The intermediate Jacobian of the cubic threefold

By C. HERBERT CLEMENS and PHILLIP A. GRIFFITHS

TABLE OF CONTENTS

0. Introduction

The purpose of this paper is to study the cubic threefold, that is, the hypersurface of degree three in complex projective four-space. Our principal tool in this study will be the *intermediate Jacobian* of the threefold, an abelian variety which has a role in the analysis of algebraic curves on the threefold similar to the role of the Jacobian variety in the study of divisors on an algebraic curve. Much of what we will do is motivated by analogy with properties of curves and their Jacobian varieties, so we shall begin by recalling some of these properties together with some general facts about abelian varieties (see [17]).

A positive *polarization* on a complex torus A is given equivalently by:

(i) a skew-symmetric form

$$Q\colon H_1(A) \otimes H_1(A) \longrightarrow \mathbf{Z}$$

which satisfies the Riemann bilinear relations;

(ii) a non-degenerate divisor θ on A, taken up to numerical equivalence (see [17; Chapter 1]). Since $H^2(A) \approx \mathrm{Hom}_{\mathbf{Z}}(\wedge^2 H_1(A), \mathbf{Z})$, if we are given θ, then the dual cohomology class $\Omega \in H^2(A)$ defines the corresponding form Q. Conversely, given Q, the Riemann bilinear relations permit the construction of the non-degenerate divisor θ. A complex torus is called an abelian variety if it admits a positive polarization.

A positive polarization on a complex torus is *principal* if equivalently:

(i) Q is unimodular;

(ii) $\dim|\theta| = 0$ so that θ is determined up to translation by its homology class.

Because of this last property, principal positive polarizations are especially useful in geometry. If we define a morphism

$$\varphi\colon (A, \theta_A) \longrightarrow (B, \theta_B)$$

between polarized abelian varieties to be given by a homomorphism $\varphi\colon A \to B$ such that $\varphi^*(\Omega_B) = \Omega_A$, then the principally polarized abelian varieties form a category having very strong semi-simplicity properties (see (3.6) and (3.20)).

Given a smooth projective variety W, there are associated to W two abelian varieties, the *Albanese variety* $\mathrm{Alb}(W)$ characterized by the existence of a mapping $\lambda\colon W \to \mathrm{Alb}(W)$ such that any rational mapping of W into an abelian variety factors uniquely through λ, and the *Picard variety* $\mathrm{Pic}(W)$ which is the group of divisors algebraically equivalent to zero modulo divisors of rational functions on W. Furthermore there exist the relations:

$$\begin{aligned} &\lambda_*\colon H_1(W) \approx H_1(\mathrm{Alb}(W)) \quad \text{(modulo torsion)} \\ &\lambda^*\colon \mathrm{Pic}(\mathrm{Alb}(W)) \approx \mathrm{Pic}(W). \end{aligned} \tag{0.1}$$

If C is a smooth curve, the intersection form

$$H_1(C) \otimes H_1(C) \to \mathbf{Z}$$

induces a principal polarization on $\mathrm{Alb}(C)$ with corresponding divisor θ. The mapping $\psi\colon C \to \mathrm{Pic}(C)$ given by

$$\psi(x) = (x - x_0)$$

induces a mapping:

$$\sigma\colon \mathrm{Alb}(C) \longrightarrow \mathrm{Pic}(C)\ .$$

(Using (0.1), σ can be alternatively constructed from the mapping $\mathrm{Alb}(C) \to \mathrm{Pic}(\mathrm{Alb}(C))$ given by $a \mapsto (\theta + a)$.) It is then *Abel's theorem* that:

(0.2) The map $\sigma\colon \mathrm{Alb}(C) \to \mathrm{Pic}(C)$ is an isomorphism.

Thus in the case of curves, these two abelian varieties can be identified, the resulting variety being denoted by $J(C)$, called the *Jacobian variety* of the curve.

Continuing with the case of curves, there are induced natural mappings:

$$\kappa^{(k)}\colon C^{(k)} \longrightarrow J(C)$$

where $C^{(k)}$ is the k-fold symmetric product of C. The *Jacobi inversion theorem* states that:

(0.3) If $k \geqq g$, the genus of C, $\kappa^{(k)}$ is surjective and if $k = g$, the mapping is birational.

The *theorems of Riemann and Poincaré* state that:

(0.4) For $k \leqq g$, $\kappa^{(k)}(C^{(k)})$ has the same homology class in $H_{2k}(J(C))$ as the cycle

$$\frac{1}{(g-k)!}\underbrace{(\theta\cdot \cdots \cdot\theta)}_{(g-k)\text{-times}} .$$

In particular, since θ is determined up to translation by its homology class:

$$\kappa^{(g-1)}(C^{(g-1)}) = \theta + (\text{constant}) .$$

Finally, *Torelli's theorem* states that:

(0.5) The curve C is uniquely determined by the principally polarized abelian variety $(J(C), \theta)$.

In general, let (A, θ) be a principally polarized abelian variety of dimension g. Referring to (0.4), we say that (A, θ) has *level* k if the homology class of

$$\frac{1}{(g-k)!}\underbrace{(\theta\cdot \cdots \cdot\theta)}_{(g-k)\text{-times}}$$

contains an effective algebraic k-cycle. Thus (A, θ) is always of level $(g-1)$; and (A, θ) is of level one if and only if (A, θ) is a sum of Jacobians of smooth curves (see [14] and [16]). Now any principally polarized abelian variety has a unique direct sum decomposition into irreducible ones corresponding to the irreducible components of its theta-divisor (see Lemma 3.20). Thus given (A, θ) we can associate to it, for example, the principally polarized abelian variety (A_1, θ_1) which is the direct sum of the components of (A, θ) which are *not* of level one.

The motivation of the definition of (A_1, θ_1) is as follows: If V is a non-singular threefold such that the *Hodge numbers* $h^{1,0}(V)$ and $h^{3,0}(V)$ are zero, then there is a principally polarized abelian variety $(J(V), \theta_V)$, called the *intermediate Jacobian* of V, obtained by dividing $H^{1,2}(V)$ by a lattice generated by the third integral cohomology. The associated principally polarized abelian variety $(J(V)_1, (\theta_V)_1)$ (obtained by "throwing away" the summands of $J(V)$ which come from curves) turns out to be a *birational invariant* of V. In the case that V is a non-singular cubic threefold, $(J(V), \theta_V)$ will be shown to be irreducible and of level two but not of level one, so that V cannot be rational.

Again, let V denote a smooth, projective variety of dimension three. Given $\gamma \in H_3(V; \mathbf{Z})$ there is induced a linear mapping:

$$\begin{aligned}\gamma^*\colon (H^{3,0}(V) \oplus H^{2,1}(V)) &\longrightarrow \mathbf{C}\\ \omega &\longmapsto \int_\gamma \omega .\end{aligned}$$

Then $J(V) \approx (H^{3,0}(V) \oplus H^{2,1}(V))^*/\{\gamma^*\}$. Analogously to the case of curves, given an algebraic family $\{Z_s\}_{s\in S}$ of effective algebraic one-cycles on V, the "locus of the cycle" map

$$\phi_*\colon H_1(S) \longrightarrow H_3(V)$$

induces a homomorphism of complex tori

$$\phi\colon \mathrm{Alb}(S) \longrightarrow J(V)$$

called the *Abel-Jacobi mapping*. (S is assumed to be smooth and irreducible.) Furthermore, if $\{Z_s\}$ satisfies a mild general position requirement, then for each $s \in S$ there is defined an *incidence divisor*

$$D_s = \{t \in S\colon (Z_s \cap Z_t) \neq \varnothing\} .$$

Choosing a basepoint $s_0 \in S$, the map $\psi(s) = (D_s - D_{s_0})$ from S to $\mathrm{Pic}\,(S)$ leads to a homomorphism

$$\eta\colon \mathrm{Alb}(S) \longrightarrow \mathrm{Pic}(S) .$$

A general version of Abel's theorem says that there always is a factorization

$$\begin{tikzcd} \mathrm{Alb}(S) \arrow[d, "\eta"] \arrow[dr, "\phi"] & \\ \mathrm{Pic}(S) & J(V) \arrow[l, "\lambda"] \end{tikzcd} \tag{0.6}$$

and that if $h^{3,0}(V) = h^{1,0}(V) = 0$, $\ker \eta/\ker \phi$ is finite. This says that, up to isogeny, the equivalence relation on $\{Z_s\}$ determined by $J(V)$ is the same as that determined on $\{D_s\}$ by linear equivalence.

We now specialize to the case that V is a cubic threefold. Then $h^{3,0}(V) = h^{1,0}(V) = 0$ and $h^{2,1}(V) = 5$. Furthermore, if we denote by $\mathrm{Gr}\,(2, 5)$ the *Grass-*

mann variety of projective lines in $\mathbf{P}_4$ and put

$$S = \{s \in \mathrm{Gr}(2, 5)\colon \text{the corresponding line } L_s \subseteq V\}\,,$$

then it is a result of Fano [6] that S is a smooth irreducible surface having the numerical characters

$$h^{1,0}(S) = 5, \quad h^{2,0}(S) = 10\,. \tag{0.7}$$

Building upon results of Gherardelli [7] and Todd [19] we show:

(0.8) (Abel's theorem and the Jacobi inversion theorem). In the diagram (0.6), all three mappings are isomorphisms.

Also the natural mapping $\psi\colon S \longrightarrow \mathrm{Alb}(S) = J(V)$ is generically injective and we will show that:

(0.9) The homology class of $\psi(S)$ is the same as that of the cycle $(1/3!)\ (\theta_V \cdot \theta_V \cdot \theta_V)$.

Next, the mapping $\psi^{(2)}\colon S \times S \rightarrow J(V)$ defined by $\psi(s, t) = s - t$ is generically $6 - 1$ so that:

(0.10) (Theorems of Riemann and Poincaré). The image variety $\psi^{(2)}(S \times S)$ coincides, up to translation, with θ_V.

Thus $(J(V), \theta_V)$ is of level two. By studying the so-called Gauss map on θ_V we then derive our last two theorems:

(0.11) (Torelli theorem). The principally polarized abelian variety $(J(V), \theta_V)$ uniquely determines the cubic threefold V.

(0.12) (Non-rationality theorem). The principally polarized abelian variety $(J(V), \theta_V)$ is *not* of level one so that V cannot be rational.

Our methods of proof of (0.8)–(0.12) consist mainly in elementary geometric analysis of cubic hypersurfaces of dimensions 2, 3, and 4, applications of the theory of abelian varieties and of Picard-Lefschetz theory, and degeneration arguments, that is, reasoning based on the study of the topology of algebraic varieties acquiring some simple types of singularities. Our use of this last technique occurs wher a family V_t of cubic threefolds acquires an ordinary double point for a fixed value $t = 0$ of the parameter. The corresponding surface of lines S_0 has an ordinary double curve D_0 given by the lines on V_0 which pass through the double point. The threefold V_0 is rational and is obtained by blowing up $\mathbf{P}_3$ along a canonical embedding of D_0, a non-singular curve of genus four, and then blowing down a quadric surface. Furthermore the surface S_0 has as its normalization the second symmetric product $D_0^{(2)}$. This enables us to construct a topological model for S_t by plumbing S_0 along a tubular neighborhood of D_0.

The preceding topological analysis leads, first of all, to the conclusion that the mappings of (0.6) are all isogenies. This then allows us to conclude that two relations induced on elements of S under the mapping $S \to \mathrm{Pic}(S)$ actually already hold under $S \to \mathrm{Alb}(S)$. Before stating these relations, recall that the group law on a cubic curve is generated by the relation "three points lying on a line". In this context, if we think of $\mathrm{Alb}(S)$ as a quotient of the free abelian group generated by the lines on V, then the generating relations for the group law include the following two:

"Six lines passing through a point on V".

"Three coplanar lines on V".

The existence of these relations in $\mathrm{Alb}(S)$ are then sufficient to conclude (0.8).

A second use of the degeneration argument outlined above comes in the proof of (0.9) where we show first that the theorem is true "in the limit" and then apply the absolute irreducibility of the action of the monodromy group for a Lefschetz pencil of hyperplane sections of a cubic fourfold to conclude (0.9) for smooth V and S. For (0.10), the fact that the difference map, rather than the sum, is used to construct θ_V geometrically is intimately related to the classical "double sixes", that is, conjugate sets of six disjoint lines on a non-singular cubic surface (see § 13).

The geometric aspects of our proofs of (0.11) and (0.12) were motivated by Andreotti's proof of the Torelli theorem for curves [1]. His arguments are based on the interplay between the geometry of the canonical mapping of the curve into projective space and the Gauss mapping on the theta-divisor. In the case of the cubic threefold, we find ourselves in a situation formally analogous to that upon which Andreotti builds his proof. To explain briefly, we can define a Gauss map on $\psi(S) \subseteq J(V)$:

$$\mathcal{G}\colon \psi(S) \longrightarrow \mathrm{Gr}(2, 5)$$

where $\mathrm{Gr}(2, 5)$ should now be interpreted as the set of two-dimensional subspaces to the tangent space to $J(V)$ at the origin. The central geometric fact is then that, under suitable identification, the composition $\mathcal{G} \circ \psi$ is just the tautological inclusion coming from the definition of S as a subvariety of $\mathrm{Gr}(2, 5)$, the set of projective lines in $\mathbf{P}_4$. This essential fact, which we call the *tangent bundle theorem*, allows us to compute the branch locus of the Gauss map

$$\mathcal{G}\colon \theta_V \longrightarrow \mathbf{P}_4^* \,.$$

Analogously to the case of curves, this branch locus is the dual variety V^*

of the original cubic threefold V, from which we obtain (0.11). Finally (0.12) reduces to showing that V^* does not contain linear subspaces of dimension two so that V^* cannot coincide with the dual variety of a curve in $\mathbf{P}_4$.

By and large, the paper is self-contained. Aside from one's natural inclination to do so, the reason that we have given a self-contained development is that it was often necessary to have somewhat more precise information than was classically available. For example many computations hinge on such questions as whether the double curve D_0 of the degenerate Fano surface S_0 splits into two components under normalization, what is the normal bundle of the curve lying over D_0 in the normalization of S_0, and so forth. The length of the paper is due in large measure to this circumstance.

By way of acknowledgement, two of the central ideas in this paper were suggested to us by A. Mayer and E. Bombieri. First, it was Mayer who told us that the Gauss mapping should be used to study the subvarieties in a principally polarized abelian variety, and in particular that the Gauss map on the theta-divisor should have a branch locus with geometric significance. Secondly, it was Bombieri who suggested that the rationality of the cubic threefold should force its intermediate Jacobian to look like the Jacobian variety of a curve. After we put in the information arising from polarizations, it was exactly this notion which led eventually to the irrationality proof.

Finally, after this paper was completed, G. Lusztig pointed out to us a recent paper by A. N. Turin (Izvestia Akad. Nauk. S.S.S.R., Tom 34, no. 6, pages 1200–08). This paper, together with a subsequent one appearing in Izvestia, Tom 35 (1971), pages 498–529, seems to overlap very considerably with the geometric aspects of our study of the cubic threefold. In particular, the tangent bundle theorem is contained in an essentially equivalent form in Turin's first paper and the Torelli theorem for cubic threefolds, obtained by a different method, appears in the second paper according to a letter which we have recently received from the author. In a related letter, Y. Manin has written that he and V. Iskovskibh have recently proved the irrationality of the non-singular quartic threefold. Since some of these are unirational, this gives another counterexample to the three-dimensional Luroth problem.

Before beginning the main body of the paper, we conclude the introduction with a list of notation and conventions which will be used repeatedly throughout the paper:

Notation and Conventions

1. Dimension means *complex* dimension, and all complex manifolds will be assumed to be oriented by the form $dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n$ where $\{z_j =$

$x_j + iy_j\}_{j=1,\dots,n}$ are holomorphic local coordinates. For a complex number z, $\operatorname{Re} z$ and $\operatorname{Im} z$ will denote its real and imaginary parts respectively.

2. "Threefold" means "three-dimensional variety". If X is an algebraic variety, the expression "for generic $x \in X$" means "there exists a dense Zariski open subset U of X such that for all $x \in U$".

3. For a product $X_1 \times \cdots \times X_m$ of manifolds, $\pi_{X_i}: (X_1 \times \cdots \times X_m) \to X_i$ will denote the projection onto the factor X_i.

4. For a space X, $H_q(X) = H_q(X; \mathbf{Z})/\{\text{torsion cycles}\}$, all homology singular, with compact support. If X is a compact manifold: $H^q(X) = ($image of $H^q(X; \mathbf{Z})$ in the de Rham group $\{$closed q-forms$\}/\{$exact q-forms$\})$; $\wedge: H^*(X) \otimes H^*(X) \to H^*(X)$ denotes the cup-product operation and $\cdot: H_*(X) \otimes H_*(X) \to H_*(X)$ the dual intersection operation. If Y is an algebraic q-cycle in the algebraic manifold X, $\{Y\}$ is the element of $H_{2q}(X)$ carried by Y. "$\cdot$" will also denote intersection of algebraic cycles.

5. For a complex manifold X, $x \in X$, Y a submanifold of X: $T(X)$ is the (complex) tangent bundle of X, $T(X, x)$ its fibre at x; $T^*(X)$ is the dual cotangent bundle with fibre $T^*(X, x)$. $N(X, Y)$ will denote the normal bundle to Y in X. For a divisor D on the algebraic manifold X, $L(D)$ is the associated line bundle and $\mathcal{O}(D)$ its sheaf of holomorphic sections.

6. $\mathbf{P}_n$ will denote complex projective n-space; $\operatorname{Gr}(k+1, n+1)$ will denote the Grassmann manifold of $(k+1)$-dimensional subspaces of complex $(n+1)$-space. Then $\mathbf{P}_n = \operatorname{Gr}(1, n+1)$ and $\mathbf{P}_n^*$ will denote $\operatorname{Gr}(n, n+1)$, the dual projective space to $\mathbf{P}_n$. For $A \subseteq \mathbf{P}_n^*$, let $[A] =$ the subspace of $\mathbf{P}_n$ determined by the linear subspace $\bigcap\{h\colon h\colon h \in A\}$, for $B \subseteq \mathbf{P}_n$, let $[B] = \{h \in \mathbf{P}_n^*\colon B \subseteq [h]\}$.

7. $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{C}$ will denote respectively the integers, the rational, real and complex number fields.

Part one: Intermediate Jacobians of threefolds

1. Algebraic correspondences and homology relations

Let X and Y be smooth irreducible complex projective varieties of dimension m and n respectively, and let T be an algebraic r-cycle in $(X \times Y)$. Let

$$K\colon H_*(X) \otimes H_*(Y) \longrightarrow H_*(X \times Y)$$

be the Künneth isomorphism. Then T induces a homology mapping $\varphi(X, Y; T)$ defined by the composition:

$$H_*(X) \xrightarrow{\otimes\{Y\}} H_*(X) \otimes H_{2n}(Y) \xrightarrow{K} H_{*+2n}(X \times Y)$$
$$\xrightarrow{\cdot\{T\}} H_{*+2r-2m}(X \times Y) \xrightarrow{(\pi_Y)_*} H_{*+2r-2m}(Y) .$$

We have immediately a purely formal numerical relation:

$$(1.1) \qquad (\varphi(X, Y; T)(\gamma) \cdot \alpha)_Y = (\gamma \cdot (\varphi(Y, X; T)(\alpha)))_X$$

where $\gamma \in H_q(X)$, $\alpha \in H_{2(m+n-r)-q}(Y)$.

