in E$ for infinitely many integers $j > 0$, then $\mu(E_0) = 0$. Set $F = E \backslash E_0$ and $F_n = F \cap \left(T^{-n}(F) \backslash \bigcup_{j \geq n+1} T^{-j}(F)\right)$, $n \geq 0$. Then the F_n's are disjoint wandering sets and $F = \bigcup_0^{\infty} F_n$. Hence

$$\begin{aligned}
\int_{\pi(D)} \sum_{-\infty}^{\infty} \chi_E(T^j(x))\, d\tilde{\mu} &= \int_{\pi(D)} \sum_{j=-\infty}^{\infty} \sum_{n=1}^{\infty} \chi_{E_0 \cup F_n}(T^j(x))\, d\tilde{\mu} \\
&= \sum_{n=1}^{\infty} \int_{\pi(D)} \sum_{j=-\infty}^{\infty} \chi_{F_n}(T^j(x))\, d\tilde{\mu} \\
&\quad + \int_{\pi(D)} \sum_{-\infty}^{\infty} \chi_{E_0}(T^j(x))\, d\tilde{\mu} \\
&= \sum_{1}^{\infty} \mu(F_n) = \mu(E) = \int_D \chi_E\, d\mu,
\end{aligned}$$

which completes the proof of Proposition 2.3.

3. The selfadjoint Riccati tensor

Let M be a complete Riemannian manifold. Let $Z \subset SM$ be a T_t-invariant set of vectors $v \in SM$ such that the geodesic γ_v has no pair of conjugate points. One can construct (see [6]) a tensor U defined on Z, such that for every $v \in Z$, $U_v = U(v)$ is a selfadjoint linear operator on $\{w \in SM,\ \langle w, v \rangle = 0\}$ and U satisfies the Riccati equation

$$\dot{U} + U^2 + R = 0. \tag{3.1}$$

Here, the derivative is defined by $\langle \dot{U}_{v_s}(x(s)), y(s) \rangle = \frac{d}{ds} \langle U_{v_s}(x(s)), y(s) \rangle$ for every pair of parallel fields $x(s), y(s)$ along γ_v, where $v_s = T_s(v)$. The tensor $R = R_v$ is the curvature tensor given by $R_v(w) = R(v, w)v$. In fact, if γ_v has no points conjugate to $\gamma(0)$ on $(0, \infty)$, then U_{v_s} is defined for all $s \in (0, \infty)$ (see [5, Proposition 3]).

Taking the trace of (3.1) and integrating, we obtain

$$\operatorname{tr} U_{v_s} - \operatorname{tr} U_{v_t} + \int_t^s \operatorname{tr} U_{v_\lambda}^2\, d\lambda + \int_t^s \operatorname{tr} R_{v_\lambda}\, d\lambda = 0.$$

Write $u(s) = \operatorname{tr} U_{v_s}$. Since $(\operatorname{tr} U)^2 \leq (n-1)\operatorname{tr}(U^2)$ it follows that, for every $\varepsilon > 0$,

$$(3.2)\quad u(s) - u(t) + \frac{\varepsilon}{n-1}\int_t^s u^2(\lambda)\,d\lambda + \int_t^s ((1-\varepsilon)\operatorname{tr} U^2 + \operatorname{tr} R)(v_\lambda)\,d\lambda \leq 0.$$

The following result is crucial to the proof of Theorem A.

3.3. Proposition. *Let* $u\colon [0, \infty) \to R$ *and* $v\colon (-\infty, \infty) \to R$ *be continuous functions. Then, for every* $a > 0$:

(i) $\limsup_{s\to+\infty}\left(u(s) + a\int_0^s u^2(t)\,dt\right) \geq 0$ *where the equality holds only if* u *is identically zero.*

(ii) $\limsup_{s\to+\infty}\left(v(s) - v(-s) + a\int_{-s}^s v^2(t)\,dt\right) \geq 0$ *where the equality holds only if* v *is identically zero.*

Proof. (i) Taking $u = af$, we may assume that $a = 1$. Suppose that

$$\limsup_{s\to+\infty}\left(u(s) + \int_0^s u^2(t)\,dt\right) < -\delta < 0.$$

Then there exists $s_0 > 0$ such that

$$(3.4)\qquad u(s) + \int_0^s u^2(t)\,dt < -\delta$$

for every $s \geq s_0$. Set

$$g(s) = \exp\left(-\int_0^s\left(\int_0^t u^2(\lambda)\,d\lambda\right)dt\right), \qquad s \geq 0.$$

