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The Interaction between Quasilinear Elastodynamics and the Navier-Stokes Equations

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Abstract

The interaction between a viscous fluid and an elastic solid is modeled by a system of parabolic and hyperbolic equations, coupled to one another along the moving material interface through the continuity of the velocity and traction vectors. We prove the existence and uniqueness (locally in time) of strong solutions in Sobolev spaces for quasilinear elastodynamics coupled to the incompressible Navier-Stokes equations. Unlike our approach in [5] for the case of linear elastodynamics, we cannot employ a fixed-point argument on the nonlinear system itself, and are instead forced to regularize it by a *particular* parabolic artificial viscosity term. We proceed to show that with this specific regularization, we obtain a time interval of existence which is independent of the artificial viscosity; together with a priori estimates, we identify the global solution (in both phases), as well as the interface motion, as a weak limit in strong norms of our sequence of regularized problems.

1. Introduction

We establish the existence and uniqueness in Sobolev spaces of strong solutions of the unsteady fluid-structure interaction problem consisting of a nonlinear large-displacement elastic solid coupled to a viscous incompressible Newtonian fluid. The fluid motion is governed by the incompressible Navier-Stokes equations, while the solid, which can be either compressible or incompressible, is modeled by the celebrated St. Venant-Kirchhoff constitutive law (although our method can be applied to more general quasilinear hyperelastic models as those described in [3]).

The first fluid-solid interaction problems solved were for the case of a rigid body inside of a viscous flow in a bounded domain (see [7, 12, 4, 13]), and the case of a rigid body inside of a viscous flow in an infinite domain ([22, 20, 15]). Later, the elastic body was modeled with the restriction of either a finite number of modes

([8]) or a hyperviscous type law for the solid ([2, 10]), essentially by the same type of Eulerian global variational methods developed in [7]. For the steady-state problem, which is elliptic in both phases, [11] solved the case of a solid modeled as a St. Venant-Kirchhoff material. In [18], an Eulerian approach was used for the case in which the solid is a visco-hyperelastic material, which is a regularization of a hyperbolic model of solid deformation.

With the exception of our recent well-posedness result for the case of a linear elastic solid in [5], there are no known existence results for fluid-structure interaction when the solid is modeled by a standard second-order hyperbolic equation. This may be attributed to the difficulties associated with coupling a parabolic PDE for the fluid with a hyperbolic PDE for the solid through the continuity of the velocity and traction vectors across the moving material interface. As we explained in [5], an iteration scheme between fluid and solid phases fails to converge due to a regularity loss induced by the hyperbolic phase (this divergent behavior has been computationally noted as well in [14]), and so we developed a method comprised of the following new ideas: first, a functional framework which scales in a *hyperbolic* fashion for *both* the fluid and solid phases. This scaling leads to additional compatibility conditions in the fluid phase (when compared to the use of the classical parabolic framework), and is absolutely crucial for obtaining consistent energy estimates. Second, we developed a regularity theory founded upon central *trace* estimates for the velocity vector restricted to the interface, rather than traditional interior regularity arguments which do not work for our problem. Third, we were forced to bypass the use of the frozen (or constant) coefficient basic linear problem, which requires estimates on one more time derivative of the pressure function than the initial data allows, and created a new method wherein the solution was found as a limit of a sequence of penalized problems set in the Lagrangian framework. The penalization scheme approximates the divergence-free constraint, whereas the Lagrangian framework alleviates the difficulties associated with the lack of *a priori* estimates in the solid phase for the frozen coefficient problem; this method indeed differs significantly from the classical methods used in fluid-fluid interface problems (see for instance [21, 1]).

The fundamental difficulty in extending our result to the case of nonlinear elasticity is the absence of any method of analysis for quasilinear elastodynamics which is compatible with the general scheme of [5], involving a global Lagrangian variational formulation and the use of difference quotients to track the regularity of interface data. We remind the reader that unlike the analysis of elastostatic motion, direct inverse function theorem arguments cannot be applied directly to the case of quasilinear elastodynamics due to the fact that the perturbation term arising from the nonlinear operator is not an element of the appropriate function space for optimal regularity. Alternatively, it is possible to attempt a fixed-point approach, wherein a portion of the nonlinear elasticity operator is viewed as a forcing function coming from a given velocity v, and then try to solve a linear problem for an unknown w. The difficulty in this approach stems from the fact that we need to find exact time derivatives of elastic energies for the forcing term associated with the elasticity operator, which is complicated by the inner-product of a term involving $\int_0^t v$ and a term involving w. This difficulty is overcome in [6], by a clever and essential use of the Dirichlet boundary condition in order to reformulate the problem in a non-standard way. As it turns out, the various known methods that have been used in the well-studied area of quasilinear elasticity, such as those in [6] and [17] for the Dirichlet boundary condition, or those in [19] and [9] for the Neumann boundary condition, require *a priori* knowledge of the boundary data regularity, and are hence intrinsically incompatible with fluid-structure interaction analysis (in fact, the methods devised for Dirichlet conditions do not work for Neumann conditions and vice versa). Indeed, of these various methodologies, only [6] and [17] use a variational approach; the others employ either semi-group techniques as in the early work of [16] in the full space, or technical paradifferential calculus as in [19] for the two-dimensional Neumann case.

In this paper, we develop a new method for quasilinear elastodynamics, variational in nature, which is compatible with our method in [5]. We proceed in two steps. First, we add a *specific* artificial viscosity to the solid phase which regularizes the system, thus converting our hyperbolic PDE into a parabolic one and transforming the fluid-structure interaction into a fluid-fluid interface-type problem for which existence and uniqueness of solutions is already known on a time interval that a priori shrinks to zero as the artificial viscosity κ tends to zero. Second, and this is where the primary difficulty rests, we prove that our *specific* choice of parabolic smoothing renders the time interval (on which a unique solution exists) independent of κ; furthermore, our *a priori* estimates allow us to construct a solution by weak convergence in strong norms. We note that the use of higher-order operators in the artificial viscosity term, while providing the necessary *a priori* control of the regularity of the moving interface, would not yield the κ-independent estimates which are essential here. Also, as our parabolic regularization method is not specialized to any particular boundary condition, it thus provides a *unified* approach to the classical problem of quasilinear elastodynamics when the solid is not coupled to a fluid.

We now proceed to the formulation of our problem. Let $\Omega \subset \mathbb{R}^3$ denote an open, bounded, connected and smooth domain with smooth boundary $\partial\Omega$ which represents the fluid container in which both the solid and fluid move. Let $\overline{\Omega^s(t)} \subset \Omega$ denote the closure of an open and bounded subset representing the solid body at each instant of time $t \in [0, T]$ with $\Omega^f(t) := \Omega/\overline{\Omega^s(t)}$ denoting the fluid domain at each $t \in [0, T]$. Note that in our analysis $\Omega^s(t)$ is not necessarily connected, which allows us to handle the case of several elastic bodies moving in the fluid.

Remark 1. If a function u is defined on all of Ω, we will denote $u^f = u\, 1_{\overline{\Omega_0^f}}$ and $u^s = u\, 1_{\overline{\Omega_0^s}}$. This allows us to indicate from which phase the traces on

$$\Gamma(0) := \overline{\Omega^f(0)} \cap \overline{\Omega^s(0)}$$

of various discontinuous terms arise, and also to specify functions that are associated with the fluid and solid phases.

For each $t \in (0, T]$, we wish to find the location of these domains inside Ω, the divergence-free velocity field $u^f(t,\cdot)$ of the fluid, the fluid pressure function $p(t,\cdot)$ on $\Omega^f(t)$, the fluid Lagrangian volume-preserving configuration $\eta^f(t,\cdot)$:

$\Omega^f(0) = \Omega_0^f \to \Omega^f(t)$, and the elastic Lagrangian configuration field $\eta^s(t,\cdot)$: $\Omega^s(0) = \Omega_0^s \to \Omega^s(t)$ such that $\Omega = \eta^s(t, \overline{\Omega_0^s}) \cup \eta^f(t, \Omega_0^f)$, where $\eta_t^f(t,x) = u^f(t, \eta^f(t,x))$, u^f solves the Navier-Stokes equations in $\Omega^f(t)$:

$$u_t^f + (u^f \cdot \nabla)u^f = \operatorname{div} T^f + f_f\,,$$
$$\operatorname{div} u^f = 0\,,$$

with

$$T^f = \nu \ \operatorname{Def} u^f - p\,\mathrm{I}\,, \tag{1}$$

and η^s solves the elasticity equations on $\Omega^s(0)$

$$\ddot{\eta}^s = \operatorname{div} T^s + f_s,$$

with $T^s = \frac{\lambda}{2}\,\mathrm{Tr}((\nabla\eta^s)^T\nabla\eta^s - I)\ I\ + \mu\ ((\nabla\eta^s)^T\nabla\eta^s -\ I)$, and where the equations are coupled together by the continuity of the normal component of stress along the material interface $\Gamma(t) := \overline{\Omega^s}(t) \cap \overline{\Omega^f}(t)$ expressed in the Lagrangian representation on $\Gamma_0 := \Gamma(0)$ as

$$T^s N = \Big[T^f \circ \eta^f\Big]\Big[(\nabla\eta^f)^{-1}N\Big],$$

and the continuity of particle displacement fields along Γ_0

$$\eta^f = \eta^s\,,$$

together with the initial conditions $u(0,x) = u_0(x)$, $\eta(0,x) = x$ and the Dirichlet (no-slip) condition on the boundary $\partial\Omega$ of the container $u^f = 0$, where $\nu > 0$ is the kinematic viscosity of the fluid, $\lambda > 0$ and $\mu > 0$ denote the Lamé constants of the elastic material, N is the outward unit normal to Γ_0 and Def u is twice the rate of the deformation tensor of u, given in coordinates by $u^i{}_{,j} + u^j{}_{,i}$. Note that Latin indices run through 1, 2, 3, the Einstein summation convention is employed, and indices after commas denote partial derivatives.

We now briefly outline the proof. As the solid and fluid phases are naturally expressed in the Lagrangian and Eulerian framework, respectively, we begin by transforming the fluid phase into Lagrangian coordinates, which leads us to the system of equations (4) and, as in [5], we work in an hyperbolic framework in order to accomodate the dual nature of the problem (parabolic in the fluid and hyperbolic in the solid).

In order to solve (4), in Section 7, we first add a *particular* form of artificial viscosity to the quasilinear hyperbolic equation in the solid, transforming the hyperbolic phase into a parabolic one; specifically, we add the term $-\kappa L(\eta_t)$, where L denotes the linearized (about the identity) elasticity operator and η_t is the material velocity. We hence obtain an interface problem that is *parabolic* in nature in both phases, and can be thought of as a fluid-fluid parabolic interface problem for which well-posedness is classical (note that both phases are required to scale in an hyperbolic fashion). The time interval of existence $[0, T_\kappa]$ for this parabolic system a priori *shrinks* to zero as $\kappa \to 0$.

In Section 8, we establish κ-independent estimates on the solutions v_κ of the regularized parabolic problem on the time interval $[0, T_\kappa]$ by identifying exact time derivatives of elastic energies, and establish regularity of the *interface*. A direct fixed-point approach for (4) does not appear to yield these exact time derivatives for the elastic energy, whereas the regularized problem (14) does indeed lead to them. An essential *key* for obtaining estimates independent of κ inside the solid is Lemma 1. Whereas the trace estimates could be carried with other choices of artificial viscosity, we absolutely need the special choice made in our analysis in order to recover the regularity inside the solid independently of κ. In particular, a different choice of a regularizing operator either of the same order such as $-\Delta\eta_t$ or of higher order such as $L^2\eta$ or $L^2\eta_t$ would not provide κ-independent estimates.

In Section 9, we then explain how our estimates allow the construction of solutions v_κ on a time interval independent of κ, still with energy estimates independent of κ. The existence of a solution of (4) then follows by weak convergence as $\kappa \to 0$.

Uniqueness is established in Section 11 in the same functional framework used for existence.

As our method seemingly requires more regularity on the initial data in the solid than it should, due to the artificial viscosity in the compatibility conditions, we explain in Section 12 how this extra regularity can be removed, thus leading to the result with optimal regularity.

Section 13 is dedicated to the case where the incompressibility constraint is added to the solid. The additional difficulty with respect to the compressible case comes from the fact that we control the velocity uniformly in κ in function spaces which possess less regularity than in the fluid, whereas the pressure is controlled uniformly in the same regularity spaces in both phases. Also, we cannot use Lemma 1 in the most optimal form for the regularity of the pressure in the solid phase.

2. Notational simplification

Although a fluid with a Neumann (free-slip) boundary condition indeed obeys the constitutive law (1), we will replace for notational convenience (1) with

$$T^f = \nu \nabla u^f - p\mathrm{I}; \tag{2}$$

this amounts to replacing the energy $\int_{\Omega_0^f} \mathrm{Def}\, u^f : \mathrm{Def}\, v$ by $\int_{\Omega_0^f} \nabla u^f : \nabla v$, which is not a problem mathematically due to the well-known Korn inequality. Henceforth, we shall take (2) as the fluid constitutive law.

3. Lagrangian formulation of the problem

With regards to the forcing functions, we shall use the convention of denoting both the fluid forcing f_f and the solid forcing f_s by the same letter f. Since f_f has to be defined in Ω (because of the composition with η), and f_s must be defined in Ω_0^s, we will assume that the forcing f is defined over the entire domain Ω.

Let

$$a(x) = [\mathrm{Cof}\nabla\eta^f(x)]^T, \tag{3}$$

where $(\nabla\eta^f(x))^i_j = \partial(\eta^f)^i/\partial x^j(x)$ denotes the matrix of partial derivatives of η^f. Clearly, the matrix a depends on η and we shall sometimes use the notation $a^i_j(\eta)$ to denote formula (3).

Let $v = u \circ \eta$ denote the Lagrangian or material velocity field, $q = p \circ \eta$ is the Lagrangian pressure function (in the fluid), and $F = f^f \circ \eta^f$ is the fluid forcing function in the material frame. Then, as long as no collisions occur between the solids (if there are initially more than one) or between a solid and $\partial\Omega$, the problem can be reformulated as

$$\eta_t = v \quad \text{in } (0,T)\times\Omega, \tag{4a}$$
$$v^i_t - \nu(a^j_l a^k_l v^i,_k),_j + (a^k_i q),_k = F^i \quad \text{in } (0,T)\times\Omega^f_0, \tag{4b}$$
$$a^k_i v^i,_k = 0 \quad \text{in } (0,T)\times\Omega^f_0, \tag{4c}$$
$$v_t - c^{mjkl}[(\eta,_m\cdot\eta,_j - \delta_{mj})\eta,_k],_l = f \quad \text{in } (0,T)\times\Omega^s_0, \tag{4d}$$
$$c^{mjkl}\,(\eta,_m\cdot\eta,_j - \delta_{mj})\eta^i,_k\, N_l = \nu\, v^i,_k\, a^k_l a^j_l N_j - q a^j_i N_j \quad \text{on } (0,T)\times\Gamma_0, \tag{4e}$$
$$v(t,\cdot) \in H^1_0(\Omega;\mathbb{R}^3) \quad \text{a.e. in } (0,T), \tag{4f}$$
$$v = u_0 \quad \text{on } \Omega\times\{t=0\}, \tag{4g}$$
$$\eta = \mathrm{Id} \quad \text{on } \Omega\times\{t=0\}, \tag{4h}$$

where N denotes the outward-pointing unit normal to Γ_0 (pointing into the solid phase), and

$$c^{ijkl} = \lambda\delta^{ij}\delta^{kl} + \mu(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}).$$

Throughout the paper, all Greek indices run through 1, 2 and all Latin indices run through 1, 2, 3. Note that the continuity of the velocity along the interface is satisfied in the sense of traces on Γ_0 by condition (4f), whereas the continuity of the normal stress along the interface is represented by (4e).

4. Notation and conventions

We begin by specifying our notation for certain vector and matrix operations.

- We write the Euclidean inner-product between two vectors x and y as $x\cdot y$, so that $x\cdot y = x^i\, y^i$.
- The transpose of a matrix A will be denoted by A^T, i.e., $(A^T)^i_j = A^j_i$.
- We write the product of a matrix A and a vector b as $A\,b$, i.e., $(A\,b)^i = A^i_j b^j$.
- The product of two matrices A and S will be denoted by $A\cdot S$, i.e., $(A\cdot S)^i_j = A^i_k\, S^k_j$.

For $T > 0$ and $k \in \mathbb{N}$, we set

$$V_f^k(T) = \{w \in L^2(0,T; H^k(\Omega_0^f; \mathbb{R}^3)) \mid \partial_t^n w \in L^2(0,T; H^{k-n}(\Omega_0^f; \mathbb{R}^3)),\\ n = 1, ..., k-1\},$$

where $V_s^k(T)$ is defined with Ω_0^s replacing Ω_0^f.

In order to specify the initial data for the weak formulation, we introduce the space

$$L^2_{div,f} = \{\psi \in L^2(\Omega; \mathbb{R}^3) \mid \operatorname{div}\psi = 0 \text{ in } \Omega_0^f, \quad \psi \cdot N = 0 \text{ on } \partial\Omega\},$$

which is endowed with the $L^2(\Omega; \mathbb{R}^3)$ scalar product.

The space of velocities, X_T, where the solution of (4) exists, is defined as the following separable Hilbert space:

$$X_T = \left\{ v \in L^2(0,T; H_0^1(\Omega; \mathbb{R}^3)) \mid \left(v^f, \int_0^\cdot v^s \right) \in V_f^4(T) \times V_s^4(T) \right\}, \tag{5}$$

endowed with its natural Hilbert norm

$$\|v\|_{X_T}^2 = \|v\|^2_{L^2(0,T;H_0^1(\Omega;\mathbb{R}^3))} + \sum_{n=0}^{3} \left[\|\partial_t^n v\|^2_{L^2(0,T;H^{4-n}(\Omega_0^f;\mathbb{R}^3))} + \left\| \partial_t^n \int_0^\cdot v \right\|^2_{L^2(0,T;H^{4-n}(\Omega_0^s;\mathbb{R}^3))} \right].$$

We also need the space

$$Y_T = \{(v,q) \in X_T \times L^2(0,T; H^3(\Omega_0^f; \mathbb{R}))|\\ \partial_t^n q \in L^2(0,T; H^{3-n}(\Omega_0^f; \mathbb{R}))(n = 1, 2)\},$$

endowed with its natural Hilbert norm

$$\|(v,q)\|_{Y_T}^2 = \|v\|_{X_T}^2 + \sum_{n=0}^{2} \|\partial_t^n q\|^2_{L^2(0,T;H^{3-n}(\Omega_0^f;\mathbb{R}))}.$$

We shall also need L^∞-in-time control of certain norms of the velocity, which necessitates the use of the following closed subspace of X_T:

$$W_T = \{ v \in X_T |\ v_{ttt} \in L^\infty(0,T; L^2(\Omega; \mathbb{R}^3)),\\ \partial_t^n \int_0^\cdot v \in L^\infty(0,T; H^{4-n}(\Omega_0^s; \mathbb{R}^3))(n = 0, 1, 2, 3)\},$$

endowed with the following norm

$$\|v\|_{W_T}^2 = \|v\|_{X_T}^2 + \|v_{ttt}\|^2_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))} + \sum_{n=0}^{3} \|\partial_t^n \int_0^\cdot v\|^2_{L^\infty(0,T;H^{4-n}(\Omega_0^s;\mathbb{R}^3))}.$$

Finally, we will also make use of the space

$$Z_T = \{(v,q) \in W_T \times L^2(0,T;\,H^3(\Omega_0^f;\mathbb{R}))|\\ \partial_t^n q \in L^2(0,T;\,H^{3-n}(\Omega_0^f;\mathbb{R})),\ (n=1,2)|\ q_{tt} \in L^\infty(0,T;\,L^2(\Omega_0^f;\mathbb{R}))\},$$

endowed with its natural norm

$$\|(v,q)\|^2_{Z_T} = \|v\|^2_{W_T} + \sum_{n=0}^{2} \|\partial_t^n q\|^2_{L^2(0,T;H^{3-n}(\Omega_0^f;\mathbb{R}))} + \|\partial_t^2 q\|^2_{L^\infty(0,T;L^2(\Omega_0^f;\mathbb{R}))}\,.$$

Remark 2. Note that our functional framework does not make use of the third-time derivative of the pressure q_{ttt}, even though we do use the third-time derivative of velocity w_{ttt}; this functional framework is necessitated by the fact that the Dirichlet boundary condition together with the limited regularity of w_{ttt} does not allow us to obtain q_{ttt} with the appropriate regularity. Note also that we have added the L^∞-in-time control of q_{tt} in the definition of Z_T mostly for a more convenient way to prove our theorems, rather than out of absolute necessity.

Throughout the paper, we shall use C to denote a generic constant, which may possibly depend on the coefficients ν, λ, μ, or on the initial geometry given by Ω and Ω_0^f (such as a Sobolev constant or an elliptic constant). For the sake of notational convenience, we will also write $u(t)$ for $u(t,\cdot)$.

5. The first theorem

We now state our first theorem. We impose greater regularity requirements on the initial data than is optimal so as to avoid technical difficulties associated with a particular type of initial data regularization that would otherwise be necessitated. We consider the case of optimal regularity on the initial data in Theorem 2.

Theorem 1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class H^4, and let Ω_0^s be an open set (with a finite number $\geqq 1$ of connected components) of class H^4 such that $\overline{\Omega_0^s} \subset \Omega$ and such that the distance between two distinct connected components of Ω_0^s (if there are multiple solid components) is greater than zero. Denote $\Omega_0^f = \Omega \cap (\overline{\Omega_0^s})^c$ and let $\nu > 0$, $\lambda > 0$, $\mu > 0$ be given. Let*

$$(f, f_t, f_{tt}, f_{ttt}) \in L^2(0,\bar{T};\,H^3(\Omega;\mathbb{R}^3) \times H^2(\Omega;\mathbb{R}^3) \times H^1(\Omega;\mathbb{R}^3)\\ \times L^2(\Omega;\mathbb{R}^3)), \tag{6a}$$

$$f(0) \in H^4(\Omega;\mathbb{R}^3)\,,\ \ f_t(0) \in H^4(\Omega;\mathbb{R}^3). \tag{6b}$$

Assume that the initial data satisfies

$$u_0 \in H^6(\Omega_0^f;\mathbb{R}^3) \cap H^6(\Omega_0^s;\mathbb{R}^3) \cap H^1_0(\Omega;\mathbb{R}^3) \cap L^2_{div,f},$$

as well as the compatibility conditions

$$[\nabla u_0^f\ N]_{\tan} = 0 \text{ on } \Gamma_0, \quad w_1 = 0 = w_2 \text{ on } \partial\Omega, \quad \nu\triangle u_0^f - \nabla q_0 = 0 \text{ on } \Gamma_0, \tag{7a}$$

$$[(\nu[(a_l^k a_l^j) w^f,_k^i]_t(0)N_j)_{i=1}^3]_{\tan} - [(q_0\, a_i^j{}_t(0)N_j)_{i=1}^3]_{\tan}$$
$$= [c^{mjkl}\big[(\eta^s,_m \cdot \eta^s,_j - \delta_{mj})\eta^s,_k\big]_t(0)N_l]_{\tan} \text{ on } \Gamma_0, \tag{7b}$$

$$[(\nu[(a_l^k a_l^j) w^f,_k^i]_{tt}(0)N_j)_{i=1}^3]_{\tan} - [(2q_1\, a_i^j{}_t(0)N_j + q_0\, a_i^j{}_{tt}(0)N_j)_{i=1}^3]_{\tan}$$
$$= [c^{mjkl}\big[(\eta^s,_m \cdot \eta^s,_j - \delta_{mj})\eta^s,_k\big]_{tt}(0)N_l]_{\tan} \text{ on } \Gamma_0, \tag{7c}$$

$$\nu\triangle w_1^f + \nu((a_l^j a_l^k)_t(0)u_0^f,_k),_j + F_t(0) - [((a_i^j)_t(0)q_0),_j + q_1,_i]_{i=1}^3$$
$$= f_t(0) + c^{mjkl}\big[[(\eta^s,_m \cdot \eta^s,_j - \delta_{mj})\eta^s,_k],_l\big]_t(0) \text{ on } \Gamma_0\ , \tag{7d}$$

where the time derivatives appearing in these equations and in the following ones are computed from any w satisfying $w(0) = u_0$, $\partial_t^n w(0) = w_n$ ($n = 1, 2$), and from any q satisfying $\partial_t^n q(0) = q_n$ ($n = 0, 1, 2$), where the quantities w_n and q_n are defined in the following way. First, $q_0 \in H^3(\Omega_0^f; \mathbb{R})$ is defined by

$$\triangle q_0 = \operatorname{div} f(0) + (a_i^j)_t(0)u_0^i,_j \quad \text{in } \Omega_0^f, \tag{8a}$$

$$q_0 = \nu[\nabla u_0^f\ N]\cdot N \quad \text{on } \Gamma_0, \tag{8b}$$

$$\frac{\partial q_0}{\partial N} = f(0)\cdot N + \nu\triangle u_0^f \cdot N \quad \text{on } \partial\Omega, \tag{8c}$$

and $w_1 \in H_0^1(\Omega; \mathbb{R}^3) \cap H^4(\Omega_0^s; \mathbb{R}^3) \cap H^4(\Omega_0^f; \mathbb{R}^3)$ by

$$w_1 = \nu\triangle u_0 - \nabla q_0 + f(0) \quad \text{in } \Omega_0^f, \tag{9a}$$

$$w_1 = f(0) \quad \text{in } \Omega_0^s\ . \tag{9b}$$

Note that $w_1 \in H^4(\Omega_0^f; \mathbb{R}^3)$ since $\triangle w_1 \in H^2(\Omega_0^f; \mathbb{R}^3)$ and $w_1 = 0$ on $\partial\Omega$, $w_1 = f(0)$ on Γ_0. We also have $q_1 \in H^3(\Omega_0^f; \mathbb{R})$ defined by

$$\triangle q_1 = \operatorname{div}[\nu\triangle w_1 + F_t(0) + [\nu((a_l^j a_l^k)_t(0)u_0^i,_k),_j - ((a_i^j)_t(0)q_0),_j]_{i=1}^3]$$
$$+2(a_i^j)_t(0)w_1^i,_j + (a_i^j)_{tt}(0)u_0^i,_j \text{ in } \Omega_0^f\ , \tag{10a}$$

$$q_1 = \nu[\nabla w_1^f\ N\cdot N + (a_l^k a_l^j)_t(0)u_0^{f\,i},_k N_j N_i] - q_0\, a_i^j{}_t(0)N_j N_i$$
$$-c^{mjkl}\big[(\eta^s,_m \cdot \eta^s,_j - \delta_{mj})\eta^s,_k\big]_t(0)N_l \cdot N \text{ on } \Gamma_0, \tag{10b}$$

$$\frac{\partial q_1}{\partial N} = F_t(0)\cdot N - [(a_i^j)_t(0)q_0],_j N_i + \nu\triangle w_1\cdot N + \nu((a_l^j a_l^k)_t(0)u_0^i,_k),_j N_i$$
$$\text{on } \partial\Omega\ , \tag{10c}$$

and $w_2 \in H_0^1(\Omega; \mathbb{R}^3) \cap H^4(\Omega_0^s; \mathbb{R}^3) \cap H^2(\Omega_0^f; \mathbb{R}^3)$ by

$$w_2^i = \nu\triangle w_1^i + \nu((a_l^j a_l^k)_t(0)u_0^i,_k),_j + F_t^i(0) - ((a_i^j)_t(0)q_0),_j - q_1,_i$$
$$\text{in } \Omega_0^f, \tag{11a}$$

$$w_2 = f_t(0) + c^{mjkl}\big[[(\eta,_m \cdot \eta,_j - \delta_{mj})\eta,_k],_l\big]_t(0)$$
$$\text{in } \Omega_0^s. \tag{11b}$$

Finally, $q_2 \in H^1(\Omega_0^f;\mathbb{R})$ *is defined by*

$$\triangle q_2 = \mathrm{div}\Big[(f\circ\eta)_{tt}(0) + \nu\,[(a_l^j a_l^k w,_k\,,_j\,]_{tt}(0) - [((a_i^j)_{tt}(0)q_0 + 2(a_i^j)_t(0)q_1),_j]_{i=1}^3\Big] + 3(a_i^j)_t(0)w_2^i,_j + 3(a_i^j)_{tt}(0)w_1^i,_j + (a_i^j)_{ttt}(0)u_0^i,_j \ \ in\ \Omega_0^f\ , \tag{12a}$$

$$q_2 = \nu\left[(a_l^k a_l^j)w^{f\,i},_k\right]_{tt}(0)N_jN_i - c^{mjkl}\left[(\eta^s,_m\cdot\eta^s,_j - \delta_{mj})\eta^s,_k\right]_{tt}(0)N_l\cdot N - q_0\, a^j_{i\ tt}(0)N_jN_i - 2q_1\, a^j_{i\ t}(0)N_jN_i \ \ on\ \Gamma_0, \tag{12b}$$

$$\frac{\partial q_2}{\partial N} = (f\circ\eta)_{tt}(0)\cdot N - 2\left[(a_i^j)_t(0)q_1\right],_j N_i - \left[(a_i^j)_{tt}(0)q_0\right],_j N_i + \nu\triangle w_2\cdot N + 2\nu((a_l^j a_l^k)_t(0)w_1^i,_k),_j N_i + \nu((a_l^j a_l^k)_{tt}(0)u_0^i,_k),_j N_i \ \ on\ \partial\Omega\ . \tag{12c}$$

Then there exists $T\in(0,\bar T)$ *depending on* u_0*,* f*, and* Ω_0^f*, such that there exists a unique solution* $(v,q)\in Z_T$ *of problem* (4)*. Furthermore,* $\eta\in C^0([0,T];H^4(\Omega_0^f;\mathbb{R}^3)\cap H^4(\Omega_0^s;\mathbb{R}^3)\cap H^1(\Omega;\mathbb{R}^3))$.