If we let $\sum_{i=0}^{2r} a_i \otimes b_{2r-i}$ be the decomposition of $\{T\}$ according to the Künneth isomorphism K and if we let X_1 and X_2 be two copies of the manifold X, then we can define an algebraic cycle M in $X_1 \times Y \times X_2$ such that:

$$(1.2) \qquad \{M\} = \left(\sum_i a_i \otimes b_{2r-i} \otimes \{X_2\}\right) \cdot \left(\sum_j \{X_1\} \otimes b_{2r-j} \otimes a_j\right)$$

(again we use the Künneth isomorphism, this time on $X_1 \times Y \times X_2$). Define the mapping $\eta(X; T)$ as the composition:

$$H_*(X) = H_*(X_1) \xrightarrow{\otimes\{Y\}\otimes\{X_2\}} H_{*+2(m+r)}(X_1 \times Y \times X_2)$$
$$\xrightarrow{\cdot\{M\}} H_{*+4r-2(m+n)}(X_1 \times Y \times X_2)$$
$$\xrightarrow{(\pi_{X_2})*} H_{*+4r-2(m+n)}(X) .$$

Again, by a purely formal calculation using the relation between the Künneth formula and the intersection pairing, one obtains:

$$(1.3) \qquad \eta(X; T) = \varphi(Y, X; T) \circ \varphi(X, Y; T) .$$

Also notice that if we view T as a correspondence $T\colon X \to Y$ and let $T^*\colon Y \to X$ be the dual correspondence (also defined by $T \subseteq (X \times Y)$) then

$$M = (T^* \circ T)\colon X \to X .$$

If T is an effective algebraic cycle (components with multiplicity one), then:

$$(1.4) \qquad \{M\} = \{(\pi_{X_1 \times Y})^{-1}(T)\} \cdot \{(\pi_{Y \times X_2})^{-1}(T)\} .$$

2. Families of algebraic curves on a threefold

For the purposes of this paper, we will be interested in the formalism of § 1 only in a very restricted setting. Let V be a smooth irreducible complex projective variety of complex dimension three.

Definition 2.1. An *algebraic family of algebraic curves* on V is a non-singular projective variety S (called the parameter space) together with an algebraic subvariety $T \subseteq (S \times V)$ such that:

$$Z_s = \pi_V((\{s\} \times V) \cap T)$$

is an algebraic curve in V for each $s \in S$.

(Note: Multiplicities of the components of Z_s are given by the multiplicities of the components of $(\{s\} \times V)\cdot T$. Thus since all components of T will be counted with multiplicity 1, the same will be true of Z_s for generic $s \in S$. This setting, while not the most general, is adequate for our present purposes.)

For $x \in V$ define:

$$W_x = \pi_S\big((S \times \{x\}) \cap T\big)\ . \tag{2.2}$$

Thus W_x is the set of $s \in S$ such that Z_s passes through x. Now the family $\{W_x\}_{x \in X}$ does not have to be equidimensional, however it will be of use to make some restriction in this direction, which we will incorporate in the following definition:

Definition 2.3. The family $\{Z_s\}_{s \in S}$ is said to be a *covering family of curves* for V if:

(i) S is irreducible, $\dim S = 2$, and Z_s is irreducible for generic s;

(ii) for all but a finite number of points of V,

$$\dim W_x = 0\ ;$$

(iii) for generic $x \in V$, W_x has more than one element;

(iv) for generic x, if s_1 and s_2 are distinct points of W_x then Z_{s_1} and Z_{s_2} have transverse intersection at x and have no other common points.

A consequence of this definition is that if $\{Z_s\}$ is a covering family, we can define an effective divisor I, called the *incidence divisor of* $(S \times S)$, by putting:

$$I = \big(\text{union of all components of dimension three of the set } \{(s_1, s_2) \in (S \times S)\colon (Z_{s_1} \cap Z_{s_2}) \neq \varnothing\}\big)\ . \tag{2.4}$$

Furthermore, if we define M as in (1.2) for the triple $(S, V; T)$ it follows that

$$(\pi_{S\times S})_*\big(\{M\}\big) = \{I\}\ ,$$

so that, again using the notation of § 1:

$$\eta(S;\, T) = \varphi(S, S;\, I)\ . \tag{2.5}$$

Notice also that if we define

$$D_s = \pi_S\big((\{s\} \times S) \cap I\big) \tag{2.6}$$

where π_S means projection on the second factor, then it is a consequence of Definition 2.3 that D_s is an effective divisor on S for each $s \in S$. (Set-theoretically D_s is just the set of all s' such that Z_s and $Z_{s'}$ intersect on V, but as with Z_s, we will want to count components of D_s with multiplicities given by those of the components of $\big((\{s\} \times S)\cdot I\big)$. For generic s, D_s is counted with multiplicity one.) The algebraic family of divisors $\{D_s\}_{s \in S}$ will be called the *family of incidence divisors on* S.

Recalling the definitions in § 1, put:

$$(2.7)\qquad \begin{aligned} \varphi_* &= \varphi(S, V; T)\colon H_1(S) \longrightarrow H_3(V) \\ \lambda_* &= \varphi(V, S; T)\colon H_3(V) \longrightarrow H_3(S) \\ \eta_* &= \eta(S; T)\colon \quad H_1(S) \longrightarrow H_3(S) . \end{aligned}$$

Using § 1 and (2.5), these mappings can be thought of gemetrically as follows:

$\varphi(\gamma)$ = three-cycle traced out on V by Z_s as s traces out the one-cycle γ;
$\lambda(\alpha)$ = three-cycle traced out by W_x as x traces out the three-cycle α;
$\eta(\gamma)$ = three-cycle traced out by D_s as s traces out the one-cycle γ.

Furthermore, since T and I are algebraic cycles, the dual mappings

$$(2.8)\qquad \begin{aligned} &\varphi^*\colon H^3(V) \longrightarrow H^1(S) \quad \text{given by} \quad \int_\gamma \varphi^*(\omega) = \int_{\varphi_*(\gamma)} \omega \\ &\lambda^*\colon H^3(S) \longrightarrow H^3(V) \quad \text{given by} \quad \int_\alpha \lambda^*(\beta) = \int_{\lambda_*(\alpha)} \beta \\ &\eta^*\colon H^3(S) \longrightarrow H^1(S) \quad \text{given by} \quad \int_\gamma \eta^*(\beta) = \int_{\eta_*(\gamma)} \beta \end{aligned}$$

respect the Hodge decompositions [13; § 15] of $H^*(S) \otimes \mathrm{C}$ and $H^*(V) \otimes \mathrm{C}$ according to the following rules:

$$(2.9)\qquad \begin{aligned} &\varphi^*\colon H^{p,q}(V) \longrightarrow H^{p-1,q-1}(S) \\ &\qquad (\text{so, in particular, } \varphi^* |_{H^{3,0}(V)} \equiv 0) ; \\ &\lambda^*\colon H^{p,q}(S) \longrightarrow H^{p,q}(V) ; \\ &\eta^*\colon H^{p,q}(S) \longrightarrow H^{p-1,q-1}(S) . \end{aligned}$$

Finally for a covering family $\{Z_s\}$ of curves for V there is a relation between the "positivity" of the cycles Z_s and D_s which is conveniently introduced at this point:

Definition 2.10. An effective algebraic one-cycle Z on V is called *numerically positive* if for any effective divisor W on V:

$$(W \cdot Z) > 0 .$$

PROPOSITION 2.11. *If $\{Z_s\}$ is a covering family for V and if Z_s is numerically positive, then the divisors D_s are ample on S.*

Proof. Let C be any effective divisor on S. Since $\{Z_s\}$ is a covering family, there is an effective divisor W on V such that

$$\varphi(S, V; T)\big(\{C\}\big) = \{W\} \qquad \text{(see § 1).}$$

But

$$\big(\varphi(S, V; T)(\{C\}) \cdot \varphi(S, V; T)(\{s\})\big) = \big(C \cdot \eta(S; T)(\{s\})\big)$$

by (1.1), that is $(W \cdot Z_s) = (C \cdot D_s)$. Since Z_s is numerically positive, $(C \cdot D_s) > 0$, and so by the criterion of Moishezon-Nakai [11; page 30], D_s is ample.

COROLLARY 2.12. *If V has Picard number $= 1$, then D_s is ample.* (Recall that if $H^2(V)$ has rank one, which is the case, for example, when V is a complete intersection, then the Picard number of $V = 1$.)

3. The intermediate Jacobian and its polarizing class

As was mentioned in § 2, one has the Hodge decomposition of the group $H^3(V) \otimes \mathbf{C}$:

$$H^3(V) \otimes \mathbf{C} \approx H^{3,0}(V) \oplus H^{2,1}(V) \oplus H^{1,2}(V) \oplus H^{0,3}(V).$$

If we project the subgroup $H^3(V)$ of $H^3(V) \otimes \mathbf{C}$ into the subspace $W = H^{1,2}(V) \oplus H^{0,3}(V)$, the image is a lattice U_V in W and there is an isomorphism

$$\rho\colon H^3(V) \longrightarrow U_V \tag{3.1}$$

such that for $\alpha, \beta \in H^3(V)$:

$$\int_V \alpha \wedge \beta = 2\,\mathrm{Re} \int_V \rho(\alpha) \wedge \overline{\rho(\beta)}\ . \tag{3.2}$$

Now there is a non-degenerate hermitian form $\mathcal{H}_V$ defined on W by the formula:

$$\mathcal{H}_V(\omega_1, \omega_2) = 2i \int_V \omega_1 \wedge \bar{\omega}_2\ . \tag{3.3}$$

By (3.2), $\mathrm{Im}\,\mathcal{H}_V$ corresponds under the isomorphism (3.1) to the cup product pairing on $H^3(V)$. Thus $\mathrm{Im}\,\mathcal{H}_V$ is unimodular on U_V. Also if Ω is the fundamental class of a Kähler metric for V, then for appropriate choice of local coordinates z_1, z_2, z_3 around $x \in V$

$$\Omega\,|_x = \frac{\sqrt{-1}}{2\pi}\Big\{\textstyle\sum_{k=1}^3 dz_k \wedge d\bar{z}_k \in \big(T^*(V, x) \wedge \bar{T}^*(V, x)\big)\Big\}\ .$$

If $\omega \in H^{1,2}(V)$ is such that $(\omega \wedge \Omega)\,|_x = 0$, then

$$\omega\,|_x = a_1 dz_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3 + a_2 d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_3 + a_3 d\bar{z}_1 \wedge d\bar{z}_2 \wedge dz_3$$

so that

$$\omega \wedge \bar{\omega} = r(-2i)^3(-1)dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \wedge dx_3 \wedge dy_3$$

where $z_k = x_k + iy_k$ and, if $\omega\,|_x \neq 0$, r is a positive real number. Thus we have:

LEMMA 3.4 *Let E be a subgroup of $H^3(V)$ such that for $E_{\mathbf{C}} = (E \otimes \mathbf{C}) \subseteq (H^3(V) \otimes \mathbf{C})$:*

(i) $E_{\mathbf{C}} = \big(E_{\mathbf{C}} \cap H^{2,1}(V)\big) \oplus \big(E_{\mathbf{C}} \cap H^{1,2}(V)\big)$;

(ii) $\omega \wedge \Omega = 0$ *(i.e., ω is primitive) for each $\omega \in E_{\mathbf{C}}$;*

then $\mathcal{H}_V$ *is positive definite on* $(E_{\mathbf{C}} \cap H^{1,2}(V))$.

The preceding discussion suggests the following formalism (see also [17; Chapter I]):

Definition 3.5. Let W be a finite-dimensional complex vector space, U a lattice (of maximal real rank) in W, and $\mathcal{H}$: $(W \times W) \to \mathbf{C}$ a nondegenerate hermitian form on W such that $\operatorname{Im} \mathcal{H}$ is integral-valued and unimodular on U. The triple $(W, U, \mathcal{H})$ is called a *principally polarized complex torus.*

In the category of principally polarized complex tori, the notion of morphism will be the strong one, namely

$$(W_1, U_1, \mathcal{H}_1) \xrightarrow{\sigma} (W_2, U_2, \mathcal{H}_2)$$

means a linear transformation σ: $W_1 \to W_2$ such that $\sigma(U_1) \subseteq U_2$ and $\mathcal{H}_1 = (\mathcal{H}_2)_\sigma$, the "pullback" of $\mathcal{H}_2$ under σ. This implies that σ: $W_1 \to W_2$ is injective and that $\sigma(U_1)$ is a direct summand of U_2 such that $\operatorname{Im} \mathcal{H}_2$ is also unimodular on $U_1^{\perp} = (\sigma(U_1))^{\perp \operatorname{Im} \mathcal{H}_2} \subseteq U_2$. Since $(\sigma(W_1))^{\perp \mathcal{H}_2} \subseteq (\sigma(W_1))^{\perp \operatorname{Im} \mathcal{H}_2}$, we have by dimension that:

$$(\sigma(W_1))^{\perp \mathcal{H}_2} = (\sigma(W_1))^{\perp \operatorname{Im} \mathcal{H}_2} .$$

Denoting this last complex vector space by $W_1^{\perp}$, we obtain a direct sum decomposition:

$$(W_2, U_2, \mathcal{H}_2) \approx (W_1, U_1, \mathcal{H}_1) \oplus (W_1^{\perp}, U_1^{\perp}, \mathcal{H}_2 \,|\, W_1^{\perp}) . \tag{3.6}$$

We have just seen that every non-singular threefold V has associated a principally polarized complex torus

$$\mathcal{J}(V) = (W_V, U_V, \mathcal{H}_V) . \tag{3.7}$$

If $h^{3,0}(V) = h^{1,0}(V) = 0$, then by Lemma 3.4, $\mathcal{H}_V$ is positive definite so that $\mathcal{J}(V)$ becomes a *principally polarized abelian variety.* It is of course a standard fact that if C is a non-singular projective curve, $W_C = H^{0,1}(C)$, $U_C =$ the projection into $H^{0,1}(C)$ of the subgroup $H^1(C)$ of $H^1(C) \otimes \mathbf{C}$, and

$$\mathcal{H}_C(\omega_1, \omega_2) = -\, 2i \int_C \omega_1 \wedge \bar{\omega}_2 ,$$

then $\mathcal{H}_C$ is positive definite on W_C and

$$\mathcal{J}(C) = (W_C, U_C, \mathcal{H}_C) \tag{3.8}$$

is a principally polarized abelian variety, called the *Jacobian variety* of C.

Suppose now that λ: $V_1 \to V_2$ is a birational morphism between non-singular threefolds. Then for $\omega_1, \omega_2 \in W_{V_2}$:

$$\int_{V_2} \omega_1 \wedge \bar{\omega}_2 = \int_{V_1} (\lambda^*\omega_1) \wedge \overline{(\lambda^*\omega_2)} \tag{3.9}$$

so that there is induced a morphism

$$\mathcal{J}(V_2) \longrightarrow \mathcal{J}(V_1)$$

and hence by (3.6) a direct sum decomposition:

$$\mathcal{J}(V_1) \approx \mathcal{J}(V_2) \oplus \mathcal{J}(V_2)^{\perp} . \tag{3.10}$$

If we suppose further that V_1 is the monoidal transform of V_2 obtained by blowing up V_2 along a non-singular curve C, then for $\mathcal{J}(C)$ as in (3.8) we have:

LEMMA 3.11. $\mathcal{J}(V_1) \approx \mathcal{J}(V_2) \oplus \mathcal{J}(C)$.

Proof. Put $T = \{(s, x) \in (C \times V_1)\colon \lambda(x) = s\}$, and define $\varphi_* = \varphi(C, V_1; T)$: $H_1(C) \to H_3(V_1)$ as in § 1. One checks immediately that the sequence

$$0 \longrightarrow H^3(V_2) \otimes \mathbf{C} \xrightarrow{\lambda^*} H^3(V_1) \otimes \mathbf{C} \xrightarrow{\pi^*} H^3(T) \otimes \mathbf{C}$$

is exact where $\pi = \pi_{V_1}\colon T \to V_1$ is the natural projection. By the Thom isomorphism and the fact that as in (2.9) all maps respect the Hodge decomposition of cohomology, we have the following diagram in which the horizontal sequence is exact:

$$\begin{tikzcd} 0 \arrow[r] & W_{V_2} \arrow[r, "\lambda^*"] & W_{V_1} \arrow[r, "\pi^*"] \arrow[dr, "\varphi^*"'] & H^{1,2}(T) \arrow[d, phantom, "\wr\wr"] \\ & & & H^{0,1}(C) . \end{tikzcd}$$

The lemma will follow if we can show that

(i) φ^* is onto;

(ii) if $(\mathcal{H}_C)_{\varphi^*}$ denotes the pull-back of $\mathcal{H}_C$, then

$$(\mathcal{H}_C)_{\varphi^*} = (\mathcal{H}_{V_1})\,|_{\lambda^*(W_{V_2})^{\perp}} .$$

We obtain both (i) and (ii) at once by proving: If $\gamma_1, \gamma_2 \in H_1(C)$, then:

$$\big(\varphi_*(\gamma_1)\cdot\varphi_*(\gamma_2)\big)_{V_1} = -\,(\gamma_1\cdot\gamma_2)_C . \tag{3.12}$$

To prove (3.12), first notice that if γ_1 and γ_2 can be represented by cycles α_1 and α_2 which have no common point, then so can $\varphi_*(\gamma_1)$ and $\varphi_*(\gamma_2)$. It therefore suffices to check the formula in the case where $(\gamma_1\cdot\gamma_2) = 1$ and their representatives α_1 and α_2 are part of a standard basis for $H_1(C)$. Thus α_1 and α_2 meet transversely at one point $s_0 \in C$ and have no other common point. Let W be a non-singular surface in V_2 which meets C transversely. Then $\widetilde{W} = \lambda^{-1}(W)$ is a non-singular surface in V_1 with exceptional curves of the first kind above each point of $(W \cap C)$. Suppose now that $s_0 \in (W \cap C)$. Then:

$$\big(\varphi_*(\gamma_1)\cdot\varphi_*(\gamma_2)\big)_{V_1} = \big(\{\lambda^{-1}(s_0)\}\cdot\{\lambda^{-1}(s_0)\}\big)_{\widetilde{W}} .$$

But it is a standard fact in the theory of algebraic surfaces that $\lambda^{-1}(s)$ has self-intersection $= (-1)$ in $\tilde{W}$. So (3.12) follows and so does the lemma.

Given a principally polarized complex torus

$$\mathcal{T} = (W, U, \mathcal{H})$$

there is a natural identification

$$U \approx H_1(W/U) \tag{3.13}$$

so that $\operatorname{Im} \mathcal{H}$ can be interpreted as a linear mapping on $\big(H_1(W/U) \wedge H_1(W/U)\big)$. Since

$$\wedge^2 H_1(W/U) \approx H_2(W/U) ,$$

$(-\operatorname{Im} \mathcal{H})$ corresponds naturally to an element

$$\Omega(\mathcal{T}) \in H^2(W/U)$$

called the *polarizing class* of $\mathcal{T}$. If $\mathcal{H}$ is positive definite, that is, if $\mathcal{T}$ is a *principally polarized abelian variety* (see [17; page 30]), then there is an effective divisor $\theta(\mathcal{T})$ on (W/U) such that $\{\theta(\mathcal{T})\}$ is the Poincaré dual of $\Omega(\mathcal{T})$. $\theta(\mathcal{T})$ is uniquely determined up to translation in (W/U) and is called the *theta divisor* of $\mathcal{T}$. One has the formula for $\gamma_1, \gamma_2 \in U$:

$$\operatorname{Im} \mathcal{H}(\gamma_1, \gamma_2) = -\big((\gamma_1 \times \gamma_2)\cdot\theta(\mathcal{T})\big)_{(W/U)} \tag{3.14}$$

where the identification (3.13) is assumed and "$\times$" denotes the Pontrjagin product.