Then $g'(s) = -g(s)\cdot\int_0^s u^2(t)\,dt \leq 0$ and

$$g''(s) = \left[\int_0^s u^2(t)\,dt + u(s)\right]\left[\int_0^s u^2(t)\,dt - u(s)\right] g(s).$$

From (3.4) it follows that $-u(s) > \delta$ on $[s_0, \infty)$, so $\int_0^s u^2(t)\,dt - u(s) > \delta$. Hence

$$g''(s) < -\delta^2 g(s) < 0 \quad \text{for every } s \in [s_0, \infty).$$

But this is a contradiction, because g is positive and nonincreasing. Hence we have proved that

$$\limsup_{s\to+\infty}\left(u(s) + a\int_0^s u^2(t)\,dt\right) \geq 0.$$

Furthermore,

$$\begin{aligned}\limsup_{s\to+\infty}&\left(u(s)+a\int_0^s u^2(t)\,dt\right)\\&=\limsup_{s\to+\infty}\left(u(s)+\frac{a}{2}\int_0^s u^2(t)\,dt\right)+\frac{a}{2}\int_0^\infty u^2(t)\,dt\\&\geq\frac{a}{2}\int_0^\infty u^2(t)\,dt\geq 0.\end{aligned}$$

Now, if $\limsup_{s\to+\infty}\left(u(s)+a\int_0^s u^2(t)\,dt\right)=0$, then $\int_0^\infty u^2(t)\,dt=0$ and so u is identically zero.

In order to prove (ii), we note that for every $s\in R$

$$\int_{-s}^s v^2(t)\,dt=\int_0^s (v^2(t)+v^2(-t))\,dt\geq\frac{1}{2}\int_0^s (v(t)-v(-t))^2\,dt.$$

Then by item (i) we have

$$\begin{aligned}\limsup_{s\to+\infty}&\left(v(s)-v(-s)+a\int_{-s}^s v^2(t)\,dt\right)\\&\geq\limsup_{s\to+\infty}\left(v(s)-v(-s)+\frac{a}{2}\int_0^s (v(t)-v(-t))^2\,dt\right)\geq 0.\end{aligned}$$

Now, if $\limsup_{s\to+\infty}\left(v(s)-v(-s)+a\int_{-s}^s v^2(t)\,dt\right)=0$, then, by item (i), $v(s)-v(-s)=0$ identically. Thus, $\int_{-\infty}^\infty v^2(t)\,dt=0$ and so $v=0$. q.e.d.

The following result due to Ambrose [1] follows from (3.2) by taking $t=t_0>0$ and applying Proposition 3.3(i).

3.5. Corollary. *If* $\lim_{s\to+\infty}\int_0^s \operatorname{Ric}(\dot\gamma(t))\,dt=+\infty$, *then* $\gamma(s)$ *is conjugate to* $\gamma(0)$ *for some* $s>0$.

The following result is an improvement of Theorem 1.2 of Chicone and Ehrlich [3], and is an immediate consequence of (3.2) and Proposition 3.3(ii).

3.6. Corollary. *If* $\gamma\colon(-\infty,\infty)\to M$ *has no pair of conjugate points, then*

$$\limsup_{s\to+\infty}\int_{-s}^s \operatorname{Ric}(\dot\gamma(t))\,dt\leq 0,$$

where the equality holds only if the curvature tensor $R_{\dot\gamma}$ *is identically zero along* γ.

4. Proof of Theorem A

If M has no conjugate points, then U is defined in all of SM. We will show that

$$\int_{SM} \operatorname{tr} R \, d\mu \le -\int_{SM} \operatorname{tr} U^2 \, d\mu. \tag{4.1}$$

Theorem A will follow from (4.1). To see this, let S be the scalar curvature of M, let $\omega_{n-1} = \operatorname{vol}(S^{n-1})$, and let m be the Lebesgue measure on M. Then by Fubini's theorem, we obtain

$$\int_M S \, dm = \frac{1}{\omega_{n-1}} \int_{SM} \operatorname{tr} R \, d\mu \le -\frac{1}{\omega_{n-1}} \int_{SM} \operatorname{tr} U^2 \, d\mu \le 0.$$

Furthermore, if $\int_M S \, dm = 0$, then U^2 is identically zero and so $R = 0$ on SM.