Remark 3. The remarks appearing in [5] at the end of Section 5 concerning the compatibility conditions and forcing functions for the linear elasticity case still hold in this setting with the necessary adjustments. In particular, we do not need the forcing functions to have the same regularity in both phases.

6. Preliminary result

In the remainder of the paper, we set

$$L(u)^i = [c^{ijkl}(u^k,_l + u^l,_k)],_j\ .$$

In our limit process as the artificial viscosity tends to zero, we will make use in a crucial way of the basic following result:

Lemma 1. *Let* $g\in C^0([0,T];L^2(\Omega_0^s;\mathbb{R}^3))$ *and* u *be such that* $u_t\in L^2(0,T;H^2(\Omega_0^s;\mathbb{R}^3))$ *and*

$$\varepsilon L(u_t) + L(u) = g\ on\ [0,T]\times\Omega_0^s. \tag{13}$$

Then, independently of $\varepsilon>0$*,*

$$\|L(u)\|_{L^\infty(0,T;L^2(\Omega_0^s;\mathbb{R}^3))} \leqq \|g\|_{L^\infty(0,T;L^2(\Omega_0^s;\mathbb{R}^3))} + \|L(u_0)\|_{L^2(\Omega_0^s;\mathbb{R}^3)}.$$

Proof. Since $L(u)\in C^0(0,T;L^2(\Omega_0^s;\mathbb{R}^3))$, let $T'\in[0,T]$ be such that

$$\|L(u(T'))\|_{L^2(\Omega_0^s;\mathbb{R}^3)} = \sup_{[0,T]}\|L(u)\|_{L^2(\Omega_0^s;\mathbb{R}^3)}.$$

If $T'=0$, then the statement of the Lemma is satisfied. Now, let us assume that $T'\in(0,T]$. Let $\delta\in(0,T')$ be arbitrary. From (13), we infer that

$$\varepsilon^2\int_{T'-\delta}^{T'}\|L(u_t)\|^2_{L^2(\Omega_0^s;\mathbb{R}^3)} + \int_{T'-\delta}^{T'}\|L(u)\|^2_{L^2(\Omega_0^s;\mathbb{R}^3)} + \varepsilon\,[\|L(u(T'))\|^2_{L^2(\Omega_0^s;\mathbb{R}^3)} - \|L(u(T'-\delta))\|^2_{L^2(\Omega_0^s;\mathbb{R}^3)}] = \int_{T'-\delta}^{T'}\|g\|^2_{L^2(\Omega_0^s;\mathbb{R}^3)}.$$

From the definition of T' we then infer that for any $\delta \in (0, T')$,

$$\int_{T'-\delta}^{T'} \|L(u)\|^2_{L^2(\Omega_0^s;\mathbb{R}^3)} \leqq \int_{T'-\delta}^{T'} \|g\|^2_{L^2(\Omega_0^s;\mathbb{R}^3)},$$

which after division by δ gives at the limit $\delta \to 0$:

$$\|L(u(T'))\|^2_{L^2(\Omega_0^s;\mathbb{R}^3)} \leqq \|g(T')\|^2_{L^2(\Omega_0^s;\mathbb{R}^3)},$$

which concludes the proof of the Lemma. □

Remark 4. It should be clear that Lemma 1 applies to a more general class of linear operators than L.

7. The smoothed problem and its basic linear problem

As we described in the introduction, we cannot find an appropriate linear problem whose fixed-point provides a solution of (4). We are thus lead to introduce the following (parabolic) regularization of (4), with the artificial viscosity $\kappa > 0$:

$$\begin{aligned}
v^i_t - \nu(a^j_l a^k_l v^i,_k),_j + (a^k_i q),_k &= f^i \circ \eta && \text{in } (0,T)\times\Omega_0^f, && \text{(14a)}\\
a^k_i v^i,_k &= 0 && \text{in } (0,T)\times\Omega_0^f, && \text{(14b)}\\
v^i_t - \kappa[c^{ijkl} v^k,_l],_j + N(\eta)^i &= f^i + \kappa h^i && \text{in } (0,T)\times\Omega_0^s, && \text{(14c)}\\
\kappa\, c^{ijkl} v^k,_l\ N_j + G(\eta)^i &= \nu\, v^i,_k a^k_l a^j_l N_j && && \\
&\quad -q a^j_i N_j + \kappa g^i && \text{on } (0,T)\times\Gamma_0, && \text{(14d)}\\
v(t,\cdot) &\in H^1_0(\Omega;\mathbb{R}^3) && \text{a.e. in } (0,T), && \text{(14e)}\\
v &= u_0 && \text{on } \Omega\times\{t=0\}, && \text{(14f)}
\end{aligned}$$

where

$$N(\eta) = -c^{mjkl}[(\eta,_m \cdot \eta,_j - \delta_{mj})\eta^i,_k],_l \text{ in } \Omega_0^s, \tag{15a}$$

$$G(\eta) = c^{mjkl}[(\eta,_m \cdot \eta,_j - \delta_{mj})\eta^i,_k] N_l \text{ on } \Gamma_0, \tag{15b}$$

and

$$h^i(t,\cdot) = -\left[c^{ijkl}\left(u_0 + t w_1 + \frac{t^2}{2} w_2\right)^k,_l\right],_j \text{ in } \Omega_0^s, \tag{16a}$$

$$g^i(t,\cdot) = \left[c^{ijkl}\left(u_0 + t w_1 + \frac{t^2}{2} w_2\right)^k,_l\right] N_j \text{ on } \Gamma_0. \tag{16b}$$

Solutions of (4) will be obtained as the limit (as $\kappa \to 0$) of solutions of (14).

Suppose that $v \in W_T$ is given. Let $\eta = Id + \int_0^\cdot v$ and let a^j_i be the quantity associated with η through (3).

We are concerned with the following time-dependent linear problem, whose fixed-point $w = v$ provides a solution of (14):

$$
\begin{aligned}
w_t^i - \nu(a_l^j a_l^k w^i,_k),_j + (a_i^k q),_k &= f^i \circ \eta && \text{in } (0,T)\times\Omega_0^f, && (17a)\\
a_i^k w^i,_k &= 0 && \text{in } (0,T)\times\Omega_0^f, && (17b)\\
w_t^i - \kappa[c^{ijkl} w^k,_l],_j + N(\eta)^i &= f^i + \kappa h^i && \text{in } (0,T)\times\Omega_0^s, && (17c)\\
\kappa\, c^{ijkl} w^k,_l \ N_j + G(\eta)^i &= \nu\, w^i,_k a_l^k a_l^j N_j \\
&\quad -q a_i^j N_j + \kappa g^i && \text{on } (0,T)\times\Gamma_0, && (17d)\\
w(t,\cdot) &\in H_0^1(\Omega;\mathbb{R}^3) && \text{a.e. in } (0,T), && (17e)\\
w &= u_0 && \text{on } \Omega\times\{t=0\}. && (17f)
\end{aligned}
$$

Remark 5. The two forcing functions (16a) are introduced for compatibility conditions at time $t = 0$, allowing the solution of (17) to satisfy $(w_t(0), w_{tt}(0)) \in H_0^1(\Omega;\mathbb{R}^3)^2$ and even to satisfy the same initial conditions as solutions of (4) would.

In the following, for the sake of notational convenience, we will denote by $N(u_0, (w_i)_{i=1}^3)$ a generic smooth function depending only on $\sum_{i=0}^{3}[\|w_{3-i}\|_{H^i(\Omega_0^s;\mathbb{R}^3)}+\|w_{3-i}\|_{H^i(\Omega_0^f;\mathbb{R}^3)}]$ (with the convention that $u_0 = w_0$), by $N((q_i)_{i=0}^2)$ a generic smooth function depending only on $\sum_{i=0}^{2}\|q_{2-i}\|_{H^i(\Omega_0^f;\mathbb{R}^3)}$ and by $M(f,\kappa g,\kappa h)$ a generic smooth function depending only on $\|f\|_{V_f^3(T)}+\|f\|_{V_s^3(T)}+\kappa[\|u_0\|_{H^4(\Omega_0^s;\mathbb{R}^3)} +\|w_1\|_{H^4(\Omega_0^s;\mathbb{R}^3)} + \|w_2\|_{H^4(\Omega_0^s;\mathbb{R}^3)}]$. Then, let $w_3 \in L^2(\Omega;\mathbb{R}^3)$ be defined by

$$
\begin{aligned}
w_3^i &= \nu[(a_l^j a_l^k w^i,_k),_j]_{tt}(0) + (f\circ\eta)_{tt}(0) - [a_i^j q,_j]_{tt}(0) && \text{in } \Omega_0^f, && (18a)\\
w_3^i &= f_{tt}^i(0) + c^{mjkl}\big[[(\eta,_m\cdot\eta,_j - \delta_{mj})\eta,_k],_l\big]_{tt}(0) && \text{in } \Omega_0^s, && (18b)
\end{aligned}
$$

where the time derivatives are computed with any $\eta(0,x) = x$, $w = \eta_t(0) = u_0$, $\partial_t^n w(0) = w_n$, $(n = 1,2)$, $\partial_t^n q(0) = q_n$, $(n = 0,1,2)$.

Let us now define

$$b_\kappa(\phi) = \kappa(c^{ijkl} {w_2}^k,_l, \phi^i,_j)_{L^2(\Omega_0^s;\mathbb{R})}, \tag{19a}$$

$$c_\kappa(\phi) = \kappa(c^{ijkl} w_1^k,_l, \phi^i,_j)_{L^2(\Omega_0^s;\mathbb{R})}, \tag{19b}$$

$$d_\kappa(\phi) = \kappa(c^{ijkl} {u_0}^k,_l, \phi^i,_j)_{L^2(\Omega_0^s;\mathbb{R})}\,. \tag{19c}$$

By proceding as in [5], we can establish the existence of a fixed-point for system (14). This follows the lines of [5] by first approximating by a penalty scheme the divergence-free constraint in the fluid in our Lagrangian setting, and by performing a regularity analysis of the solution of (17), allowing the use of the Tychonoff fixed-point theorem. Given the estimates obtained in [5], no new difficulties arise, since the parabolic artificial viscosity in the solid controls the forcing coming from the quasilinear part on a short time which is a priori shrinking to zero, and for this reason the proof is omitted here.

This leads us to the following

Lemma 2. *There exists $T_\kappa > 0$ depending a priori on κ and on a given expression of the type $N_0(u_0,(w_i)_{i=1}^3) + N_0((q_i)_{i=0}^2) + M_0(f,\kappa g,\kappa h)$, so that there exists a unique solution $(w_\kappa, q_\kappa) \in Z_{T_\kappa}$ of the regularized problem* (14). *Moreover, $w_\kappa \in V_s^4(T_\kappa)$.*

In the next section we will study the limit of these solutions of the smoothed problems as $\kappa \to 0$; this is problematic since the solutions of these regularized problems are a priori defined on a time interval shrinking to zero as $\kappa \to 0$.

Moreover, the following variational equations (for $n = 0, 1, 2$) are satisfied for any test function $\phi \in L^2(0, T_\kappa; H_0^1(\Omega; \mathbb{R}^3))$:

$$\begin{aligned}
&\int_0^{T_\kappa} (\partial_t^n (w_\kappa)_t, \phi)_{L^2(\Omega;\mathbb{R}^3)}\, dt + \nu \int_0^{T_\kappa} (\partial_t^n (a_k^r a_k^s w_\kappa,_r),\ \phi,_s)_{L^2(\Omega_0^f;\mathbb{R}^3)}\, dt \\
&\quad + \kappa \int_0^{T_\kappa} (c^{ijkl} \partial_t^n w_\kappa{}^k,_l , \phi^i,_j)_{L^2(\Omega_0^s;\mathbb{R})}\, dt - \int_0^{T_\kappa} (\partial_t^n (a_k^l q_\kappa),\ \phi^k,_l)_{L^2(\Omega_0^f;\mathbb{R})} dt \\
&\quad + \int_0^{T_\kappa} (c^{ijkl} \partial_t^n [(\eta,_i \cdot \eta,_j - \delta_{ij})\eta,_l], \phi,_k)_{L^2(\Omega_0^s;\mathbb{R}^3)}\, dt \\
&= \int_0^{T_\kappa} (\partial_t^n F, \phi)_{L^2(\Omega_0^f;\mathbb{R}^3)} + (\partial_t^n f, \phi)_{L^2(\Omega_0^s;\mathbb{R}^3)} + \partial_t^n \left[\frac{t^2}{2}\right] b_\kappa(\phi) + \partial_t^n [t] c_\kappa(\phi) \\
&\quad + \partial_t^n [1] d_\kappa(\phi)\, dt\ ,
\end{aligned} \tag{20}$$

together with the initial conditions $w_\kappa(0) = u_0$, $(w_\kappa)_t(0) = w_1$, $(w_\kappa)_{tt}(0) = w_2$ and $q_\kappa(0) = q_0$, $(q_\kappa)_t(0) = q_1$, $(q_\kappa)_{tt}(0) = q_2$. Moreover for the third-time differentiated problem in time, we also have that a.e. in $(0, T_\kappa)$,

$$\begin{aligned}
&\frac{1}{2} \|(w_\kappa)_{ttt}(t)\|^2_{L^2(\Omega;\mathbb{R}^3)} + \nu \int_0^t ((a_k^r a_k^s w_\kappa,_r)_{ttt},\ (w_\kappa)_{ttt},_s)_{L^2(\Omega_0^f;\mathbb{R}^3)} \\
&\quad + \kappa \int_0^t (c^{ijkl} (w_\kappa)_{ttt}{}^k,_l (t), (w_\kappa)_{ttt}{}^i,_j (t))_{L^2(\Omega_0^s;\mathbb{R})} \\
&\quad - \int_0^t \int_{\Omega_0^f} (q_\kappa)_{tt}\ [3(a_i^j)_{tt} (w_\kappa)^i_t,_j + 3(a_i^j)_t (w_\kappa)^i_{tt},_j + (a_i^j)_{ttt} w_\kappa{}^i,_j]_t \\
&\quad + \int_{\Omega_0^f} (q_\kappa)_{tt}(t)\ [3(a_i^j)_{tt} (w_\kappa)^i_t,_j + 3(a_i^j)_t (w_\kappa)^i_{tt},_j + (a_i^j)_{ttt} w_\kappa{}^i,_j](t) \\
&\quad - \int_0^t \int_{\Omega_0^f} [3(a_i^j)_{tt} (q_\kappa)_t (w_\kappa)^i_{ttt},_j + 3(a_i^j)_t (q_\kappa)_{tt} (w_\kappa)^i_{ttt},_j + (a_i^j)_{ttt} q_\kappa (w_\kappa)^i_{ttt},_j] \\
&\quad + \int_0^t \int_{\Omega_0^s} c^{ijkl} [(\eta,_i \cdot \eta,_j - \delta_{ij})\eta,_l]_{ttt} \cdot (w_\kappa)_{ttt},_k \\
&\leqq N(u_0, (w_i)_{i=1}^3) \\
&\quad + N((q_i)_{i=0}^2) + \int_0^t [(F_{ttt}, (w_\kappa)_{ttt})_{L^2(\Omega_0^f;\mathbb{R}^3)} + (f_{ttt}, (w_\kappa)_{ttt})_{L^2(\Omega_0^s;\mathbb{R}^3)}],
\end{aligned} \tag{21}$$

where we recall that C does not depend on the artificial viscosity κ. The following result will be fundamental to our proof that the time interval of existence of solutions of (14) is in fact κ-independent.

Lemma 3. *The mapping* $\gamma : t \to \|(w_\kappa, q_\kappa)\|_{Z_t}$ *is continuous on* $[0, T_\kappa]$.

Proof. The continuity with respect to t of the terms of the type $L^2(0,t;H^s)$ is obvious, and since $w_\kappa \in V_s^4(T_\kappa)$ (due to our artificial viscosity), so is the continuity of $\sum_{n=0}^3 \|\partial_t^n \eta_\kappa\|^2_{L^\infty(0,t;H^{4-n}(\Omega_0^s;\mathbb{R}^3))}$. The only terms that remain are $\|\partial_t^3 w_\kappa\|_{L^\infty(0,t;L^2(\Omega;\mathbb{R}^3))}$ and $\|\partial_t^2 q_\kappa\|_{L^\infty(0,t;L^2(\Omega_0^f;\mathbb{R}))}$.

In order to treat them, we will invoke the fact that due to our artificial viscosity in the solid, we actually have $\partial_t^4 w_\kappa \in L^2(0, T_\kappa; L^2(\Omega;\mathbb{R}^3))$, which provides $\partial_t^3 w_\kappa \in \mathcal{C}^0([0,T_\kappa]; L^2(\Omega;\mathbb{R}^3))$. For the second-time derivative of the pressure, we notice that from the variational form, which is true almost everywhere on $[0, T_\kappa]$ for any $\phi \in H_0^1(\Omega;\mathbb{R}^3)$,

$$\begin{aligned}
&(\partial_t^3 w_\kappa, \phi)_{L^2(\Omega;\mathbb{R}^3)} + \nu(\partial_t^2(a_k^r a_k^s w_\kappa{,_r}),\ \phi{,_s})_{L^2(\Omega_0^f;\mathbb{R}^3)} \\
&+\kappa(c^{ijkl}\partial_t^2 w_\kappa{}^k{,_l}, \phi^i{,_j})_{L^2(\Omega_0^s;\mathbb{R})} - (\partial_t^2(a_k^l q_\kappa) - a_k^l q_{\kappa tt},\ \phi^k{,_l})_{L^2(\Omega_0^f;\mathbb{R})} \\
&+(c^{ijkl}\partial_t^2[(\eta,_i\cdot\eta,_j - \delta_{ij})\eta,_l], \phi,_k)_{L^2(\Omega_0^s;\mathbb{R}^3)} - (\partial_t^2 F, \phi)_{L^2(\Omega_0^f;\mathbb{R}^3)} \\
&-(\partial_t^2 f, \phi)_{L^2(\Omega_0^s;\mathbb{R}^3)} - b_\kappa(\phi) \\
&\quad = (a_k^l q_{\kappa tt},\ \phi^k{,_l})_{L^2(\Omega_0^f;\mathbb{R})},
\end{aligned}$$

and from the Lagrange multiplier Lemma 13 of [5] associated with the continuity results previously established, we have the continuity of $t \to \|q_{\kappa tt}\|_{L^\infty(0,t;L^2(\Omega_0^f;\mathbb{R}))}$ on $[0, T_\kappa]$.

We now explain briefly why such a control on the fourth-time derivative of $\tilde{w}_\kappa$ holds, and is possible only with the addition of the artificial viscosity in the solid. In particular, this norm cannot be controlled as $\kappa \to 0$, which is not crucial for our purposes in any case. In order to understand the idea, we return to the level of the setting of the fixed-point argument, where we assume that v in an appropriate convex set of $V_f^4(T) \times V_s^4(T)$ is given, and search for a solution w of (17) by a Galerkin approximation on a penalized problem (for the pressure), in a way similar to [5]. The penalization parameter $\varepsilon > 0$ is given, and we denote $q_\varepsilon^n = \displaystyle\sum_{n=0}^{2} \frac{t^n}{n!} q_n - \frac{1}{\varepsilon} a_i^j (w_\varepsilon^n)^i{,_j}$, where w_ε^n is solution of the Galerkin approximation at rank n, and where a_i^j is computed from η associated with the given v. Our interest will be with the first problem that appears in our methodology in [5]; namely, the highest-order time-differentiated problem is multiplied by $\partial_t^4 w_\varepsilon^n$ (which is permitted since it belongs to the appropriate finite dimensional space), and then integrated from 0 to t. We obtain

$$\begin{aligned}
&\int_0^t \|\partial_t^4 w_\varepsilon^n\|^2_{L^2(\Omega;\mathbb{R}^3)} + \nu \int_0^t (\partial_t^3(a_k^r a_k^s w_\varepsilon^n{,_r}),\ \partial_t^4(w_\varepsilon^n){,_s})_{L^2(\Omega_0^f;\mathbb{R}^3)} \\
&\quad + \left[\frac{\kappa}{2}(c^{ijkl}\partial_t^3(w_\varepsilon^n)^k{,_l}, \partial_t^3(w_\varepsilon^n)^i{,_j})_{L^2(\Omega_0^s;\mathbb{R})}\right]_0^t
\end{aligned}$$

$$
\begin{aligned}
&-\int_0^t (\partial_t^3(a_k^l q_\varepsilon^n),\ \partial_t^4 (w_\varepsilon^n)^k,_l)_{L^2(\Omega_0^f;\mathbb{R})}\\
&-\int_0^t (c^{ijkl}\partial_t^4[(\eta,_i\cdot\eta,_j-\delta_{ij})\eta,_l],\partial_t^3(w_\varepsilon^n),_k)_{L^2(\Omega_0^s;\mathbb{R}^3)}\\
&+\left[\int_{\Omega_0^s} c^{ijkl}\partial_t^3[(\eta,_i\cdot\eta,_j-\delta_{ij})\eta,_l]\cdot\partial_t^3(w_\varepsilon^n),_k\right]_0^t\\
=&\int_0^t\left[\int_{\Omega_0^f}\partial_t^3 F\cdot\partial_t^4 w_\varepsilon^n+\int_{\Omega_0^s}\partial_t^3 f\cdot\partial_t^4 w_\varepsilon^n\right],
\end{aligned}
$$

leading us for a time small enough depending on the artificial viscosity κ (but not on n and ε) to an inequality of the type,

$$
\begin{aligned}
&\int_0^t \|\partial_t^4 w_\varepsilon^n\|^2_{L^2(\Omega;\mathbb{R}^3)}\\
&\quad+\sup_{[0,t]}[\|\partial_t^3 w_\varepsilon^n\|^2_{H^1(\Omega_0^f;\mathbb{R}^3)}+\kappa\|\partial_t^3 w_\varepsilon^n\|^2_{H^1(\Omega_0^s;\mathbb{R}^3)}+\varepsilon\|\partial_t^3 q_\varepsilon^n\|^2_{L^2(\Omega_0^f;\mathbb{R})}]\\
&\quad\leqq C_\varepsilon[N(u_0,(w_i)_{i=1}^3)+N((q_i)_{i=0}^2)+N(f)],
\end{aligned}
$$

where C_ε depends a priori on ε. By proceding in a way inspired by our methodology in Section 9 of [5], we can then prove that we have control, independently of ε, on the first three norms. Taking the limit first as $n\to\infty$ and then as $\varepsilon\to 0$, indeed provides us with $\partial_t^4 w_\kappa\in L^2(0,T_\kappa;L^2(\Omega_0^f;\mathbb{R}^3))$ as announced. □

We note that this latter regularity property in the solid is only possible with the artificial viscosity $\kappa>0$.

8. An estimate for the solutions of (17) independent of κ

In this section, we will denote $(w_\kappa,q_\kappa)=(\tilde{w},\tilde{q})$ and denote the corresponding quantities a_i^j by $\tilde{a}_i^j$. In what follows, $\delta>0$ is a given positive number to be made precise later when it will be chosen to be sufficiently small.

8.1. Energy estimate for $\tilde{w}_{ttt}$ independent of κ

We are now going to use the regularity result $(\tilde{w},\tilde{q})\in Z_{T_\kappa}$ in the energy inequality (21) (which was established independently of the artificial viscosity); this time we interpolate and use the energy properties of the nonlinear elasticity operator, in order to get an estimate independent of the artificial viscosity.