For a principally polarized complex torus $\mathcal{T}$, we define

$$\dim \mathcal{T} = \dim W .$$

Definition 3.15. Let $\mathcal{T}$ be a principally polarized abelian variety of dimension q. $\mathcal{T}$ is said to be of level k $(1 \leqq k \leqq q-1)$ if

$$\underbrace{\Omega(\mathcal{T}) \wedge \cdots \wedge \Omega(\mathcal{T})}_{q-k\text{-times}}/(q-k)\,!$$

is the Poincaré dual of an effective algebraic k-cycle in (W/U).

The condition of the definition is of course vacuous if $k = q-1$. It is a theorem of Matsusaka (see [16] and [14]) that $\mathcal{T}$ is of level one if and only if $\mathcal{T} = \mathcal{J}(C)$ where C is a (possibly reducible) non-singular algebraic curve (in which case the image of C in $J(C) = (W_C/U_C)$ is the desired algebraic one-cycle). If $\mathcal{T}$ is of level one, then $\mathcal{T}$ is of level k for each $k = 1, \cdots, q-1$, since the image of the k-fold symmetric product $C^{(k)}$ of C in $J(C)$ is a k-cycle satisfying the required condition in Definition 3.15. Finally, we note that in the case of the cubic threefold V which will be studied in detail later on, $\mathcal{J}(V)$ will be shown to be *not* of level one. We will prove, however, that $\mathcal{J}(V)$ is of level two (see § 13).

Next we let $\mathcal{C}$ denote the set of isomorphism classes of principally polarized complex tori. We define an equivalence relation $(\sim_1)$ in $\mathcal{C}$ as follows:

(3.16) $\mathcal{T} \sim_1 \mathcal{T}'$ if there exist non-singular (possibly reducible) curves C and C' such that there are morphisms

$$\mathcal{T} \longrightarrow \mathcal{T}' \oplus \mathcal{J}(C')$$
$$\mathcal{T}' \longrightarrow \mathcal{T} \oplus \mathcal{J}(C) .$$

This equivalence relation has the properties:

(3.17) If $\mathcal{T}_1 \sim_1 \mathcal{T}_1'$ and $\mathcal{T}_2 \sim_1 \mathcal{T}_2'$, then $(\mathcal{T}_1 \oplus \mathcal{T}_2) \sim_1 (\mathcal{T}_1' \oplus \mathcal{T}_2')$.

(3.18) If $\mathcal{T}$ is a principally polarized abelian variety and $\mathcal{T} \sim_1 \mathcal{T}'$, then $\mathcal{T}'$ is a principally polarized abelian variety.

The set of equivalence classes $\mathcal{C}/\{\sim_1\}$ with the operator $\oplus$ forms a commutative semi-group with identity element equal to the equivalence class of Jacobian varieties of curves. Let **G** denote this semi-group. It would seem that the structure of **G** is quite complicated. However if we define:

(3.19) $\mathbf{A} = ($semi-group in $\mathbf{G}$ given by $\{\mathcal{T}: \mathcal{T}$ is a principally polarized abelian variety$\})$

we can give the structure of **A** explicitly with the help of the following lemma which was pointed out to the authors by G. Shimura (see also [14]):

LEMMA 3.20. *Let $(W, U, \mathcal{H})$ be a principally polarized abelian variety and suppose that $\theta(\mathcal{T})$ is reducible, that is $\theta(\mathcal{T}) = \sum_{i=1}^{n} m_i\theta_i$ where each θ_i is effective and irreducible and each m_i is a positive integer. Then all the $m_i = 1$ and there exist principally polarized abelian varieties $\mathcal{T}_i = (W_i, U_i, \mathcal{H}_i)$ such that:*

(i) $\mathcal{T} \approx \bigoplus \mathcal{T}_i$,

(ii) *under the isomorphism* (i), *θ_i corresponds to* $(W_1/U_1) \times \cdots \times (W_{i-1}/U_{i-1}) \times \theta(\mathcal{T}_i) \times (W_{i+1}/U_{i+1}) \times \cdots \times (W_n/U_n)$.

Proof. Define:

$$\Psi\colon (W/U)^n \longrightarrow \operatorname{Pic}(W/U)$$
$$(a_1, \cdots, a_n) \longmapsto \Big(\Big(\sum m_i(a_i + \theta_i)\Big) - \theta\Big)$$

where θ is a particular divisor representing $\theta(\mathcal{T})$ and

$$\theta = \sum m_i\theta_i .$$

Now $\dim H^0\big((W/U), \mathcal{O}(\sum m_i(a_i + \theta_i))\big) = 1$ for each $(a_1, \cdots, a_n) \in (W/U)^n$ and $(W/U)^n$ is connected. Thus:

(3.21) $\Psi(a_1, \cdots, a_n) = \Psi(b_1, \cdots, b_n)$ if and only if $a_i + \theta_i = b_i + \theta_i$ for all i (where equality means equality as divisors).

Since Ψ is a homomorphism of abelian varieties and Ψ is clearly onto, dim (ker Ψ) $= (n-1)q$ where $q = \dim \mathcal{T}$. Let $(W/U)_i$ be the subvariety of $(W/U)^n$ corresponding to the inclusion of the i-th factor into the product. Then if $A_i = (\ker \Psi) \cap (W/U)_i$ we have by (3.21) that:

$$\ker \Psi = \sum A_i \ .$$

Now consider each A_i as a subvariety of (W/U). By (3.21):

$$\bigcap_i A_i = \{(0, \cdots, 0)\} \ .$$

If we put $B_i = \bigcap_{j \neq i} A_j$, then an easy dimension count gives that

$$\dim A_i + \dim B_i = q \ .$$

But $\big(\{A_i\} \cdot \{B_i\}\big)_{(W/U)} = 1$ since $\bigcap_i A_i$ has only one point. Thus $(W/U) \approx A_i \oplus B_i$, and it follows immediately that:

$$(W/U) = \bigoplus_{i=1}^{n} B_i \ .$$

For $a \in B_i$, $a + \theta_j = \theta_j$ for all $j \neq i$ so that

$$\big(\{\theta\} \cdot \{B_i\}\big) = \big(\{m_i \theta_i\} \cdot \{B_i\}\big) \ .$$

But θ induces a principal polarization on B_i since B_i is a direct summand of (W/U) so that $m_i = 1$ and the lemma is proved.

Definition 3.22. A principally polarized complex torus $\mathcal{T}$ is *irreducible* if for any morphism

$$\rho \colon \mathcal{T}' \longrightarrow \mathcal{T}$$

either $\mathcal{T}' = 0$ or ρ is an isomorphism.

Since the direct summands of $\mathcal{T}$ in Lemma 3.20 were constructed intrinsically from the components of $\theta(\mathcal{T})$, we have:

COROLLARY 3.23. *If $\mathcal{T}$ is a principally polarized abelian variety, $\mathcal{T}$ has a unique decomposition into the direct sum of irreducible principally polarized abelian varieties. $\mathcal{T}$ itself is irreducible if and only if $\theta(\mathcal{T})$ is irreducible.*

Now the semi-group $\mathbf{A}$ defined in (3.19) can be characterized as follows:

COROLLARY 3.24. $\mathbf{A} \approx$ *(free abelian semi-group on the set of all irreducible principally polarized abelian varieties $\mathcal{T}$ which are not of level one).*

If V is a non-singular algebraic threefold, we denote by $\mathbf{g}(V)$ the element of the semi-group $\mathbf{G}$ corresponding to the principally polarized complex torus $\mathcal{J}(V)$, called the *intermediate Jacobian* of V (see (3.7)). Motivation for the definition of $\mathbf{g}(V)$ is contained in the following:

THEOREM 3.25. **g** *is a birational invariant.*

Proof. Let $V \to V'$ be a birational map. Then by [12; page 140], there exists a sequence

$$\tilde{V} = V_n \longrightarrow V_{n-1} \longrightarrow \cdots \longrightarrow V_0 = V$$

such that each $V_i \to V_{i-1}$ is obtained by blowing up a point or along a non-singular curve, and such that the composition

$$\tilde{V} \to V \to V'$$

is a (birational) morphism. Since blowing up a point has no effect on third homology and hence no effect on the intermediate Jacobian, we conclude by Lemma 3.11:

$$\mathcal{J}(\tilde{V}) \approx \mathcal{J}(V) \oplus \mathcal{J}(C)$$

for some (possibly reducible) non-singular curve C. By (3.9) and (3.10), there exists a morphism

$$\mathcal{J}(V') \longrightarrow \mathcal{J}(\tilde{V}) .$$

Similarly there exists $\tilde{V}'$ such that $\mathcal{J}(\tilde{V}') \approx \mathcal{J}(V') \oplus \mathcal{J}(C')$ and $\mathcal{J}(V) \to \mathcal{J}(\tilde{V}')$. So $\mathcal{J}(V) \sim_1 \mathcal{J}(V')$ and the theorem follows.

An essential role in what follows will be played by:

COROLLARY 3.26. *If there exists a birational mapping between V and $\mathbf{P}_3$, then $\mathcal{J}(V) = \mathcal{J}(C)$ for some (possibly reducible) non-singular curve C.*

Proof. By the previous theorem $\mathcal{J}(V) \sim_1 (0)$. In particular by (3.18), $\mathcal{J}(V)$ is an abelian variety. Now use Corollary 3.23.

4. The Abel-Jacobi mapping

We wish to combine the considerations of § 2 and § 3. Let V be a non-singular algebraic threefold, and let $\{Z_s\}_{s \in S}$ be a covering family of algebraic curves for V. Let

$$\mathcal{J}(V) = (W_V, U_V, \mathcal{H}_V)$$

as in (3.7). W_V is canonically the dual vector space of $H^{3,0}(V) \oplus H^{2,1}(V)$. Furthermore, if for $\alpha \in H_3(V)$ we define

$$\begin{aligned} \alpha^*: \ \left(H^{3,0}(V) \oplus H^{2,1}(V)\right) &\longrightarrow \mathbf{C} \\ \omega &\longmapsto \int_\alpha \omega \end{aligned}$$

then we have a natural isomorphism of pairs:

$$(W_V, U_V) \approx \left((H^{3,0}(V) \oplus H^{2,1}(V))^*, \{\alpha^*: \alpha \in H_3(V)\}\right) .$$

Define

$$J(V) = (W_V/U_V) \approx \big(H^{3,0}(V) \oplus H^{2,1}(V)\big)^* \,/\, H_3(V) \tag{4.1}$$

where $H_3(V)$ is identified with the corresponding lattice in $\big(H^{3,0}(V) \oplus H^{2,1}(V)\big)^*$. Notice that under this last identification we can write elements of $J(V)$ in the form

$$\sum a_i\alpha_i \quad (a_i \text{ real})$$

where $\{\alpha_i\}$ is some basis for $H_3(V)$, and (3.14) gives:

$$\int_{\alpha\times\beta} \Omega\big(\mathcal{J}(V)\big) = -\,(\alpha\cdot\beta)_V \tag{4.2}$$

where $(\alpha \times \beta)$ denotes the Pontrjagin product of α and β considered as elements of $H_1\big(J(V)\big)$.

Similarly, if $p + q = m$, any cycle $\gamma \in H_m(S)$ can be identified with a linear functional:

$$\begin{aligned} \gamma^*\colon H^{p,q}(S) &\longrightarrow \mathbf{C} \\ \omega &\longmapsto \int_\gamma \omega\,. \end{aligned}$$

We define the *Albanese variety* of S:

$$\mathrm{Alb}(S) = \big(H^{1,0}(S)\big)^* \,/\, H_1(S)\,,$$

and the *Picard variety* of S:

$$\mathrm{Pic}(S) = \big(H^{2,1}(S)\big)^* \,/\, H_3(S)\,.$$

By (2.7) and (2.9) we clearly have the following commutative diagram of induced homomorphisms:

$$\begin{tikzcd} \mathrm{Alb}(S) \arrow[dr, "\varphi"] \arrow[dd, "\eta"'] & \\ & J(V)\,. \arrow[dl, "\lambda"] \\ \mathrm{Pic}(S) & \end{tikzcd} \tag{4.3}$$

Again under the appropriate identifications, we write:

$$\begin{aligned} \varphi\big(\textstyle\sum c_j\gamma_j\big) &= \textstyle\sum c_j\varphi_*(\gamma_j) \\ \lambda\big(\textstyle\sum a_i\alpha_i\big) &= \textstyle\sum a_i\lambda_*(\alpha_i) \\ \eta\big(\textstyle\sum c_j\gamma_j\big) &= \textstyle\sum c_j\eta_*(\gamma_j)\,. \end{aligned}$$

LEMMA 4.4. *Let W be an ample divisor on V. Suppose that $\big(\varphi_*(\gamma)\cdot W\big) = 0$ in $H_1(V)$ for each $\gamma \in H_1(S)$. Then the intersection pairing on V is non-degenerate on $\varphi_*\big(H_1(S)\big)$.*

Proof. The statement is an immediate corollary of (4.2), (2.9), and

Lemma 3.4.

The mapping $\varphi\colon \mathrm{Alb}(S) \to J(V)$ is called the *Abel-Jacobi mapping*. We are now in a position to prove an analogue of the classical Abel's theorem for curves:

THEOREM 4.5. *Suppose that there is an ample divisor W on V such that $(\varphi_*(\gamma)\cdot W) = 0$ for all $\gamma \in H_1(S)$. Then in the commutative diagram*

$$\begin{tikzcd} \mathrm{Alb}(S) \arrow[d, "\eta"] \arrow[dr, "\varphi"] & \\ \mathrm{Pic}(S) & J(V) \arrow[l, "\lambda"] \end{tikzcd}$$

$(\ker\varphi)^\circ = (\ker\eta)^\circ$ *where* $(^\circ)$ *denotes the component of the identity.*

Proof. We need only check that λ_* is injective on $\varphi_*(H_1(S))$. If $\lambda_*(\varphi_*(\gamma')) = 0$ then by (1.1)

$$(\varphi_*(\gamma)\cdot\varphi_*(\gamma')) = 0$$

for all $\gamma \in H_1(S)$. Now use Lemma 4.4.

Geometrically Theorem 4.5 means that the equivalence relation on the curves $\{Z_s\}_{s\in S}$ which is induced by the intermediate Jacobian of V is, up to isogeny, the same as linear equivalence on the incidence divisors $\{D_s\}_{s\in S}$. There is a general discussion of incidence divisors and Abel's theorem in [9; §1–4].

For each choice of a basepoint $s \in S$ we have a canonical morphism:

$$\alpha_s\colon S \longrightarrow \mathrm{Alb}(S) \tag{4.6}$$

given by $\alpha_s(s') = \left(\omega \mapsto \int_s^{s'} \omega\right)$. In the remainder of this section let us suppose that there is a principally polarized subtorus

$$\mathcal{T} = (W_1, U_1, \mathcal{H}_1) \longrightarrow \mathcal{J}(V)$$

such that:

(i) $\varphi(\mathrm{Alb}(S)) \subseteqq (W_1/U_1)$

(ii) $\mathcal{H}_1$ is positive definite on W_1.

Let θ be an effective divisor on (W_1/U_1). Then there is associated to θ a homomorphism

$$\mu\colon \mathrm{Alb}(S) \longrightarrow \mathrm{Pic}(S)$$

which can be constructed as follows. Let θ_φ be the divisor given on $\mathrm{Alb}(S)$ by $\varphi^{-1}(\theta)$. (Notice that θ_φ may be zero but otherwise is an effective divisor. We suppose that $\theta_\varphi \neq 0$ since otherwise everything is trivial.) Now θ_φ induces an algebraic family of divisors on S:

$$\{E_s\}_{s \in S}$$

such that $L(E_s)$ is the pull-back to S of the line bundle $L(\theta)$ under the mapping α_s. The mapping $\rho_{s_0}\colon S \to \mathrm{Pic}(S)$ defined by $\rho_{s_0}(s) = (E_s - E_{s_0})$ has the following properties:

(i) The homomorphism $\mu\colon \mathrm{Alb}(S) \to \mathrm{Pic}(S)$ such that $\mu \circ \alpha_{s_0} = \rho_{s_0}$ is independent of the choice of s_0.

(ii) Let $\mu_*\colon H_1(S) \to H_3(S)$ be the homomorphism induced by $\mu\colon \mathrm{Alb}(S) \to \mathrm{Pic}(S)$ and the identifications $H_1(\mathrm{Alb}(S)) = H_1(S)$, $H_1(\mathrm{Pic}(S)) = H_3(S)$. Then continuing with these same identifications, we obtain the formula:

$$(4.7) \qquad \begin{aligned} \big(\gamma_1 \cdot \mu_*(\gamma_2)\big)_S &= \big((\gamma_1 \times \gamma_2)\cdot \theta_\varphi\big)_{\mathrm{Alb}(S)} \\ &= \big(\big(\varphi_*(\gamma_1) \times \varphi_*(\gamma_2)\big)\cdot \theta\big)_{(W_1/U_1)} \end{aligned}$$

where "$\times$" as before denotes Pontrjagin product.

Now suppose that θ is a specific representative of the theta-divisor $\theta(\mathcal{T})$ determined up to translation. Recalling the definition of η in (4.3), we have:

THEOREM 4.8. $\mu = -\eta\colon \mathrm{Alb}(S) \to \mathrm{Pic}(S)$.

Proof. For $\gamma_1, \gamma_2 \in H_1(S)$:

$$\begin{aligned} \big(\gamma_1 \cdot \eta_*(\gamma_2)\big)_S &\overset{(1.1)}{=} \big(\varphi_*(\gamma_1)\cdot \varphi_*(\gamma_2)\big)_V \\ &\overset{(4.2)}{=} -\big(\big(\varphi_*(\gamma_1) \times \varphi_*(\gamma_2)\big)\cdot \theta\big)_{(W_1/U_1)} \\ &\overset{(4.7)}{=} -\big(\gamma_1 \cdot \mu_*(\gamma_2)\big)_S . \end{aligned}$$

PART TWO: GEOMETRY OF CUBIC HYPERSURFACES

5. The dual mapping, Lefschetz hypersurfaces

Let V be a hypersurface in complex projective n-space $\mathbf{P}_n$.

Definition 5.1. V will be called a *Lefschetz hypersurface* if V is either non-singular or has at most one ordinary double point.