To prove (4.1), let $r^+ = (\operatorname{tr} R)^+$ and $r^- = (\operatorname{tr} R)^-$ be the positive and the negative parts of $\operatorname{tr} R$, respectively. If $\int_{SM} r^- \, d\mu = +\infty$, then $\int_{SM} \operatorname{tr} R \, d\mu = -\infty$, because in this case $\int_{SM} r^+ \, d\mu < \infty$. So (4.1) holds. Hence we may suppose $\int_{SM} r^- \, d\mu < \infty$. Let $\varepsilon > 0$ and let f be an integrable function on SM such that

$$0 \le f \le (1 - \varepsilon) \operatorname{tr} U^2 + r^+.$$

Thus it suffices to show that

$$\int_{SM} f \, d\mu \le \int_{SM} r^- \, d\mu. \tag{4.2}$$

Indeed, from this, we have

$$\int_{SM} ((1 - \varepsilon) \operatorname{tr} U^2 + r^+) \, d\mu \le \int_{SM} r^- \, d\mu,$$

and (4.1) by letting $\varepsilon \to 0$.

To prove (4.2) we will verify that the integral of $(f - r^-)$ is nonpositive in each one of the sets D, C^+, and $C^- \backslash C^+$. Note that SM is the disjoint union $SM = D \cup C^+ \cup (C^- \backslash C^+)$. From inequality (3.2) it follows that

$$u(s) - u(t) + \frac{\varepsilon}{n-1} \int_t^s u^2(\lambda) \, d\lambda + \int_t^s (f - r^-)(T_\lambda(v)) \, d\lambda \le 0 \tag{4.3}$$

for every $v \in SM$ and $s, t \in \mathbf{R}$. By Proposition 3.3(ii) we obtain immediately

$$\int_{-\infty}^{\infty} (f - r^-)(T_t(v)) \, dt \le 0$$

for almost all $v \in D$. So, by Proposition 2.3 we conclude that

$$\int_D (f - r^-)\, d\mu = \int_{\pi(D)} \int_{-\infty}^{\infty} (f - r^-)(T_t(v))\, dt\, d\tilde{\mu} \le 0.$$

In order to compute $\int_{C^+}(f - r^-)\, d\mu$, let $\delta > 0$ and let $f_0 > 0$ be integrable on SM. Define $g = f - r^- - \delta f_0$. From (4.3) we have

$$\delta \int_0^s f_0(T_t(v))\, dt - u(0) + u(s) + \frac{\varepsilon}{n-1} \int_0^s u^2(t)\, dt + \int_0^s g(T_t(v))\, dt \le 0$$

for every $v \in SM$ and $s \in \mathbf{R}$. If $v \in C^+$, using Proposition 3.3(i) we conclude that $\liminf_{s\to\infty} \int_0^s g(T_t(v))\, dt = -\infty$. Then by the Maximal Ergodic Theorem 1.1 we obtain

$$\int_{C^+} (f - r^- - \delta f_0)\, d\mu = \int_{C^+} g\, d\mu \le 0$$

and therefore

$$\int_{C^+} (f - r^-)\, d\mu \le 0,$$

by letting $\delta \to 0$. In order to compute $\int_{C^+\setminus C^-}(f - r^-)\, d\mu$ we derive from (4.3) that

$$\delta \int_{-s}^0 f_0(T_t(v))\, dt + u(0) - u(-s) + \frac{\varepsilon}{n-1} \int_{-s}^0 u^2(t)\, dt + \int_{-s}^0 g(T_t(v))\, dt \le 0$$

for every $v \in SM$ and $s \in R$. Hence

$$\delta \int_0^s f_0(T_{-t}(v))\, dt + u(0) - u(-s) + \frac{\varepsilon}{n-1} \int_0^s u^2(-t)\, dt + \int_0^s g(T_{-t}(v))\, dt \le 0. \tag{4.4}$$

If $v \in C^-$, then $\liminf_{s\to+\infty} \int_0^s g(T_{-t}(v))\, dt = -\infty$ by (4.4) and Proposition 3.3(i). Thus using the Maximal Ergodic Theorem 1.1 we obtain

$$\int_{C^-\setminus C^+} (f - r^- - \delta f_0)\, d\mu = \int_{C^-\setminus C^+} g\, d\mu \le 0.$$

Hence letting $\delta \to 0$ leads to

$$\int_{C^-\setminus C^+} (f - r^-)\, d\mu \le 0,$$

which completes the proof of Theorem A.

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