Step 1. Let $I_1=\displaystyle\int_0^t\int_{\Omega_0^s} c^{ijkl}(\tilde{\eta},_i\cdot\tilde{\eta},_j-\delta_{ij})\tilde{w}_{tt},_l\cdot\tilde{w}_{ttt},_k$. An integration by parts in time shows that

$$
\begin{aligned}
I_1=&-\frac{1}{2}\int_0^t\int_{\Omega_0^s} c^{ijkl}(\tilde{\eta},_i\cdot\tilde{\eta},_j-\delta_{ij})_t\tilde{w}_{tt},_l\cdot\tilde{w}_{tt},_k\\
&+\frac{1}{2}\int_{\Omega_0^s} c^{ijkl}(\tilde{\eta},_i\cdot\tilde{\eta},_j-\delta_{ij})\tilde{w}_{tt},_l\cdot\tilde{w}_{tt},_k(t)\ ,
\end{aligned}
$$

and thus with the properties of the Bochner integral in $H^2(\Omega_0^s;\mathbb{R})$,

$$\tilde\eta,_i\cdot\tilde\eta,_j(t)-\delta_{ij}=\int_0^t[\tilde\eta,_i\cdot\tilde w,_j+\tilde w,_i\cdot\tilde\eta,_j],$$

we deduce

$$\begin{aligned}|I_1|&\leqq Ct\ \|\tilde w_{tt}\|^2_{L^\infty(0,t;H^1(\Omega_0^s;\mathbb{R}^3))}\|\tilde w\|_{L^\infty(0,t;H^3(\Omega_0^s;\mathbb{R}^3))}\|\tilde\eta\|_{L^\infty(0,t;H^3(\Omega_0^s;\mathbb{R}^3))}\\&\leqq Ct\ \|\tilde w\|^4_{W_t}.\end{aligned}\tag{22}$$

Step 2. Let $I_2=3\displaystyle\int_0^t\int_{\Omega_0^s}c^{ijkl}(\tilde\eta,_i\cdot\tilde\eta,_j-\delta_{ij})_t\tilde w_t,_l\cdot\tilde w_{ttt},_k$. Similarly,

$$\begin{aligned}I_2=&-3\int_0^t\int_{\Omega_0^s}c^{ijkl}[(\tilde\eta,_i\cdot\tilde\eta,_j-\delta_{ij})_t\tilde w_{tt},_l\cdot\tilde w_{tt},_k+(\tilde\eta,_i\cdot\tilde\eta,_j-\delta_{ij})_{tt}\tilde w_t,_l\cdot\tilde w_{tt},_k]\\&+3\left[\int_{\Omega_0^s}c^{ijkl}(\tilde\eta,_i\cdot\tilde\eta,_j-\delta_{ij})_t\tilde w_t,_l\cdot\tilde w_{tt},_k(t)\right]_0^t.\end{aligned}$$

By the same type of argument used in the previous step, we then get

$$\begin{aligned}|I_2|\leqq\ &Ct\ \|\tilde w_{tt}\|^2_{L^\infty(0,t;H^1(\Omega_0^s;\mathbb{R}^3))}\|\tilde w\|_{L^\infty(0,t;H^3(\Omega_0^s;\mathbb{R}^3))}\|\tilde\eta\|_{L^\infty(0,t;H^3(\Omega_0^s;\mathbb{R}^3))}\\&+Ct\ \|\tilde w_{tt}\|_{L^\infty(0,t;H^1(\Omega_0^s;\mathbb{R}^3))}\|\tilde w\|^2_{L^\infty(0,t;H^3(\Omega_0^s;\mathbb{R}^3))}\|\tilde w_t\|_{L^\infty(0,t;H^1(\Omega_0^s;\mathbb{R}^3))}\\&+Ct\ \|\tilde w_{tt}\|_{L^\infty(0,t;H^1(\Omega_0^s;\mathbb{R}^3))}\|\tilde w_t\|^2_{L^\infty(0,t;H^2(\Omega_0^s;\mathbb{R}^3))}\|\tilde\eta\|_{L^\infty(0,t;H^3(\Omega_0^s;\mathbb{R}^3))}\\&+C\left\|c^{ijkl}\left[(u_0^i,_j+u_0^j,_i)w_1,_l+\int_0^\cdot((\tilde\eta,_i\cdot\tilde\eta,_j-\delta_{ij})_t\tilde w_t,_l)_t\right]\right\|_{L^\infty(0,t;L^2(\Omega_0^s;\mathbb{R}^3))}\\&\quad\times\|\tilde w,_{ktt}\|_{L^\infty(0,t;L^2(\Omega_0^s;\mathbb{R}^3))}\\&+N(u_0,(w_i)_{i=1}^3),\end{aligned}$$

and thus,

$$|I_2|\leqq\delta\|\tilde w\|^2_{W_t}+Ct\|\tilde w\|^4_{W_t}+C_\delta N(u_0,(w_i)_{i=1}^3).\tag{23}$$

Step 3. Let $I_3=3\displaystyle\int_0^t\int_{\Omega_0^s}c^{ijkl}(\tilde\eta,_i\cdot\tilde\eta,_j-\delta_{ij})_{tt}\tilde w,_l\cdot\tilde w_{ttt},_k$. By an integration by parts in time,

$$\begin{aligned}I_3=\ &-3\int_0^t\int_{\Omega_0^s}c^{ijkl}[(\tilde\eta,_i\cdot\tilde\eta,_j-\delta_{ij})_{tt}\tilde w_t,_l\cdot\tilde w_{tt},_k+(\tilde\eta,_i\cdot\tilde\eta,_j-\delta_{ij})_{ttt}\tilde w,_l\cdot\tilde w_{tt},_k]\\&+3\left[\int_{\Omega_0^s}c^{ijkl}(\tilde\eta,_i\cdot\tilde\eta,_j-\delta_{ij})_{tt}\tilde w,_l\cdot\tilde w_{tt},_k(t)\right]_0^t.\end{aligned}$$

Similarly as before, we get

$$
\begin{aligned}
|I_3| \leqq Ct\ & \|\tilde{w}_{tt}\|_{L^\infty(0,T;H^1(\Omega_0^s;\mathbb{R}^3))} \|\tilde{w}\|^2_{L^\infty(0,t;H^3(\Omega_0^s;\mathbb{R}^3))} \|\tilde{w}_t\|_{L^\infty(0,t;H^1(\Omega_0^s;\mathbb{R}^3))} \\
& + Ct\ \|\tilde{w}_{tt}\|_{L^\infty(0,T;H^1(\Omega_0^s;\mathbb{R}^3))} \|\tilde{w}_t\|^2_{L^\infty(0,t;H^2(\Omega_0^s;\mathbb{R}^3))} \|\tilde{\eta}\|_{L^\infty(0,t;H^3(\Omega_0^s;\mathbb{R}^3))} \\
& + Ct\ \|\tilde{w}_{tt}\|^2_{L^\infty(0,T;H^1(\Omega_0^s;\mathbb{R}^3))} \|\tilde{w}\|_{L^\infty(0,t;H^3(\Omega_0^s;\mathbb{R}^3))} \|\tilde{\eta}\|_{L^\infty(0,t;H^3(\Omega_0^s;\mathbb{R}^3))} \\
& + Ct\ \|\tilde{w}_{tt}\|_{L^\infty(0,T;H^1(\Omega_0^s;\mathbb{R}^3))} \|\tilde{w}_t\|_{L^\infty(0,t;H^1(\Omega_0^s;\mathbb{R}^3))} \|\tilde{w}\|^2_{L^\infty(0,t;H^3(\Omega_0^s;\mathbb{R}^3))} \\
& + C \Big\| c^{ijkl} ((\tilde{\eta},_i \cdot \tilde{\eta},_j)_{tt}(0) w_1,_l + \int_0^\cdot ((\tilde{\eta},_i \cdot \tilde{\eta},_j - \delta_{ij})_{tt} \tilde{w},_l)_t) \Big\|_{L^\infty(0,t;L^2(\Omega_0^s;\mathbb{R}^3))} \\
& \quad \times \|\tilde{w}_{tt},_k\|_{L^\infty(0,t;L^2(\Omega_0^s;\mathbb{R}^3))} \\
& + N(u_0, (w_i)_{i=1}^3),
\end{aligned}
$$

and therefore

$$
|I_3| \leqq \delta \|\tilde{w}\|^2_{W_t} + Ct\|\tilde{w}\|^4_{W_t} + C_\delta N(u_0, (w_i)_{i=1}^3). \tag{24}
$$

Step 4. Let $I_4 = \int_0^t \int_{\Omega_0^s} c^{ijkl} (\tilde{\eta},_i \cdot \tilde{\eta},_j - \delta_{ij})_{ttt} \tilde{\eta},_l \cdot \tilde{w}_{ttt},_k$. By the symmetry of c, we notice that

$$
\begin{aligned}
I_4 = & \frac{1}{2} \int_0^t \int_{\Omega_0^s} c^{ijkl} (\tilde{\eta},_i \cdot \tilde{\eta},_j)_{ttt} (\tilde{\eta},_l \cdot \tilde{\eta},_k)_{tttt} \\
& - \int_0^t \int_{\Omega_0^s} c^{ijkl} [4(\tilde{\eta},_i \cdot \tilde{\eta},_j)_{ttt} (\tilde{w},_l \cdot \tilde{w}_{tt},_k) + 3(\tilde{\eta},_i \cdot \tilde{\eta},_j)_{ttt} (\tilde{w}_t,_l \cdot \tilde{w}_t,_k)],
\end{aligned}
$$

and thus,

$$
\begin{aligned}
& \Big| I_4 - \frac{1}{4} \int_{\Omega_0^s} c^{ijkl} (\tilde{\eta},_i \cdot \tilde{\eta},_j)_{ttt} (\tilde{\eta},_l \cdot \tilde{\eta},_k)_{ttt}(t) \Big| \\
& \leqq Ct\ \|\nabla \tilde{w}_{tt}\|^2_{L^\infty(0,t;L^2(\Omega_0^s;\mathbb{R}^9))} \|\nabla \tilde{w}\|_{L^\infty(0,t;H^2(\Omega_0^s;\mathbb{R}^9))} \|\nabla \tilde{\eta}\|_{L^\infty(0,t;H^2(\Omega_0^s;\mathbb{R}^9))} \\
& \quad + Ct \|\nabla \tilde{w}_t\|_{L^\infty(0,t;L^2(\Omega_0^s;\mathbb{R}^9))} \|\nabla \tilde{w}_{tt}\|_{L^\infty(0,t;L^2(\Omega_0^s;\mathbb{R}^9))} \|\nabla \tilde{w}\|^2_{L^\infty(0,t;H^2(\Omega_0^s;\mathbb{R}^9))} \\
& \quad + Ct \|\nabla \tilde{w}_t\|^2_{L^\infty(0,t;H^1(\Omega_0^s;\mathbb{R}^9))} \|\nabla \tilde{w}_{tt}\|_{L^\infty(0,t;L^2(\Omega_0^s;\mathbb{R}^9))} \|\nabla \tilde{\eta}\|_{L^\infty(0,t;H^2(\Omega_0^s;\mathbb{R}^9))} \\
& \quad + Ct \|\nabla \tilde{w}\|_{L^\infty(0,t;H^2(\Omega_0^s;\mathbb{R}^9))} \|\nabla \tilde{w}_t\|^3_{L^\infty(0,t;H^1(\Omega_0^s;\mathbb{R}^9))} + N(u_0, (w_i)_{i=1}^3) \\
& \leqq N(u_0, (w_i)_{i=1}^3) + Ct\ \|\tilde{w}\|^4_{W_t}.
\end{aligned} \tag{25}
$$

Now, Let $I = \frac{1}{4} \int_{\Omega_0^s} c^{ijkl} (\tilde{\eta},_i \cdot \tilde{\eta},_j)_{ttt} (\tilde{\eta},_l \cdot \tilde{\eta},_k)_{ttt}(t)$. By expanding the integrand with respect to the time derivatives and using the relation in $H^3(\Omega_0^s; \mathbb{R}^3)$: $\tilde{\eta}(t, \cdot) = \mathrm{Id} + \int_0^t \tilde{w}(t', \cdot)\, dt'$ and estimates similar as in the previous steps, we find that

$$
\left| I - \int_{\Omega_0^s} c^{ijkl} \tilde{w}^j_{tt},_i\ \tilde{w}^k_{tt},_l (t) \right| \leqq C_\delta N(u_0, (w_i)_{i=1}^3) + \delta \|\tilde{w}\|^2_{W_t} + Ct\|\tilde{w}\|^4_{W_t}. \tag{26}
$$

Step 5. By using (25) and (26) we find that

$$\left| I_4 - \int_{\Omega_0^s} c^{ijkl} \tilde{w}_{tt}^j,_i \, \tilde{w}_{tt}^k,_l \, (t) \right| \leqq C_\delta N(u_0, (w_i)_{i=1}^3) + \delta \|\tilde{w}\|_{W_t}^2 + Ct \|\tilde{w}\|_{W_t}^4. \quad (27)$$

Step 6. By proceding in a way similar to [5], except that we replace the constants $C(M)$ appearing there by appropriate powers of $\|(\tilde{w}, \tilde{q})\|_{Z_t}$, we find that the integrals set in the fluid domain are bounded by

$$\delta \|(\tilde{w}, \tilde{q})\|_{Z_t}^2 + C_\delta \left[N((q_i)_{i=0}^2) + N(u_0, (w_i)_{i=1}^3) + M(f, \kappa g, \kappa h) + t^{\frac{1}{4}} \|(\tilde{w}, \tilde{q})\|_{Z_t}^6 \right].$$

Step 7. Thus, from (21), and Steps 1–6, we then obtain on $[0, T_\kappa]$:

$$\begin{aligned} &\sup_{[0,t]} \|\tilde{w}_{ttt}\|_{L^2(\Omega;\mathbb{R}^3)}^2 + \int_0^t \left[\|\tilde{w}_{ttt}\|_{H^1(\Omega_0^f;\mathbb{R}^3)}^2 + \kappa \|\tilde{w}_{ttt}\|_{H^1(\Omega_0^s;\mathbb{R}^3)}^2 \right] \\ &\quad + \sup_{[0,t]} \|\tilde{w}_{tt}\|_{H^1(\Omega_0^s;\mathbb{R}^3)}^2 \\ &\qquad \leqq C_\delta [N(u_0, (w_i)_{i=1}^3) + M(f, \kappa g, \kappa h) + N((q_i)_{i=0}^2) + t^{\frac{1}{4}} \|(\tilde{w}, \tilde{q})\|_{Z_t}^6] \\ &\qquad\quad + C\delta \|(\tilde{w}, \tilde{q})\|_{Z_t}^2. \end{aligned} \quad (28)$$

Step 8. The estimate of $\tilde{q}_{tt}$ in $L^\infty(L^2)$, independent of κ, will require some adjustments with respect to the methodology of [5]. To this end, we notice that we can apply a Lagrange multiplier Lemma similar to Lemma 13 of [5], but corresponding to the case $a_i^j = \delta_i^j$, to the variational form true on $[0, T_\kappa]$: for all $\phi \in H_0^1(\Omega; \mathbb{R}^3)$,

$$\begin{aligned} &(\tilde{w}_{ttt}, \phi)_{L^2(\Omega;\mathbb{R}^3)} \\ &\quad + \nu((\tilde{a}_k^r \tilde{a}_k^s \tilde{w},_r)_{tt}, \ \phi,_s)_{L^2(\Omega_0^f;\mathbb{R}^3)} + \kappa(c^{ijkl} \tilde{w}_{tt}^k,_l, \phi^i,_j)_{L^2(\Omega_0^s;\mathbb{R})} \\ &\quad + (c^{ijkl}[(\tilde{\eta},_i \cdot \tilde{\eta},_j - \delta_{ij})\tilde{\eta},_l]_{tt}, \phi,_k)_{L^2(\Omega_0^s;\mathbb{R}^3)} - ((\tilde{a}_k^l - \delta_{kl})\tilde{q}_{tt}, \ \phi^k,_l)_{L^2(\Omega_0^f;\mathbb{R})} \\ &\quad - ((\tilde{a}_k^l \tilde{q})_{tt} - \tilde{a}_k^l \tilde{q}_{tt}, \phi^k,_l)_{L^2(\Omega_0^f;\mathbb{R})} - (F_{tt}, \phi)_{L^2(\Omega_0^f;\mathbb{R}^3)} - (f_{tt}, \phi)_{L^2(\Omega_0^s;\mathbb{R}^3)} - b_\kappa(\phi) \\ &= (\tilde{q}_{tt}, \ \operatorname{div} \phi)_{L^2(\Omega_0^f;\mathbb{R})}, \end{aligned}$$

which provides for any $t \in [0, T_\kappa]$,:

$$\begin{aligned} \|\tilde{q}_{tt}\|_{L^2(\Omega_0^f;\mathbb{R})} \leqq \ & C \Big[\|\tilde{w}_{ttt}\|_{L^2(\Omega;\mathbb{R}^3)} + \|(\tilde{a}_k^r \tilde{a}_k^s \tilde{w},_r)_{tt}\|_{L^2(\Omega_0^f;\mathbb{R}^3)} + \kappa \|\tilde{w}_{tt}\|_{H^1(\Omega_0^s;\mathbb{R})} \\ &+ \|[(\tilde{\eta},_i \cdot \tilde{\eta},_j - \delta_{ij})\tilde{\eta},_l]_{tt}\|_{L^2(\Omega_0^s;\mathbb{R}^3)} + \|(\tilde{a}_k^l \tilde{q})_{tt} - \tilde{a}_k^l \tilde{q}_{tt}\|_{L^2(\Omega;\mathbb{R})} \\ &+ \|\tilde{a} - \mathrm{Id}\|_{H^2(\Omega_0^f;\mathbb{R}^9)} \|\tilde{q}_{tt}\|_{L^2(\Omega_0^f;\mathbb{R})} + N(u_0, (w_i)_{i=1}^3) + M(f, \kappa g, \kappa h) \Big]. \end{aligned}$$

By using (28) for the first four terms of the right-hand side of this inequality and remembering that the $L^\infty(0, t; L^2(\Omega_0^f; \mathbb{R}))$ norm of $\tilde{q}_{tt}$ is part of the norm Z_t for the next two terms of this inequality, we get

$$\begin{aligned} \sup_{[0,t]} \|\tilde{q}_{tt}\|_{L^2(\Omega_0^f;\mathbb{R})}^2 \leqq C_\delta & \left[N(u_0, (w_i)_{i=1}^3) + M(f, \kappa g, \kappa h) + N((q_i)_{i=0}^2) \right] \\ & + C_\delta t^{\frac{1}{4}} \|(\tilde{w}, \tilde{q})\|_{Z_t}^6 + C\delta \|(\tilde{w}, \tilde{q})\|_{Z_t}^2. \end{aligned} \quad (29)$$

8.2. Estimate on w_{tt} and w_t

From the previous estimates, and the arguments that we will see hereafter for the case of $\tilde{w}$, we have

$$\begin{aligned}
&\|\tilde{w}_{tt}\|^2_{L^2(0,t;H^2(\Omega_0^f;\mathbb{R}^3))} + \|\tilde{q}_{tt}\|^2_{L^2(0,t;H^1(\Omega_0^f;\mathbb{R}))} + \|\tilde{w}_t\|^2_{L^\infty(0,t;H^2(\Omega_0^s;\mathbb{R}^3))} \\
&\leqq C_\delta \left[N(u_0, (w_i)_{i=1}^3) + M(f, \kappa g, \kappa h) + N((q_i)_{i=0}^2) \right] \\
&\quad + C_\delta t^{\frac{1}{4}} \|(\tilde{w}, \tilde{q})\|^6_{Z_t} + C\delta \|(\tilde{w}, \tilde{q})\|^2_{Z_t}. \qquad (30)
\end{aligned}$$

Similarly, we infer from (30) that

$$\begin{aligned}
&\|\tilde{w}_t\|^2_{L^2(0,t;H^3(\Omega_0^f;\mathbb{R}^3))} + \|\tilde{q}_t\|^2_{L^2(0,t;H^2(\Omega_0^f;\mathbb{R}))} + \|\tilde{w}\|^2_{L^\infty(0,t;H^3(\Omega_0^s;\mathbb{R}^3))} \\
&\leqq C_\delta \left[N(u_0, (w_i)_{i=1}^3) + M(f, \kappa g, \kappa h) + N((q_i)_{i=0}^2) \right] \\
&\quad + C_\delta t^{\frac{1}{4}} \|(\tilde{w}, \tilde{q})\|^6_{Z_t} + C\delta \|(\tilde{w}, \tilde{q})\|^2_{Z_t}. \qquad (31)
\end{aligned}$$

8.3. Estimate on $\tilde{w}$

We will denote $\mathbb{R}^3_+ = \{x \in \mathbb{R}^3 |\ x_3 > 0\}$, $\mathbb{R}^3_- = \{x \in \mathbb{R}^3 |\ x_3 < 0\}$, and $B_-(0,r) = B(0,r) \cap \mathbb{R}^3_-$. We denote by Ψ an H^4 diffeomorphism from $B(0,1)$ into a neighborhood V of a point $x_0 \in \Gamma_0$ such that $\Psi(B(0,1) \cap \mathbb{R}^3_+) = V \cap \Omega_0^f$, $\Psi(B(0,1) \cap \mathbb{R}^3_-) = V \cap \Omega_0^s$, $\Psi(B(0,1) \cap \mathbb{R}^2 \times \{0\}) = V \cap \Gamma_0$, with $\det \nabla\Psi = 1$. We consider a cut-off function ζ compactly supported in $B(0,1)$, and equal to 1 in $B(0, \frac{1}{2})$.

With the use of test functions $\phi_p = -[\rho_p \star (\zeta^2\ \tilde{w} \circ \Psi)],_{\alpha_1\alpha_1\alpha_2\alpha_2\alpha_3\alpha_3} \circ \Psi^{-1}$ (which is in $L^2(0, T_\kappa; H^1_0(\Omega; \mathbb{R}^3)))$ in (20) for $n = 0$, and by denoting $W = \tilde{w} \circ \Psi$, $Q = \tilde{q} \circ \Psi$, $E = \tilde{\eta} \circ \Psi$, we get after integrating by parts appropriately and letting $p \to \infty$,

$$\begin{aligned}
&\frac{1}{2} \|\zeta W,_{\alpha_1\alpha_2\alpha_3}(t)\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^3)} \\
&+ \int_0^t (W_t,_{\alpha_1\alpha_2\alpha_3},\ [\zeta^2 W],_{\alpha_1\alpha_2\alpha_3} - \zeta^2 W,_{\alpha_1\alpha_2\alpha_3})_{L^2(\mathbb{R}^3;\mathbb{R}^3)} \\
&+ \nu \int_0^t ([\tilde{b}^r_k \tilde{b}^s_k W,_r],_{\alpha_1\alpha_2\alpha_3},\ [\zeta^2 W],_{s\alpha_1\alpha_2\alpha_3})_{L^2(\mathbb{R}^3_+;\mathbb{R}^3)} \\
&- \int_0^t \int_{\mathbb{R}^3_+} [Q\, \tilde{b}^j_i],_{\alpha_1\alpha_2\alpha_3} [\zeta^2 W]^i,_{j\alpha_1\alpha_2\alpha_3} \\
&+ \kappa \int_0^t ([C^{ijkl}(W,_i \cdot \Psi,_j)\, \Psi,_l],_{\alpha_1\alpha_2\alpha_3},\ [\zeta^2 W],_{k\alpha_1\alpha_2\alpha_3})_{L^2(\mathbb{R}^3_-;\mathbb{R}^3)} \\
&+ \int_0^t ([C^{ijkl}(E,_i \cdot E,_j - \Psi,_i \cdot \Psi,_j)\, E,_l],_{\alpha_1\alpha_2\alpha_3},\ [\zeta^2 W],_{k\alpha_1\alpha_2\alpha_3})_{L^2(\mathbb{R}^3_-;\mathbb{R}^3)}
\end{aligned}$$

$$\leqq C\ N(u_0, (w_i)_{i=1}^3) + \lim_{p\to\infty} \int_0^t [\ (F, \phi_p)_{L^2(\Omega_0^f;\mathbb{R}^3)} + (f, \phi_p)_{L^2(\Omega_0^s;\mathbb{R}^3)}]$$
$$+ \lim_{p\to\infty} \int_0^t [\frac{t'^2}{2} b_\kappa(\phi_p) + t' c_\kappa(\phi_p) + d_\kappa(\phi_p)\ ], \tag{32}$$

where $C^{ijkl} = c^{mnop} g_m^i g_n^j g_o^k g_p^l \in H^3(B(0,1);\mathbb{R})$, $g = [\nabla\Psi]^{-1} \in H^3(B(0,1);\mathbb{R}^9)$, $\tilde{b}_l^j = \tilde{a}_l^k(\Psi) g_k^j$.

Remark 6. Note that this limit process as $p \to \infty$ for the nonlinear elastic energy is possible because $\partial_t^n \tilde{w} \in L^2(0, T_\kappa; H^{4-n}(\Omega_0^s;\mathbb{R}^3))$ $(n = 0, 1)$ due to our artificial viscosity in the solid. Whereas we could also use difference quotients, it appears that the product rules are less cumbersome with the use of horizontal derivatives instead, which is permitted since we already know at this stage the regularity of $\tilde{w}$ and $\tilde{q}$. Also, the limits on the right-hand side of (32) do not present any difficulties, given the regularity of the forcing functions and three integrations by parts with respect to horizontal variables.

Remark 7. Since ζ is compactly supported in $B(0,1)$, the integrals set on $\mathbb{R}^3$, $\mathbb{R}^3_-$, $\mathbb{R}^3_+$ do not depend on the extension that we chose for W, E or Q, and simply represent a more convenient way to write these integrals.