It follows that, if V is Lefschetz, then V is given locally in $\mathbf{P}_n$ by either the equation $z_n = 0$ (simple point) or $(z_1^2 + \cdots + z_n^2) = 0$ (double point) for appropriately chosen holomorphic local coordinates in $\mathbf{P}_n$. Suppose that V is defined by an irreducible homogeneous polynomial of degree d:

$$(5.2) \qquad F(X_0, \cdots, X_n) .$$

Let $\mathbf{P}_n^*$ be the Grassmann manifold of hyperplanes in $\mathbf{P}_n$ and let $(\mathbf{C}^{n+1})^*$ denote the dual vector space to $\mathbf{C}^{n+1}$. Using the Plücker coordinates in $(\mathbf{C}^{n+1})^*$, we define a mapping

$$\mathcal{E}_V\colon \mathbf{C}^{n+1} \longrightarrow (\mathbf{C}^{n+1})^* \tag{5.3}$$

by the formula:

$$\begin{aligned} \mathcal{E}_V(y_0, \cdots, y_n) \\ = \big((\partial F/\partial X_0)(y_0, \cdots, y_n), \cdots, (\partial F/\partial X_n)(y_0, \cdots, y_n)\big) . \end{aligned} \tag{5.4}$$

Then $\mathcal{E}_V$ induces a rational mapping:

$$\mathcal{D}_V\colon \mathbf{P}_n \longrightarrow \mathbf{P}_n^* . \tag{5.5}$$

This mapping is called the *dual* or *polar mapping* associated to V. It is an elementary exercise to show that:

(5.6) $(1, 0, \cdots, 0)$ is a singular point of V if and only if $(\partial^2 F/\partial X_0 \partial X_i)\,|_{(1\,0,\cdots,0)} = 0$ for all i (use Euler's formula), and $(1, 0, \cdots, 0)$ is an ordinary double point if and only if, in addition,

$$\det\big(((\partial^2 F/\partial X_i \partial X_j)\,|_{(1\,0,\cdots,0)})\; 1 \leqq i, j \leqq n\big) \neq 0 .$$

Now suppose that V is Lefschetz. Let $\mathbf{P}_V$ denote the closure of the graph of $\mathcal{D}_V$ in $(\mathbf{P}_n \times \mathbf{P}_n^*)$ and let $\tilde{V}$ denote the closure of the graph of $\mathcal{D}_V\,|_V$. The projections onto the factors of $(\mathbf{P}_n \times \mathbf{P}_n^*)$ give the standard commutative diagram:

$$\begin{tikzcd} & \mathbf{P}_n \arrow[d, "\mathcal{D}_V"] \\ \mathbf{P}_V \arrow[ur, "\pi"] \arrow[dr, "\mathcal{D}"'] & \mathbf{P}_n^* \\ & \end{tikzcd}$$

LEMMA 5.7. *$\mathbf{P}_V$ and $\tilde{V}$ are non-singular. If V is non-singular, π is an isomorphism. If V has double point x_0, $\pi\colon (\mathbf{P}_V - \pi^{-1}(x_0)) \longrightarrow (\mathbf{P}_n - \{x_0\})$ is an isomorphism and so is $\mathcal{D}\colon \pi^{-1}(x_0) \to [x_0]$ where $[x_0] = \{h \in \mathbf{P}_n^*\colon x_0 \in [h]\}$. Thus $\mathbf{P}_V \approx (\mathbf{P}_n$ blown up at $x_0)$.*

Proof. If V is non-singular, everything is obvious. If V has double point $x_0 = (1, 0, \cdots, 0) \in \mathbf{P}_n$, define a mapping $f\colon \mathbf{C}^n \to (\mathbf{C}^{n+1})^*$ by $f(y_1, \cdots, y_n) = \big((\partial F/\partial X_0)(1, y_1, \cdots, y_n), \cdots, (\partial F/\partial X_n)(1, y_1, \cdots, y_n)\big)$. If $(\mathbf{C}^n)_0 = (\mathbf{C}^n$ blown up at the origin) and $(\mathbf{C}^{n+1})_0^* = \big((\mathbf{C}^{n+1})^*$ blown up at the origin$\big)$, then by (5.6) f induces a regular mapping $f_0\colon(\mathbf{C}^n)_0 \to (\mathbf{C}^{n+1})_0^*$ which takes the exceptional set in $(\mathbf{C}^n)_0$ isomorphically onto a linear subspace of the exceptional set of $(\mathbf{C}^{n+1})_0^*$. Let g_1 denote the composition $(\mathbf{C}^n)_0 \to \mathbf{C}^n \to \mathbf{P}_n$ and g_2 the composition $(\mathbf{C}^n)_0 \xrightarrow{f_0} (\mathbf{C}^{n+1})_0^* \to \mathbf{P}_n^*$. The mapping $g_1 \times g_2\colon \mathbf{C}_0^n \to (\mathbf{P}_n \times \mathbf{P}_n^*)$ is injective and everywhere of maximal rank and is onto a neighborhood of $\pi^{-1}(x_0)$ in $\mathbf{P}_V$. The assertions of the lemma now follow immediately.

COROLLARY 5.8. $Q_0 = \big(\tilde{V} \cap (\pi^{-1}(x_0))\big)$ *is a non-singular quadratic hyper-*

surface of $\pi^{-1}(x_0)$.

Given a Lefschetz hypersurface V and $h \in \mathbf{P}_n^*$, then $V_h = (V \cap [h])$ is a hypersurface in the projective space $[h]$. In general, of course, V_h will not be Lefschetz; however we now show that if h is chosen in a sufficiently generic manner, V_h continues to be Lefschetz. First of all, let H denote the subvariety of $\mathbf{P}_n$ defined by

$$\det\left((\partial^2 F/\partial X_i \partial X_j)_{0 \leqq i, j \leqq n}\right) = 0 .$$

H is a hypersurface of degree $(n+1)(d-2)$ in $\mathbf{P}_n$, called the Hessian subvariety of $\mathbf{P}_n$ associated to V.

LEMMA 5.9. *Let V be a hypersurface, x a simple point of V. The following statements are equivalent*:

(i) $x \notin H$;

(ii) $\mathfrak{D}_V \colon \mathbf{P}_n \longrightarrow \mathbf{P}_n^*$ *is of maximal rank at* x;

(iii) $\mathfrak{D}_V|_V \colon V \longrightarrow \mathbf{P}_n^*$ *is of maximal rank at* x;

(iv) *if* $h = \mathfrak{D}_V(x)$, *the tangent hyperplane to V at x, then V_h has an ordinary double point at x.*

Proof. The equivalence of (i) and (ii) is obvious. To get the rest, assume $x = (1, 0, \cdots, 0)$ and that the tangent hyperplane to V at x is given by $X_n = 0$. Using that

$$\partial F/\partial X_j = (1/d-1) \sum X_i (\partial^2 F/\partial X_i \partial X_j)$$

one has that at the point x:

$$(5.10) \qquad (\partial^2 F/\partial X_0 \partial X_j) = 0 \ \text{ if } j \neq n, \quad = (d-1) \ \text{ if } j = n .$$

Then (iii) is just the condition that

$$\det\left((\partial^2 F/\partial X_i \partial X_j)_{\substack{0 \leqq i \leqq n-1 \\ 1 \leqq j \leqq n}}\right)$$

be non-zero at x. So (iii) and (5.10) give (ii). By (5.6), (iv) is equivalent to the condition that

$$\det\left((\partial^2 F/\partial X_i \partial X_j)_{1 \leqq i, j \leqq n-1}\right)$$

be non-zero at x so that again by (5.10), we have the equivalence of (iii) and (iv).

In the following, V will always be a Lefschetz hypersurface and x_0 will always stand for the double point of V if there is one. All statements should be taken as applying both to the case V non-singular and to the double point case, unless it is explicitly indicated to the contrary. Also we shall use the following notation:

(5.11) If W is a subvariety of $\mathbf{P}_n$, then $\tilde{W}$ will denote its proper transform in $\mathbf{P}_V$. Furthermore if W has dimension r:

$m_0(W)$ = multiplicity of x_0 in W
= intersection multiplicity at x_0 of W with a generic $(n-r)$-plane passing through x_0.

Now for $k>0$: $H_k(\mathbf{P}_V) \approx H_k(\mathbf{P}_n) \oplus H_k(\pi^{-1}(x_0))$. For k odd, these groups are all zero, and for $k=2r$ a basis for $H_k(\mathbf{P}_V)$ is given by $\{\{L\}, \{L_0\}\}$ where L is an r-dimensional linear subspace of $\mathbf{P}_n$ and L_0 is an r-dimensional linear subspace of $(\pi^{-1}(x_0))$. For linear subspaces L and L' (of dimension r) in $\mathbf{P}_n$, one has in $\mathbf{P}_V$ the homology relation:

$$(\{\tilde{L}\} - \{\tilde{L}'\}) = (m_0(L') - m_0(L))\{L_0\} . \tag{5.12}$$

If $x_0 \notin L$, then an immediate consequence of (5.4) is that $\deg(\mathfrak{D}_*(\{\tilde{L}\})) = (d-1)^r$, where $\mathfrak{D}_*$ is the homology map associated to $\mathfrak{D}\colon \mathbf{P}_V \to \mathbf{P}_n^*$. By Lemma 5.7, $\deg(\mathfrak{D}_*(\{L_0\})) = 1$. Taken together these last two facts give the formula:

$$\deg(\mathfrak{D}_*(\{\tilde{W}\})) = (d-1)^r \deg(\{W\}) - m_0(W) \tag{5.13}$$

where W is any r-dimensional subvariety of $\mathbf{P}_n$.

COROLLARY 5.14. *If $d>2$, $\mathfrak{D}\colon \mathbf{P}_V \to \mathbf{P}_n^*$ is finite-to-one.*

Proof. If not, there would be an algebraic curve in $\mathbf{P}_V$ which $\mathfrak{D}$ sends to a point. By (5.13), the curve must lie in $\pi^{-1}(x_0)$. But that is impossible by Lemma 5.7.

LEMMA 5.15. *If $d>2$, $\mathfrak{D}|_{\tilde{V}}$ is generically injective.*

Proof. Let $V^* = \mathfrak{D}(\tilde{V})$. Then V^* (considered as a subvariety of $\mathbf{P}_n^*$ and taken with multiplicity one) has its own dual mapping $\mathfrak{D}_{V^*}\colon \mathbf{P}_n^* \to \mathbf{P}_n^{**} = \mathbf{P}_n$. At a generic point $x \in V$, $\mathfrak{D}_V|_V$ must be of maximal rank since $\mathfrak{D}|_{\tilde{V}}$ is equidimensional by Corollary 5.14. Therefore there is an open neighborhood U of x in $\mathbf{P}_n$ such that $\mathfrak{D}_V(U \cap V)$ is smooth at $\mathfrak{D}_V(x)$. If $\sum a_j X_j = 0$ gives the tangent hyperplane to $\mathfrak{D}_V(U \cap V)$ at $\mathfrak{D}_V(x)$, then $\sum (a_j(\partial^2 F/\partial X_j \partial X_i)(x))X_i$ must give the tangent hyperplane to V at x, since $\mathfrak{D}_V$ is of maximal rank at x by Lemma 5.9. Therefore at x:

$$\begin{aligned}(\textstyle\sum a_j(\partial^2 F/\partial X_j \partial X_0), \cdots, \sum a_j(\partial^2 F/\partial X_j \partial X_n)) \\ = ((\partial F/\partial X_0), \cdots, (\partial F/\partial X_n)) .\end{aligned}$$

On the other hand if $x = (y_0, \cdots, y_n)$, then at x:

$$\begin{aligned}(\textstyle\sum y_j(\partial^2 F/\partial X_j \partial X_0), \cdots, \sum y_j(\partial^2 F/\partial X_j \partial X_n)) \\ = ((d-1)(\partial F/\partial X_0), \cdots, (d-1)(\partial F/\partial X_n)) .\end{aligned}$$

Thus as projective points $(a_0, \cdots, a_n) = (y_0, \cdots, y_n)$ which means that, restricted to V, $(\mathfrak{D}_{V^*}\circ\mathfrak{D}_V)$ is the identity map. The lemma is then clear.

PROPOSITION 5.16. *Let V be a Lefschetz hypersurface. Then a generic pencil of hyperplane sections of V contains only Lefschetz hypersurfaces.*

Proof. By what we have done above we know that there is a Zariski open subset $U \subseteqq \tilde{V}$ such that $(\mathfrak{D}(U) \cap \mathfrak{D}(\tilde{V} - U)) = \varnothing$ and $\mathfrak{D}|_U$ is injective and of maximal rank. Also there exists a Zariski open subset U_0 of $[x_0]$ such that the points of U_0 correspond to hyperplane sections of V which have at x_0 only an ordinary double point. Also $\mathfrak{D}(\tilde{V}) \not\subseteqq [x_0]$ by (5.13) and Lemma 5.15. So the subvariety $\mathfrak{D}(\tilde{V} - U) \cup (\mathfrak{D}(\tilde{V}) \cap [x_0]) \cup ([x_0] - U_0)$ has codimension $\geqq 2$ in $\mathbf{P}_n^*$. The proposition follows.

Definition 5.17. A pencil of hyperplane sections of a hypersurface V will be called *Lefschetz* if a generic element of the pencil is non-singular and each element of the pencil is Lefschetz.

COROLLARY 5.18. *Let V be a Lefschetz hypersurface. Then a generic pencil of hyperplane sections of V is a Lefschetz pencil.*

Finally, it is an easy exercise in differential topology to show that all the non-singular hyperplane sections of V are diffeomorphic, in fact they can be deformed one onto another along paths in $(\mathbf{P}_n^* - V^*)$. Furthermore, since V^* is irreducible, there is a connected Zariski open subet U^* of V^* containing only smooth points of V^* and such that for $h \in U^*$, V_h is Lefschetz and for $h_1, h_2 \in U^*$, V_{h_1} can be deformed homeomorphically onto V_{h_2} along a path in U^*. (We shall need this fact in § 11—see discussion preceding Lemma 11.23.)

6. Cubic hypersurfaces

We shall now restrict our attention to hypersurfaces of degree three. As in § 5, V will be Lefschetz. We begin by deriving a fact about the dual mapping in this case which will be of central importance in Part three of the paper.

LEMMA 6.1. *Let K^* be a linear subspace of $\mathbf{P}_n^*$. Suppose $W \subseteqq V$ and $\mathfrak{D}(\tilde{W}) = K^*$. Then*

$$[K^*] = \bigcap \{[h]: h \in K^*\}$$

is tangent to V at each point of W.

Proof. If $\dim K^* = 0$, the assertion is trivial. If $\dim K^* > 0$, let x and y be distinct points of $(W - \{x_0\})$ such that $\mathfrak{D}(x) \neq \mathfrak{D}(y)$. Let L^* be the line in K^* connecting $\mathfrak{D}(x)$ and $\mathfrak{D}(y)$ and let $C = \pi(\mathfrak{D}^{-1}(L^*))$ (see (5.7)). By (5.13), $\mathfrak{D}\colon \tilde{C} \to L^*$ is a covering by $(2\deg(\{C\}) - m_0(C))$ sheets. For the point $y \in C$:

$$\begin{aligned}(\{[\mathfrak{D}(y)]\}\cdot\{C\}) &\geqq 2(2\deg(\{C\}) - m_0(C)) \\ &\geqq 2\deg(\{C\})\end{aligned}$$

since $[\mathfrak{D}(y)]$ is tangent to C at $(2\deg(\{C\}) - m_0(C))$ points. So $C \subseteqq [\mathfrak{D}(y)]$ and in particular $x \in [\mathfrak{D}(y)]$, which implies that $W \subseteqq [K^*]$. But $[K^*] \subseteqq [\mathfrak{D}(x)]$ for $x \in W$. The lemma follows.

COROLLARY 6.2. *If* $\dim K^* = r$ *in Lemma* 6.1, *then* $2r \leqq (n-1)$.

Proof. $\mathfrak{D}$ is finite-to-one on $\tilde{V}$. But $\mathfrak{D}(\tilde{W}) \subseteqq \{h\colon W \subseteqq [h]\}$. So $r = \dim W \leqq \dim\{h\colon W \subseteqq [h]\} \leqq (n - (r+1))$.

We next turn our attention to the algebraic family of projective lines (effective algebraic curves of degree one in $\mathbf{P}_n$) which lie inside V. For $n = 2$, $V \subseteqq \mathbf{P}_2$ contains no lines. However, if $n > 2$, then V does contain lines. It is a classical fact, for example, that:

(6.3) If V is a non-singular cubic hypersurface in $\mathbf{P}_3$, then V contains exactly 27 lines. (See [18].)

Suppose that V has double point x_0. Let L be a line in $\mathbf{P}_n$ which passes through x_0. Then either L lies in V or L intersects V in precisely one more point. Thus the projection $\mathbf{P}_n \to \mathbf{P}_{n-1}$ centered at x_0 gives a birational morphism:

$$\rho\colon \tilde{V} \longrightarrow \mathbf{P}_{n-1}\,. \tag{6.4}$$

Furthermore, if W_0 is the cone formed by the lines through x_0 which lie in V, then

$$\rho\colon (\tilde{V} - \tilde{W}_0) \longrightarrow (\mathbf{P}_{n-1} - \rho(\tilde{W}_0))$$

is an isomorphism. As for the structure of W_0:

LEMMA 6.5. *Let* $[h]$ *be any hyperplane in* $\mathbf{P}_n$ *which does not contain* x_0. *Then there is a non-singular complete intersection* C *of type* (2,3) *in* $[h]$ *such that* W_0 *is the cone over* C *with vertex* x_0.

(Recall that a complete intersection of type (2,3) in a projective space $[h]$ is the intersection of a quadratic and a cubic hypersurface of $[h]$.)

Proof. Let Y_0 be the tangent cone to V at x_0. If $x_0 \notin [h]$, then $(Y_0 \cap [h])$ is a non-singular quadratic hypersurface of $[h]$. Also if L is a ray of Y_0, then L lies in V if and only if L meets V in one point outside of x_0. Thus $W_0 =$ (cone over $([h] \cap Y_0 \cap V)$), and we need only check that $([h] \cap W_0)$ is non-singular. But if this variety were singular, it would be singular independently of the choice of $[h]$ so that Y_0 and V would have to be tangent along an entire ray L of W_0. This in turn would imply that $\mathfrak{D}(\tilde{L}) =$ (point) which contradicts Corollary 5.14.

We can summarize our discussion of the case where V has a double point x_0 as follows: V can be transformed (birationally) into $\mathbf{P}_{n-1}$ by blowing up V at x_0 and then blowing down the proper transforms of the lines through x_0. So, for instance, if V is a surface, $\widetilde{V}$ is obtained from $\mathbf{P}_2$ by blowing up six points lying on a non-singular quadric curve. From this it follows easily that V contains exactly 21 lines, six of which pass through the double point x_0. Furthermore no three of the lines through x_0 are coplanar.

The Chow variety of projective lines lying in a Lefschetz cubic hypersurface V was studied classically by Fano [6] and also was treated extensively by Bombieri and Swinnerton-Dyer in [2]. Our presentation in § 6 – § 10 borrows heavily from these two works. We begin by classifying the lines in $\mathbf{P}_n$ into types depending on the behavior of the dual mapping $\mathfrak{D}_V$ along the line. By (5.13), for any line L, $\deg\big(\mathfrak{D}_*(\{\widetilde{L}\})\big) \leqq 2$ so that we can make the following classification:

Definition 6.6. Let L be a projective line in $\mathbf{P}_n$.

(i) L will be called a line *of first type* if $\mathfrak{D}(\widetilde{L})$ is a non-singular (plane) quadric curve;

(ii) L will be called *of second type* if either

(a) $x_0 \notin L$ but $\mathfrak{D}(\widetilde{L})$ is a projective line (so that $\mathfrak{D}\colon \widetilde{L} \to \mathfrak{D}(\widetilde{L})$ is a ramified two-sheeted covering),

(b) $x_0 \in L$ and $\mathfrak{D}|_{\widetilde{L}}$ is an isomorphism onto a projective line in $[x_0] \subseteqq \mathbf{P}_n^*$.

An easy corollary of (5.13) is that possibilities (i), (ii) (a), and (ii) (b) are mutually exclusive and exhaust all possible cases for a line L in $\mathbf{P}_n$. For any L, the dimension of the linear subspace $[\mathfrak{D}(\widetilde{L})]$ of $\mathbf{P}_n$ is always $\geqq (n-3)$ and we have:

LEMMA 6.7. *A line $L \subseteqq V$ is of second type if and only if there is a (unique) $(n-2)$-plane tangent to V along L. This $(n-2)$-plane is $[\mathfrak{D}(\widetilde{L})]$. If $L \subseteqq V$ is of first type, $[\mathfrak{D}(\widetilde{L})]$ is an $(n-3)$-plane tangent to V along L.*

For a line L lying in V, we can characterize the local behavior of V near L according to the type of L. For this discussion we assume that L is given by:

$$X_2 = \cdots = X_n = 0\ . \tag{6.8}$$

Then $F(X_0, \cdots, X_n) = \sum\{X_i Q_i(X_0, X_1)\colon\ i = 2, \cdots, n\}$ + (terms involving higher powers in $X_2, \cdots, X_n$), where degree $Q_i = 2$ for each i. Thus $\mathfrak{D}_V|_L$ is given by the formula:

$$\mathfrak{D}_V(y_0, y_1, 0, \cdots, 0) = \big(0, 0, Q_2(y_0, y_1), \cdots, Q_n(y_0, y_1)\big)\ .$$

L is of first type if and only if the Q_i span the entire three-dimensional vector space of quadratic forms in two variables. In this case, a linear change of coordinates involving only $X_2, \cdots, X_n$ reduces $F(X_0, \cdots, X_n)$ to the form:

$$X_2X_0^2 + X_3X_0X_1 + X_4X_1^2 + (\text{terms with higher powers of } X_2, \cdots, X_n) . \tag{6.9}$$

If L is of second type, the Q_i span a two-dimensional vector space and we can put $F(X_0, \cdots, X_n)$ into the form:

$$X_2Q_2(X_0, X_1) + X_3Q_3(X_0, X_1) + (\text{terms with higher powers of } X_2, \cdots, X_n) .$$

If $x_0 \notin L$, Q_2 and Q_3 have no common zero and one can make a coordinate change involving X_0, X_1 to get $(1, 0, \cdots, 0)$ and $(0, 1, 0, \cdots, 0)$ as the ramification points of $\mathfrak{D}|_{\tilde{L}}$. Then a change involving only X_2, X_3 reduces $F(X_0, \cdots, X_n)$ to the form:

$$X_2X_0^2 + X_3X_1^2 + (\text{terms with higher powers of } X_2, \cdots, X_n) . \tag{6.10}$$

Finally, if $x_0 \in L$, then Q_2 and Q_3 have x_0 as their only common zero and a change of coordinates involving X_0, X_1 followed by one involving X_2, X_3 reduces $F(X_0, \cdots, X_n)$ to the form:

$$X_2X_0X_1 + X_3X_1^2 + X_0Q_0(X_2, \cdots, X_n) + X_1Q_1(X_2, \cdots, X_n) + P(X_2, \cdots, X_n) \tag{6.11}$$

where degree $Q_i = 2$ and degree $P = 3$. The double point x_0 then becomes the point $(1, 0, \cdots, 0)$ and by (5.6):

$$\det\left((\partial Q_0/\partial X_i \partial X_j)_{3 \leq i, j \leq n}\right) \neq 0 . \tag{6.12}$$

Let $\mathrm{Gr}(2, n+1)$ be the Grassmann variety of projective lines in $\mathbf{P}_n$. We wish to describe the family of lines in V locally around L. To do this, let H_i be the hyperplane in $\mathbf{P}_n$ given by the equation $X_i = 0$. Put $u_j = (X_j/X_0)$, $z_j = (X_j/X_1)$ for $j = 2, \cdots, n$. Then $(u_2, \cdots, u_n)$ gives affine coordinates for $(H_1 - (H_0 \cap H_1))$ and $(z_2, \cdots, z_n)$ gives affine coordinates for $(H_0 - (H_0 \cap H_1))$. Furthermore:

$$(u_2, \cdots, u_n, z_2, \cdots, z_n) \tag{6.13}$$

give local coordinates in $\mathrm{Gr}(2, n+1)$ around the point corresponding to the line L. To see which lines lie in V, one forms the polynomial

$$F\left(\lambda(1, 0, u_2, \cdots, u_n) + \mu(0, 1, z_2, \cdots, z_n)\right)$$

and sets the coefficients of λ^3, $\lambda^2\mu$, $\lambda\mu^2$ and μ^3 each equal to zero. If L is of first type, we use (6.9) and get the local equations:

$$
\begin{aligned}
&u_2 + \text{(higher powers)} = 0\\
&u_3 + z_2 + \text{(higher powers)} = 0\\
&u_4 + z_3 + \text{(higher powers)} = 0\\
&z_4 + \text{(higher powers)} = 0\,.
\end{aligned}
\tag{6.14}
$$

If L is of second type and $x_0 \notin L$, use (6.10) to get:

$$
\begin{aligned}
&u_2 + \text{(higher powers)} = 0\\
&z_2 + \text{(higher powers)} = 0\\
&u_3 + \text{(higher powers)} = 0\\
&z_3 + \text{(higher powers)} = 0\,.
\end{aligned}
\tag{6.15}
$$

If L is of second type and $x_0 \in L$, use (6.11) to get:

$$
\begin{aligned}
&Q_0(u_2, \cdots, u_n) + \text{(higher powers)} = 0\\
&u_2 + \text{(higher powers)} = 0\\
&u_3 + z_2 + \text{(higher powers)} = 0\\
&z_3 + \text{(higher powers)} = 0\,.
\end{aligned}
\tag{6.16}
$$

If L is of first type, there is a way to define a family of curves in $\mathrm{Gr}(2, n+1)$ which will be quite helpful later on in studying the tangent space to the variety defined by (6.14) at the point

$$u_2 = \cdots = u_n = z_2 = \cdots = z_n = 0\,.$$

For $(\alpha_0, \alpha_1) \in \mathbf{P}_1$, let $B(\alpha_0, \alpha_1)$ be the closed irreducible curve in $\mathrm{Gr}(2, n+1)$ which is given (in terms of the local coordinates (6.13)) by the equations:

$$
\begin{gathered}
u_2 + z_4 = 0\\
u_3 + z_2 = 0\\
u_4 + z_3 = 0\\
u_5 = \cdots = u_n = z_2 = \cdots = z_n = 0
\end{gathered}
$$

and

$$\alpha_0 u_3 + \alpha_1 u_4 = 0\,.$$

(Compare this with (6.14).) For each $s \in B(\alpha_0, \alpha_1)$ let L_s be the corresponding line in $\mathbf{P}_n$ and define:

$$Q(\alpha_0, \alpha_1) = \bigcup\{L_s\colon s \in B(\alpha_0, \alpha_1)\}\,. \tag{6.17}$$

Then $Q(\alpha_0, \alpha_1)$ is a non-singular quadric surface which spans a three-plane $M(\alpha_0, \alpha_1)$ in $\mathbf{P}_n$. Assuming L is given by (6.8) and V by (6.9), we let $h(\alpha_0, \alpha_1)$ be the element of $\mathbf{P}_n^*$ such that $[h(\alpha_0, \alpha_1)]$ is spanned by $M(\alpha_0, \alpha_1)$ and $[\mathfrak{D}(\tilde{L})]$. Then:

LEMMA 6.18. (i) *$Q(\alpha_0, \alpha_1)$ is tangent to V along L.* (ii) *The mapping*

$(\alpha_0, \alpha_1) \mapsto h(\alpha_0, \alpha_1)$ *is an isomorphism from* $\mathbf{P}_1$ *onto* $\mathfrak{D}(\tilde{L})$.

Proof: (i) follows immediately from (6.11) and the definition of $Q(\alpha_0, \alpha_1)$. For (ii), we see from (6.9) that the tangent hyperplane to V at the point $(\beta_0, \beta_1, 0, \cdots, 0)$ on L is given by:

$$\beta_0^2 X_2 + \beta_0\beta_1 X_3 + \beta_1^2 X_4 = 0 .$$

But $[h(\alpha_0, \alpha_1)]$ is spanned by the points $(1, 0, 0, \alpha_1, -\alpha_0, 0, \cdots, 0)$ and $(0, 1, -\alpha_1, \alpha_0, 0, \cdots, 0)$ together with the $(n-3)$-plane given by $X_2 = X_3 = X_4 = 0$. Thus $[h(\alpha_0, \alpha_1)]$ is given by the equation

$$c_2X_2 + c_3X_3 + c_4X_4 = 0$$

where $c_2\alpha_1 - c_3\alpha_0 = 0$ and $-c_2\alpha_1 + c_3\alpha_0 = 0$. Normalizing things by picking $c_2 = \alpha_0^2$, we get $c_3 = \alpha_0\alpha_1$ and $c_4 = \alpha_1^2$ and the lemma is proved.

PROPOSITION 6.19. *Let V be a Lefschetz cubic hypersurface, L a line in V such that $x_0 \notin L$. Let (n) denote the line bundle of degree n on L. If L is of first type, the normal bundle $N(V, L)$ to L in V is given by*

$$(0) \oplus (0) \oplus (1) \oplus \cdots \oplus (1) ;$$

if L is of second type, $N(V, L) \approx (-1) \oplus (1) \oplus (1) \oplus \cdots \oplus (1)$.

Proof. If L is of first type, normalize the equations for L and V as in (6.8) and (6.9). Using (6.14) and the definition of $Q(\alpha_0, \alpha_1)$ in (6.17), one checks immediately that $Q(1,0)$, $Q(0, 1)$, and $[\mathfrak{D}(\tilde{L})]$ meet transversely along L. However by Lemmas 6.7 and 6.18, $Q(1, 0)$, $Q(0, 1)$ and $[\mathfrak{D}(\tilde{L})]$ are all tangent to V along L. Thus $N(V, L) \approx N(Q(1, 0), L) \oplus N(Q(0, 1), L) \oplus N([\mathfrak{D}(\tilde{L})], L)$. If L is of second type, the $(n-2)$-plane $[\mathfrak{D}(\tilde{L})]$ is tangent to V along L. Normalize the equations for L and V as in (6.8) and (6.10). Then $[\mathfrak{D}(\tilde{L})]$ is given by the equations

$$X_2 = X_3 = 0 .$$

The tangent hyperplane to V at $(\beta_0, \beta_1, 0, \cdots, 0)$ is given by $\beta_0^2 X_2 + \beta_1^2 X_3 = 0$. Let $L(\beta_0, \beta_1)$ be the line spanned by $(\beta_0, \beta_1, 0, \cdots, 0)$ and $(0, 0, -\beta_1^2, \beta_0^2, 0, \cdots, 0)$ and let

$$B = \bigcup\{L(\beta_0, \beta_1) \colon (\beta_0, \beta_1) \in \mathbf{P}_1\} .$$

Then B is non-singular along L and tangent to V there. Also $[\mathfrak{D}(\tilde{L})]$ and B meet transversely along L. Thus:

$$N(V, L) \approx N([\mathfrak{D}(\tilde{L})], L) \oplus N(B, L) .$$

Since degree $N(V, L) = (n-4)$ and degree $N([\mathfrak{D}(\tilde{L})], L) = (n-3)$, it follows that degree $N(B, L) = -1$ and the proposition is proved.

The algebraic structure of the set of lines in a Lefschetz cubic hypersurface is now accessible. The purpose of the next section is to examine that structure.

7. The variety of lines on a cubic hypersurface

For each $s \in \mathrm{Gr}(2, n+1)$, let L_s be the associated line $\subseteqq \mathbf{P}_n$. For a Lefschetz cubic hypersurface V in $\mathbf{P}_n$ we define:

$$\begin{aligned} S &= S_V = \{s \in \mathrm{Gr}(2, n+1)\colon L_s \subseteqq V\} \\ D &= D_V = \{s \in S_V\colon L_s \text{ is of second type}\} \qquad (7.1) \\ T &= T_V = \{(s, x) \in (S_V \times V)\colon x \in L_s\}\,. \end{aligned}$$

Then the projection

$$\pi_S\colon T \longrightarrow S \qquad (7.2)$$

is an algebraic fibre bundle with fibre a projective line. We also have a projection

$$\pi_V\colon T \longrightarrow V\,. \qquad (7.3)$$

If V has double point x_0, define

$$D_0 = \{s \in S_V\colon x_0 \in L_s\}\,. \qquad (7.4)$$

By Lemma 6.5, D_0 is isomorphic to a non-singular complete intersection of type (2,3) in $\mathbf{P}_{n-1}$. By Proposition 5.16, therefore, the generic fibre of π_V in (7.3) must be isomorphic to a non-singular complete intersection of type (2, 3) in $\mathbf{P}_{n-2}$. Next let D_1 be the union of the components of D_V which do not lie in D_0. We wish to bound the dimension of D_1. To do this, let R_1 be the set of all $(s, x) \in T_V$ such that $s \in D_1$, $x_0 \notin L_s$, and x is not one of the two ramification points of the mapping $\mathfrak{D}\,|_{\tilde{L}_s}$.

LEMMA 7.5 *$\pi_V\colon R_1 \longrightarrow V$ is finite-to-one.*

Proof. Suppose there is an irreducible (open) curve C in R_1 such that $\pi_V(C) = \{y\}$. R_1 possesses an involution i which is induced from the involutions $i_s\colon L_s \longrightarrow L_s$ given by the two-sheeted mapping $\mathfrak{D}\colon \tilde{L}_s \longrightarrow \mathfrak{D}(\tilde{L}_s)$. Also $(\pi_S \circ i) = i$ and $(\mathfrak{D}_V \circ \pi_V \circ i) = (\mathfrak{D}_V \circ \pi_V)$. So $(\mathfrak{D}_V \circ \pi_V)(i(C)) = \{\mathfrak{D}_V(y)\}$ and since $D_V\,|_{(V-\{x_0\})}$ is finite-to-one, there exists $z \in V$ such that $\pi_V(i(C)) = \{z\}$. Since π_V is injective on fibres of $\pi_S\colon T_V \longrightarrow S_V$, $z \neq y$. So for any $(s, x) \in C$, L_s must be the line passing through y and z. The lemma follows.

COROLLARY 7.6. $\dim R_1 \leqq (n-2)$ and $\dim D_1 \leqq (n-3)$.

Proof. The first statement follows from the fact that $\mathfrak{D}_V$ is at least two-to-one on $\pi_V(R_1)$ together with Lemma 5.15. The second statement then follows from Lemma 7.5.

We have already studied the local structure of S_V in (6.14)–(6.16). Using (6.14), (6.15), and the Jacobian criterion for non-singularity we get:

LEMMA 7.7. *Let V be a Lefschetz cubic hypersurface in $\mathbf{P}_n$. Then $(S_V - D_0)$ is non-singular and has pure dimension $2(n-3)$.*

(Notice that it follows from Proposition 6.19 that if $x_0 \notin L$, $H^1(L, \mathcal{O}(N(V, L))) = 0$ and $H^0(L, \mathcal{O}(N(V, L))) = 2(n-3)$. By a theorem of Kodaira [15; page 150], the Chow variety of lines in V is smooth at L and the tangent space to this Chow variety is $H^0(L, \mathcal{O}(N(V, L)))$. This yields an alternate proof of Lemma 7.7.)

If $s \in D_0$, the local equations for S_V around s are given in (6.16). If L_s is given by (6.8) and V by (6.11), the double point x_0 is given by $(1, 0, \cdots, 0)$ so that the local equations for D_0 are given by (6.16) together with the additional conditions

$$u_2 = \cdots = u_n = 0 .$$

By (6.12), $Q_0(u_2, \cdots, u_n)$ is a non-degenerate quadratic form. Taken together, these conditions mean that there is a neighborhood U_s of s in S_V which is biholomorphic to the analytic variety $(P_{n-3} \times Q_{n-3})$ where P_{n-3} is an $(n-3)$-dimensional polydisc and

$$Q_{n-3} = \{(u_1, \cdots, u_{n-2})\colon \textstyle\sum u_j^2 = 0\, ,\ \sum u_j \bar{u}_j < 1\} .$$

This biholomorphism carries $(D_0 \cap U_s)$ onto $(P_{n-3} \times \{0\})$. To describe this situation, we say that D_0 is an *ordinary double variety* of S_V.

THEOREM 7.8. *Let S_V be a Lefschetz cubic hypersurface. Then S_V is a projective variety of pure dimension $2(n-3)$. If V is non-singular, so is S_V. If V has a double point, S_V is non-singular except along D_0 which is a non-singular $(n-3)$-dimensional ordinary double variety for S_V.*

This theorem will allow us to get rather precise information about the topology of S_V. But first we need a general proposition about non-singularity of various subvarieties of S_V. In order to give geometric proofs of non-singularity of such subvarieties it will be convenient to construct a "linear" subvariety $T_s \subseteqq \mathrm{Gr}(2, n+1)$ for each non-singular point $s \in S_V$ to play the analogous role to that of the tangent hyperplane in the case of hypersurfaces in $\mathbf{P}_n$. If L_s is of second type, define:

$$T_s = \{t \in \mathrm{Gr}(2, n+1)\colon L_t \subseteqq [\mathfrak{D}(\tilde{L}_s)]\, ,\ \text{the } (n-2)\text{-plane tangent to } V \text{ along } L_s\} . \tag{7.9}$$

T_s is a non-singular $2(n-3)$-dimensional (Schubert) subvariety of Gr $(2, n+1)$ and we have the equality of tangent spaces:

$$T(S_V, s) = T(T_s, s)$$

(considered as subspaces of $T(\mathrm{Gr}(2, n+1), s)$). If L_s is of first type, our construction of T_s depends on the choice (6.13) of local coordinates for $\mathrm{Gr}(2, n+1)$. Define:

$$(7.10) \qquad T_s = \big(\text{closure in } \mathrm{Gr}(2, n+1) \text{ of the variety given in local coordinates (6.13) by the equations } u_2 = 0\,,\ u_3 + z_2 = 0\,,\ u_4 + z_3 = 0\,,\ z_4 = 0\big)\,.$$

Then T_s is irreducible of dimension $2(n-3)$ and by (6.14):

$$T(T_s, s) = T(S_V, s)\,.$$

Let R be an $(n-2)$-plane in $\mathbf{P}_n$ and let Y be a non-singular cubic hypersurface in $\mathbf{P}_n$. Define

$$\begin{aligned} & Y_h = (Y \cap [h]) \ \text{ for } \ h \in [R]; \\ (7.11) \qquad & A_R = \{(s, h) \in (S_Y \times [R]) \colon L_s \subseteqq [h]\}\,; \\ & S_h = \pi_{[R]}^{-1}(h) \ \text{ where } \ \pi_{[R]} \colon A_R \to [R] \ \text{ is the natural projection.} \end{aligned}$$

For generic choice of $[R]$, the family $\{Y_h\}$ is a Lefschetz pencil of hyperplane sections of Y and since we can assume that R itself meets Y transversely the double points which occur in Y_h never lie on R. Therefore to show that for generic R, A_R is non-singular, let

$$\begin{aligned} B_h &= \{s \in \mathrm{Gr}(2, n+1) \colon L_s \subseteqq [h]\} \\ B_R &= \{s \in \mathrm{Gr}(2, n+1) \colon (L_s \cap R) \neq \varnothing\} \end{aligned}$$

and we prove:

PROPOSITION 7.12. (i) *For generic R, if $s \in (B_R \cap S_Y)$ and $L_s \nsubseteq R$, then $(B_R \cap S_Y)$ is non-singular at s.* (ii) *For generic R, if $h \in [R]$ and $s \in S_h$ such that L_s contains only non-singular points of Y_h, then B_h and S_Y meet transversely at s.*

Proof. (i) At s, the local coordinates (6.13) can be constructed so that B_R is given by linear equations. Thus it suffices to show that if $s \in (B_R \cap S_Y)$, $T_s \nsubseteq B_R$. If L_s is of first type and $T_s \subseteqq B_R$, fix

$$t_0 \in (B(1, 0) - \{s\}), \quad t_1 \in (B(0, 1) - \{s\})$$

where $B(\alpha_0, \alpha_1)$ is as in (6.17). Then t_0, t_1 and $\{t \colon L_t \subseteqq [\mathfrak{D}(\tilde{L}_s)]\}$ all lie in T_s and R is determined by $(R \cap L_{t_0})$, $(R \cap L_{t_1})$, and $(R \cap [\mathfrak{D}(\tilde{L}_s)])$. Thus $\dim \{R \colon T_s \subseteqq B_R\} = (n-1)$. If L_s is of second type, $T_s \subseteqq B_R$ if and only if $\dim([\mathfrak{D}(\tilde{L}_s)] \cap R) \geqq (n-3)$. A dimension argument now yields (i).