Step 1. Let $L_1 = \kappa \int_0^t (\ [C^{ijkl} W,_i \cdot \Psi,_j\ \ \Psi,_l\ ],_{\alpha_1\alpha_2\alpha_3}, [\zeta^2\ W],_{k\alpha_1\alpha_2\alpha_3})_{L^2(\mathbb{R}^3_-;\mathbb{R}^3)}$. By using the H^3 regularity of the coefficients C^{ijkl},

$$\begin{aligned}
L_1 = \kappa \int_0^t & (C^{ijkl} W,_{i\alpha_1\alpha_2\alpha_3} \cdot \Psi,_j\ \ \Psi,_l\ ,\ \zeta^2\ W,_{k\alpha_1\alpha_2\alpha_3})_{L^2(\mathbb{R}^3_-;\mathbb{R}^3)} \\
& +\kappa \int_0^t \Big([C^{ijkl} W,_i \cdot \Psi,_j\ \ \Psi,_l\ ],_{\alpha_1\alpha_2\alpha_3} - C^{ijkl} W,_{i\alpha_1\alpha_2\alpha_3} \cdot \Psi,_j\ \ \Psi,_l\ , \\
& \quad \zeta^2\ W,_{k\alpha_1\alpha_2\alpha_3} \Big)_{L^2(\mathbb{R}^3_-;\mathbb{R}^3)} \\
& +\kappa \int_0^t \Big([C^{ijkl} W,_i \cdot \Psi,_j\ \ \Psi,_l\ ],_{\alpha_1\alpha_2\alpha_3}\ , \\
& \quad [\zeta^2\ W,_k\ ],_{\alpha_1\alpha_2\alpha_3} - \zeta^2\ W,_{k\alpha_1\alpha_2\alpha_3} \Big)_{L^2(\mathbb{R}^3_-;\mathbb{R}^3)} \\
\geqq C\kappa \int_0^t & (C^{ijkl} W,_{i\alpha_1\alpha_2\alpha_3} \cdot \Psi,_j\ \ \Psi,_l\ ,\ \zeta^2\ W,_{k\alpha_1\alpha_2\alpha_3})_{L^2(\mathbb{R}^3_-;\mathbb{R}^3)} \\
& -C\kappa \int_0^t \|\tilde{w}\|^2_{H^3(\Omega_0^s;\mathbb{R}^3)} \\
\geqq C\kappa \int_0^t & \|W,_{\alpha_1\alpha_2\alpha_3}\|^2_{H^1(B_-(0,\frac{1}{2});\mathbb{R}^3)} - C\kappa t \|\tilde{w}\|^2_{W_t}.
\end{aligned} \tag{33}$$

Step 2. Let

$$L_2 = \int_0^t ([C^{ijkl}(E,_i \cdot E,_j - \Psi,_i \cdot \Psi,_j\ )\ E,_l\ ],_{\alpha_1\alpha_2\alpha_3},\ [\zeta^2 E_t],_{k\alpha_1\alpha_2\alpha_3})_{L^2(\mathbb{R}^3_-;\mathbb{R}^3)}.$$

With Σ_3 denoting the set of permutations of $\{1, 2, 3\}$, we have

$$
\begin{aligned}
L_2 = &\int_0^t (C^{ijkl}(E,_i \cdot E,_j - \Psi,_i \cdot \Psi,_j)\, E,_{l\alpha_1\alpha_2\alpha_3},\ \zeta^2 E_{t},_{k\alpha_1\alpha_2\alpha_3})_{L^2(\mathbb{R}^3_-;\mathbb{R}^3)} \\
&+2\int_0^t (\zeta^2 C^{ijkl}(E,_{i\alpha_1\alpha_2\alpha_3} \cdot E,_j)\, E,_l,\ E_{t},_{k\alpha_1\alpha_2\alpha_3})_{L^2(\mathbb{R}^3_-;\mathbb{R}^3)} \\
&+\int_0^t (\zeta^2 [[C^{ijkl}(E,_i \cdot E,_j)],_{\alpha_1\alpha_2\alpha_3} - 2C^{ijkl}(E,_{i\alpha_1\alpha_2\alpha_3} \cdot E,_j)] E,_l, \\
&\quad E_{t},_{k\alpha_1\alpha_2\alpha_3})_{L^2(\mathbb{R}^3_-;\mathbb{R}^3)} \\
&+\int_0^t ([\zeta^2 (C^{ijkl}\Psi,_i \cdot \Psi,_j),_{\alpha_1\alpha_2\alpha_3}\ E,_l]_t,\ E,_{k\alpha_1\alpha_2\alpha_3})_{L^2(\mathbb{R}^3_-;\mathbb{R}^3)} \\
&-\big[(\zeta^2 (C^{ijkl}\Psi,_i \cdot \Psi,_j),_{\alpha_1\alpha_2\alpha_3}\ E,_l,\ E,_{k\alpha_1\alpha_2\alpha_3})_{L^2(\mathbb{R}^3_-;\mathbb{R}^3)}\big]_0^t \\
&-\sum_{\sigma\in\Sigma_3}\int_0^t ([\zeta^2 [C^{ijkl}(E,_i \cdot E,_j - \Psi,_i \cdot \Psi,_j)],_{\alpha_{\sigma(1)}}\, E,_{\alpha_{\sigma(2)}\alpha_{\sigma(3)}l}],_{\alpha_1}, \\
&\quad E_{t},_{\alpha_2\alpha_3 k})_{L^2(\mathbb{R}^3_-;\mathbb{R}^3)} \\
&-\sum_{\sigma\in\Sigma_3}\int_0^t ([\zeta^2 [C^{ijkl}(E,_i \cdot E,_j - \Psi,_i \cdot \Psi,_j)],_{\alpha_{\sigma(1)}\alpha_{\sigma(2)}}\, E,_{\alpha_{\sigma(3)}l}],_{\alpha_1}, \\
&\quad E_{t},_{\alpha_2\alpha_3 k})_{L^2(\mathbb{R}^3_-;\mathbb{R}^3)} \\
&+\int_0^t ([C^{ijkl}(E,_i \cdot E,_j - \Psi,_i \cdot \Psi,_j)\, E,_l],_{\alpha_1\alpha_2\alpha_3}, \\
&\quad [\zeta^2 E_t],_{\alpha_1\alpha_2\alpha_3 k} - \zeta^2 E_{t},_{\alpha_1\alpha_2\alpha_3 k})_{L^2(\mathbb{R}^3_-;\mathbb{R}^3)}\ .
\end{aligned}
$$

From the regularity of $\tilde{w}$ and the H^4 regularity of Ψ, we then infer

$$
\begin{aligned}
L_2 = &\int_0^t (C^{ijkl}(E,_i \cdot E,_j - \Psi,_i \cdot \Psi,_j)\, E,_{\alpha_1\alpha_2\alpha_3 l},\ \zeta^2 E_{t},_{\alpha_1\alpha_2\alpha_3 k})_{L^2(\mathbb{R}^3_-;\mathbb{R}^3)} \\
&+2\int_0^t (C^{ijkl} E,_{\alpha_1\alpha_2\alpha_3 i} \cdot E,_j\ \zeta^2 E,_l,\ E_{t},_{\alpha_1\alpha_2\alpha_3 k})_{L^2(\mathbb{R}^3_-;\mathbb{R}^3)} \\
&+\int_0^t (C^{ijkl},_{\alpha_1\alpha_2\alpha_3}\, E,_i \cdot E,_j\ \zeta^2 E,_l,\ E_{t},_{\alpha_1\alpha_2\alpha_3 k})_{L^2(\mathbb{R}^3_-;\mathbb{R}^3)} + L_2^r,
\end{aligned}
$$

with

$$
|L_2^r| \leqq \delta \|\tilde{w}\|^2_{W_t} + Ct\ \|\tilde{w}\|^4_{W_t} + C_\delta N(u_0, (w_i)_{i=1}^3). \tag{34}
$$

By integrating by parts in time, we deduce

$$
\begin{aligned}
L_2 = &-\frac{1}{2}\int_0^t (C^{ijkl}(E,_i \cdot E,_j - \Psi,_i \cdot \Psi,_j)_t\, E,_{\alpha_1\alpha_2\alpha_3 l},\ \zeta^2 E,_{\alpha_1\alpha_2\alpha_3 k})_{L^2(\mathbb{R}^3_-;\mathbb{R}^3)} \\
&+\Big[\frac{1}{2}(C^{ijkl}(E,_i \cdot E,_j - \Psi,_i \cdot \Psi,_j)\, E,_{\alpha_1\alpha_2\alpha_3 l},\ \zeta^2 E,_{\alpha_1\alpha_2\alpha_3 k})_{L^2(\mathbb{R}^3_-;\mathbb{R}^3)}\Big]_0^t \\
&-2\int_0^t (C^{ijkl} E,_{\alpha_1\alpha_2\alpha_3 i} \cdot E,_j\ \zeta^2 E_{t},_l,\ E,_{\alpha_1\alpha_2\alpha_3 k})_{L^2(\mathbb{R}^3_-;\mathbb{R}^3)}
\end{aligned}
$$

$$+\big[(C^{ijkl}E,_{\alpha_1\alpha_2\alpha_3 i}\cdot E,_j\ \zeta^2 E,_l\ ,\ E,_{\alpha_1\alpha_2\alpha_3 k})_{L^2(\mathbb{R}^3_-;\mathbb{R}^3)}\big]_0^t$$
$$-\int_0^t (C,^{ijkl}_{\alpha_1\alpha_2\alpha_3}(E,_i\cdot E,_j\ \zeta^2 E,_l)_t,\ E,_{\alpha_1\alpha_2\alpha_3 k})_{L^2(\mathbb{R}^3_-;\mathbb{R}^3)}$$
$$+\big[(C,^{ijkl}_{\alpha_1\alpha_2\alpha_3}E,_i\cdot E,_j\ \zeta^2 E,_l\ ,\ E,_{\alpha_1\alpha_2\alpha_3 k})_{L^2(\mathbb{R}^3_-;\mathbb{R}^3)}\big]_0^t + L_2^r,$$

which implies in turn

$$\big|L_2-(C^{ijkl}\zeta^2 E,_{\alpha_1\alpha_2\alpha_3 i}(t)\cdot\Psi,_j\ \Psi,_l\ ,\ E,_{\alpha_1\alpha_2\alpha_3 k}(t))_{L^2(\mathbb{R}^3_-;\mathbb{R}^3)}\big|$$
$$\leqq \delta\|\tilde{w}\|^2_{W_t}+C_\delta N(u_0,(w_i)_{i=1}^3)+Ct\ \|\tilde{w}\|^4_{W_t}. \tag{35}$$

With e_k ($k=1,2,3$) denoting the canonical vectors of $\mathbb{R}^3$, let

$$P(t)=\|\ 2c^{ijkl}(\tilde{\eta},_{mnil}\cdot e_j)\ e_k-[c^{ijkl}(\tilde{\eta},_i\cdot\tilde{\eta}_j-\delta_{ij})\tilde{\eta},_k],_{lmn}\|_{L^2(\Omega_0^s;\mathbb{R}^3)}(t),$$

where m and n are arbitrarily fixed in $\{1,2,3\}$. We then have

$$P(t)\leqq P_1(t)+P_2(t),$$

with

$$P_1(t)=\|\ 2c^{ijkl}(\tilde{\eta},_{ilmn}\cdot e,_j)\ e_k-2[c^{ijkl}(\tilde{\eta},_{mni}\cdot\tilde{\eta},_j)\tilde{\eta},_k],_l\|_{L^2(\Omega_0^s;\mathbb{R}^3)}(t),$$
$$P_2(t)=\|-[c^{ijkl}(\tilde{\eta},_i\cdot\tilde{\eta},_j-\delta_{ij})\tilde{\eta},_k],_{lmn}+2[c^{ijkl}(\tilde{\eta},_{mni}\cdot\tilde{\eta},_j)\tilde{\eta},_k],_l\|_{L^2(\Omega_0^s;\mathbb{R}^3)}(t).$$

We first notice that

$$P_1(t)\leqq\|\ 2c^{ijkl}[(\tilde{\eta},_{ilmn}\cdot e_j)\ e_k-(\tilde{\eta},_{ilmn}\cdot\tilde{\eta},_j)\tilde{\eta},_k]\|_{L^2(\Omega_0^s;\mathbb{R}^3)}(t)$$
$$+\|\ 2c^{ijkl}[\tilde{\eta},_{imn}\cdot(\tilde{\eta},_j]\ \tilde{\eta},_k),_l\|_{L^2(\Omega_0^s;\mathbb{R}^3)}(t),$$

Where $[u.(v]w),_l=u.v,_l\ w+u.vw,_l$
Next, by writing $\tilde{\eta}(t)=\mathrm{Id}+\int_0^t\tilde{w}$ and $[\tilde{\eta},_{mni}\cdot(\tilde{\eta},_j]\,\tilde{\eta},_k),_l(t)=[\tilde{\eta},_{mni}\cdot(\tilde{\eta},_j]\,\tilde{\eta},_k),_l(0)+\int_0^t[[\tilde{\eta},_{mni}\cdot(\tilde{\eta},_j]\ \tilde{\eta},_k),_l]_t$ respectively in $H^3(\Omega_0^s;\mathbb{R}^3)$ and $L^2(\Omega_0^s;\mathbb{R}^3)$, we obtain

$$P_1(t)\leqq C[\int_0^t\|\tilde{w}\|_{H^3(\Omega_0^s;\mathbb{R}^3)}]\ \sup_{[0,t]}[\|\tilde{\eta}\|^2_{H^4(\Omega_0^s;\mathbb{R}^3)}+\|\tilde{\eta}\|_{H^4(\Omega_0^s;\mathbb{R}^3)}]$$
$$+N(u_0,(w_i)_{i=1}^3)+C\int_0^t\|\tilde{w}\|_{H^3(\Omega_0^s;\mathbb{R}^3)}\sup_{[0,t]}\|\tilde{\eta}\|^2_{H^4(\Omega_0^s;\mathbb{R}^3)}$$
$$\leqq N(u_0,(w_i)_{i=1}^3)+Ct\ \|\tilde{w}\|^3_{W_t}.$$

Next, we see that

$$P_2(t)\leqq\|c^{ijkl}[(\tilde{\eta},_i\cdot\tilde{\eta},_j-\delta_{ij})\tilde{\eta},_{kmn}+(\tilde{\eta},_i\cdot\tilde{\eta},_j),_m\tilde{\eta},_{kn}$$
$$+(\tilde{\eta},_i\cdot\tilde{\eta},_j),_n\tilde{\eta},_{km}],_l\|_{L^2(\Omega_0^s;\mathbb{R}^3)}(t)$$
$$+\|c^{ijkl}[(\tilde{\eta},_{mi}\cdot\tilde{\eta},_{nj}+\tilde{\eta},_{in}\cdot\tilde{\eta},_{jm})\tilde{\eta},_k],_l\|_{L^2(\Omega_0^s;\mathbb{R}^3)}(t),$$

and by the same type of arguments as for $P_1(t)$,

$$P_2(t) \leqq N(u_0, (w_i)_{i=1}^3) + Ct\,\|\tilde{w}\|_{W_t}^3,$$

implying that

$$P(t) \leqq N(u_0, (w_i)_{i=1}^3) + Ct\,\|\tilde{w}\|_{W_t}^3. \tag{36}$$

Now, from the definition of a solution of the smoothed problem (14),

$$\begin{aligned}&\|\,\kappa c^{ijkl}(\tilde{\eta}_t,_{ilmn}\cdot e_j)\,e_k + [c^{ijkl}(\tilde{\eta},_i\cdot\tilde{\eta}_j - \delta_{ij})\tilde{\eta},_k],_{mnl}\,\|_{L^2(\Omega_0^s;\mathbb{R}^3)}(t)\\ &\quad = \|(\tilde{w}_t - f - \kappa h),_{mn}(t)\|_{L^2(\Omega_0^s;\mathbb{R}^3)},\end{aligned}$$

which implies with (36) that

$$\begin{aligned}\left\|\frac{\kappa}{2}L(\tilde{w},_{nm}) + L(\tilde{\eta},_{nm})\right\|_{L^2(\Omega_0^s;\mathbb{R}^3)}(t) \leqq{}& \|(\tilde{w}_t - f - \kappa h),_{nm}(t)\|_{L^2(\Omega_0^s;\mathbb{R}^3)}\\ &+N(u_0, (w_i)_{i=1}^3) + Ct\,\|\tilde{w}\|_{W_t}^3.\end{aligned}$$

Since this inequality also holds for any $t' \in (0,t)$, Lemma 1 provides

$$\begin{aligned}\|L(\tilde{\eta},_{nm})\|_{L^\infty(0,t;L^2(\Omega_0^s;\mathbb{R}^3))} \leqq{}& C\|\tilde{w}_t,_{mn}\|_{L^\infty(0,t;L^2(\Omega_0^s;\mathbb{R}^3))} + CM(f,\kappa g,\kappa h)\\ &+N(u_0, (w_i)_{i=1}^3) + Ct\,\|\tilde{w}\|_{W_t}^3,\end{aligned}$$

which with the estimate on w_t from the previous subsection leads to

$$\begin{aligned}\|L(\tilde{\eta},_{nm})\|_{L^\infty(0,t;L^2(\Omega_0^s;\mathbb{R}^3))} \leqq{}& C_\delta\left[N(u_0, (w_i)_{i=1}^3) + M(f,\kappa g,\kappa h) + N((q_i)_{i=0}^2)\right]\\ &+C_\delta t^{\frac{1}{4}}\|(\tilde{w},\tilde{q})\|_{Z_t}^6 + C\delta\|(\tilde{w},\tilde{q})\|_{Z_t}^2.\end{aligned} \tag{37}$$

Step 3. From the estimates on $L_1 - L_2$, and similar estimates that we could get in the fluid as in [5], but this time by replacing $C(M)$ by appropriate powers of $\|(\tilde{w},\tilde{q})\|_{Z_t}$, we then deduce that for all $t \in [0,\check{T}]$,

$$\begin{aligned}&\frac{1}{2}\,\|\zeta W,_{\alpha_1\alpha_2\alpha_3}(t)\|_{L^2(\mathbb{R}_+^3;\mathbb{R}^3)}^2 + \nu\int_0^t (\zeta^2 b_k^r W,_{\alpha_1\alpha_2\alpha_3 r}\,,\ b_k^s W,_{\alpha_1\alpha_2\alpha_3 s})_{L^2(\mathbb{R}_+^3;\mathbb{R}^3)}\\ &\quad\leqq C_\delta\left[N(u_0, (w_i)_{i=1}^3) + M(f,\kappa g,\kappa h) + N((q_i)_{i=0}^2)\right]\\ &\quad\quad+C_\delta t^{\frac{1}{4}}\|(\tilde{w},\tilde{q})\|_{Z_t}^6 + C\delta\|(\tilde{w},\tilde{q})\|_{Z_t}^2.\end{aligned}$$

By the trace theorem, we then get

$$\begin{aligned}\int_0^t \|W\|_{H^{3.5}(S;\mathbb{R}^3)}^2 \leqq{}& C_\delta\left[N(u_0, (w_i)_{i=1}^3) + M(f,\kappa g,\kappa h) + N((q_i)_{i=0}^2)\right]\\ &+C_\delta t^{\frac{1}{4}}\|(\tilde{w},\tilde{q})\|_{Z_t}^6 + C\delta\|(\tilde{w},\tilde{q})\|_{Z_t}^2,\end{aligned}$$

where $S = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid |x_1| \leqq \frac{1}{2},\ |x_2| \leqq \frac{1}{2},\ x_3 = 0\right\}$. By a finite covering argument, we then get

$$\int_0^t \|\tilde{w}\|^2_{H^{3.5}(\Gamma_0;\mathbb{R}^3)} \leqq C_\delta \left[N(u_0, (w_i)_{i=1}^3) + M(f, \kappa g, \kappa h) + N((q_i)_{i=0}^2)\right] + C_\delta t^{\frac{1}{4}} \|(\tilde{w}, \tilde{q})\|^6_{Z_t} + C\delta \|(\tilde{w}, \tilde{q})\|^2_{Z_t}. \tag{38}$$

From the estimate (31) on $\tilde{w}_t$ and the trace estimate (38), we infer in a way similar to [5] by elliptic regularity arguments that

$$\|\tilde{w}\|^2_{L^2(0,t;H^4(\Omega_0^f;\mathbb{R}^3))} + \|\tilde{q}\|^2_{L^2(0,t;H^3(\Omega_0^f;\mathbb{R}))} \leqq C_\delta \left[N(u_0, (w_i)_{i=1}^3) + M(f, \kappa g, \kappa h) + N((q_i)_{i=0}^2)\right] + C_\delta t^{\frac{1}{4}} \|(\tilde{w}, \tilde{q})\|^6_{Z_t} + C\delta \|(\tilde{w}, \tilde{q})\|^2_{Z_t}. \tag{39}$$

Similarly, from (37), and the trace estimate (38), elliptic regularity yields

$$\|\tilde{\eta}\|^2_{L^\infty(0,t;H^4(\Omega_0^s;\mathbb{R}^3))} \leqq C_\delta \left[N(u_0, (w_i)_{i=1}^3) + M(f, \kappa g, \kappa h) + N((q_i)_{i=0}^2)\right] + C_\delta t^{\frac{1}{4}} \|(\tilde{w}, \tilde{q})\|^6_{Z_t} + C\delta\, \|(\tilde{w}, \tilde{q})\|^2_{Z_t}. \tag{40}$$

9. Time of existence independent of κ

From (28), (29), (30), (31), (39) and (40), we then have for any $t \in [0, T_\kappa]$,

$$\|(\tilde{w}, \tilde{q})\|^2_{Z_t} \leqq C_\delta \left[N_0(u_0, (w_i)_{i=1}^3) + M_0(f, \kappa g, \kappa h)) + N_0((q_i)_{i=0}^2)\right] + C_\delta t^{\frac{1}{4}} \|(\tilde{w}, \tilde{q})\|^6_{Z_t} + C_0\delta\, \|(\tilde{w}, \tilde{q})\|^2_{Z_t}.$$

The subscripts 0 in C_0, N_0, M_0 mean that we no longer consider generic constants from now on.

Now, let $\delta_0 > 0$ be such that $C_0\delta_0 = \frac{1}{2}$. For $\kappa > 0$ small enough and $t \in (0, T_\kappa)$ we have

$$\|(\tilde{w}, \tilde{q})\|^2_{Z_t} \leqq 4C_{\delta_0}\left[N_0(u_0, (w_i)_{i=1}^3) + M_0(f) + N_0((q_i)_{i=0}^2)\right] + 2C_{\delta_0} t^{\frac{1}{4}} \|(\tilde{w}, \tilde{q})\|^6_{Z_T}, \tag{41}$$

where $M_0(f) = M_0(f, 0, 0)$. For conciseness, we will denote $C_1 = 2C_{\delta_0}$ and $N_1 = 4C_{\delta_0}[N_0(u_0, (w_i)_{i=1}^3) + M_0(f) + N_0((q_i)_{i=0}^2)]$.

Now for $t \in (0, T_\kappa)$ fixed, let $\alpha_t(x) = x^3 - \dfrac{x}{C_1 t^{\frac{1}{4}}} + \dfrac{N_1}{C_1 t^{\frac{1}{4}}}$, so that

$$\alpha_t(\|(\tilde{w}, \tilde{q})\|^2_{Z_t}) \geqq 0.$$

Now let $t_1 = \left[\frac{2}{27 C_1 N_1^2}\right]^4 > 0$, *which does not depend on* κ, and let $\check{T} = \min(T_\kappa, t_1)$. From now on, we assume that $t \in (0, \check{T})$. We then have $\alpha_t\left((3C_1 t^{\frac{1}{4}})^{-\frac{1}{2}}\right) < 0$

which implies that α_t has three real roots z_1, z_2, z_3, with $z_1 < -(3C_1 t^{\frac{1}{4}})^{-\frac{1}{2}} < z_2 < (3C_1 t^{\frac{1}{4}})^{-\frac{1}{2}} < z_3$. From the product $z_1 z_2 z_3 = -\dfrac{N_1}{C_1 t^{\frac{1}{4}}}$ and $\alpha_t(3N_1) < 0$, we infer that $0 < z_2 < 3N_1 < z_3$. From (41) and the *continuity* of $t \to \|(\tilde{w}, \tilde{q})\|_{Z_t}$ (established in Lemma 3) we then infer since $\|(\tilde{w}, \tilde{q})\|^2_{Z_0} \leqq N_1 < z_3$ that we have

$$\forall t \in (0, \check{T}],\ \|(\tilde{w}, \tilde{q})\|^2_{Z_t} \leqq z_2 \leqq 3N_1. \tag{42}$$

This implies that $\tilde{\eta}(\check{T}) \in H^4(\Omega_0^f; \mathbb{R}^3) \cap H^4(\Omega_0^s; \mathbb{R}^3)$, $\tilde{w}(\check{T}) \in H^1_0(\Omega; \mathbb{R}^3) \cap H^3(\Omega_0^s; \mathbb{R}^3) \cap H^3(\Omega_0^s; \mathbb{R}^3)$, $\tilde{w}_t(\check{T}) \in H^1_0(\Omega; \mathbb{R}^3) \cap H^2(\Omega_0^s; \mathbb{R}^3) \cap H^2(\Omega_0^s; \mathbb{R}^3)$, $\tilde{w}_{tt}(\check{T}) \in H^1_0(\Omega; \mathbb{R}^3)$, $\tilde{w}_{ttt}(\check{T}) \in L^2(\Omega; \mathbb{R}^3)$, $\tilde{q}(\check{T}) \in H^2(\Omega_0^f; \mathbb{R})$, $\tilde{q}_t(\check{T}) \in H^1(\Omega_0^f; \mathbb{R})$, $\tilde{q}_{tt}(\check{T}) \in L^2(\Omega_0^f; \mathbb{R})$, with a bound that depends only on the right-hand side of (42). The compatibility conditions for the smoothed problem (14) at $\check{T}$ are also satisfied by the definition of a solution, which means that we do not have any new terms of the type b_κ, c_κ or d_κ associated with $\tilde{w}(\check{T})$ to add to the already existing forcing terms coming from $t = 0$.

We can thus build a solution of the smoothed problem (14) defined on $[\check{T}, \check{T} + \delta T]$, δT depending solely on the right-hand side of (42), which we will still denote $(\tilde{w}, \tilde{q})$. It is then readily seen that $(\tilde{w}, \tilde{q}) \in Z_{\check{T}+\delta T}$ and is a solution of the approximated problem (14) on $[0, \check{T} + \delta T]$. If $\check{T} = t_1$, we have our solution defined on the κ independent time interval $[0, t_1]$, with the κ independent estimate (42). Otherwise, if $\check{T} < t_1$, we can also assume that $\check{T} + \delta T \leqq t_1$, which implies, in the same fashion as we got (41),

$$\forall t \in [0, \tilde{T} + \delta T],\ \|(\tilde{w}, \tilde{q})\|^2_{Z_t} \leqq N_1 + C_1 t^{\frac{1}{4}}\ \|(\tilde{w}, \tilde{q})\|^6_{Z_t}. \tag{43}$$

This implies in turn that $\tilde{\eta}(\check{T}+\delta T)$, $\tilde{w}(\check{T}+\delta T)$, $\tilde{w}_t(\check{T}+\delta T)$, $\tilde{w}_{tt}(\check{T}+\delta T)$, $\tilde{w}_{ttt}(\check{T}+\delta T)$, $\tilde{q}(\check{T}+\delta T)$, $\tilde{q}_t(\check{T}+\delta T)$, $\tilde{q}_{tt}(\check{T}+\delta T)$ are in the same spaces as their respective counterparts at time $\check{T}$, with the same bound as well, since we could from (43) repeat the same argument leading to (42), this time on $[0, \check{T} + \delta T]$. Since the compatibility conditions at $\check{T} + \delta T$ are also automatically satisfied, we can thus build a solution of the approximated problem (14) defined on $[\check{T} + \delta T, \check{T} + 2\delta T]$, the time of existence being the same as starting from $\check{T}$ from the similarity of the bound that we obtain on $\tilde{\eta}(\check{T} + \delta T)$, $\partial_t^n \tilde{w}(\check{T} + \delta T)(n = 0, 1, 2, 3)$, $\partial_t^n \tilde{q}(\check{T} + \delta T)(n = 0, 1, 2)$ and their respective counterparts at time $\check{T}$. We will still denote this solution $(\tilde{w}, \tilde{q})$. It is then readily seen that $(\tilde{w}, \tilde{q}) \in Z_{\check{T}+2\delta T}$ and is a solution of the approximated problem on $[0, \check{T} + 2\delta T]$. We then have in the same fashion as we got (41),

$$\forall t \in [0, \tilde{T} + 2\delta T],\ \|(\tilde{w}, \tilde{q})\|^2_{Z_t} \leqq N_1 + C_1 t^{\frac{1}{4}}\ \|(\tilde{w}, \tilde{q})\|^6_{Z_t}.$$

By induction, we then see that we get a solution $(\tilde{w}, \tilde{q})$ defined on $[0, t_1]$,satisfying the estimate

$$\forall t \in [0, t_1],\ \|(\tilde{w}, \tilde{q})\|_{Z_t} \leqq 3N_1 = 12C_{\delta_0}[N_0(u_0, (w_i)_{i=1}^3) + M_0(f) + N_0((q_i)_{i=0}^2)], \tag{44}$$

establishing the independence of the time of existence respectively to κ, since t_1 does not depend on κ. In the following we will note $T = t_1$.