(ii) At s, B_h is given by linear equations in the coordinates (6.13), so it suffices to show that:

$$\dim_s (T_s \cap B_h) \leqq (2(n-3)-2)\,. \tag{7.13}$$

Let L_s be of first type (for Y in $\mathbf{P}_n$). If $[\mathfrak{D}(\tilde{L}_s)] \not\subseteqq [h]$, then (7.13) is clearly satisfied since T_s contains all lines lying in $[\mathfrak{D}(\tilde{L}_s)]$. If $[\mathfrak{D}(\tilde{L}_s)] \not\subseteqq [h]$ and (7.13) were not satisfied then there would exist $(\alpha_0, \alpha_1) \in \mathbf{P}_1$ such that $B(\alpha_0, \alpha_1) \subseteqq B_h$ (see (6.17)). Thus $h = h(\alpha_0, \alpha_1)$ and by Lemma 6.18 (ii), L_s passes through the point where $[h]$ and Y are tangent. This gives (7.13) in case L_s is of first type. If L_s is of second type, it suffices to show that $[\mathfrak{D}(\tilde{L}_s)] \not\subseteqq [h]$. But if this were not the case, $[\mathfrak{D}(\tilde{L}_s)]$ would be a hyperplane of $[h]$ which was tangent to Y_h along all of L_s. But this contradicts the fact that Y_h is Lefschetz and therefore has a finite-to-one dual mapping. The proposition is proved.

If V is a non-singular, cubic hypersurface in $\mathbf{P}_{n-1}$, there is a non-singular cubic hypersurface $Y \subseteqq \mathbf{P}_n$ such that V is a hyperplane section of Y. Let

$$U = \{h \in \mathbf{P}_n^*\colon\ Y_h \text{ is non-singular}\}\,.$$

Since U is a Zariski open subset of $\mathbf{P}_n^*$ it is connected. Then Proposition 7.12 (ii) and a standard elementary argument from differential topology gives:

LEMMA 7.14. *S_h is diffeomorphic to S_V for all $h \in U$.*

PART THREE: THE CUBIC THREEFOLD

8. The Fano surface of lines on a cubic threefold, the double point case

We now restrict ourselves to the case of central interest in this paper, namely the case in which V is a Lefschetz cubic hypersurface in $\mathbf{P}_4$. For $x \in V$, let $W_x = \{s \in S_V\colon x \in L_s\}$ (see (2.2)).

LEMMA 8.1. *There are at most a finite number of points x on V such that* $\dim W_x > 0$. *If x is a simple point on V and* $\dim W_x > 0$, *then W_x is a cone over a non-singular plane curve of degree* 3.

Proof. If $\dim W_x > 0$, let $W = \bigcup\{L_s\colon s \in W_x\}$. Clearly there is a plane tangent to V along any ray of W so that by Lemma 6.7, L_s is of second type for each $s \in W_x$. Lemma 7.5 and Corollary 7.6 then give the first statement. The proof of the second statement parallels exactly the proof of Lemma 6.5.

The simple points x such that W_x is infinite should be called "Eckardt points" (see [18; page 6]). For instance, the cubic threefold given by the equation

$$X_0^3 + \cdots + X_4^3 = 0$$

has all points of the form $(y_0, \cdots, y_4)$ such that all but two of the y_j's are 0 as Eckardt points. There are 30 such points. (From results of § 10, it will follow that 30 is the maximal number of Eckardt points for a non-singular cubic threefold.) Assuming for a moment the irreducibility of S_V, we conclude:

COROLLARY 8.2. *If V is a non-singular cubic threefold, the family $\{L_s\}_{s \in S_V}$ is a covering family of curves for V.* (See Definition 2.3.)

Suppose that V has a double point x_0. If $x_0 \notin [h]$, let

$$\rho\colon \widetilde{V} \to [h] \approx \mathbf{P}_3$$

be the birational morphism defined as in (6.4). S_V has double curve $D_0 = \{s \in S_V\colon x_0 \in L_s\}$, and if $x_0 \notin [h]$ Lemma 6.5 gives that the mapping $s \mapsto (L_s \cap [h])$ is an isomorphism of D_0 onto a non-singular space curve of genus four. By the adjunction formula, this embedding

$$\kappa\colon D_0 \longrightarrow [h] \approx \mathbf{P}_3 \tag{8.3}$$

is canonical. Since $\mathfrak{D}\colon \widetilde{V} \to \mathbf{P}_4^*$ is finite-to-one, V contains no planes so that each $(t, t') \in D_0^{(2)}$ determines a unique point $\lambda(t, t')$ in S_V such that

$$L_t + L_{t'} + L_{\lambda(t,t')}$$

is a plane section of V. The morphism

$$\lambda\colon D_0^{(2)} \longrightarrow S_V$$

clearly restricts to an isomorphism from $(D_0^{(2)} - \lambda^{-1}(D_0))$ onto $(S_V - D_0)$. Also associated to κ in (8.3) there is a canonical morphism

$$\kappa^{(2)}\colon D_0^{(2)} \longrightarrow \mathrm{Gr}\,(2, 4) \subseteqq \mathbf{P}_{\binom{4}{2}-1} \tag{8.4}$$

which assigns to (t, t') the line through $\kappa(t)$ and $\kappa(t')$. It is clear that for $(t, t') \in (D_0^{(2)} - \lambda^{-1}(D_0))$:

$$L_{\kappa^{(2)}(t,t')} = \rho(\widetilde{L}_{\lambda(t,t')})\ . \tag{8.5}$$

For $(t, t') \in D_0^{(2)}$, let $K_{(t,t')}$ be the unique plane such that $L_t + L_{t'} +$ (third line) is the section of V by $K_{(t,t')}$.

LEMMA 8.6. *Let Q_0 be the non-singular quadric surface given in Corollary 5.8. The following conditions are equivalent for* $(t, t') \in D_0^{(2)}$:

(i) $(t, t') \in \lambda^{-1}(D_0)$;

(ii) $L_{\kappa^{(2)}(t,t')}$ *is a trisecant of* $\kappa(D_0)$;

(iii) $(\widetilde{K}_{(t,t')} \cap Q_0) =$ *one of the lines on* Q_0.

Proof. The equivalence of (i) and (ii) is clear. If $(t, t') \in \lambda^{-1}(D_0)$, then $(\{\widetilde{K}_{(t,t')}\} \cdot \{Q_0\})_{\widetilde{V}} \geqq 3$ so that (iii) must hold. Conversely, if (iii) holds, then every line in $K_{(t,t')}$ which passes through x_0 lies in the tangent cone to V at x_0. Thus if L is a line in $K_{(t,t')}$ not passing through x_0, the three lines connecting x_0 with the three points of $(L \cap V)$ must lie in V. This gives the lemma.

For $(t, t') \in D_0^{(2)}$, define:

$$J_{(t,t')} = \begin{cases} \widetilde{L}_{\lambda(t,t')} & \text{if } \lambda(t, t') \notin D_0\,, \\ \widetilde{L}_{\lambda(t,t')} + (\widetilde{K}_{(t,t')} \cap Q_0) & \text{if } \lambda(t, t') \in D_0\,. \end{cases} \tag{8.7}$$

$J_{(t,t')}$ is an algebraic one-cycle on $\widetilde{V}$ and under the mappings

$$\begin{array}{ccc} & \widetilde{V} & \\ {}^{\pi}\swarrow & & \searrow^{\rho} \\ V & & \mathbf{P}_2\,. \end{array}$$

$J_{(t,t')}$ behaves according to the formulas:

$$\pi(J_{(t,t')}) = L_{\lambda(t,t')}\,; \quad \rho(J_{(t,t')}) = L_{\kappa^{(2)}(t,t')}\,. \tag{8.8}$$

LEMMA 8.9. *The family* $\{J_{(t,t')}\}_{(t,t') \in D_0^{(2)}}$ *is a covering family of curves for* $\widetilde{V}$.

Proof. Use Lemma 8.1 and the fact that $(\pi \times \rho)$ gives an embedding of $\widetilde{V}$ into $(\mathbf{P}_4 \times \mathbf{P}_3)$.

It follows from the preceding discussion that $\lambda^{-1}(D_0)$ must have two components, D_1 and D_2, corresponding to the two rulings of Q_0. Also the map $\lambda\colon D_0^{(2)} \longrightarrow S_V$ is just the standard desingularization of the variety S_V. Each component D_i of $\lambda^{-1}(D_0)$ for $i = 1, 2$ must be isomorphic to D_0 under the mapping:

$$\lambda_i = \lambda|_{D_i}\,, \qquad i = 1, 2\,. \tag{8.10}$$

It is interesting to apply the considerations of Part one to the covering family $\{J_{(t,t')}\}_{(t,t') \in D_0^{(2)}}$ for $\widetilde{V}$. For $t \in D_0$, let

$$E_t = \{(t', t'') \in D_0^{(2)}\colon\ t' = t\}\,. \tag{8.11}$$

If we let $\{D_{(t,t')}\}_{(t,t') \in D_0^{(2)}}$ be the family of incidence divisors (see (2.6)) associated to $\{J_{(t,t')}\}$, then it is immediate from the definition of $J_{(t,t')}$ that:

$$\begin{aligned} &\text{For } (t, t') \in D_1\colon\ D_{(t,t')} = E_{\lambda(t,t')} + D_2\,. \\ &\text{For } (t, t') \in D_2\colon\ D_{(t,t')} = E_{\lambda(t,t')} + D_1\,. \end{aligned} \tag{8.12}$$

LEMMA 8.13. (i) $\{D_1\} = \{D_2\}$ *in* $H_2(D_0^{(2)})$. *So in particular* $(\{D_i\} \cdot \{D_i\}) = 0$ *for* $i = 1, 2$;

(ii) $(\{E_t\}\cdot\{D_i\}) = 2$ *for* $i = 1, 2$.

Proof. (i) is clear from (8.12). Given $t_0 \in D_0$, there is exactly one point $(t, t') \in D_1$ such that $t_0 = \lambda(t, t')$. So $(E_{t_0} \cap D_1) = \{(t, t_0), (t', t_0)\}$. That this intersection is generically transverse follows immediately from the fact that E_t is given by the curve $(D_0 \times \{t\})$ in $(D_0 \times D_0)$. This gives (ii).

Next let φ: $\mathrm{Alb}(D_0^{(2)}) \to J(\tilde{V})$ be the Abel-Jacobi mapping associated to $\{J_{(t\ t')}\}_{(t\ t') \in D_0^{(2)}}$ (see (4.3)). For $i = 1, 2$, $t \in D_0$, the inclusions

$$\mu_i\colon D_i \longrightarrow D_0^{(2)}\ ;$$
$$\mu_t\colon E_t \longrightarrow D_0^{(2)}$$

induce homomorphisms of the corresponding Albanese varieties (which we denote by the same symbols μ_i and μ_t respectively). Also, since the mapping ρ: $\tilde{V} \to \mathbf{P}_3$ is just the monoidal transform which blows up the curve $\kappa(D_0)$ in $\mathbf{P}_3$, there is induced by Lemma 3.11 an isomorphism

$$\hat{\rho}\colon J(D_0) \longrightarrow J(\tilde{V})\ .$$

PROPOSITION 8.14. (i) *For* $t \in D_0$, *the diagram*

$$\begin{array}{ccc} J(E_t) & = & J(D_0) \\ {\scriptstyle \mu_t}\big\downarrow & & \big\downarrow{\scriptstyle \hat{\rho}} \\ \mathrm{Alb}(D_0^{(2)}) & \xrightarrow{\varphi} & J(\tilde{V}) \end{array}$$

is anticommutative.

(ii) *For* $i = 1, 2$ *and* λ_i *as in* (8.10), *the diagram*

$$\begin{array}{ccc} J(D_i) & \xrightarrow{\lambda_i} & J(D_0) \\ {\scriptstyle \mu_i}\big\downarrow & & \big\downarrow{\scriptstyle \hat{\rho}} \\ \mathrm{Alb}(D_0^{(2)}) & \xrightarrow{\varphi} & J(\tilde{V}) \end{array}$$

is commutative.

Proof. For $(t, t') \in D_i$, $J_{(t\ t')} = (\tilde{L}_{\lambda(t,t')} + L_i(t, t'))$ where $L_i(t, t')$ is a line in Q_0. The algebraic family $\{\tilde{L}_t + L_i(\lambda_i^{-1}(t))\}_{t \in D_0}$ of algebraic one-cycles induces a homology map (see § 1) σ_i: $H_1(D_0) \to H_3(\tilde{V})$. Under the natural identification $H_1(D_0) = H_1(J(D_0))$, $H_3(\tilde{V}) = H_1(J(\tilde{V}))$, σ_1 is the homology map corresponding to $(\varphi \circ \mu_i \circ \lambda_i^{-1})$. But the homology map $H_1(D_0) \to H_3(\tilde{V})$ induced by the family $\{L_i(\lambda_i^{-1}(t))\}_{t \in D_0}$ is the zero map since it factors through $H_3(Q_0) = 0$. Thus $\sigma_i = \sigma$: $H_1(D_0) \to H_3(\tilde{V})$ where σ is the homology map induced by the family $\{\tilde{L}_t\}_{t \in D_0}$, which is the homology map corresponding to $\hat{\rho}$. This gives (ii). For (i) notice that $(\varphi \circ \mu_{t_1})$ corresponds to the homology map induced by the family $\{J_{(t\ t_1)}\}_{t \in D_0}$. Thus it suffices to show that the homology map

$H_1(D_0^{(2)}) \to H_3(\tilde{V})$ induced by the family $\{\tilde{L}_t + \tilde{L}_{t'} + J(t, t')\}$ is the zero map. Let $G = \mathrm{Gr}(2, 4)$, the Grassmann variety of lines in $\mathbf{P}_3$. Then G is simply connected ([3; page 70]) and G parametrizes an algebraic family of algebraic one-cycles on $\tilde{V}$ whose generic element is $\rho^{-1}(L_u)$, $u \in G$. If $u = \kappa^{(2)}(t, t')$ (see (8.4)), then

$$\rho^{-1}(L_u) = (\tilde{L}_t + \tilde{L}_{t'} + J_{(t\ t')} + Q_{(t,t')})$$

where $Q_{(t\ t')}$ is an algebraic one-cycle in Q_0. As above the homology map $H_1(D_0^{(2)}) \to H_3(\tilde{V})$ induced by the family $\{Q_{(t\ t')}\}_{(t,t') \in D_0^{(2)}}$ must be zero since it factors through $H_3(Q_0) = 0$. But the homology map induced by the family $\{\rho^{-1}(\kappa^{(2)}(t, t'))\}_{(t,t') \in D_0^{(2)}}$ must also be zero since it factors through $H_1(G) = 0$. The proposition follows.

COROLLARY 8.15. *The mapping φ: $\mathrm{Alb}(D_0^{(2)}) \to J(\tilde{V})$ is an isomorphism and if θ is a representative of the theta-divisor $\theta(\mathcal{J}(\tilde{V}))$:*

$$(\{\theta\}\cdot\{\theta\})/2\,! = \{\varphi(D_0^{(2)})\}$$

in $H_4(J(\tilde{V}))$.

(Compare this with § 3, especially Definition 3.15. Also notice that we have used the fact that the automorphism $a \mapsto (-a)$ on $J(\tilde{V})$ induces the identity map in even dimensional homology.)

PROPOSITION 8.16. *Let V_1 and V_2 be two Lefschetz cubic threefolds, each with a double point. If $\mathcal{J}(\tilde{V}_1) \approx \mathcal{J}(\tilde{V}_2)$ then $\tilde{V}_1 \approx \tilde{V}_2$.*

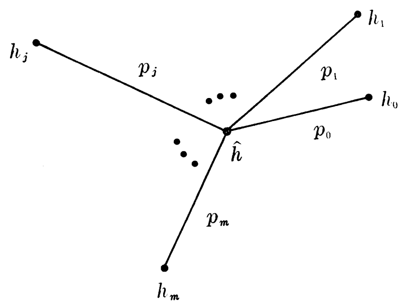
Proof. Let D_{j0} be the double curve of S_{V_j}, $j = 1, 2$. Then $\mathcal{J}(D_{10}) \approx \mathcal{J}(D_{20})$ so that by the classical Torelli theorem [1], $D_{10} \approx D_{20}$. Thus a canonical embedding of D_{10} into $\mathbf{P}_3$ can differ from one for D_{20} by at most a linear automorphism of $\mathbf{P}_3$. Since $\tilde{V}_j$ is obtained from $\mathbf{P}_3$ by blowing up along the canonically embedded D_{j0}, the proposition follows.

Our purpose in much of the remainder of the paper is to examine the analogues of Proposition 8.14 (i), Corollary 8.15 and Proposition 8.16 in the case of non-singular cubic threefolds. Our task is made more difficult by the fact that in this case $\mathcal{J}(V)$ will turn out *not* to be the Jacobian of a curve and much less is known about principally polarized abelian varieties which are not of level one (see Definition 3.15).

9. A topological model for the Fano surface, the non-singular case

Let V be a non-singular cubic threefold. As in § 7 let Y be a non-singular cubic fourfold such that V is a hyperplane section of Y. Let R be a generic three-plane in $\mathbf{P}_5$. Let A_R, Y_h, and S_h $(h \in [R])$ be as in (7.11). Since R is assumed to be transverse to Y, double points of Y_h never lie on R and so, using Proposition 7.12, A_R is easily seen to be non-singular. Moreover S_h is non-singular for almost all h, and for the remaining values $h_0, \cdots, h_m$ of h, S_h is the surface of lines for a cubic threefold with one double point. We will construct a topological model for S_h from our knowledge of the topology of S_{h_j}, $j = 0, 1, \cdots, m$. Our techniques are essentially those of the classical Lefschetz theory ([21; Chapter VI]).

Fix $\hat{h} \in [R]$, $\hat{h} \notin \{h_0, \cdots, h_m\}$. Connect $\hat{h}$ to each h_j by a path p_j in $[R]$:



Let $B_j = \bigcup\{S_h: h \in p_j\}$. We now proceed to analyze the structure of B_j for fixed j. For convenience of notation in this part of the argument, fix j and put $S_{\hat{h}} = S$, $S_{h_j} = S_0$ with double curve D_0, and put $B_j = B$. Then we have as in § 8:

$$\kappa\colon D_0^{(2)} \longrightarrow S_0$$

and we let D_1 and D_2 be the two components of $\lambda^{-1}(D_0)$. There is a retraction theorem for this situation ([5; page 42]). Since the normal bundles to D_1 and D_2 in $D_0^{(2)}$ are topologically trivial by Lemma 8.13, we can state the retraction theorem in this case as follows:

(9.1) For $i = 1, 2$ let N_i be a tubular neighborhood of D_i. Then there exists a C^∞ normal fibration

$$\nu_i\colon N_i \longrightarrow D_i \approx D_0$$

and trivializations

$$N_1 \xrightarrow{\tau_1} D_0 \times \{z \in \mathbf{C}\colon |z| < 2\} \xleftarrow{\tau_2} N_2$$

such that:

(i) If $M_i = \tau_i^{-1}(D_0 \times \{z\colon |z| \leqq 1/2\})$, then S is obtained (as a differentiable manifold) from

$$\left(D_0^{(2)} - (M_1 \cup M_2)\right)$$

by identifying $a \in (N_1 - M_1)$ with $b \in (N_2 - M_2)$ whenever, for $\tau_1(a) = (t_1, z_1)$ and $\tau_2(b) = (t_2, z_2)$:

$$t_1 = t_2$$

and

$$z_1 z_2 = 1 .$$

Thus S is irreducible. Also we have a quotient mapping

$$\psi\colon \big(D_0^{(2)} - (M_1 \cup M_2)\big) \longrightarrow S .$$

(ii) Let $r\colon [1/2, 2] \to [0, 1]$ be a C^∞-function which $= 0$ on $[1/2, 1]$, is increasing on $[1, 2]$ and $= 1$ on some neighborhood $(2 - \varepsilon, 2]$ of 2. Define $\rho\colon S \to S_0$ by:

$$\rho(a) = \begin{cases} \lambda\big(\psi^{-1}(a)\big) & \text{if } a \in \psi\big(D_0^{(2)} - (N_1 \cup N_2)\big); \\ \lambda\big(\tau_1^{-1}(t, r(|z|)z)\big) & \text{whenever, for } b \in \big(N_1 \cap \psi^{-1}(a)\big), \\ & \tau_1(b) = (t, z) \quad \text{with } |z| \geqq 1; \\ \lambda\big(\tau_2^{-1}(t, r(|z|)z)\big) & \text{whenever, for } b \in \big(N_2 \cap \psi^{-1}(a)\big), \\ & \tau_2(b) = (t, z) \text{ with } |z| \geqq 1. \end{cases}$$

Then B is homeomorphic to the mapping cylinder of ρ.

Let $X = \rho^{-1}(D_0)$. Then $X \approx \big(D_0 \times \text{(circle)}\big)$. Using Lemma 8.13, Proposition 8.14, and 9.1 (i), it is easily shown that:

(9.2) The natural mapping $H_q(X) \to H_q(S)$ takes $H_q(X)$ isomorphically onto a direct summand of $H_q(S)$ for all q.

In the remainder of the chapter let $H^q(\ )$ denote q-th integral cohomology *not* modulo torsion. If M_ρ is the mapping cylinder of ρ, and if X_ρ is the mapping cylinder of $\rho|_X$, then using excision and the Thom-Gysin isomorphism we have

$$H^q(M_\rho, S) \approx H^q(X_\rho \cup S, S) \approx H^q(X_\rho, X_\rho \cap S) \approx H^{q-2}(D_0) .$$

Thus there is an exact cohomology sequence associated to ρ:

$$\begin{aligned} \cdots \longrightarrow H^q(S_0) \xrightarrow{\rho^*} H^q(S) \xrightarrow{\mu} H^{q-1}(D_0) \\ \longrightarrow H^{q+1}(S_0) \xrightarrow{\rho^*} H^{q+1}(S) \longrightarrow \cdots \end{aligned}$$

where μ is the composition of $H^q(S) \to H^q(X)$ with the Gysin map $H^q(X) \to H^{q-1}(D_0)$. By (9.2) therefore, $\mu\colon H^q(S) \to H^{q-1}(D_0)$ is onto for $q > 0$. So:

LEMMA 9.3. *The sequence*

$$0 \longrightarrow H^q(S_0) \xrightarrow{\rho^*} H^q(S) \xrightarrow{\mu} H^{q-1}(D_0) \longrightarrow 0$$

is exact for all q.

Next let K be the quotient space obtained from $D_0^{(2)} + \big(D_0 \times [0, 1]\big)$ by identifying

$$(t, t') \in D_1 \text{ with } \big(\lambda(t, t'), 0\big)$$
$$(t, t') \in D_2 \text{ with } \big(\lambda(t, t'), 1\big).$$

Then K has the same homotopy type as S_0 and using the exact cohomology sequence for the pair $(K, D_0^{(2)})$ and the Thom-Gysin isomorphism, we obtain the following exact sequence associated to the mapping λ:

$$\cdots \longrightarrow H^q(S_0) \xrightarrow{\lambda^*} H^q(D_0^{(2)}) \longrightarrow H^q(D_0)$$
$$\xrightarrow{\sigma} H^{q+1}(S_0) \xrightarrow{\lambda^*} H^{q+1}(D_0^{(2)}) \longrightarrow \cdots.$$

The mapping $H^q(D_0^{(2)}) \to H^q(D_0)$ in this sequence is given by $(\omega_2^* - \omega_1^*)$ where $\omega_i = \lambda|_{D_i}^{-1}\colon D_0 \to D_0^{(2)}$ for $i = 1, 2$. Again by Lemma 8.13 and Proposition 8.14, it is easily seen that $\omega_1^* = \omega_2^*$. Therefore:

LEMMA 9.4. *The sequence*

$$0 \longrightarrow H^{q-1}(D_0) \xrightarrow{\sigma} H^q(S_0) \xrightarrow{\lambda^*} H^q(D_0^{(2)}) \longrightarrow 0$$

is exact for all q.

Taking Lemmas 9.3 and 9.4 together we have:

COROLLARY 9.5. $H^1(S) \approx \mathbf{Z}^{10}$, $H^1(S_0) \approx \mathbf{Z}^9$.

Also, for any one-cycle γ in D_0, there is a two-chain β in $D_0^{(2)}$ such that $\partial\beta = \gamma_2 - \gamma_1$ where γ_i lies in D_i for $i = 1, 2$ and $\lambda(\gamma_1) = \lambda(\gamma_2) = \gamma$ (see Proposition 8.14). Thus $\lambda(\beta)$ is a two-cycle in S_0. If $\eta \in H^1(S_0)$ is such that $\langle\gamma, \eta\rangle \neq 0$, then for non-zero $\xi \in H^0(D_0)$ the definition of σ gives that $\langle\lambda(\beta), \sigma(\xi) \cup \eta\rangle \neq 0$.

LEMMA 9.6. *The cup product mapping*

$$H^1(S_0) \wedge H^1(S_0) \longrightarrow H^2(S_0)$$

is injective.

Proof. If ξ generates $H^0(D_0)$ and $\eta \in \big(H^1(S_0) - (\ker\lambda^*)\big)$, one can find a two-chain β as in the discussion just previous such that $\langle\lambda(\beta), \sigma(\xi) \cup \eta\rangle \neq 0$. Thus

$$\big(H^1(S_0)/(\ker\lambda^*)\big) \longrightarrow H^2(S_0)$$
$$\eta \longmapsto \eta \cup \sigma(\xi)$$

is injective. Now use Lemma 9.4 together with the fact that the cupproduct mapping $H^1(D_0^{(2)}) \wedge H^1(D_0^{(2)}) \to H^2(D_0^{(2)})$ is injective.

Let $\{D_s\}_{s\in S}$ denote the family of incidence divisors for $\{L_s\}_{s\in S}$ (see § 2). By (8.12) we have that under the mapping $\rho_*\colon H_2(S) \to H_2(S_0)$:

(9.7) $$\rho_*(\{D_s\}) = \lambda_*(\{E_{t_0}\}) + \{D_0\} = \lambda_*(\{E_{t_0} + D_i\})$$
$$\text{for } s \in S,\ t_0 \in D_0,\ \text{and } i = 1, 2.$$

Choose a basis $\eta'_1, \cdots, \eta'_8$ for $H^1(D_0^{(2)})$ such that:

$$\begin{aligned} &\langle\{E_{t_0}\}, \eta'_k \cup \eta'_{4+k}\rangle = 1 && \text{for } k = 1, \cdots, 4\,; \\ &\langle\{E_{t_0}\}, \eta'_k \cup \eta'_l\rangle = 0 && \text{for all other } l > k\,. \end{aligned} \tag{9.8}$$

Then by Proposition 8.14:

$$\begin{aligned} &\langle\{D_i\}, \eta'_k \cup \eta'_{4+k}\rangle = 1 && \text{for } k = 1, \cdots, 4 \text{ and } i = 1, 2\,; \\ &\langle\{D_i\}, \eta'_k \cup \eta'_l\rangle = 0 && \text{for all other } l > k\,. \end{aligned} \tag{9.9}$$

Next let $\eta^0_1, \cdots, \eta^0_8 \in H^1(S_0)$ be such that $\lambda^*(\eta^0_k) = \eta'_k$ for all k and put

$$\eta_k = \rho^*(\eta^0_k) \in H^1(S)\,.$$

A consequence of (9.7)–(9.9) is:

$$\begin{aligned} &\langle\{D_s\}, \eta_k \cup \eta_{4+k}\rangle = 2 && \text{for } k = 1, \cdots, 4; \\ &\langle\{D_s\}, \eta_k \cup \eta_l\rangle = 0 && \text{for all other } l > k\,. \end{aligned} \tag{9.10}$$

Let χ' be a generator for $H^0(D_0)$ and put

$$\chi = \rho^* \circ \sigma(\chi') \in H^1(S)\,.$$

Then by (9.7):

$$\langle\{D_s\}, \chi \cup \eta_k\rangle = 0 \qquad \text{for } k = 1, \cdots, 8\,. \tag{9.11}$$

To see what the cocycle χ is geometrically, let U be a regular tubular neighborhood of the circle bundle X in S. If ω is the orientation class for the bundle $(U, \partial U)$ over X, then (up to sign), χ is the image of the generator of $H^0(X)$ under the composition:

$$H^0(X) \xrightarrow{\cup\omega} H^1(U, \partial U) \longrightarrow H^1(S)\,.$$

By Corollary 2.12, D_s is an ample divisor on S. Using this fact and sequence (9.3), pick $\delta \in H^1(S)$ such that:

(9.12) (i) $\mu(\delta) =$ generator of $H^0(D_0)$;

(ii) $\langle\{D_s\}, \delta \cup \eta_k\rangle = 0$ for all $k = 1, \cdots, 8$;

(iii) $\langle\{D_s\}, \chi \cup \delta\rangle > 0$.

Then the collection $\{\chi, \delta, \eta_1, \cdots, \eta_8\}$ is a basis for $H^1(S)$.

Suppose now that in $H^2(S)$ we have the relation:

$$\xi = e\chi \cup \delta + \chi \cup \left(\sum a_k\eta_k\right) + \delta \cup \left(\sum b_k\eta_k\right) + \sum c_{kl}\eta_k \cup \eta_l = 0\,.$$

Using the mapping $H^2(S) \to H^2(X) \approx H^2\big(D_0 \times \text{(circle)}\big)$ we conclude that all the $b_k = 0$ and $\sum_{k=1}^4 c_{k(4+k)} = 0$. Then by restricting ξ to D_s we have also $e = 0$. Then by Lemmas 9.3 and 9.6 we have:

LEMMA 9.13. *The cupproduct mapping*

$$H^1(S) \wedge H^1(S) \longrightarrow H^2(S)$$

is injective.

(Notice that by Lemma 7.14 this result is valid for the surface of lines associated to *any* non-singular cubic threefold.)

Another result about the topology of S is gotten from the fact that by direct computation (see for example [1; Section 1]) the topological Euler characteristic $\chi(D_0^{(2)}) = 15$. For M_i as in (9.1) (i), $\chi(M_i) = -6$ and $\chi(\partial M_i) = 0$. Thus by (9.1) (i):

$$\chi(S) = 27\,. \tag{9.14}$$

Finally, we will need in § 13 a series of integration formulas on S. We have on $D_0^{(2)}$:

$$\begin{aligned} &\langle\{D_0^{(2)}\}, \eta'_k \cup \eta'_{4+k} \cup \eta'_l \cup \eta'_{4+l}\rangle = 1 && \text{for } 1 \leqq k < l \leqq 4 \\ &\langle\{D_0^{(2)}\}, \eta'_p \cup \eta'_q \cup \eta'_r \cup \eta'_s\rangle = 0 && \text{for } \{p, q, r, s\} \neq \{k, 4+k, l, 4+l\}\,. \end{aligned} \tag{9.15}$$

(To see this look, for instance, at the image of $D_0^{(2)}$ in $J(D_0)$ and compute its Poincaré dual.) Also by Lemma 9.4, $\langle\{S_0\}, \sigma(\chi) \cup \eta_k^0 \cup \eta_l^0 \cup \eta_m^0\rangle = 0$ for all k, l, m. This gives the formulas on S:

$$\begin{aligned} &\langle\{S\}, \eta_k \cup \eta_{4+k} \cup \eta_l \cup \eta_{4+l}\rangle = 1 && \text{for } l \neq k\,; \\ &\langle\{S\}, \eta_p \cup \eta_q \cup \eta_r \cup \eta_s\rangle = 0 && \text{for } \{p, q, r, s\} \neq \{k, 4+k, l, 4+l\}\,; \\ &\langle\{S\}, \chi \cup \eta_k \cup \eta_l \cup \eta_m\rangle = 0 && \text{for all } k, l, m\,. \end{aligned} \tag{9.16}$$

Leaving the orientation check for later (see the end of § 11), the definition of χ and the map μ (see Lemma 9.3) give easily that:

$$\begin{aligned} &\langle\{S\}, \chi \cup \delta \cup \eta_k \cup \eta_{4+k}\rangle = \pm 1 && \text{for } k = 1, \cdots, 4\,; \\ &\langle\{S\}, \chi \cup \delta \cup \eta_k \cup \eta_l\rangle = 0 && \text{for other } l > k\,. \end{aligned} \tag{9.17}$$

Besides checking the sign in (9.17) we also want to compute $\langle\{S\}, \delta \cup \eta_k \cup \eta_l \cup \eta_m\rangle$ and $\langle\{D_s\}, \chi \cup \delta\rangle$. For this we need more information about the divisor D_s which we shall obtain in § 10 and § 11. In any case we can make some steps in the computation now:

LEMMA 9.18. *In* $H^3(S)$ *we have* (modulo torsion) *the relations*:

(i) $(\eta_k \cup \eta_{4+k} \cup \eta_l)$ *is independent of* $k = 1, \cdots, 4$ *as long as* $l \notin \{k, 4+k\}$;

(ii) $(\eta_k \cup \eta_l \cup \eta_m) = 0$ *unless, for some* $\{a, b\} \subseteqq \{k, l, m\}$, $a - b = 4$.

Proof. By (9.15) and duality the relations obtained on $D_0^{(2)}$ by replacing η by the corresponding η' are valid. Also the subspace A of $H^3(S_0)$ generated by $\{\eta_k^0\}_{k=1,\cdots,8}$ cannot have rank 9. This can be seen as follows. Using Lemma 8.13 and the homology sequence for the pair $(K, D_0^{(2)})$ used in the proof of Lemma 9.4 one computes immediately the exactness of the sequence

$$0 \longrightarrow H_3(D_0^{(2)}) \xrightarrow{\lambda_*} H_3(S_0) \xrightarrow{\tilde{\sigma}} H_2(D_0) \longrightarrow 0 .$$

Then the pairing $H_3(D_0^{(2)}) \times H^3(D_0^{(2)}) \to \mathbf{Z}$ is non-singular and unimodular so that there is a three-cycle β on S_0 such that

(i) $\tilde{\sigma}(B) =$ generator of $H_2(D_0)$

(ii) $\langle\beta, \omega\rangle = 0$ for all $\omega \in A$.

If ξ generates $H^2(D_0)$, $\langle\beta, \sigma(\xi)\rangle = \pm 1$, so rank $A = 8$. Thus modulo torsion $\lambda^*: A \to H^3(D_0^{(2)})$ is injective. Thus the relations (i) and (ii) of the lemma hold on S_0 if we replace η by η^0. The lemma then follows.

(From now on we return to the convention that all cohomology is *modulo torsion*.)

10. Distinguished divisors on the Fano surface

From now on, unless explicitly indicated to the contrary, V will be a non-singular cubic threefold, and $S = S_V$, $D = D_V$, and $T = T_V$ will be as in (7.1). Since in this case, the dual mapping $\mathfrak{D}_V: \mathbf{P}_4 \to \mathbf{P}_4^*$ is everywhere defined, we can identify $\mathfrak{D}: \mathbf{P}_V \to \mathbf{P}_4^*$ with $\mathfrak{D}_V$ (see § 5). We write simply

$$\mathfrak{D}: \mathbf{P}_4 \longrightarrow \mathbf{P}_4^* .$$

Let K be a plane in $\mathbf{P}_4$. $K \not\subseteq V$ since $\mathfrak{D}$ is finite-to-one. We therefore have a family of divisors on S:

$$\{D_K\}_{K \in \mathrm{Gr}(3,5)} , \quad D_K = \{s \in S: (L_s \cap K) \neq \emptyset\} .$$

Each K determines a hyperplane section of Gr(2, 5) under the standard Plücker embedding

$$\mathrm{Gr}(2, 5) \subseteq \mathbf{P}_9 , \tag{10.1}$$

and the set $\{[h_K]\}_{K \in \mathrm{Gr}(3,5)}$ so determined spans the projective space $\mathbf{P}_9^*$. Since for all K, $S \not\subseteq [h_K]$, we conclude:

LEMMA 10.2. *Under the Plücker embedding* $S \subseteq \mathrm{Gr}(2, 5) \subseteq \mathbf{P}_9$, *no hyperplane of* $\mathbf{P}_9$ *contains* S.

A fundamental result of Fano for the family $\{D_K\}$ is:

PROPOSITION 10.3. *For a plane* K *in* $\mathbf{P}_4$, D_K *is a canonical divisor on* S.

Proof. For a generic line L in $\mathbf{P}_4$, $V_h = (V \cap [h])$ is a Lefschetz cubic surface for all but a finite set N of values of $h \in [L] \subseteq \mathbf{P}_4^*$. Furthermore for any $h \in [L]$, the number of lines of S which lie in V_h is finite (since we can choose L so that $[L]$ does not contain $\mathfrak{D}(x)$ for any Eckardt point $x \in V$). Let $S' = S - \{s: L_s \subseteq V_h$ for some $h \in N\}$ and we have an induced finite-to-one morphism:

$$\tau \colon S' \longrightarrow ([L] - N) .$$

By (6.3), $\tau^{-1}(h)$ has 27 elements if V_h is non-singular and by the discussion following Lemma 6.5, $\tau^{-1}(h)$ has 21 elements if V_h has a double point. Therefore τ ramifies only over $h \in (\mathfrak{D}(V) \cap [L])$. Furthermore τ ramifies at $s \in S'$ if and only if there is a plane K in $\mathbf{P}_4$ such that

(i) $[K] \subseteqq [L] \subseteqq \mathbf{P}_4^*$;

(ii) s is a singular point of $\tau^{-1}([K]) = (D_K \cap S')$.

Let $M_K = \{t \in \mathrm{Gr}(2, 5) \colon (L_t \cap K) \neq \varnothing\}$. Then (ii) is equivalent to the condition that M_K and S be tangent at s. Using local coordinates (6.14) and (6.15), it is easily seen that if K is tangent to V at some point of L_s, then M_K and S are tangent at s. This means that if there is an $x \in L_s$ such that $L \subseteqq [\mathfrak{D}(x)]$, the tangent hyperplane to V at x, then τ is ramified at s. Thus if V_h has double point x_h $(h \in ([L] - N))$, then τ must ramify at each of the six points s such that $x_h \in L_s$. Since $\tau^{-1}(h)$ for such an h has 21 elements, we conclude that the ramification occurs only at the six points mentioned and that the branching at each of the points is simple. Thus if $W_x = \{s \in S \colon x \in L_s\}$ and C_L is the divisor on S given by the set

$$\bigcup \{W_x \colon \mathfrak{D}(x) \in [L]\} ,$$

the divisor $(C_L - 3D_K)$ is a canonical divisor on S. However for a generic plane $[L] \subseteqq \mathbf{P}_4^*$, $\mathfrak{D}^{-1}([L])$ is a complete intersection of type $(2, 2)$ in $\mathbf{P}_4$ (by (5.4)) which meets V transversely except possibly at a finite set (by Lemma 5.9 and Corollary 5.14). Therefore C_L is linearly equivalent to the divisor $4D_K$ and the proposition is proved.