10. Existence for (4)

Proof. We can here choose to take $\kappa = \frac{1}{n}$, and let $n \to \infty$. By the bound (44) independent of κ on $[0, T]$, we then have the existence of a weakly convergent subsequence of $(\tilde{w}, \tilde{q})$ in the reflexive Hilbert space Y_T, to a limit that we call (v, q), which also belongs to Z_T and satisfies the estimate

$$\|(v, q)\|_{Z_T} \leqq 3N_1 = 12C_{\delta_0}\left[N_0(u_0, (w_i)_{i=1}^3) + M_0(f) + N((q_i)_{i=0}^2)\right].$$

The usual compactness theorems ensure at this stage that (v, q) is a solution of (4) on $[0, T]$. The smoothness of our solution ensures that the solids do not collide with each other (if there is more than one) or the boundary (for an eventually smaller time), which establishes the existence part of Theorem 1. □

11. Uniqueness for (4)

Proof. Since we cannot use a contractive mapping scheme for our problem, we have to establish uniqueness separately. Let then $(\bar{v}, \bar{q})$ denote another solution of (4) in Z_T. Then, taking $v - \bar{v}$ as a test function in the variational formulation of the difference between the systems (4) associated with each solution yields for $t \in [0, T]$,

$$\begin{aligned} &\frac{1}{2}\,\|(v-\bar{v})(t)\|^2_{L^2(\Omega;\mathbb{R}^3)} + \nu\int_0^t (a_k^r a_k^s v,_r - \bar{a}_k^r \bar{a}_k^s \bar{v},_r\,,\ v,_s - \bar{v},_s)_{L^2(\Omega_0^f;\mathbb{R}^3)} \\ &\quad + \int_0^t (c^{ijkl}[(\eta,_i \cdot \eta,_j - \delta_{ij})\eta,_k - (\bar{\eta},_i \cdot \bar{\eta},_j - \delta_{ij})\bar{\eta},_k],\ v,_l - \bar{v},_l)_{L^2(\Omega_0^s;\mathbb{R}^3)} \\ &\quad - \int_0^t (a_i^j q - \bar{a}_i^j \bar{q},\ v^i,_j - \bar{v}^i,_j)_{L^2(\Omega_0^f;\mathbb{R})} \\ &= \int_0^t (f\circ\eta - f\circ\bar{\eta},\ v - \bar{v})_{L^2(\Omega_0^f;\mathbb{R}^3)}. \end{aligned} \tag{45}$$

For the viscous term in the fluid, we write

$$a_k^r a_k^s v,_r - \bar{a}_k^r \bar{a}_k^s \bar{v},_r = a_k^r a_k^s (v,_r - \bar{v},_r) + (a_k^r a_k^s - \bar{a}_k^r \bar{a}_k^s)\bar{v},_r\,,$$

which with the $L^\infty(0, T; H^3(\Omega_0^f; \mathbb{R}^3))$ control of $\bar{v}$ and v provides us with an estimate of the type (where C denotes once again a generic constant)

$$\begin{aligned} \int_0^t (a_k^r a_k^s v,_r - \bar{a}_k^r \bar{a}_k^s \bar{v},_r\,,\ v,_s - \bar{v},_s)_{L^2(\Omega_0^f;\mathbb{R}^3)} &\geqq C\int_0^t \|v-\bar{v}\|^2_{H^1(\Omega_0^f;\mathbb{R}^3)} \\ &\quad - C\int_0^t\int_0^{t'} \|v-\bar{v}\|^2_{H^1(\Omega_0^f;\mathbb{R}^3)}. \end{aligned} \tag{46}$$

Concerning the forcing term in the fluid, we first notice that if we still denote $E(\Omega)(f)$ as f,

$$\begin{aligned} &f(t,\bar\eta(t,x))-f(t,\eta(t,x))\\ &\quad=\int_0^1 f,_i(t,(\eta+t'(\bar\eta-\eta))(t,x))dt'\,(\bar\eta^i(t,x)-\eta^i(t,x)), \end{aligned}$$

which leads us to

$$\begin{aligned} &\|f(t,\bar\eta(t,\cdot))-f(t,\eta(t,\cdot))\|_{L^{1.5}(\Omega_0^f;\mathbb{R}^3)}\\ &\quad\leqq C\|\bar\eta(t,\cdot)-\eta(t,\cdot)\|_{L^6(\Omega_0^f;\mathbb{R}^3)}\left[\sum_{i=1}^3\int_0^1\int_{\Omega_0^f} f,_i^2(t,\phi(t',t,x))dxdt'\right]^{0.5}, \end{aligned}$$

with $\phi(t',t,x)=\eta(t,x)+t'(\bar\eta(t,x)-\eta(t,x))$. We have $\phi(t',t,\cdot)\in\mathcal{C}^0(\overline{\Omega};\mathbb{R}^3)\cap\mathcal{C}^1(\Omega\cap\Gamma_0^c;\mathbb{R}^3)$. Moreover $\phi(t',t,\partial\Omega)=\partial\Omega$. We then have by invariance by homotopy of the Brouwer degree (for the parameter t)

$$\forall z\in\Omega,\ \deg(\phi(t',t,\cdot),\Omega,z)=\deg(\phi(t',0,\cdot),\Omega,z)=\deg(\mathrm{Id},\Omega,z)=1,$$

which together with the regularity of $\phi(t',t,\cdot)$ establishes that $\phi(t',t,\cdot)(\Omega)=\Omega$ and that $\mathrm{Card}\{\phi^{-1}(t',t,\cdot)(x)\}=1$ for almost all $x\in\Omega$. Thus,

$$\int_{\Omega_0^f} f,_i^2(t,\phi(t',t,x))dx=\int_{\phi(t',t,\Omega_0^f)} f,_i^2(t,y)|\det\nabla\phi(t',t,\phi^{-1}(t',t,y))|^{-1}\,dy,$$

which with the $L^\infty(0,T;H^4(\Omega_0^f;\mathbb{R}^3))$ control of η and $\bar\eta$ yields

$$\int_{\Omega_0^f} f,_i^2(t,\phi(t',t,x))dx\leqq C\int_\Omega f,_i^2(t,y)dy.$$

Consequently,

$$\begin{aligned} &\|f(t,\bar\eta(t,\cdot))-f(t,\eta(t,\cdot))\|_{L^{1.5}(\Omega_0^f;\mathbb{R}^3)}\\ &\quad\leqq C\|\bar\eta(t,\cdot)-\eta(t,\cdot)\|_{H^1(\Omega_0^f;\mathbb{R}^3)}\|f\|_{H^1(\Omega;\mathbb{R}^3)}, \end{aligned}$$

implying

$$\begin{aligned} &\left|\int_0^t(f\circ\eta-f\circ\bar\eta,v-\bar v)_{L^2(\Omega_0^f;\mathbb{R}^3)}\right|\\ &\quad\leqq C\sqrt{t}\|f\|_{L^2(0,t;H^1(\Omega;\mathbb{R}^3))}\|v-\bar v\|^2_{L^2(0,t;H^1(\Omega_0^f;\mathbb{R}^3))}. \end{aligned}\tag{47}$$

Concerning the elastic term,

$$\begin{aligned} &\int_0^t(c^{ijkl}[(\eta,_i\cdot\eta,_j-\delta_{ij})\eta,_k-(\bar\eta,_i\cdot\bar\eta,_j-\delta_{ij})\bar\eta,_k],\ v,_l-\bar v,_l)_{L^2(\Omega_0^s;\mathbb{R}^3)}\\ &\quad=I_1+I_2+I_3, \end{aligned}$$

with

$$
\begin{aligned}
I_1 &= \int_0^t (c^{ijkl}(\eta,_i \cdot \eta,_j - \delta_{ij})(\eta,_k - \bar\eta,_k),\ v,_l - \bar v,_l)_{L^2(\Omega_0^s;\mathbb{R}^3)} \\
&= \frac{1}{2}(c^{ijkl}(\eta,_i \cdot \eta,_j - \delta_{ij})(\eta,_k - \bar\eta,_k),\ \eta,_l - \bar\eta,_l)_{L^2(\Omega_0^s;\mathbb{R}^3)}(t) \\
&\quad -\frac{1}{2}\int_0^t (c^{ijkl}(\eta,_i \cdot \eta,_j - \delta_{ij})_t(\eta,_k - \bar\eta,_k),\ \eta,_l - \bar\eta,_l)_{L^2(\Omega_0^s;\mathbb{R}^3)} \\
&\geqq -Ct\|\eta(t) - \bar\eta(t)\|^2_{H^1(\Omega_0^s;\mathbb{R}^3)} - C\int_0^t \|\eta - \bar\eta\|^2_{H^1(\Omega_0^s;\mathbb{R}^3)},
\end{aligned}
$$

where we have used the $L^\infty(0, T;\ H^3(\Omega_0^s;\ \mathbb{R}^3))$ control of v and $\bar v$ for the inequality.

Next, for the same reasons,

$$
\begin{aligned}
I_2 &= \int_0^t (c^{ijkl}(\eta,_i - \bar\eta,_i) \cdot \eta,_j\ \bar\eta,_k,\ v,_l - \bar v,_l)_{L^2(\Omega_0^s;\mathbb{R}^3)} \\
&= \int_0^t (c^{ijkl}(\eta,_i - \bar\eta,_i) \cdot (\eta,_j - \bar\eta,_j)\ \bar\eta,_k,\ v,_l - \bar v,_l)_{L^2(\Omega_0^s;\mathbb{R}^3)} \\
&\quad + \int_0^t (c^{ijkl}(\eta,_i - \bar\eta,_i) \cdot \bar\eta,_j\ \bar\eta,_k,\ v,_l - \bar v,_l)_{L^2(\Omega_0^s;\mathbb{R}^3)} \\
&= \int_0^t (c^{ijkl}(\eta,_i - \bar\eta,_i) \cdot (\eta,_j - \bar\eta,_j)\ \bar\eta,_k,\ v,_l - \bar v,_l)_{L^2(\Omega_0^s;\mathbb{R}^3)} \\
&\quad + \frac{1}{2}(c^{ijkl}(\eta,_i - \bar\eta,_i) \cdot \bar\eta,_j\ \bar\eta,_k,\ \eta,_l - \bar\eta,_l)_{L^2(\Omega_0^s;\mathbb{R}^3)}(t) \\
&\quad - \int_0^t (c^{ijkl}(\eta,_i - \bar\eta,_i) \cdot \bar\eta,_j\ \bar v,_k,\ \eta,_l - \bar\eta,_l)_{L^2(\Omega_0^s;\mathbb{R}^3)}.
\end{aligned}
$$

We then write for the second term on the right-hand side of the last equality $\bar\eta,_i(t, \cdot) = e_i + \int_0^t \bar v,_i$, to get by Korn's inequality ,

$$
\begin{aligned}
I_2 &\geqq C[\|\eta(t) - \bar\eta(t)\|^2_{H^1(\Omega_0^s;\mathbb{R}^3)} - \|\eta(t) - \bar\eta(t)\|^2_{L^2(\Omega_0^s;\mathbb{R}^3)}] \\
&\quad -Ct \sup_{[0,t]} \|\eta - \bar\eta\|^2_{H^1(\Omega_0^s;\mathbb{R}^3)}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
I_3 &= \int_0^t (c^{ijkl}(\eta,_j - \bar\eta,_j) \cdot \bar\eta,_i\ \bar\eta,_k,\ v,_l - \bar v,_l)_{L^2(\Omega_0^s;\mathbb{R}^3)} \\
&\geqq C[\ \|\eta(t) - \bar\eta(t)\|^2_{H^1(\Omega_0^s;\mathbb{R}^3)} - \|\eta(t) - \bar\eta(t)\|^2_{L^2(\Omega_0^s;\mathbb{R}^3)}] \\
&\quad -C\int_0^t \|\eta - \bar\eta\|^2_{H^1(\Omega_0^s;\mathbb{R}^3)}.
\end{aligned}
$$

Thus,

$$\int_0^t (c^{ijkl}[(\eta,_i \cdot\eta,_j -\delta_{ij})\eta,_k -(\bar{\eta},_i \cdot\bar{\eta},_j -\delta_{ij})\bar{\eta},_k],\ v,_l -\bar{v},_l)_{L^2(\Omega_0^s;\mathbb{R}^3)}$$
$$\geqq C\Big[\ \|\eta(t)-\bar{\eta}(t)\|^2_{H^1(\Omega_0^s;\mathbb{R}^3)} - \|\eta(t)-\bar{\eta}(t)\|^2_{L^2(\Omega_0^s;\mathbb{R}^3)}\Big]$$
$$-C\int_0^t \|\eta-\bar{\eta}\|^2_{H^1(\Omega_0^s;\mathbb{R}^3)}. \tag{48}$$

Concerning the pressure term, with $a_i^j q - \bar{a}_i^j\bar{q} = (a_i^j - \bar{a}_i^j)q + \bar{a}_i^j(q-\bar{q})$ and the $L^\infty(0,T;\,H^2(\Omega_0^f;\mathbb{R}))$ control of the pressure, we get

$$-\int_0^t (a_i^j q - \bar{a}_i^j\bar{q},\, v^i,_j -\bar{v}^i,_j)_{L^2(\Omega_0^f;\mathbb{R})}$$
$$\geqq -C\Big[\sqrt{t}\|q-\bar{q}\|_{L^\infty(0,t;L^2(\Omega_0^f;\mathbb{R}))}\|v-\bar{v}\|_{L^2(0,t;H^1(\Omega_0^f;\mathbb{R}^3))}$$
$$+t\|v-\bar{v}\|^2_{L^2(0,t;H^1(\Omega_0^f;\mathbb{R}^3))}\Big]. \tag{49}$$

In order to get the estimate of $q-\bar{q}$ in $L^2(\Omega_0^f;\mathbb{R})$, we have to introduce the time differentiated problem. By taking $v_t - \bar{v}_t$ in the variational formulation associated to the difference between the time differentiated systems, we obtain

$$\frac{1}{2}\ \|(v_t-\bar{v}_t)(t)\|^2_{L^2(\Omega;\mathbb{R}^3)} + \nu\int_0^t ([a_k^r a_k^s v,_r -\bar{a}_k^r\bar{a}_k^s\bar{v},_r]_t,\ [v,_s -\bar{v},_s]_t)_{L^2(\Omega_0^f;\mathbb{R}^3)}$$
$$+\int_0^t (c^{ijkl}[(\eta,_i \cdot\eta,_j -\delta_{ij})\eta,_k -(\bar{\eta},_i \cdot\bar{\eta},_j -\delta_{ij})\bar{\eta},_k]_t,\ [v,_l -\bar{v},_l]_t)_{L^2(\Omega_0^s;\mathbb{R}^3)}$$
$$-\int_0^t ([a_i^j q - \bar{a}_i^j\bar{q}]_t, [v^i,_j -\bar{v}^i,_j]_t)_{L^2(\Omega_0^f;\mathbb{R})}$$
$$=\int_0^t ([f\circ\eta - f\circ\bar{\eta}]_t, v_t-\bar{v}_t)_{L^2(\Omega_0^f;\mathbb{R}^3)}. \tag{50}$$

For the fluid viscous term, we easily find with the $L^2(0,T;\,H^3(\Omega_0^f;\mathbb{R}^3))$ control of the first-time derivative of the velocity that

$$\int_0^t ([a_k^r a_k^s v,_r -\bar{a}_k^r\bar{a}_k^s\bar{v},_r]_t,\ v_t,_s -\bar{v}_t,_s)_{L^2(\Omega_0^f;\mathbb{R}^3)}$$
$$\geqq\ C(1-t)\int_0^t \|v_t-\bar{v}_t\|^2_{H^1(\Omega_0^f;\mathbb{R}^3)}. \tag{51}$$

Concerning the forcing term in the fluid, since $(f\circ\eta)_t = (f_t + v^i f,_i)(\eta)$ (with a similar formula for $\bar{v}$), we then deduce in a way similar to the steps leading to (47) that

$$\left| \int_0^t ([f \circ \eta - f \circ \bar{\eta}]_t, [v - \bar{v}]_t)_{L^2(\Omega_0^f;\mathbb{R}^3)} \right|$$
$$\leqq C\sqrt{t}[\|f_t\|_{L^2(0,t;H^1(\Omega;\mathbb{R}^3))} + \|f\|_{L^2(0,t;H^2(\Omega;\mathbb{R}^3))}] \; \|v_t - \bar{v}_t\|^2_{L^2(0,t;H^1(\Omega_0^f;\mathbb{R}^3))}. \tag{52}$$

For the elastic term, we can also essentially reproduce the arguments leading to (48), leading us to

$$\begin{aligned}\int_0^t (c^{ijkl}[(\eta,_i \cdot \eta,_j - \delta_{ij})\eta,_k - (\bar{\eta},_i \cdot \bar{\eta},_j - \delta_{ij})\bar{\eta},_k]_t, \; [v,_l - \bar{v},_l]_t)_{L^2(\Omega_0^s;\mathbb{R}^3)} \\ \geqq C[\; \|v(t) - \bar{v}(t)\|^2_{H^1(\Omega_0^s;\mathbb{R}^3)} - \|v(t) - \bar{v}(t)\|^2_{L^2(\Omega_0^s;\mathbb{R}^3)}] \\ -Ct \sup_{[0,t]} \|v - \bar{v}\|^2_{H^1(\Omega_0^s;\mathbb{R}^3)}. \end{aligned} \tag{53}$$

The pressure term will require more care since we want to avoid the introduction of $q_t - \bar{q}_t$, which the most direct method would lead to. To do so, we notice that

$$\int_0^t ([a_i^j q - \bar{a}_i^j \bar{q}]_t, [v^i,_j - \bar{v}^i,_j]_t)_{L^2(\Omega_0^f;\mathbb{R})} = I_4 + I_5 + I_6,$$

with

$$\begin{aligned} I_4 &= \int_0^t ([(a_i^j)_t q - (\bar{a}_i^j)_t \bar{q}], [v^i,_j - \bar{v}^i,_j]_t)_{L^2(\Omega_0^f;\mathbb{R})}, \\ I_5 &= \int_0^t (a_i^j (q_t - \bar{q}_t), [v^i,_j - \bar{v}^i,_j]_t)_{L^2(\Omega_0^f;\mathbb{R})}, \\ I_6 &= \int_0^t ([a_i^j - \bar{a}_i^j]\bar{q}_t, [v^i,_j - \bar{v}^i,_j]_t)_{L^2(\Omega_0^f;\mathbb{R})}. \end{aligned}$$

For I_4, we have in a way similar to (49),

$$\begin{aligned} |I_4| \leqq C \Big[\sqrt{t} \|q - \bar{q}\|_{L^\infty(0,t;L^2(\Omega_0^f;\mathbb{R}))} \|v_t - \bar{v}_t\|_{L^2(0,t;H^1(\Omega_0^f;\mathbb{R}^3))} \\ + t\|v_t - \bar{v}_t\|^2_{L^2(0,t;H^1(\Omega_0^f;\mathbb{R}^3))} \Big]. \end{aligned}$$

For I_6, the $L^2(0, T; H^2(\Omega_0^f; \mathbb{R}))$ control of $\bar{q}_t$ provides us with

$$|I_6| \leqq Ct \; \|v_t - \bar{v}_t\|^2_{L^2(0,t;H^1(\Omega_0^f;\mathbb{R}^3))}.$$

For I_5 we have:

$$\begin{aligned} I_5 &= \int_0^t (q_t - \bar{q}_t, a_i^j v_t^i,_j - \bar{a}_i^j \bar{v}_t^i,_j)_{L^2(\Omega_0^f;\mathbb{R})} - \int_0^t (q_t - \bar{q}_t, (a_i^j - \bar{a}_i^j)\bar{v}_t^i,_j)_{L^2(\Omega_0^f;\mathbb{R})} \\ &= \int_0^t (\bar{q}_t - q_t, (a_i^j)_t v^i,_j - (\bar{a}_i^j)_t \bar{v}^i,_j)_{L^2(\Omega_0^f;\mathbb{R})} \\ &\quad - \int_0^t (q_t - \bar{q}_t, (a_i^j - \bar{a}_i^j)\bar{v}_t^i,_j)_{L^2(\Omega_0^f;\mathbb{R})}, \end{aligned}$$

where we have used the relations $a_i^j v^i,_j = 0 = \bar{a}_i^j \bar{v}^i,_j$ in Ω_0^f for the first integral. By integrating by parts in time,

$$
\begin{aligned}
I_5 = & \int_0^t (q-\bar{q}, [(a_i^j)_t v^i,_j - (\bar{a}_i^j)_t \bar{v}^i,_j]_t)_{L^2(\Omega_0^f;\mathbb{R})} \\
& + \int_0^t (q-\bar{q}, [(a_i^j - \bar{a}_i^j)\bar{v}^i_t,_j]_t)_{L^2(\Omega_0^f;\mathbb{R})} \\
& + (\bar{q}-q, (a_i^j)_t v^i,_j - (\bar{a}_i^j)_t \bar{v}^i,_j)_{L^2(\Omega_0^f;\mathbb{R})}(t) \\
& + (\bar{q}-q, (a_i^j - \bar{a}_i^j)\bar{v}^i_t,_j)_{L^2(\Omega_0^f;\mathbb{R})}(t).
\end{aligned}
$$

With the $L^2(0,T;H^3(\Omega_0^f;\mathbb{R}^3))$ control of v_t we have

$$
\begin{aligned}
& \left| \int_0^t (q-\bar{q}, [(a_i^j)_t v^i,_j - (\bar{a}_i^j)_t \bar{v}^i,_j]_t)_{L^2(\Omega_0^f;\mathbb{R})} \right| \\
& \quad + \left| \int_0^t (q-\bar{q}, [(a_i^j)_t - (\bar{a}_i^j)_t] \bar{v}^i_t,_j)_{L^2(\Omega_0^f;\mathbb{R})} \right| \\
& \leqq C\sqrt{t}\, \|q-\bar{q}\|_{L^\infty(0,t;L^2(\Omega_0^f;\mathbb{R}))} \|v_t - \bar{v}_t\|_{L^2(0,t;H^1(\Omega_0^f;\mathbb{R}^3))}, \\
& \quad \left| (q-\bar{q}, (a_i^j)_t v^i,_j - (\bar{a}_i^j)_t \bar{v}^i,_j)_{L^2(\Omega_0^f;\mathbb{R})}(t) \right| \\
& \leqq C\sqrt{t}\, \|q(t)-\bar{q}(t)\|_{L^2(\Omega_0^f;\mathbb{R})} \|v_t - \bar{v}_t\|_{L^2(0,t;H^1(\Omega_0^f;\mathbb{R}^3))}.
\end{aligned}
$$

The remaining terms are more delicate. We first have

$$
\begin{aligned}
& \left| \int_0^t (q-\bar{q}, (a_i^j - \bar{a}_i^j)\, \bar{v}^i_{tt},_j)_{L^2(\Omega_0^f;\mathbb{R})} \right| + \left| (q-\bar{q}, (a_i^j - \bar{a}_i^j)\, \bar{v}^i_t,_j)_{L^2(\Omega_0^f;\mathbb{R})}(t) \right| \\
& \leqq C \int_0^t \|q-\bar{q}\|_{L^2(\Omega_0^f;\mathbb{R})} \|a-\bar{a}\|_{L^4(\Omega_0^f;\mathbb{R}^9)} \|\nabla \bar{v}_{tt}\|_{L^4(\Omega_0^f;\mathbb{R}^9)} \\
& \quad + \|q(t)-\bar{q}(t)\|_{L^2(\Omega_0^f;\mathbb{R})} \|a(t)-\bar{a}(t)\|_{L^4(\Omega_0^f;\mathbb{R}^9)} \|\nabla \bar{v}_t(t)\|_{L^4(\Omega_0^f;\mathbb{R}^9)}. \qquad (54)
\end{aligned}
$$

The apparent problem here is that $a-\bar{a}$ is estimated in $L^2(\Omega_0^f;\mathbb{R}^9)$ in terms of $v-\bar{v}$ in $H^1(\Omega_0^f;\mathbb{R}^3)$. Now, a bound of this quantity in $L^4(\Omega_0^f;\mathbb{R}^9)$ will require a bound of $v-\bar{v}$ in $H^2(\Omega_0^f;\mathbb{R}^3)$. In order to get such an estimate, we will bound $v-\bar{v}$ in $H^2(\Omega_0^f;\mathbb{R}^3)$ by lower-order terms in $v-\bar{v}$. To do so, let us first estimate the trace of $v-\bar{v}$ on Γ_0 by using the test function $-[\zeta^2\,(v-\bar{v})\circ\Psi],_{\alpha\alpha}\circ\Psi^{-1}$ in the difference between the variational problems satisfied by v and $\bar{v}$. By proceeding as in Section 10, we would then get an estimate of the type, where $\delta>0$ is given:

$$
\begin{aligned}
& \int_0^t \|\zeta\nabla[(v-\bar{v})\circ\Psi],_\alpha\|^2_{L^2(\mathbb{R}^3_+;\mathbb{R}^9)} + \|\zeta\nabla[(\eta-\bar{\eta})\circ\Psi],_\alpha(t)\|^2_{L^2(\mathbb{R}^3_-;\mathbb{R}^9)} \\
& \leqq C[\sqrt{t}+\delta] \int_0^t \left[\|v-\bar{v}\|^2_{H^2(\Omega_0^f;\mathbb{R}^3)} + \|q-\bar{q}\|^2_{H^1(\Omega_0^f;\mathbb{R})} \right] \\
& \quad + C_\delta \int_0^t \|v_t-\bar{v}_t\|^2_{L^2(\Omega;\mathbb{R}^3)} + C\int_0^t \|\eta-\bar{\eta}\|^2_{H^2(\Omega_0^s;\mathbb{R}^3)} \\
& \quad + C\int_0^t \|v-\bar{v}\|^2_{H^1(\Omega;\mathbb{R}^3)},
\end{aligned}
$$

which by patching all the charts defining Γ_0 leads to an estimate of $v - \bar{v}$ in $L^2(0, t; H^{1.5}(\Gamma_0; \mathbb{R}^3))$ yielding by elliptic regularity:

$$\begin{aligned}\int_0^t [\|v-\bar{v}\|^2_{H^2(\Omega_0^f;\mathbb{R}^3)} + \|q-\bar{q}\|^2_{H^1(\Omega_0^f;\mathbb{R})}] + \|\eta(t)-\bar{\eta}(t)\|^2_{H^2(\Omega_0^s;\mathbb{R}^3)} \\ \leqq C[\sqrt{t}+\delta]\int_0^t [\|v-\bar{v}\|^2_{H^2(\Omega_0^f;\mathbb{R}^3)} + \|q-\bar{q}\|^2_{H^1(\Omega_0^f;\mathbb{R})}] \\ +C_\delta \int_0^t \|v_t-\bar{v}_t\|^2_{L^2(\Omega;\mathbb{R}^3)} + C\int_0^t \|\eta-\bar{\eta}\|^2_{H^2(\Omega_0^s;\mathbb{R}^3)} \\ +C\int_0^t \|v-\bar{v}\|^2_{H^1(\Omega;\mathbb{R}^3)}.\end{aligned}$$

Thus, with a choice of $\delta > 0$ small enough, we have for t small enough by the use of Gronwall's inequality,

$$\begin{aligned}\|\eta(t)-\bar{\eta}(t)\|^2_{H^2(\Omega_0^s;\mathbb{R}^3)} + \int_0^t [\|v-\bar{v}\|^2_{H^2(\Omega_0^f;\mathbb{R}^3)} + \|q-\bar{q}\|^2_{H^1(\Omega_0^f;\mathbb{R})}] \\ \leqq C\int_0^t \|v_t-\bar{v}_t\|^2_{L^2(\Omega;\mathbb{R}^3)} + C\int_0^t \|v-\bar{v}\|^2_{H^1(\Omega;\mathbb{R}^3)}.\end{aligned}$$

By using this estimate in (54), we then get for a time small enough,

$$\begin{aligned}\left|\int_0^t (q-\bar{q}, (a_i^j-\bar{a}_i^j)\, \bar{v}^i_{tt},_j)_{L^2(\Omega_0^f;\mathbb{R})}\right| + \left|(q-\bar{q}, (a_i^j-\bar{a}_i^j)\, \bar{v}^i_t,_j)_{L^2(\Omega_0^f;\mathbb{R})}(t)\right| \\ \leqq C\sqrt{t}\left[\int_0^t \|v_t-\bar{v}_t\|^2_{L^2(\Omega;\mathbb{R}^3)} + \int_0^t \|v-\bar{v}\|^2_{H^1(\Omega;\mathbb{R}^3)} \right. \\ \left. +\|q-\bar{q}\|^2_{L^\infty(0,t;L^2(\Omega_0^f;\mathbb{R}))}\right].\end{aligned}$$

By putting together the estimates on I_4, I_5 and I_6, we have

$$\begin{aligned}\left|\int_0^t ([a_i^j q - \bar{a}_i^j\bar{q}]_t, [v^i,_j - \bar{v}^i,_j]_t)_{L^2(\Omega_0^f;\mathbb{R})}\right| \\ \leqq C\sqrt{t}\left[\int_0^t \|v_t-\bar{v}_t\|^2_{L^2(\Omega;\mathbb{R}^3)} + \int_0^t \|v-\bar{v}\|^2_{H^1(\Omega;\mathbb{R}^3)} \right. \\ \left. +\|q-\bar{q}\|^2_{L^\infty(0,t;L^2(\Omega_0^f;\mathbb{R}))}\right]. \qquad (55)\end{aligned}$$

Now, by considering the difference between the two variational forms satisfied rep-ectively by (v, q) and $(\bar{v}, \bar{q})$, and writing the difference between the pressure terms as

$$\int_{\Omega_0^f} (a_i^j q - \bar{a}_i^j\bar{q})\phi^i,_j = \int_{\Omega_0^f} a_i^j (q-\bar{q})\phi^i,_j + \int_{\Omega_0^f} (a_i^j - \bar{a}_i^j)\bar{q}\phi^i,_j\,,$$

the Lagrange multiplier Lemma 13 of [5] yields for all $t \in [0, T]$,

$$\|q(t) - \bar{q}(t)\|_{L^2(\Omega_0^f;\mathbb{R})} \leqq C[\ \|(v_t - \bar{v}_t)(t)\|_{L^2(\Omega;\mathbb{R}^3)} + \|(v - \bar{v})(t)\|_{H^1(\Omega_0^f;\mathbb{R}^3)} + \|(\eta - \bar{\eta})(t)\|_{H^1(\Omega_0^s;\mathbb{R}^3)} + \sqrt{t}\|v - \bar{v}\|_{L^2(0,t;H^1(\Omega_0^f;\mathbb{R}^3))}]. \tag{56}$$

By putting together the estimates (45)–(56), we then obtain for $t_u > 0$ small enough an inequality of the type:

$$\|v_t - \bar{v}_t\|^2_{L^\infty(0,t_u;L^2(\Omega;\mathbb{R}^3))} + \int_0^{t_u} \|v_t - \bar{v}_t\|^2_{H^1(\Omega_0^f;\mathbb{R}^3)} + \|v - \bar{v}\|^2_{L^\infty(0,t_u;H^1(\Omega_0^s;\mathbb{R}^3))} \leqq 0,$$

which shows that $(v, q) = (\bar{v}, \bar{q})$ on $[0, t_u]$. Let

$$T_u = \sup\{t \in [0, T]|\ (v, q) = (\bar{v}, \bar{q}) \text{ on } [0, t]\}.$$

If $T_u < T$, we can repeat the same procedure with T_u replacing 0, which would lead to uniqueness for $[T_u, T_u + \delta t)$ as well. Thus, we have $T_u = T$, which concludes the proof of the theorem. □

12. Optimal regularity on the initial data

We first recall some extensions and regularization results on domains:

Lemma 4. *Let Ω' be a domain of class H^4. Then, there exists a linear and continuous operator $E(\Omega')$ from $H^m(\Omega';\mathbb{R}^3)$ into $H^m(\mathbb{R}^3;\mathbb{R}^3)$ (for each $0 \leqq m \leqq 4$) such that $E(\Omega')(u) = u$ in Ω'. Also, if the H^4 norms of a family of domains stay bounded, the norms of the corresponding linear operators also stay bounded.*

Lemma 5. *Since Ω_0^s is of class H^4, let $\psi^m \in H^4(B_-(0,1);\mathbb{R}^3)$ $(m = 1, ..., N)$ be a collection of charts defining a neighborhood of its boundary. We note*

$$\|\Omega_0^f\|_{H^4} = \sum_{m=1}^N \|\psi^m\|_{H^4(B_-(0,1);\mathbb{R}^3)}.$$

Then, there exists a sequence of domains $(\Omega_0^{s,n})$ of class C^∞, so that $\Omega_0^s \subset \Omega_0^{s,n}$, and the domains are defined with a collection of charts $\psi^{m,n} \in H^4(B_-(0,1);\mathbb{R}^3)$ $(m = 1, ..., N)$ so that $\sum_{m=1}^N \|\psi^m - \psi^{m,n}\|_{H^4(B_-(0,1);\mathbb{R}^3)} \to 0$ as $n \to \infty$. Then denote the complementary of $\overline{\Omega}_0^{s,n}$ in Ω by $\Omega_0^{f,n}$ and $\Gamma_0^n = \partial\Omega_0^{s,n}$. Assume that n is large enough so that the different connected components of $\Omega_0^{s,n}$ (if there is more than one solid) do not intersect each other or the boundary of Ω. Denote $\alpha_n = \|\Omega^{s,n}\|_{H^6}$.

We now state the optimal regularity assumptions needed in our analysis, and explain the adjustements required to the previous proofs.

Theorem 2. *With the same assumptions as in Theorem 1, except for the following concerning the regularity of the initial data:*

$$u_0 \in H^6(\Omega_0^f;\mathbb{R}^3)\cap H^3(\Omega_0^s;\mathbb{R}^3)\cap H_0^1(\Omega;\mathbb{R}^3)\cap L^2_{div,f}, \quad (57a)$$

$$\begin{aligned} f_s(0) &\in H^2(\Omega_0^s;\mathbb{R}^3)\cap H^{3.5}(\Gamma_0;\mathbb{R}^3),\\ (f_s)_t(0) &\in H^1(\Omega_0^s;\mathbb{R}^3),\\ (f_s)_{tt}(0) &\in L^2(\Omega;\mathbb{R}^3), \end{aligned} \quad (57b)$$

the conclusion of Theorem 1 still holds.

Remark 8. We have chosen here to take different forcings for the fluid, which we still denote as f with the same assumptions as in Theorem 1, and the solid, in order to point out that the higher-order regularity required indeed comes from the hyperbolic scaling of the Navier-Stokes equations. The somewhat not-so-natural condition $f_s(0)\in H^{3.5}(\Gamma_0;\mathbb{R}^3)$ is set in order to get $w_1\in H^4(\Omega_0^f;\mathbb{R}^3)$ associated with the condition $w_3\in L^2(\Omega_0^f;\mathbb{R}^3)$.

Proof. The idea is to first regularize the domains and initial data, modify the forcings in an appropriate way, and then pass to the limit.

Given $0\leqq\rho\in\mathcal{D}(B(0,1))$ with $\int_{B(0,1)}\rho=1$, we define as usual $\rho_n(x)=n^3\rho(nx)$.

We first notice that u_0, w_1, q_0 and q_1 still have the same regularity in Ω_0^f as in Theorem 1. We first define in $\Omega_0^{f,n}$, $u_0^n=u_0$ and $w_1^n=w_1$, $q_0^n=q_0$, $q_1^n=q_1$, which is permitted since $\Omega_0^{f,n}\subset\Omega_0^f$. We next define w_2^n in $\Omega_0^{f,n}$,

$$\begin{aligned} -\nu\Delta w_2^n+\nabla q_2^n &= \rho_n\star E(\Omega_0^f)(-\nu\Delta w_2+\nabla q_2) && \text{in } \Omega_0^{f,n}, && (58a)\\ \operatorname{div} w_2^n &= -[(a_i^j)_{tt}(0)u_0^i,_j+2(a_i^j)_t(0)w_1^i,_j] && \text{in } \Omega_0^{f,n}, && (58b)\\ w_2^n &= 0 && \text{on } \partial\Omega, && (58c)\\ \nu\frac{\partial w_2^n}{\partial N^n}-\tilde q_2^n N^n &= \nu\frac{\partial}{\partial N^n}\rho_n\star E(\Omega_0^f)(w_2)-\rho_n\star E(\Omega_0^f)(q_2)N^n && \text{on } \Gamma_0^n, && (58d) \end{aligned}$$

where N^n denotes the unit normal exterior to $\Omega_0^{f,n}$. Finally we define $w_3^n\in L^2(\Omega_0^{f,n};\mathbb{R}^3)$ by

$$w_3^n=[\nu(a_l^j a_l^k u,_k),_j-(a_i^j q,_j)_{i=1}^3+F]_{tt}(0) \text{ in } \Omega_0^{f,n},$$

where the time derivatives on the right-hand side are computed with the usual rules from $u(0)=u_0^n$, $\partial_t^p u(0)=w_p^n$ $(p=1,2)$, $\partial_t^p q(0)=q_p^n$ $(p=0,1,2)$.

We next define u_0^n in the solid by

$$\begin{aligned} L^2u_0^n &= L^2[\rho_n\star E(\Omega_0^s)(u_0)] && \text{in } \Omega_0^{s,n}, && (59a)\\ u_0^n &= (u_0^n)^f && \text{on } \Gamma_0^n, && (59b)\\ L(u_0^n)+\rho_n\star E(\Omega_0^s)((f_s)_t(0)) &= (w_2^n)^f && \text{on } \Gamma_0^n, && (59c) \end{aligned}$$

where the right-hand sides of the previous boundary conditions come from the fluid regularization previously carried out. Note also that

$$Lu_0^n \in H^4(\Omega_0^{s,n};\mathbb{R}^3), \tag{60}$$

(with an estimate that may blow up as $n \to \infty$) since

$$\begin{aligned} L(Lu_0^n) &= L^2[\rho_n \star E(\Omega_0^s)(u_0)] && \text{in } \Omega_0^{s,n},\\ L(u_0^n) &= -\rho_n \star E(\Omega_0^s)((f_s)_t(0)) + (w_2^n)^f && \text{on } \Gamma_0^n. \end{aligned}$$

We can then define f_0^n in $\Omega_0^{s,n}$ by

$$\begin{aligned} L^2 f_0^n &= L^2[\rho_n \star E(\Omega_0^s)(f_s(0))] \text{ in } \Omega_0^{s,n},\\ f_0^n &= (w_1^n)^f \qquad\qquad \text{on } \Gamma_0^n,\\ c^{mjki}(f_0^n,_m \cdot\mathrm{Id},_j + f_0^n,_j \cdot\mathrm{Id},_m)N_k^n &= -2c^{mjki}\,(u_0^n,_m \cdot\mathrm{Id},_j + u_0^n,_j \cdot\mathrm{Id},_m)u_0^n,{}_k^i\\ &\quad + \nu[v^i,_k a_l^k a_l^j]_{tt}(0)N_j^n - [qa_i^j]_{tt}(0)N_j^n \text{ on } \Gamma_0^n, \end{aligned}$$

with the same conventions as for the previous system for the time derivatives evaluated from $\Omega_0^{f,n}$, and $c^{mjkl}\,(u_0^n,_m \cdot\mathrm{Id},_j + u_0^n,_j \cdot\mathrm{Id},_m)u_0^n,{}_k^i\; N_l^n$ evaluated from $\Omega_0^{s,n}$. We then define in $\Omega_0^{s,n}$,

$$\begin{aligned} w_1^n &= f_0^n,\\ w_2^n &= \big[[c^{mjkl}(\eta,_m \cdot \eta,_j - \delta_{mj})\eta,_k],_l\big]_t(0) + \rho_n \star E(\Omega_0^s)((f_s)_t(0)),\\ &= L(u_0^n) + \rho_n \star E(\Omega_0^s)((f_s)_t(0)),\\ w_3^n &= \big[[c^{mjkl}(\eta,_m \cdot \eta,_j - \delta_{mj})\eta,_k],_l\big]_{tt}(0) + \rho_n \star E(\Omega_0^s)((f_s)_{tt}(0)), \end{aligned}$$

where the time derivatives on the right-hand side are evaluated with $v(0) = u_0^n$, $v_t(0) = w_1^n$. We also define the regularized forcing in the solid

$$f^n(t) = \rho_n \star E(\Omega_0^s)(f_s(t) - f_s(0)) \;+ f_0^n \text{ in } \Omega_0^{s,n}.$$

We then have u_0^n, w_1^n, w_2^n in $H_0^1(\Omega;\mathbb{R}^3) \cap H^4(\Omega_0^{f,n};\mathbb{R}^3) \cap H^4(\Omega_0^{s,n};\mathbb{R}^3)$ and $\operatorname{div} u_0^n = 0$ in $\Omega_0^{f,n}$, $w_3^n \in L^2(\Omega;\mathbb{R}^3)$, with

$$\|E(\Omega_0^{f,n})(u_0^n) - u_0\|_{H^4(\Omega_0^f;\mathbb{R}^3)} + \|E(\Omega_0^{s,n})(u_0^n) - u_0\|_{H^3(\Omega_0^s;\mathbb{R}^3)} \to 0 \text{ as } n \to \infty, \tag{61a}$$

$$\begin{aligned} &\|E(\Omega_0^{f,n})(w_1^n) - w_1\|_{H^4(\Omega_0^f;\mathbb{R}^3)} + \|E(\Omega_0^{s,n})(w_1^n) - w_1\|_{H^2(\Omega_0^s;\mathbb{R}^3)}\\ &\quad + \|w_2^n - w_2\|_{H^1(\Omega;\mathbb{R}^3)} + \|E(\Omega_0^{f,n})(w_2^n) - w_2\|_{H^2(\Omega_0^f;\mathbb{R}^3)} \to 0 \text{ as } n \to \infty, \end{aligned} \tag{61b}$$

$$\|u_0^n\|_{H^6(\Omega_0^{s,n};\mathbb{R}^3)} \leqq \beta_n,\quad \|w_1^n\|_{H^4(\Omega_0^{s,n};\mathbb{R}^3)} \leqq \beta_n,\quad \|w_2^n\|_{H^4(\Omega_0^{s,n};\mathbb{R}^3)} \leqq \beta_n, \tag{61c}$$

$$\|w_3^n - w_3\|_{L^2(\Omega;\mathbb{R}^3)} \to \infty \text{ as } n \to \infty, \tag{61d}$$

where β_n is a given polynomial expression of α_n and n. We briefly explain how those constants appear. For instance, for the first estimate of (61c), we have by elliptic regularity on (59) that $\|u_0^n\|_{H^6(\Omega_0^{s,n};\mathbb{R}^3)}$ is bounded by a sum of terms, one of which being

$P(\|\Omega_0^{s,n}\|_{H^6})\, \|(w_2^n)^f\|_{H^4(\Omega_0^{f,n};\mathbb{R}^3)}$, P being a polynomial which does not depend on n. Next, still by elliptic regularity on (58), we have that $\|(w_2^n)^f\|_{H^4(\Omega_0^{f,n};\mathbb{R}^3)}$ is bounded by a sum of terms such as $\|\rho_n \star E(\Omega_0^f)(\triangle w_2)\|_{H^2(\Omega_0^{f,n};\mathbb{R}^3)}$. This particular term, by the properties of the convolution, is in turn bounded by $n^3\|E(\Omega_0^{f,n})(w_2)\|_{H^1(\mathbb{R}^3;\mathbb{R}^3)}$. This shows that a term of the type $P(\alpha_n)n^3\|w_2\|_{H^1(\Omega_0^f;\mathbb{R}^3)}$ appears in the sum of all terms bounding $\|u_0^n\|_{H^6(\Omega_0^{s,n};\mathbb{R}^3)}$. Since the other terms in the sum can be dealt with similarly, this explains our estimate (61c).

For the pressures, we have

$$\begin{aligned} &\|E(\Omega_0^{f,n})(q_0^n) - q_0\|_{H^3(\Omega_0^f;\mathbb{R})} + \|E(\Omega_0^{f,n})(q_1^n) - q_1\|_{H^3(\Omega_0^f;\mathbb{R})} \\ &\quad + \|E(\Omega_0^{f,n})(q_2^n) - q_2\|_{H^1(\Omega_0^f;\mathbb{R})} \to 0 \text{ as } n \to \infty. \end{aligned} \tag{62}$$

Since the initial data u_0^n and forcings $f^n(0)$, $f_t^n(0)$, $f_{tt}^n(0)$ are smooth enough to ensure the regularity properties (61), we then deduce that we have similarly as for Theorem 1 the existence of a solution w_n of a system similar to (20) with f, u_0, Ω_0^f, Ω_0^s replaced by their counterparts with an exponent n, and b_κ, c_κ, d_κ replaced by b_n, c_n, d_n (with the choice $\kappa = \dfrac{1}{n(\beta_n+1)}$) given by

$$\begin{aligned} b_n(\phi) &= \frac{1}{n(\beta_n+1)}(c^{ijkl} w_2^n{}^k,_l\,, \phi^i,_j)_{L^2(\Omega_0^{s,n};\mathbb{R})}, \\ c_n(\phi) &= \frac{1}{n(\beta_n+1)}(c^{ijkl} w_1^n{}^k,_l\,, \phi^i,_j)_{L^2(\Omega_0^{s,n};\mathbb{R})} + (w_2^n - w_2, \phi)_{L^2(\Omega_0^{f,n};\mathbb{R})} \\ &\quad + (-[(a_i^j q)_t(0)N_j^n]_{i=1}^3 + [(a_l^j a_l^k)u,_k]_t(0)N_j^n,\ \phi)_{L^2(\Gamma_0^n;\mathbb{R}^3)} \\ &\quad - (c^{ijkl}[(\eta,_i\cdot\eta,_j - \delta_{ij})\eta_k]_t(0)N_l^n,\ \phi)_{L^2(\Gamma_0^n;\mathbb{R}^3)}, \\ d_n(\phi) &= \frac{1}{n(\beta_n+1)}(c^{ijkl} u_0^n{}^k,_l\,, \phi^i,_j)_{L^2(\Omega_0^{s,n};\mathbb{R})} \\ &\quad + (-[(a_i^j q)(0)N_j^n]_{i=1}^3 + [(a_l^j a_l^k)u,_k](0)N_j^n,\ \phi)_{L^2(\Gamma_0^n;\mathbb{R}^3)}, \end{aligned}$$

where the time derivatives are computed with a velocity satisfying $u(0) = u_0^n$, $u_t(0) = w_1^n$ and a pressure such that $q(0) = q_0^n$, $q_t(0) = q_1^n$. Note that by construction, the solutions w_n of these problems in Ω satisfy $w_n(0) = u_0^n$, $w_{nt}(0) = w_1^n$, $w_{ntt}(0) = w_2^n$, $w_{nttt}(0) = w_3^n$. Next, we proceed to energy estimates similar to Section 8. The bounds obtained are similar, except that this time the terms associated with b_n, c_n and d_n tend to zero as $n \to \infty$. This is clear from the convergence results (61) and (62) for the integral terms associated with the fluid. The terms associated with the solid asymptotically tend to zero by properties of the convolution. For instance, with the notations of Section 8, for $\phi_p = -[\rho_p \star (\zeta^2\ w_n \circ \Psi)],_{\alpha_1\alpha_1\alpha_2\alpha_2\alpha_3\alpha_3} \circ \Psi^{-1}$, we obtain after a change of variables, an integration by parts in time, and three integrations by parts in space:

$$\left|\int_0^t \frac{t'^2}{2n(\beta_n+1)}(c^{ijkl}w^n_{2}{}_{,l}^{\ k}, \phi^i_p{}_{,j})_{L^2(\Omega_0^{s,n};\mathbb{R})}\,dt'\right| \leqq \frac{C}{n(\beta_n+1)}\|w_2^n\|_{H^4(\Omega_0^{s,n};\mathbb{R}^3)}\|\eta_n\|_{L^\infty(0,t;H^4(\Omega_0^{s,n};\mathbb{R}^3))}.$$

Thus with our estimate (61c), we have

$$|\int_0^t \frac{t'^2}{2n(\beta_n+1)}(c^{ijkl}w^n_{2}{}_{,l}^{\ k}, \phi^i_p{}_{,j})_{L^2(\Omega_0^{s,n};\mathbb{R})}\,dt'| \leqq \frac{C}{n}\|(w_n,q_n)\|_{Z_t^n},$$

where Z_t^n denotes the same type of space as Z_t with Ω_0^s and Ω_0^f replaced by their counterparts with an exponent n. This type of estimate thus shows that this term does not change the energy inequalities in Section 8. We can thus reproduce the arguments of Section 9, establishing that (w_n, q_n) can be defined over a time T independently of n, and its norm in Z_T^n depends solely on $N(u_0^n, (w_i^n)_{i=1}^3) + N((q_i)_{i=0}^2) + M(f^n, 0, 0)$ and thus, thanks to the estimates (61a), (61b), (61d), solely on $N(u_0, (w_i)_{i=1}^3) + N((q_i)_{i=0}^2) + M(f, 0, 0)$. We can then consider the sequence $(E(\Omega)(w_n), E(\Omega_0^{f,n})(q_n))$ which is bounded in a space similar as Z_T but defined on $\mathbb{R}^3$, and extract (with respect to n) a weakly convergent sequence in a space modified from Y_T by replacing the condition $u \in H_0^1(\Omega;\mathbb{R}^3)$ by $u \in H^1(\Omega;\mathbb{R}^3)$. By the classical compactness results, we next see that the weak limit $(v, q) \in Z_T$ and is a solution of (4) with f as the forcing and $v(0) = u_0$. This solution is also unique in Z_T. □

13. The case of incompressible elasticity

In this section, we explain how to treat the supplementary difficulties appearing when the incompressibility constraint is added in the solid. This leads to the same system as (4), with the addition of the condition $\det \nabla\eta = 1$ *a.e.* in Ω_0^s, the addition of $[(a_i^k q)_{,k}]_{i=1}^3$ on the left-hand side of (4d) and the addition of $-qa_i^j N_j$ (the trace of q being from the solid phase in this new term) on the left-hand side of (4e). We now state our result and explain how to overcome the additional difficulties related to this constraint.

We first update our functional frameworks. While X_T and W_T do not change, Y_T and Z_T become respectively

$$Y_T = \Big\{(v,q) \in X_T \times L^2(0,T;L^2(\Omega;\mathbb{R}))|\ \partial_t^n q \in L^2(0,T;H^{3-n}(\Omega_0^f;\mathbb{R})), \partial_t^n q \in L^2(0,T;H^{3-n}(\Omega_0^s;\mathbb{R}))(n=0,1,2)\Big\},$$

$$Z_T = \Big\{(v,q) \in W_T \times L^2(0,T;L^2(\Omega;\mathbb{R}))|\ \partial_t^n q \in L^2(0,T;H^{3-n}(\Omega_0^f;\mathbb{R})), \partial_t^n q \in L^2(0,T;H^{3-n}(\Omega_0^s;\mathbb{R}))(n=0,1,2)|\ q_{tt} \in L^\infty(0,T;L^2(\Omega;\mathbb{R}))\Big\}.$$

Remark 9. Whereas the pressure in the solid satisfies $\partial_t^n q \in L^\infty(0,T; H^{3-n}(\Omega_0^s;\mathbb{R}))$ $(n = 0,1,2)$, it appears that the limit pressures q_κ are controlled uniformly in the norm of Z_T and seemingly not in these norms. Note also that while the velocity field is smoother in the fluid phase for the solution of our next theorem, the pressure field is actually smoother in the solid phase. Whereas our artificial viscosity smoothes the velocity field in the solid, it also interestingly makes the pressure in the solid for the regularized system less smooth than the one associated with the solution of the constrained problem, which is a source of difficulties that we shall describe later.

We now state our result:

Theorem 3. *With the same regularity assumptions as in Theorem 2 and assuming that the compatibility conditions associated with our new system at $t = 0$ hold (for the sake of conciseness we do not state them here), the conclusion of Theorem 1 holds for the case where the incompressibility constraint is added to the solid part. Furthermore, $\partial_t^n q \in L^\infty(0,T; H^{3-n}(\Omega_0^s;\mathbb{R}))(n = 0,1,2)$.*

Proof. The extra regularity (with respect to the norm of Z_T) on the pressure in the solid simply comes from the equation

$$v_t - c^{mjkl}[(\eta,_m \cdot \eta,_j - \delta_{ij})\eta,_k],_l + a_i^j q,_j = f \text{ in } (0,T)\times\Omega_0^s,$$

which once the regularity for the solution $w \in W_T$ is known provides immediately the result. We now explain how to obtain a solution in Z_T.

The beginning of the proof follows the same lines as in the compressible elasticity case. We first assume that the initial data satisfies the regularity assumptions of Theorem 1, and define the same smoothed problem as (14) with the corresponding updates for the incompressibility constraint. We then define the same fixed point linear problem as (17) where the condition $a_i^k w^i,_k = 0$ in Ω_0^s is added (the a_i^k being computed from the given v). Next we add $a_i^k q,_k$ on the left-hand side of (17c) and $-q a_i^j N_j$ (the traces being taken from Ω_0^s) on the left-hand side of (17d).

We then proceed as in [5] to construct a solution of this system by a penalty method (the penalty term being this time defined over Ω) and get the same type of regularity result. This provides us with a solution (w_κ, q_κ), which we also denote by $(\tilde{w}, \tilde{q})$, of the incompressible version of (14) on a time T_κ shrinking to zero. As in the compressible case, (w_κ, q_κ) is in Z_{T_κ}, and since our smoothed problem has a parabolic artificial viscosity, we also have for the velocity in the solid the regularity $\partial_t^n w \in L^2(0,T_\kappa; H^{4-n}(\Omega_0^s;\mathbb{R}^3))$ $(n = 0,1,2,3)$ (with estimates that blow up as $\kappa \to 0$). Thus, $(w_\kappa, q_\kappa) \in \tilde{Z}_{T_\kappa}$ with

$$\tilde{Z}_t = \left\{(w,q) \in Z_t \,|\, \partial_t^n w \in L^2(0,t; H^{4-n}(\Omega_0^s;\mathbb{R}^3))(n = 0,1,2)\right\},$$

endowed with the norm

$$\|(w,q)\|^2_{\tilde{Z}_t} = \|(w,q)\|^2_{Z_t} + \kappa^2 \sum_{n=0}^{2} \|\partial_t^n w\|^2_{L^2(0,t;H^{4-n}(\Omega_0^s;\mathbb{R}^3))}.$$

We next proceed as in Section 8 to get energy estimates, which will be carried out this time for the κ dependent norm of $\tilde{Z}_t$, independently of κ on $[0, T_\kappa]$, and for such a purpose it is important to keep the κ^2 factor in the definition of the norm. We could extend the sum to $n = 3$, though it is not necessary.

As before, the first set of estimates has to be carried out on the highest-order time derivative. Our energy inequality (21) has the same form, except that the integrals over Ω_0^f where $\tilde{q}$ appears have to be taken this time on Ω. The part over Ω_0^f is estimated as before. We now explain how to deal with the integrals set on Ω_0^s for the pressure, which indeed needs some justifications since the velocity in the solid is not controlled uniformly in κ in a space as smooth as the velocity in the fluid, while the pressure is controlled in the same type of spaces in both phases. □

13.1. Estimates on $\tilde{w}_{ttt}$

Here t denotes any time in $(0, T_\kappa)$. The most difficult integrals set in $[0, t] \times \Omega_0^s$ and associated with the incompressibility constraint in the solid are $K_1 = \int_0^t \int_{\Omega_0^s} \tilde{q}_{tt} (\tilde{a}_i^j)_t \tilde{w}^i_{ttt},_j$ and $K_2 = \int_0^t \int_{\Omega_0^s} \tilde{q} (\tilde{a}_i^j)_{ttt} \tilde{w}^i_{ttt},_j$, the others being either less difficult or similar to estimate.

Step 1. For K_1, if we denote $N^s = -N$, we have

$$
\begin{aligned}
|K_1| &= \left| -\int_0^t \int_{\Omega_0^s} (\tilde{q}_{tt}),_j (\tilde{a}_i^j)_t \tilde{w}^i_{ttt} + \int_0^t \int_{\Gamma_0} \tilde{q}_{tt} (\tilde{a}_i^j)_t \tilde{w}^i_{ttt} N_j^s \right| \\
&\leqq C\Bigg[\int_0^t \|\tilde{q}_{tt}\|_{H^1(\Omega_0^s;\mathbb{R})} \|\tilde{\eta}\|_{H^3(\Omega_0^s;\mathbb{R}^3)} \|\tilde{w}\|_{H^3(\Omega_0^s;\mathbb{R}^3)} \|\tilde{w}_{ttt}\|_{L^2(\Omega_0^s;\mathbb{R}^3)} \\
&\quad + \int_0^t \|\tilde{q}_{tt}\|_{H^{\frac{1}{2}}(\Omega_0^s;\mathbb{R})} \|\tilde{\eta}\|_{H^3(\Omega_0^s;\mathbb{R}^3)} \|\tilde{w}\|_{H^3(\Omega_0^s;\mathbb{R}^3)} \|\tilde{w}_{ttt}\|_{H^{\frac{1}{2}}(\Omega_0^f;\mathbb{R}^3)} \Bigg] \\
&\leqq C\sqrt{t} \|(\tilde{w}, \tilde{q})\|^4_{Z_t} + C t^{\frac{1}{2}} \sup_{[0,t]} \left[\|\tilde{q}_{tt}\|^{\frac{1}{2}}_{L^2(\Omega_0^s;\mathbb{R})} \|\tilde{w}_{ttt}\|^{\frac{1}{2}}_{L^2(\Omega_0^f;\mathbb{R}^3)} \right] \|(\tilde{w}, \tilde{q})\|^3_{Z_t} \qquad (63) \\
&\leqq C t^{\frac{1}{4}} \|(\tilde{w}, \tilde{q})\|^4_{Z_t}, \qquad (64)
\end{aligned}
$$

where we have used the continuity of $\tilde{w}_{ttt}$ in the sense of traces along Γ_0 to bound the $L^2(\Gamma_0; \mathbb{R}^3)$ norm of $\tilde{w}_{ttt}$ by means of the $H^{\frac{1}{2}}(\Omega_0^f; \mathbb{R}^3)$ norm. Note that we have also used the fact that the $L^\infty(L^2)$ norm of $\tilde{q}_{tt}$ is in the definition of the norm of Z_t. In order to get an estimate on this norm, we should proceed in a way similar to the one used for (29) in Section 9.

Step 2. Concerning K_2, we have by integration by parts in space:

$$K_2 = -\int_0^t \int_{\Omega_0^s} \tilde{q},_j (\tilde{a}_i^j)_{ttt} \tilde{w}^i_{ttt} + \int_0^t \int_{\Gamma_0} \tilde{q} (\tilde{a}_i^j)_{ttt} \tilde{w}^i_{ttt} N_j^s,$$

since our artificial viscosity provides the regularity $\tilde{w}_{tt} \in L^2(0, T_\kappa; H^2(\Omega_0^s; \mathbb{R}^3))$ and $\tilde{w}_{ttt} \in L^2(0, T_\kappa; H^1(\Omega_0^s; \mathbb{R}^3))$ (with estimates that may blow up as $\kappa \to 0$).

The difficulty here comes from the second integral. Whereas s for K_1 we can estimate the trace of $\tilde{w}_{ttt}$ on Γ_0 from the fluid, we have to take the norm of $\nabla\tilde{w}_{tt}$ in $H^{-0.5}(\Gamma_0;\mathbb{R}^9)$, which is problematic since the norm Z_t contains only its $L^2(\Omega_0^s;\mathbb{R}^9)$ norm. In order to circumvent this difficulty, we notice that the same formula holds if we replace $\tilde{w}_{ttt}$ by $E(\Omega_0^f)(\tilde{w}_{ttt}^f)$ (the extension to $\mathbb{R}^3$ of the velocity in the fluid). Since $\tilde{w}_{ttt} = \tilde{w}_{ttt}^f$ on Γ_0, we have $\tilde{w}_{ttt} = E(\Omega_0^f)(\tilde{w}_{ttt}^f)$ on Γ_0, which implies that

$$\begin{aligned} K_2 = &-\int_0^t\int_{\Omega_0^s}\tilde{q},_j(\tilde{a}_i^j)_{ttt}\tilde{w}_{tt}^i + \int_0^t\int_{\Omega_0^s}\tilde{q}(\tilde{a}_i^j)_{ttt}E(\Omega_0^f)(\tilde{w}_{ttt}^f),_j^i \\ &+\int_0^t\int_{\Omega_0^s}\tilde{q},_j(\tilde{a}_i^j)_{ttt}E(\Omega_0^f)(\tilde{w}_{ttt}^f)^i, \end{aligned}$$

and thus,

$$\begin{aligned} K_2 \leqq\ & C\int_0^t \|\tilde{q}\|_{H^3(\Omega_0^s;\mathbb{R})}\|\tilde{w}_{tt}\|_{H^1(\Omega_0^s;\mathbb{R}^3)}\|\tilde{\eta}\|_{H^3(\Omega_0^s;\mathbb{R}^3)}\|\tilde{w}_{ttt}\|_{L^2(\Omega;\mathbb{R}^3)} \\ &+C\int_0^t \|\tilde{q}\|_{H^3(\Omega_0^s;\mathbb{R})}\|\tilde{w}_{t}\|_{H^1(\Omega_0^s;\mathbb{R}^3)}\|\tilde{w}\|_{H^3(\Omega_0^s;\mathbb{R}^3)}\|\tilde{w}_{ttt}\|_{L^2(\Omega;\mathbb{R}^3)} \\ &+C\int_0^t \|q_0+\int_0^\cdot \tilde{q}_t\|_{H^2(\Omega_0^s;\mathbb{R})}\|\tilde{w}_{tt}\|_{H^1(\Omega_0^s;\mathbb{R}^3)}\|\tilde{\eta}\|_{H^3(\Omega_0^s;\mathbb{R}^3)}\|\tilde{w}_{ttt}\|_{H^1(\Omega_0^f;\mathbb{R}^3)} \\ &+C\int_0^t \|q_0+\int_0^\cdot \tilde{q}_t\|_{H^2(\Omega_0^s;\mathbb{R})}\|\tilde{w}_{t}\|_{H^1(\Omega_0^s;\mathbb{R}^3)}\|\tilde{w}\|_{H^3(\Omega_0^s;\mathbb{R}^3)}\|\tilde{w}_{ttt}\|_{H^1(\Omega_0^f;\mathbb{R}^3)} \\ \leqq\ & C\,\|(\tilde{w},\tilde{q})\|_{Z_t}^3\int_0^t\|\tilde{q}\|_{H^3(\Omega_0^s;\mathbb{R})} \\ &+C\,\|(\tilde{w},\tilde{q})\|_{Z_t}^2\|q_0\|_{H^2(\Omega_0^s;\mathbb{R})}\int_0^t\|\tilde{w}_{ttt}\|_{H^1(\Omega_0^f;\mathbb{R}^3)} \\ &+C\sqrt{t}\,\|(\tilde{w},\tilde{q})\|_{Z_t}^2\|\tilde{q}_t\|_{L^2(0,t;H^2(\Omega_0^s;\mathbb{R}))}\int_0^t\|\tilde{w}_{ttt}\|_{H^1(\Omega_0^f;\mathbb{R}^3)} \\ \leqq\ & C\sqrt{t}\,[\,\|(\tilde{w},\tilde{q})\|_{Z_t}^4 + N((q_i)_{i=0}^2)\,]. \end{aligned}$$

The most difficult integral set at time t on Ω_0^s and containing $\tilde{q}$ is

$$K_3 = \int_{\Omega_0^s}\tilde{q}_{tt}(\tilde{a}_i^j)_t\tilde{w}_{tt}^i,_j\,,$$

for which we apparently just have an estimate of the type $|I_3| \leqq C\|(\tilde{w},\tilde{q})\|_{Z_t}^2$ (without any small parameter in front). We now explain how to treat this difficulty.

Step 3. We first notice that

$$K_3 = -\int_{\Omega_0^s}\tilde{q}_{tt},_j(\tilde{a}_i^j)_t\tilde{w}_{tt}^i + \int_{\Gamma_0}\tilde{q}_{tt}(\tilde{a}_i^j)_t\tilde{w}_{tt}^iN_j^s.$$

If we could say that $\tilde{q}_{tt}$ is $L^\infty(H^1)$ controlled, the $L^\infty(L^2)$ control of $\tilde{w}_{tt}$ would give us a suitable bound for K_3. Whereas we have seen in the statement of our theorem that q_{tt} for the limit solution is indeed in $L^\infty(H^1)$, we cannot seemingly get such a bound on the approximate pressures $\tilde{q}_{tt}$. In order to get around this, we introduce similarly as in the previous step the extension to the solid domain of the velocity in the fluid. Since a similar integration by parts formula holds when we replace $\tilde{w}_{tt}$ by $E(\Omega_0^f)(\tilde{w}_{tt}^f)$, we deduce

$$K_3 = -\int_{\Omega_0^s} \tilde{q}_{tt},_j\, (\tilde{a}_i^j)_t \tilde{w}_{tt}^i + \int_{\Omega_0^s} \tilde{q}_{tt} (\tilde{a}_i^j)_t E(\Omega_0^f)(\tilde{w}_{tt}^f)^i,_j \\ + \int_{\Omega_0^s} \tilde{q}_{tt},_j\, (\tilde{a}_i^j)_t E(\Omega_0^f)(\tilde{w}_{tt}^f)^i. \tag{65}$$

The easier term to estimate is $K_3^2 = \displaystyle\int_{\Omega_0^s} \tilde{q}_{tt}(\tilde{a}_i^j)_t E(\Omega_0^f)(\tilde{w}_{tt}^f)^i,_j$, for which we have for an arbitrary $\delta > 0$:

$$\begin{aligned} |K_3^2| &\leqq C\|\tilde{q}_{tt}\|_{L^2(\Omega_0^s;\mathbb{R})}\|\mathrm{Id} + \int_0^t \tilde{w}\|_{H^3(\Omega_0^s;\mathbb{R}^3)}\|u_0 \\ &\quad + \int_0^t \tilde{w}_t\|^{\frac14}_{H^2(\Omega_0^s;\mathbb{R}^3)}\|\tilde{w}\|^{\frac34}_{H^3(\Omega_0^s;\mathbb{R}^3)}[\|w_2\|_{H^1(\Omega_0^f;\mathbb{R}^3)} \\ &\quad + \sqrt{t}\|\tilde{w}_{ttt}\|_{L^2(0,t;H^1(\Omega_0^f;\mathbb{R}^3))}] \\ &\leqq C\|\tilde{q}_{tt}\|_{L^2(\Omega_0^s;\mathbb{R})}[1 + t\|(\tilde{w},\tilde{q})\|_{Z_t}][N(u_0,(w_i)_{i=1}^3) \\ &\quad + t^{\frac14}\|(\tilde{w},\tilde{q})\|^{\frac14}_{Z_t}]\,\|(\tilde{w},\tilde{q})\|^{\frac34}_{Z_t}[N(u_0,(w_i)_{i=1}^3) + t^{\frac12}\|(\tilde{w},\tilde{q})\|_{Z_t}] \\ &\leqq C\|(\tilde{w},\tilde{q})\|_{Z_t}[N(u_0,(w_i)_{i=1}^3) + t^{\frac14}\|(\tilde{w},\tilde{q})\|^{\frac14}_{Z_t}]\|(\tilde{w},\tilde{q})\|^{\frac34}_{Z_t} \\ &\quad \times [N(u_0,(w_i)_{i=1}^3) + t^{\frac12}\|(\tilde{w},\tilde{q})\|_{Z_t}]^2 \\ &\leqq \delta\|(\tilde{w},\tilde{q})\|^2_{Z_t} + C_\delta N(u_0,(w_i)_{i=1}^3) + C_\delta t^{\frac14}\|(\tilde{w},\tilde{q})\|^4_{Z_t}. \end{aligned} \tag{66}$$

For the first integral, the nonlinear elastodynamics equation in Ω_0^s provides

$$\nabla\tilde{q}_{tt} = \tilde{a}^{-1}\Big[-\tilde{w}_{ttt} + \kappa L\tilde{w}_{tt} + c^{ijkl}[(\tilde{\eta},_i\cdot\tilde{\eta},_j - \delta_{ij})\tilde{\eta},_k]_{tt},_l \\ -2\tilde{a}_t\nabla\tilde{q}_t - \tilde{a}_{tt}\nabla\tilde{q} + f_{tt} + \kappa h\Big],$$

leading us for $K_3^1 = \displaystyle\int_{\Omega_0^s} \tilde{q}_{tt},_j\,(\tilde{a}_i^j)_t\tilde{w}_{tt}^i$ to (since $\tilde{a}^{-1} = \nabla\tilde{\eta}$ in virtue of $\det\nabla\tilde{\eta} = 1$),

$$K_3^1 = \int_{\Omega_0^s}\Big[\nabla\tilde{\eta}\,[-\tilde{w}_{ttt} + c^{ijkl}[(\tilde{\eta},_i\cdot\tilde{\eta},_j - \delta_{ij})\tilde{\eta},_k]_{tt},_l - 2\tilde{a}_t\nabla\tilde{q}_t - \tilde{a}_{tt}\nabla\tilde{q} \\ + f_{tt} + \kappa h]\Big]^j(\tilde{a}_i^j)_t\tilde{w}_{tt}^i + \kappa\int_{\Omega_0^s}[\nabla\tilde{\eta}\,[L\tilde{w}_{tt}]]^j(\tilde{a}_i^j)_t\tilde{w}_{tt}^i. \tag{67}$$

The integrals on the first line of this equality do not give any trouble and can be estimated in the same fashion. For instance, we have for

$$K_3^3 = \int_{\Omega_0^s}\Big[\nabla\tilde{\eta}[c^{ijkl}[(\tilde{\eta},_i\cdot\tilde{\eta},_j - \delta_{ij})\tilde{w}_t,_{lk}]\Big]^j(\tilde{a}_i^j)_t\tilde{w}_{tt}^i,$$

$$
\begin{aligned}
|K_3^3| &\leqq C\|\mathrm{Id} + \int_0^t \tilde{w}\|^4_{H^3(\Omega_0^s;\mathbb{R}^3)}\|\tilde{w}_t\|_{H^2(\Omega_0^s;\mathbb{R}^3)}\|w_2 + \int_0^t \tilde{w}_{ttt}\|_{L^2(\Omega_0^s;\mathbb{R}^3)}\|u_0 \\
&\quad + \int_0^t \tilde{w}_t\|^{\frac{1}{4}}_{H^2(\Omega_0^s;\mathbb{R}^3)}\|\tilde{w}\|^{\frac{3}{4}}_{H^3(\Omega_0^s;\mathbb{R}^3)} \\
&\leqq \delta\|(\tilde{w},\tilde{q})\|^2_{Z_t} + C_\delta t^{\frac{1}{4}}\|(\tilde{w},\tilde{q})\|^7_{Z_t} + C_\delta N(u_0,(w_i)_{i=1}^3)). \qquad (68)
\end{aligned}
$$

Now, the difficult term to handle is $K_3^4 = \kappa\int_{\Omega_0^s}[\nabla\tilde{\eta}[L\tilde{w}_{tt}]]^j(\tilde{a}_i^j)_t\tilde{w}_{tt}^i$. We first write the divergence form $L\tilde{w}_{tt}^p = \sigma,_m^{mp}(\tilde{w}_{tt})$, and integrate by parts:

$$
\begin{aligned}
K_3^4 = &-\kappa\int_{\Omega_0^s}[\nabla\tilde{\eta},_m[\sigma^{mp}(\tilde{w}_{tt})]_{p=1}^3]^j(\tilde{a}_i^j)_t\tilde{w}_{tt}^i \\
&-\kappa\int_{\Omega_0^s}[\nabla\tilde{\eta}[\sigma^{mp}(\tilde{w}_{tt})]_{p=1}^3]^j[(\tilde{a}_i^j)_t\tilde{w}_{tt}^i],_m \\
&+\kappa\int_{\Gamma_0}[\nabla\tilde{\eta}[\sigma^{mp}(\tilde{w}_{tt})]_{p=1}^3]^j(\tilde{a}_i^j)_t\tilde{w}_{tt}^i N_m^s,
\end{aligned}
$$

leading us to

$$
\left|K_3^4 - \kappa\int_{\Gamma_0}[\nabla\tilde{\eta}[\sigma^{mp}(\tilde{w}_{tt})]_{p=1}^3]^j(\tilde{a}_i^j)_t\tilde{w}_{tt}^i N_m^s\right| \leqq C\kappa\|(\tilde{w},\tilde{q})\|^5_{Z_t}, \qquad (69)
$$

and thus by putting together (67), (68) and (69),

$$
\begin{aligned}
&\left|\int_{\Omega_0^s}\tilde{q}_{tt},_j(\tilde{a}_i^j)_t\tilde{w}_{tt}^i - \kappa\int_{\Gamma_0}[\nabla\tilde{\eta}[\sigma^{mp}(\tilde{w}_{tt})]_{p=1}^3]^j(\tilde{a}_i^j)_t\tilde{w}_{tt}^i N_m^s\right| \\
&\quad\leqq C\kappa\|(\tilde{w},\tilde{q})\|^5_{Z_t} + C_\delta t^{\frac{1}{4}}\|(\tilde{w},\tilde{q})\|^7_{Z_t} + \delta\|(\tilde{w},\tilde{q})\|^2_{Z_t} \qquad (70) \\
&\qquad + C_\delta\Big[N(u_0,(w_i)_{i=1}^3) + N((q_i)_{i=0}^2) + M(f,\kappa g,\kappa h)\Big]. \qquad (71)
\end{aligned}
$$

Now, the apparent problem comes from the term $\sigma^{mp}(\tilde{w}_{tt})$ on Γ_0 that should be taken in $H^{-0.5}(\Gamma_0;\mathbb{R})$, which is troublesome since the norm in Z_t appropriate for our limit process only contains its $L^\infty(0,t;L^2(\Omega_0^s;\mathbb{R}))$ norm. In order to circumvent this, we notice that we also have, since $E(\Omega_0^f)(\tilde{w}_{tt})$ is at least as smooth as $\tilde{w}_{tt}$ in Ω_0^s,

$$
\begin{aligned}
&\left|\int_{\Omega_0^s}\tilde{q}_{tt},_j(\tilde{a}_i^j)_t E(\Omega_0^f)(\tilde{w}_{tt})^i - \kappa\int_{\Gamma_0}[\nabla\tilde{\eta}[\sigma^{mp}(\tilde{w}_{tt})]_{p=1}^3]^j(\tilde{a}_i^j)_t E(\Omega_0^f)(\tilde{w}_{tt}^f)^i N_m\right| \\
&\quad\leqq C\kappa\|(\tilde{w},\tilde{q})\|^5_{Z_t} + C_\delta t^{\frac{1}{4}}\|(\tilde{w},\tilde{q})\|^7_{Z_t} + \delta\|(\tilde{w},\tilde{q})\|^2_{Z_t} \\
&\qquad + C_\delta[N(u_0,(w_i)_{i=1}^3) + N((q_i)_{i=0}^2) + M(f,\kappa g,\kappa h)], \qquad (72)
\end{aligned}
$$

leading us, since $\tilde{w} = E(\Omega_0^f)(\tilde{w}^f)$ on Γ_0, to

$$\left| \int_{\Omega_0^s} \tilde{q}_{tt},_j\, (\tilde{a}_i^j)_t \tilde{w}_{tt}^i - \int_{\Omega_0^s} \tilde{q}_{tt},_j\, (\tilde{a}_i^j)_t E(\Omega_0^f)(\tilde{w}_{tt})^i \right|$$
$$\leqq C\kappa \|(\tilde{w}, \tilde{q})\|_{Z_t}^5 + C_\delta t^{\frac{1}{4}} \|(\tilde{w}, \tilde{q})\|_{Z_t}^7 + \delta \|(\tilde{w}, \tilde{q})\|_{Z_t}^2$$
$$+ C_\delta [N(u_0, (w_i)_{i=1}^3) + N((q_i)_{i=0}^2) + M(f, \kappa g, \kappa h)]. \tag{73}$$

Thus, by using (65), (66) and (73), we have

$$|K_3| \leqq (C\kappa + C_\delta t^{\frac{1}{4}}) \|(\tilde{w}, \tilde{q})\|_{Z_t}^7 + \delta \|(\tilde{w}, \tilde{q})\|_{Z_t}^2$$
$$+ C_\delta [N(u_0, (w_i)_{i=1}^3) + N((q_i)_{i=0}^2) + M(f, \kappa g, \kappa h)]. \tag{74}$$

Thus, we finally arrive to estimates analogous to (28) and (29), with the right-hand side being of the same type as in (74).

13.2. Estimate on $\tilde{w}_{tt}$ and $\tilde{w}_t$

With the same arguments as in the next subsection, we have for $n = 2, 1$:

$$\|\partial_t^n \tilde{w}\|^2_{L^2(0,t;H^{4-n}(\Omega_0^f;\mathbb{R}^3))} + \|\partial_t^n \tilde{q}\|^2_{L^2(0,t;H^{3-n}(\Omega_0^f;\mathbb{R}^3))} + \|\partial_t^n \tilde{\eta}\|^2_{L^\infty(0,T;H^{4-n}(\Omega_0^s;\mathbb{R}^3))}$$
$$+ \kappa^2 \|\partial_t^n \tilde{w}\|^2_{L^2(0,t;H^{4-n}(\Omega_0^s;\mathbb{R}^3))} + \|\partial_t^n \tilde{q}\|^2_{L^2(0,t;H^{3-n}(\Omega_0^s;\mathbb{R}^3))}$$
$$\leqq C_\delta [N(u_0, (w_i)_{i=1}^3) + M(f, \kappa g, \kappa h) + N((q_i)_{i=0}^2)]$$
$$+ (C\kappa + C_\delta t^{\frac{1}{4}}) \|(\tilde{w}, \tilde{q})\|_{Z_t}^7 + C\delta \|(\tilde{w}, \tilde{q})\|_{Z_t}^2. \tag{75}$$

We now explain in the case of the highest space derivative how to obtain elliptic estimates independent of κ, since the addition of the pressure term does not allow us to use Lemma 1 directly in the present case.

13.3. Estimate on $\tilde{\eta}$ in Ω_0^s

13.3.1. Regularity of the trace of $\tilde{\eta}$. First, by proceeding as in Section 8, and as for the case of the highest-order time derivative, we get an estimate for the trace similar to (38), with a majorant of the same type as in (74). We explain hereafter how to handle the estimates related to the pressure in the solid in order to get this trace estimate since difficulties, different from those in the higher-order time derivative problem, appear in the higher-order space derivative problem.

Step 1. Let $Q_1 = \int_0^t \int_{\mathbb{R}^3_-} [Q\, \tilde{b}_i^j],_{\alpha_1\alpha_2\alpha_3} [\zeta^2 W^i],_{\alpha_1\alpha_2\alpha_3 j}$.

Then,

$$Q_1 = Q_2 + Q_3 + Q_4,$$

with

$$Q_2 = \int_0^t \int_{\mathbb{R}^3_-} Q\, \tilde{b}_i^j,_{\alpha_1\alpha_2\alpha_3} [\zeta^2 W^i],_{\alpha_1\alpha_2\alpha_3 j}\,,$$

$$Q_3 = \int_0^t \int_{\mathbb{R}^3_-} \Big[[Q\, \tilde{b}_i^j],_{\alpha_1\alpha_2\alpha_3} - Q,_{\alpha_1\alpha_2\alpha_3}\ \tilde{b}_i^j - Q\, \tilde{b}_i^j,_{\alpha_1\alpha_2\alpha_3} \Big] [\zeta^2 W^i],_{\alpha_1\alpha_2\alpha_3 j}\,,$$

$$Q_4 = \int_0^t \int_{\mathbb{R}^3_-} Q,_{\alpha_1\alpha_2\alpha_3}\ \tilde{b}_i^j [\zeta^2 W^i],_{\alpha_1\alpha_2\alpha_3 j}\,.$$

For Q_2, we first notice that for $\theta = \tilde{\eta} \circ \Psi$, if ε^{ijk} is the sign of the permutation between $\{i, j, k\}$ and $\{1, 2, 3\}$ if i, j, k are distinct, or set to zero otherwise, then

$$\begin{aligned}
\tilde{b}_i^j,_{\alpha_1\alpha_2\alpha_3} W^i,_{\alpha_1\alpha_2\alpha_3 j} &= \frac{1}{2}\varepsilon^{mni}\varepsilon^{pqj}[\theta,^m_p \theta,^n_q],_{\alpha_1\alpha_2\alpha_3} W,^i_{j\alpha_1\alpha_2\alpha_3}\\
&= \varepsilon^{mni}\varepsilon^{pqj}\theta,^m_{p\alpha_1\alpha_2\alpha_3} \theta,^n_q W,^i_{j\alpha_1\alpha_2\alpha_3}\\
&\quad + \frac{1}{2}\sum_{\sigma\in\Sigma_3} \varepsilon^{mni}\varepsilon^{pqj}\theta,^m_{p\alpha_{\sigma(1)}} \theta,^n_{q\alpha_{\sigma(2)}\alpha_{\sigma(3)}} W,^i_{j\alpha_1\alpha_2\alpha_3}\\
&\quad + \frac{1}{2}\sum_{\sigma\in\Sigma_3} \varepsilon^{mni}\varepsilon^{pqj}\theta,^m_{p\alpha_{\sigma(1)}\alpha_{\sigma(2)}} \theta,^n_{q\alpha_{\sigma(3)}} W,^i_{j\alpha_1\alpha_2\alpha_3}\\
&= \frac{1}{2}\varepsilon^{mni}\varepsilon^{pqj}[\theta,^m_{p\alpha_1\alpha_2\alpha_3} \theta,^i_{j\alpha_1\alpha_2\alpha_3}]_t \theta,^n_q\\
&\quad + \frac{1}{2}\sum_{\sigma\in\Sigma_3} \varepsilon^{mni}\varepsilon^{pqj}\theta,^m_{p\alpha_{\sigma(1)}} \theta,^n_{q\alpha_{\sigma(2)}\alpha_{\sigma(3)}} W,^i_{j\alpha_1\alpha_2\alpha_3}\\
&\quad + \frac{1}{2}\sum_{\sigma\in\Sigma_3} \varepsilon^{mni}\varepsilon^{pqj}\theta,^m_{p\alpha_{\sigma(1)}\alpha_{\sigma(2)}} \theta,^n_{q\alpha_{\sigma(3)}} W,^i_{j\alpha_1\alpha_2\alpha_3}\,,
\end{aligned}$$

where we have use $\varepsilon^{mni}\varepsilon^{pqj} = \varepsilon^{nmi}\varepsilon^{qpj}$ on the second equality and $\varepsilon^{mni}\varepsilon^{pqj} = \varepsilon^{inm}\varepsilon^{jqp}$ on the third one. Thus,

$$\begin{aligned}
Q_2 &= \int_0^t \int_{\mathbb{R}^3_-} Q\tilde{b}_i^j,_{\alpha_1\alpha_2\alpha_3} [[\zeta^2 W]^i,_{\alpha_1\alpha_2\alpha_3 j} - \zeta^2 W,^i_{j\alpha_1\alpha_2\alpha_3}]\\
&\quad - \frac{1}{2}\int_0^t \int_{\mathbb{R}^3_-} \sum_{\sigma\in\Sigma_3} \varepsilon^{mni}\varepsilon^{pqj}[Q\theta,^m_{p\alpha_{\sigma(1)}} \theta,^n_{q\alpha_{\sigma(2)}\alpha_{\sigma(3)}} \zeta^2],_{\alpha_1} W,^i_{j\alpha_2\alpha_3}\\
&\quad - \frac{1}{2}\int_0^t \int_{\mathbb{R}^3_-} \sum_{\sigma\in\Sigma_3} \varepsilon^{mni}\varepsilon^{pqj}[Q\theta,^m_{p\alpha_{\sigma(1)}\alpha_{\sigma(2)}} \theta,^n_{q\alpha_{\sigma(3)}} \zeta^2],_{\alpha_1} W,^i_{j\alpha_2\alpha_3}\\
&\quad - \frac{1}{2}\int_0^t \int_{\mathbb{R}^3_-} \varepsilon^{mni}\varepsilon^{pqj}\zeta^2\theta,^m_{p\alpha_1\alpha_2\alpha_3} \theta,^i_{j\alpha_1\alpha_2\alpha_3} [Q\theta,^n_q]_t\\
&\quad + \frac{1}{2}\Big[\int_{\mathbb{R}^3_-} \varepsilon^{mni}\varepsilon^{pqj}\zeta^2\theta,^m_{p\alpha_1\alpha_2\alpha_3} \theta,^i_{j\alpha_1\alpha_2\alpha_3} Q\theta,^n_q\Big]_0^t,
\end{aligned}$$

showing that

$$|Q_2| \leqq C\sqrt{t}\|(\tilde{w},\tilde{q})\|_{Z_t}^4 + \left|\int_{\mathbb{R}^3_-} \varepsilon^{mni}\varepsilon^{pqj}\zeta^2\theta,^m_{p\alpha_1\alpha_2\alpha_3}\,\theta,^n_{q\alpha_1\alpha_2\alpha_3}\,Q\theta,^i_j\right|(t) \\ +N((q_i)_{i=0}^2).$$

In order to estimate the remaining term, we notice by integrating by parts twice for $Q_5 = \int_{\mathbb{R}^3_-} \varepsilon^{mni}\varepsilon^{pqj}\zeta^2\theta,^m_{p\alpha_1\alpha_2\alpha_3}\,\theta,^n_{q\alpha_1\alpha_2\alpha_3}\,Q\theta,^i_j$ that

$$\begin{aligned} Q_5 = \varepsilon^{mni}\varepsilon^{pqj} &\int_{\mathbb{R}^3_-} \zeta^2\theta,^m_{q\alpha_1\alpha_2\alpha_3}\,\theta,^n_{p\alpha_1\alpha_2\alpha_3}\,Q\,\theta,^i_j \\ &-\varepsilon^{mni}\varepsilon^{pqj}\int_{\mathbb{R}^3_-}\Big[(\zeta^2Q\theta,^i_j),_p\,\theta,^m_{\alpha_1\alpha_2\alpha_3}\,\theta,^n_{q\alpha_1\alpha_2\alpha_3} \\ &\qquad -(\zeta^2Q\theta,^i_j),_q\,\theta,^m_{\alpha_1\alpha_2\alpha_3}\,\theta,^n_{p\alpha_1\alpha_2\alpha_3}\Big] \\ &+\varepsilon^{mni}\varepsilon^{pqj}\int_{x_3=0}\zeta^2Q\,\theta,^m_{\alpha_1\alpha_2\alpha_3}\Big[\theta,^n_{q\alpha_1\alpha_2\alpha_3}\,\theta,^i_j\,(e_3)_p - \theta,^n_{p\alpha_1\alpha_2\alpha_3}\,\theta,^i_j\,(e_3)_q\Big]. \end{aligned}$$

Since $\varepsilon^{pqj} = -\varepsilon^{qpj}$, we then infer

$$\begin{aligned} 2Q_5 = -\varepsilon^{mni}\varepsilon^{pqj} &\int_{\mathbb{R}^3_-} \theta,^m_{\alpha_1\alpha_2\alpha_3}\,[(\zeta^2Q\theta,^i_j),_p\,\theta,^n_{q\alpha_1\alpha_2\alpha_3} - (\zeta^2Q\theta,^i_j),_q\,\theta,^n_{p\alpha_1\alpha_2\alpha_3}] \\ &+\varepsilon^{mni}\varepsilon^{pqj}\int_{x_3=0}\zeta^2Q\,\theta,^m_{\alpha_1\alpha_2\alpha_3}[\theta,^n_{q\alpha_1\alpha_2\alpha_3}\,\theta,^i_j\,(e_3)_p - \theta,^n_{p\alpha_1\alpha_2\alpha_3}\,\theta,^i_j\,(e_3)_q]. \end{aligned}$$

Now, if we note $\theta^f = E(\Omega_0^f)(\tilde{\eta}^f)\circ\Psi$, we also have for

$$\int_{\mathbb{R}^3_-} \varepsilon^{mni}\varepsilon^{pqj}\zeta^2\theta^f,^m_{p\alpha_1\alpha_2\alpha_3}\,\theta,^n_{q\alpha_1\alpha_2\alpha_3}\,Q\theta,^i_j$$

a similar formula. Since $\theta,^m_{\alpha_1\alpha_2\alpha_3} = \theta^f,^m_{\alpha_1\alpha_2\alpha_3}$ on $\{x_3 = 0\}$, we then have

$$\begin{aligned} 2Q_5 = -\varepsilon^{mni}\varepsilon^{pqj} &\int_{\mathbb{R}^3_-} (\theta-\theta^f),^m_{\alpha_1\alpha_2\alpha_3}\Big[(\zeta^2Q\theta,^i_j),_p\,\theta,^n_{q\alpha_1\alpha_2\alpha_3} \\ &\qquad -(\zeta^2Q\theta,^i_j),_q\,\theta,^n_{p\alpha_1\alpha_2\alpha_3}\Big] \\ &+2\int_{\mathbb{R}^3_-}\varepsilon^{mni}\varepsilon^{pqj}\zeta^2\theta^f,^m_{p\alpha_1\alpha_2\alpha_3}\,\theta,^n_{q\alpha_1\alpha_2\alpha_3}\,Q\theta,^i_j\,, \end{aligned}$$

leading us to

$$
\begin{aligned}
|Q_5| \leqq\ & C\left\|\mathrm{Id}+\int_0^t \tilde{w}\right\|^{\frac{1}{4}}_{H^3(\Omega_0^s;\mathbb{R}^3)} \|\tilde{\eta}\|^{\frac{7}{4}}_{H^4(\Omega_0^s;\mathbb{R}^3)} \left\|q_0+\int_0^t \tilde{q}_t\right\|_{H^2(\Omega_0^s;\mathbb{R})} \\
&\quad \left\|\mathrm{Id}+\int_0^t \tilde{w}\right\|_{H^3(\Omega_0^s;\mathbb{R}^3)} \\
&\quad + C\left\|\mathrm{Id}+\int_0^t \tilde{w}\right\|^{\frac{1}{4}}_{H^3(\Omega_0^f;\mathbb{R}^3)} \|\tilde{\eta}\|^{\frac{7}{4}}_{H^4(\Omega_0^s;\mathbb{R}^3)} \left\|q_0+\int_0^t \tilde{q}_t\right\|_{H^2(\Omega_0^s;\mathbb{R})} \\
&\quad \left\|\mathrm{Id}+\int_0^t \tilde{w}\right\|_{H^3(\Omega_0^s;\mathbb{R}^3)} \\
&\quad + C\left\|\mathrm{Id}+\int_0^t \tilde{w}^f\right\|_{H^4(\Omega_0^f;\mathbb{R}^3)} \|\tilde{\eta}\|_{H^4(\Omega_0^s;\mathbb{R}^3)} \left\|q_0+\int_0^t \tilde{q}_t\right\|_{H^2(\Omega_0^s;\mathbb{R})} \\
&\quad \left\|\mathrm{Id}+\int_0^t \tilde{w}\right\|_{H^3(\Omega_0^s;\mathbb{R}^3)} \\
&\leqq \delta\|(\tilde{w},\tilde{q})\|^2_{Z_t} + C\sqrt{t}\|(\tilde{w},\tilde{q})\|^4_{Z_t} + C_\delta[N(u_0,(w_i)_{i=1}^3) + N((q_i)_{i=0}^2)]
\end{aligned}
$$

Step 2. We see by integrating by parts with respect to the direction α_1 that we have

$$|Q_3| \leqq C\sqrt{t}\|(\tilde{w},\tilde{q})\|^4_{Z_t}.$$

Step 3. Next, $Q_4 = Q_6 + Q_7$, where

$$
\begin{aligned}
Q_6 &= \int_0^t \int_{\mathbb{R}^3_-} Q,_{\alpha_1\alpha_2\alpha_3}\ \tilde{b}_i^j \zeta^2 W^i,_{\alpha_1\alpha_2\alpha_3 j} \\
Q_7 &= \int_0^t \int_{\mathbb{R}^3_-} Q,_{\alpha_1\alpha_2\alpha_3}\ \tilde{b}_i^j \left[[\zeta^2 W]^i,_{\alpha_1\alpha_2\alpha_3 j} - \zeta^2 W^i,_{\alpha_1\alpha_2\alpha_3 j}\right].
\end{aligned}
$$

We first have

$$|Q_7| \leqq C\int_0^t [\|\tilde{q}\|_{H^3(\Omega_0^s;\mathbb{R})}\|\tilde{w}\|_{H^3(\Omega_0^s;\mathbb{R}^3)}\|\tilde{\eta}\|^2_{H^3(\Omega_0^s;\mathbb{R}^3)}] \leqq C\sqrt{t}\|(\tilde{w},\tilde{q})\|^4_{Z_t}\ .$$

For Q_6 the divergence condition $\tilde{b}_i^j W^i,_j = 0$ on *Supp* ζ implies that

$$Q_6 = \int_0^t \int_{\mathbb{R}^3_-} Q,_{\alpha_1\alpha_2\alpha_3}\,\zeta^2 \left[\tilde{b}_i^j W^i,_{\alpha_1\alpha_2\alpha_3 j} - (\tilde{b}_i^j W^i,_j),_{\alpha_1\alpha_2\alpha_3}\right],$$

which in turn provides,

$$|Q_6| \leqq C\int_0^t [\|\tilde{q}\|_{H^3(\Omega_0^s;\mathbb{R})}\|\tilde{w}\|_{H^3(\Omega_0^s;\mathbb{R}^3)}\|\tilde{\eta}\|^2_{H^4(\Omega_0^s;\mathbb{R}^3)}] \leqq C\sqrt{t}\|(\tilde{w},\tilde{q})\|^4_{Z_t}\ ,$$

which concludes the estimates on the pressure terms in the solid, justifying why we obtain a trace estimate similar as (38) with a majorant of the type of the right-hand side of (74). Now, we turn our attention to the recovery of the regularity in the solid, which will need some justifications since we cannot directly apply Lemma 1.

13.3.2. Regularity in the incompressible solid. First, with the introduction of

$$\tilde{F} = -\tilde{w}_t + c^{ijkl}[(\tilde{\eta},_i \cdot\tilde{\eta},_j - \delta_{ij})\tilde{\eta},_k],_l - L\tilde{\eta} + f + \kappa h \text{ in } \Omega_0^s,$$

and of $\tilde{r}$, the solution in Ω_0^s of

$$\frac{\kappa}{2}\tilde{r}_t + \tilde{r} = \tilde{q},$$
$$\tilde{r}(0) = q_0,$$

we have for the nonlinear elastodynamics

$$-\frac{\kappa}{2}\nabla\tilde{\eta}\ L\tilde{w} - \nabla\tilde{\eta}\ L\tilde{\eta} + \nabla\left[\frac{\kappa}{2}\tilde{r}_t + \tilde{r}\right] = \nabla\tilde{\eta}\ \tilde{F} \text{ in } \Omega_0^s,$$

i.e.,

$$\frac{\kappa}{2}[-\nabla\tilde{\eta}\ L\tilde{\eta} + \nabla\tilde{r}]_t - \nabla\tilde{\eta}\ L\tilde{\eta} + \nabla\tilde{r} = \nabla\tilde{\eta}\ \tilde{F} - \frac{\kappa}{2}\nabla\tilde{w}\ L\tilde{\eta} \text{ in } \Omega_0^s. \tag{76}$$

We now apply Lemma 1 to this equation, leading us to

$$\sup_{[0,t]}\| -\nabla\tilde{\eta}\ L(\tilde{\eta}) + \nabla\tilde{r}\|_{H^2(\Omega_0^s;\mathbb{R}^3)} \leqq \sup_{[0,t]}\|\nabla\tilde{\eta}\ \tilde{F} - \frac{\kappa}{2}\nabla\tilde{w}\ L\tilde{\eta}\|_{H^2(\Omega_0^s;\mathbb{R}^3)} + \| -L(\text{Id}) + \nabla q_0\|_{H^2(\Omega_0^s;\mathbb{R}^3)},$$

and, with $\tilde{H} = -L(\tilde{\eta}) + \nabla\tilde{r}$, to

$$\sup_{[0,t]}\|\tilde{H}\|_{H^2(\Omega_0^s;\mathbb{R}^3)} \leqq \sup_{[0,t]}[\|\nabla\tilde{\eta}\ \tilde{F} - \frac{\kappa}{2}\nabla\tilde{w}\ L\tilde{\eta}\|_{H^2(\Omega_0^s;\mathbb{R}^3)} + \|(\nabla\tilde{\eta} - \text{Id})\ L(\tilde{\eta})\|_{H^2(\Omega_0^s;\mathbb{R}^3)}] + N((q_i)_{i=0}^2). \tag{77}$$

We then want to use elliptic regularity on the system:

$$-L\tilde{\eta} + \nabla\tilde{r} = \tilde{H} \quad \text{in } \Omega_0^s, \tag{78a}$$
$$\text{div}\,\tilde{\eta} = (-\tilde{a}_i^j + \delta_{ij})\tilde{\eta}^i,_j + 3 \text{ in } \Omega_0^s, \tag{78b}$$
$$\tilde{\eta} = \tilde{\eta}_{|\Gamma_0} \quad \text{on } \Gamma_0, \tag{78c}$$

where the trace on Γ_0 is estimated as we explained in the previous subsection. Now, for the divergence condition in Ω_0^s, we notice that:

$$\left[(\tilde{a}_i^j - \delta_{ij})\tilde{\eta}^i,_j\right],_{i_1 i_2 i_3} = \tilde{a}_i^j,_{i_1 i_2 i_3}\tilde{\eta}^i,_j + (\tilde{a}_i^j - \delta_{ij})\tilde{\eta}^i,_{j i_1 i_2 i_3} + \sum_{\sigma\in\Sigma_3}\left[\tilde{a}_i^j,_{i_{\sigma(1)}}\tilde{\eta}^i,_{j i_{\sigma(2)} i_{\sigma(3)}} + \tilde{a}_i^j,_{i_{\sigma(1)} i_{\sigma(2)}}\tilde{\eta}^i,_{j i_{\sigma(3)}}\right].$$

For the apparently problematic first term on the right-hand side, we first notice that

$$\tilde{a}_i^j,_{i_1}\tilde{\eta}^i,_j = \frac{1}{2}\varepsilon^{mni}\varepsilon^{pqj}(\tilde{\eta},_p^m\tilde{\eta},_q^n),_{i_1}\tilde{\eta},_j^i = \varepsilon^{mni}\varepsilon^{pqj}\tilde{\eta},_{pi_1}^m\tilde{\eta},_q^n\tilde{\eta},_j^i$$
$$= \varepsilon^{inm}\varepsilon^{jqp}\tilde{\eta},_{ji_1}^i\tilde{\eta},_q^n\tilde{\eta},_p^m = 2\tilde{a}_i^j\tilde{\eta}^i,_{ji_1},$$

which with the condition $\tilde{a}_i^j \tilde{\eta}^i,_j = 3$, provides $0 = \tilde{a}_i^j,_{i_1} \tilde{\eta}^i,_j$, and thus

$$\tilde{a}_i^j,_{i_1 i_2 i_3} \tilde{\eta}^i,_j = -\tilde{a}_i^j,_{i_1 i_2} \tilde{\eta}^i,_{j i_3} - \tilde{a}_i^j,_{i_1 i_3} \tilde{\eta}^i,_{j i_2} - \tilde{a}_i^j,_{i_1} \tilde{\eta}^i,_{j i_2 i_3}.$$

We then deduce that

$$\begin{aligned} &\|(\tilde{a}_i^j - \delta_{ij})\tilde{\eta}^i,_j\|_{H^3(\Omega_0^s;\mathbb{R})}(t) \\ &\leqq \delta\|(\tilde{w},\tilde{q})\|_{Z_t}^2 + C_\delta N(u_0,(w_i)_{i=1}^3) + Ct\,\|(\tilde{w},\tilde{q})\|_{Z_t}^3. \end{aligned} \tag{79}$$

Now, with (77) and (79), elliptic regularity on (78) provides for

$$\|\tilde{\eta}\|^2_{L^\infty(0,t;H^4(\Omega_0^s;\mathbb{R}^3))} + \left\| \tilde{r} - \frac{1}{|\Omega_0^s|}\int_{\Omega_0^s} \tilde{r} \right\|^2_{L^\infty(0,t;H^3(\Omega_0^s;\mathbb{R}))}$$

a bound of the same type as the right-hand side of (74), with however the norms in Z_t replaced by the norms in $\tilde{Z}_t$, due to the term $\kappa\|\nabla\tilde{w}\ L\tilde{\eta}\|_{H^2(\Omega_0^s;\mathbb{R}^3)}$ appearing on the right-hand side of (77), that we bound by

$$\begin{aligned} &C\kappa\|\tilde{w}(t)\|_{H^3(\Omega_0^s;\mathbb{R}^3)}\|L(\mathrm{Id}) + \int_0^t L\tilde{w}\|_{H^2(\Omega_0^s;\mathbb{R}^3)} \\ &\leqq C\kappa\|\tilde{w}(t)\|_{H^3(\Omega_0^s;\mathbb{R}^3)}\sqrt{t}\|\tilde{w}\|_{L^2(0,t;H^4(\Omega_0^s;\mathbb{R}^3))} \\ &\leqq C\sqrt{t}\|(\tilde{w},\tilde{q})\|^2_{\tilde{Z}_t}. \end{aligned}$$

We now turn our attention to the pressure, which we just need to control in $L^2(0,t;H^3(\Omega_0^s;\mathbb{R}))$. In order to do so, we notice from (76) that we have for $\tilde{K} = \frac{\kappa}{2}[-L\tilde{w} + \nabla\tilde{r}_t]$:

$$\tilde{K} = \frac{\kappa}{2}[\nabla\tilde{\eta} - \mathrm{Id}]\ L\tilde{w} + \nabla\tilde{\eta}\ L\tilde{\eta} - \nabla\tilde{r} + \nabla\tilde{\eta}\ \tilde{F} - \frac{\kappa}{2}\nabla\tilde{w}\ L\tilde{\eta},$$

which with the previous estimate on $\tilde{\eta}$ and $\tilde{r}$ shows that we have a bound on $\|\tilde{K}\|^2_{L^2(0,t;H^2(\Omega_0^s;\mathbb{R}^3))}$ of the same type as the right-hand side of (74), but where the norms in Z_t are replaced by norms in $\tilde{Z}_t$ due to the estimate in $L^2(H^2)$ of $\kappa[\nabla\tilde{\eta} - \mathrm{Id}]\ L\tilde{w}$. Now, elliptic regularity on the system:

$$\begin{aligned} -L\kappa\tilde{w} + \nabla\kappa r_t &= 2\tilde{K} && \text{in } \Omega_0^s, \\ \operatorname{div}\kappa\tilde{w} &= \kappa(-\tilde{a}_i^j + \delta_{ij})\tilde{w}^i,_j && \text{in } \Omega_0^s, \\ \kappa\tilde{w} &= \kappa\tilde{w}^f_{|\Gamma_0} && \text{on } \Gamma_0, \end{aligned}$$

provides, after integrating in time, an estimate for

$$\kappa^2\left[\|\tilde{w}\|^2_{L^2(0,t;H^4(\Omega_0^s;\mathbb{R}^3))} + \|\tilde{r}_t - \frac{1}{|\Omega_0^s|}\int_{\Omega_0^s}\tilde{r}_t\|^2_{L^2(0,t;H^3(\Omega_0^s;\mathbb{R}))}\right]$$

with a bound similar as in (74), still with the norms in Z_t replaced by norms in $\tilde{Z}_t$.

Thus, we obtain for $\|\tilde{q} - \frac{1}{|\Omega_0^s|}\int_{\Omega_0^s}\tilde{q}\|^2_{L^2(0,t;H^3(\Omega_0^s;\mathbb{R}))}$, the same type of estimate as well. Given our estimate on $\tilde{q}_{tt}$, this also implies the same type of majoration for $\|\tilde{q}\|^2_{L^2(0,t;H^3(\Omega_0^s;\mathbb{R}))}$.

Thus, we are lead to

$$\|(\tilde{w},\tilde{q})\|^2_{\tilde{Z}_t} \leqq \left(C\kappa + C_\delta t^{\frac{1}{4}}\right)\|(\tilde{w},\tilde{q})\|^8_{\tilde{Z}_t} + \delta\|(\tilde{w},\tilde{q})\|^2_{\tilde{Z}_t} + C_\delta\left[N(u_0,(w_i)_{i=1}^3) + N((q_i)_{i=0}^2) + M(f,\kappa g,\kappa h)\right],$$

which leads as in Section 9 to the introduction of a polynomial, this time of degree 4, which does not bring any substantial change with respect to Section 9. Note that the addition of $C\kappa\|(\tilde{w},\tilde{q})\|^8_{\tilde{Z}_t}$ does not create any difficulty since a small κ_1 is chosen at the same stage as t_1, and the conclusion is similar as in Section 9 from the continuity of $\|(\tilde{w},\tilde{q})\|_{\tilde{Z}_t}$ on $[0,T_\kappa]$ which is established in the same way as the continuity of $\|(\tilde{w},\tilde{q})\|_{Z_t}$. We then infer that there is a time of existence of κ for our smoothed problems, with a bound on $\|(\tilde{w},\tilde{q})\|_{\tilde{Z}_T}$ and thus on $\|(\tilde{w},\tilde{q})\|_{Z_T}$ independent of κ. Existence follows then by weak convergence in Y_T and uniqueness can be established similarly as for the compressible case in Section 11.

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