We next turn our attention to $\{D_s\}_{s \in S}$, the family of incidence divisors on S. By Corollary 2.12:

LEMMA 10.4. *The family $\{D_s\}$ is a family of ample divisors on S.*

LEMMA 10.5. *For generic $s \in S$, D_s is non-singular.*

Proof. Let $M_s = \{t \in \mathrm{Gr}(2, 5) \colon (L_t \cap L_s) \neq \varnothing\}$. If $s' \in D_s$ and $L_{s'}$ is of first type, we can assume local coordinates (6.13) to be constructed at s' such that L_s meets $(H_0 \cap H_1)$ (given by $X_0 = X_1 = 0$). Since M_s is then given by linear equations, it follows easily that M_s and S meet transversely along D_s in a neighborhood of s'. Similarly if $L_{s'}$ is of second type, M_s and S meet transversely along D_s in a neighborhood of s' except if:

(10.6) L_s lies in the plane $[\mathfrak{D}(L_{s'})]$ tangent to V along $L_{s'}$.

Thus s' is a singular point of D_s only if $L_{s'}$ is of second type and (10.6) holds. A dimension argument then gives the lemma since a generic $s \in S$ corresponds to a line L_s of first type.

LEMMA 10.7. *L_s is of second type if and only if $s \in D_s$.*

Proof. For $s' \in D_s$, $s' \neq s$, let $K_{s'}$ be the plane spanned by $(L_s \cup L_{s'})$. If $s \in D_s$, $\lim_{s' \to s} K_{s'}$ is a plane tangent to V along L_s. So by Lemma 6.7, L_s is of second type. Conversely, for L_s of second type, the set of planes K such that $(K \cdot V) = (L_s + L_{s_1} + L_{s_2})$ is connected and as K approaches $[\mathfrak{D}(L_s)]$, either L_{s_1} or L_{s_2} (or both) must approach L_s. The lemma follows.

We can now compute a series of numerical invariants associated to the surface S (see, for instance, [13; page 154]). By Lemmas 10.4 and 10.5, D_s is a non-singular irreducible curve for generic s. Since two skew lines in a non-singular cubic surface have exactly five common incident lines ([18; pages 3–5]):

$$(\{D_s\} \cdot \{D_s\})_S = 5 \,. \tag{10.8}$$

Picking a plane K such that $(K \cdot V) = L_{s_1} + L_{s_2} + L_{s_3}$, $(s_i \neq s_j$ for $i \neq j)$, we have

$$D_K \sim (D_{s_1} + D_{s_2} + D_{s_3}) \tag{10.9}$$

where "$\sim$" denotes linear equivalence. Then by Proposition 10.3 and the adjunction formula for surfaces:

$$\text{genus}\,(D_s) = 11 \,. \tag{10.10}$$

We have already seen that the second *Chern number* $c_2[S] = 27$ in (9.14). To get the first Chern number, use Proposition 10.3:

$$c_1^2[S] = (\{D_K\} \cdot \{D_K\})_S = 45 \,. \tag{10.11}$$

Since $12(h^{0,0}(S) - h^{1,0}(S) + h^{2,0}(S)) = (c_1^2 + c_2)$:

$$h^{2,0}(S) = 10 \,. \tag{10.12}$$

Then by Lemma 10.2 and Proposition 10.3:

LEMMA 10.13. *The Plücker embedding $S \subseteqq \mathrm{Gr}(2, 5) \to \mathbf{P}_9$ is a canonical embedding of S.*

Since $\chi(S) = 27 = (2 - 2\beta_1(S) + 2\beta_2(S))$, where β_j denotes the j-th Betti number, $\beta_2(S) = 45$ and so by Lemma 9.13:

(10.14) The natural map $(H^1(S_V) \otimes \mathbf{Q}) \wedge (H^1(S_V) \otimes \mathbf{Q}) \to H^2(S_V) \otimes \mathbf{Q}$ is an isomorphism.

In the final portion of this chapter, we treat the algebraic subvariety D of S, where $D = D_V$ is the set of points on S which correspond to lines of second type on V. From Corollary 7.6, $\dim D \leqq 1$. In order to characterize the divisor D in S, we need several lemmas.

LEMMA 10.15. *Let* $E = \{s \in D\colon ([\mathfrak{D}(L_s)] \cap V) = 3L_s\}$. *Then E is a finite set.*

Proof. Suppose E contains a curve C. By Lemma 7.5 $\dim W = 2$ where $W = \pi_V(\pi_S^{-1}(C))$. Since $\mathfrak{D}$ is finite-to-one, $\mathfrak{D}|_W$ must be of maximal rank at a generic simple point x of W. Let $x = \pi_V(s, x)$, s a simple point of E. By (6.10) we can normalize the equation for V to the form:

$$X_2X_0^2 + X_3X_1^2 + X_0 \sum \{b_{0jk}X_jX_k\colon 2 \leqq j \leqq k \leqq 4\} \\ + X_1 \sum \{b_{1jk}X_jX_k\colon 2 \leqq j \leqq k \leqq 4\} + P(X_2, X_3, X_4)$$

where L_s is given by $X_2 = X_3 = X_4 = 0$. The condition that $s \in E$ is precisely the condition that $b_{044} = b_{144} = 0$. This implies that the (5×3)-matrix

$$\big((\partial^2 F/\partial X_i \partial X_0)_x \quad (\partial^2 F/\partial X_i \partial X_1)_x \quad (\partial^2 F/\partial X_i \partial X_4)_x\big)_{i=0,1,\cdots,4}$$

is not of maximal rank. But it is immediate from (6.15) that the plane given by $X_2 = X_3 = 0$ is the tangent plane to W at a generic point of L_s. Thus $\mathfrak{D}|_W$ cannot be of maximal rank at x which gives the desired contradiction.

Recall that the tangent cone C_x to V at a point x is the union of all lines in $\mathbf{P}_4$ which have contact of order $\geqq 3$ with V at x.

LEMMA 10.16. *Suppose for some $s \in D$*

$$(C_x \cdot V) = (3L_s + L_{t_1} + L_{t_2} + L_{t_3})$$

for generic $x \in L_s$. Then

$$([\mathfrak{D}(L_s)] \cdot V) = 3L_s\,.$$

Proof. Put the equation for V in normal form with respect to L_s as in the proof of Lemma 10.15. At the point $(1, 0, \cdots, 0)$ one calculates immediately that the tangent cone $C_{(1,0,\cdots,0)}$ is either

(i) the union of two planes K and K';

(ii) a plane K;

(iii) the tangent hyperplane to V at $(1, 0, \cdots, 0)$.

In cases (i) and (ii), $(C_{(1,0,\cdots,0)} \cdot V) = (3L_s + \cdots)$ so that K (or K') must be the unique hyperplane tangent to V along L_s. Thus in any case $[\mathfrak{D}(L_s)] \subseteqq C_{(1,0,\cdots,0)}$. Similarly $[\mathfrak{D}(L_s)] \subseteqq C_{(0,1,0,\cdots,0)}$. Suppose

$$([\mathfrak{D}(L_s)] \cdot V) = (2L_s + L_t)$$

where $t \neq s$, and suppose, for example, that $(1, 0, \cdots, 0) \notin L_t$. Then any line joining $(1, 0, \cdots, 0)$ with a point on L_t has contact of order $\geqq 4$ with V and so must lie in V. Since V contains no planes, we have a contradiction to the assumption that $t \neq s$.

Next consider the mapping $\pi_V\colon T \to V$ given in (7.3). π_V has fibre

$(W_x \times \{x\})$ where $W_x = \{s: x \in L_s\}$. Let V' be the set obtained by deleting from V the (finite) number of points x for which $\dim W_x > 0$. Putting $T' = \pi_V^{-1}(V')$, we have that the mapping

$$\pi' \colon T' \to V' \tag{10.17}$$

is proper, finite-to-one and generically six-to-one.

LEMMA 10.18. *π' is unramified at (s, x) if and only if L_s is of first type. Furthermore the ramification of π' is simple along a Zariski open subset of each component of $\pi_S^{-1}(D)$.*

Proof. One checks easily from equations (6.14) that if L_s is of first type, then there is a tubular neighborhood of $(\{s\} \times L_s)$ in T on which π_V is an immersion. Therefore π' is unramified at each point of $(\{s\} \times L_s)$. If L_s is of second type, we proceed as follows. Let $M = \{(t, x) \in (\mathrm{Gr}(2, 5) \times \mathbf{P}_4): x \in L_t\}$. Let $\pi_G \colon M \to \mathrm{Gr}(2, 5)$ and $\pi_{\mathbf{P}_4} \colon M \to \mathbf{P}_4$ be the standard projections. If $x \in L_s$, let

$$E_x = \{t \in \mathrm{Gr}(2, 5): x \in L_t \subseteqq [\mathfrak{D}(L_s)]\} .$$

Then E_x is tangent to S at s by equations (6.15). Let E'_x be a non-singular curve on S which is tangent to E_x at s. Since the surfaces $\pi_G^{-1}(E_x)$ and $\pi_G^{-1}(E'_x)$ are tangent along $(\{s\} \times L_s)$, $\pi_{\mathbf{P}_4}|_{\pi_G^{-1}(E'_x)}$ is not of maximal rank at (s, x); so π' cannot be either. The rest of the proof now follows immediately from Lemmas 10.15 and 10.16.

We are now ready to characterize $D = D_V$. For a hyperplane section $V_h = (V \cap [h])$ of V, let

$$S(h) = \pi_V^{-1}(V_h) . \tag{10.19}$$

The Bertini theorems give that for generic h, $S(h)$ is non-singular. So for generic h, $S(h)$ is isomorphic to S blown up at the twenty-seven points s such that $L_s \subseteqq [h]$, and the monoidal transformation is given by the projection $\pi_S \colon S(h) \to S$. Let $E(h)$ denote the exceptional divisor on $S(h)$. If K_X denotes the canonical divisor on the algebraic manifold X and "$\sim$" denotes linear equivalence, we have

$$K_V \sim (-2V_h)$$

and so by Lemma 10.18:

$$K_T \sim (\pi_S^{-1}(D) - 2S(h)) .$$

By the adjunction formula, we have on $S(h)$:

$$\begin{aligned} K_{S(h)} &\sim (S(h) \cdot (K_T + S(h))) \\ &\sim (S(h) \cdot (\pi_S^{-1}(D) - S(h'))) \end{aligned}$$

where h' is any element of $\mathbf{P}_4^*$. On the other hand, Lemma 10.3 gives that on $S(h)$:

$$K_{S(h)} \sim \big((S(h)\cdot S(h')) + 2E(h)\big) .$$

Thus $\big(\pi_S^{-1}(D)\cdot S(h)\big) \sim 2\big((S(h)\cdot S(h')) + E(h)\big)$ on $S(h)$ so that on S:

(10.20) $$D \sim 2D_K$$

where $K = \big([h] \cap [h']\big)$. This gives the result of Fano:

PROPOSITION 10.21. *The set D is of pure dimension one and, considered as a divisor on S, is linearly equivalent to twice the canonical divisor.*

PART FOUR: THE INTERMEDIATE JACOBIAN OF THE CUBIC THREEFOLD

11. The Gherardelli-Todd isomorphism

We now return to the general considerations of Part one in order to apply them to the case in which V is a non-singular cubic hypersurface in $\mathbf{P}_4$. Our analysis of the geometry of V and its surface of lines S will allow us to characterize these varieties entirely in terms of the principally polarized complex torus $\mathcal{J}(V)$. (Notice that $h^{1,0}(V) = 0 = h^{3,0}(V)$, so $\mathcal{J}(V)$ is in fact a principally polarized abelian variety.)

From § 4, we have a homomorphism of abelian varieties

(11.1) $$\varphi\colon \mathrm{Alb}(S) \longrightarrow J(V) .$$

By § 9, $\dim \mathrm{Alb}(S) = 5$, and by [8; page 488], $h^{3,0}(V) = 0$ and $h^{2,1}(V) = 5$. We devote this chapter to showing that φ is actually an isomorphism of abelian varieties. This result is also a corollary of work of Todd [19; page 183]. We begin with a result of Gherardelli ([7]):

LEMMA 11.2. $\varphi\colon \mathrm{Alb}(S) \to J(V)$ *is an isogeny.*

Proof. Let $T = T_V$ be as in (7.1), and $\pi_S\colon T \to S$ the projective line bundle given in (7.2). By the Gysin sequence for sphere bundles, there is a vector space isomorphism:

(11.3) $$H^{2,1}(S) \oplus H^{1,0}(S) \xrightarrow{\pi_S^* + \chi} H^{2,1}(T)$$

where $\chi\colon H^{1,0}(S) \to H^{2,1}(T)$ is given by $\chi(\alpha) = \pi_S^*(\alpha) \wedge \eta$ where η is the Poincaré dual to $\{S(h)\}$ in T (see (10.19)). Since π_V is generically finite-to-one, $\pi_V^*\colon H^{2,1}(V) \to H^{2,1}(T)$ is injective, and since $h^{2,1}(V) = h^{1,0}(S)$, it suffices to show that in $H^{2,1}(V)$:

$$\big((\text{image } \pi_V^*) \cap (\text{image } \pi_S^*)\big) = \{0\} .$$

To see this, choose h as in § 10 so that $S(h)$ is non-singular. For $\omega \in H^{2,1}(V)$, decompose $\pi_V^*(\omega)$ according to (11.3):

$$\pi_V^*(\omega) = \pi_S^*(\beta) + \big(\pi_S^*(\alpha) \wedge \eta\big) .$$

If $\alpha = 0$ and $\beta \neq 0$, there would exist a three-cycle γ on the surface $S(h)$ with $\int_\gamma \pi_V^*(\omega) \neq 0$. However

$$\int_\gamma \pi_V^*(\omega) = \int_{\pi_{V*}(\gamma)} \omega = 0$$

since π_{V*} factors through $H_3(V_h)$, the zero group. This gives the lemma.

The proof of Lemma 11.2 shows that for any $\omega \in H^3(V)$:

$$\pi_V^*(\omega)\,|_{S(h)} = 0 .$$

Thus if $\pi_V^*(\omega) = \pi_S^*(\beta) + \big(\pi_S^*(\alpha) \wedge \eta\big)$ as in (11.3):

$$\beta = -\,\alpha \wedge c_1$$

where $c_1 \in H^{1,1}(S)$ is the *first Chern class* of the bundle π_S. By a theorem of Chern [4; page 571], one has the relation on T:

$$\eta \wedge \eta = \eta \wedge \pi_S^*(c_1) - \pi_S^*(c_2) .$$

So for any $\omega, \omega' \in H^3(V)$:

$$\begin{aligned}
\int_V \omega \wedge \omega' &= (1/6) \int_T \pi_V^*(\omega) \wedge \pi_V^*(\omega') \\
&= (1/6) \int_T \big(\pi_S^*(\alpha) \wedge \eta - \pi_S^*(\alpha \wedge c_1)\big) \wedge \big(\pi_S^*(\alpha') \wedge \eta - \pi_S^*(\alpha' \wedge c_1)\big) \\
&= -\,(1/6) \int_T \pi_S^*(\alpha \wedge \alpha' \wedge c_1) \wedge \eta \\
&= -\,(1/6) \int_S \alpha \wedge \alpha' \wedge c_1 \\
&= -\,(1/6) \int_{D_K} \alpha \wedge \alpha' \\
&= -\,(1/2) \int_{D_s} \alpha \wedge \alpha'
\end{aligned} \tag{11.4}$$

by Proposition 10.3 and (10.9). (By (2.8), $\alpha = \varphi^*(\omega)$, $\alpha' = \varphi^*(\omega')$.) Under the natural identification $H^3(V) \approx H^1(J(V))$ we therefore have that if γ is the image of $\{D_s\}$ under the mapping $(\varphi\circ\alpha_s)_*\colon H_2(S) \to H_2(J(V))$, then

$$\int_\gamma \xi \wedge \xi' = -\,2 \operatorname{Im} \mathcal{H}_V(\xi, \xi')$$

for any $\xi, \xi' \in H^1(J(V))$ (see (3.1) and (3.3)).

On the other hand, if $\Omega_V = \Omega\big(\mathcal{J}(V)\big)$ is the polarizing class of $\mathcal{J}(V)$ as in § 3, it is an easy exercise in linear algebra to show that for $\xi, \xi' \in H^1(J(V))$:

$$\int_{J(V)} \xi \wedge \xi' \wedge \Omega_V^4 = -\,4!\ \mathrm{Im}\,\mathcal{H}_V(\xi, \xi')\ .$$

Thus we have:

LEMMA 11.5. *$(\varphi \circ \alpha_s)_*(\{D_s\})$ has as its Poincaré dual on $J(V)$ the class* $(2\Omega_V^4/4\,!)$.

(Compare this result with Definition 3.15.)

Another corollary of Lemma 11.2 is the following:

LEMMA 11.6. *Let $\lambda : J(V) \to \mathrm{Pic}(S)$ be as in* (4.3). *Then λ is an isogeny and*

$$\deg(\lambda) = \deg(\varphi)\ .$$

(The degree of an isogeny is the cardinality of its kernel.)

Proof. The homology maps φ_*, λ_*, and η_* associated to φ, λ, and $\eta = (\lambda \circ \varphi)$ are given in (2.7). By (1.1) there is an intersection formula:

$$\big(\varphi_*(\gamma)\cdot\varphi_*(\gamma')\big)_V = \big(\gamma\cdot\eta_*(\gamma')\big)_S\ .$$

Let E denote the bilinear form on $H_1(S)$ induced under φ_* from the intersection pairing on $H_3(V)$. Then $\deg(\varphi) = |\det E|^{1/2}$ and by the intersection formula for η_*, $\deg(\eta) = |\det E|$. Thus $\deg(\lambda) = |\det E|^{1/2}$.

We wish to show, of course, that $\deg(\varphi) = 1$. By (4.3) and standard facts about curves we have a commutative diagram:

(11.7)
$$\begin{tikzcd}
\mathrm{Alb}\,(D_s) \arrow[r, "\kappa_s"] \arrow[d, equal] & \mathrm{Alb}(S) \arrow[dd, "\eta"] \arrow[dr, "\varphi"] & \\
J(D_s) \arrow[d, equal] & & J(V) \arrow[dl, "\lambda"] \\
\mathrm{Pic}\,(D_s) & \mathrm{Pic}\,(S) \arrow[l, "\mu_s"'] &
\end{tikzcd}$$

where D_s is any non-singular incidence divisor and κ_s and μ_s are induced by the inclusion $D_s \to S$. Let σ denote the composition $(\kappa_s \circ \mu_s \circ \eta)$.

LEMMA 11.8. *The homorphism σ: $\mathrm{Alb}\,(S) \to \mathrm{Alb}\,(S)$ is given by*

$$\sigma(u) = -\,2u\ .$$

Proof. Since S is connected and for all $s \in S$ the ample divisor D_s is connected, it is immediate that

$$\{\text{planes } K\text{: } (K\cdot V) \text{ is a sum of lines}\}$$

is connected. Also, if $(K\cdot V) = (L_{s_1} + L_{s_2} + L_{s_3})$, then $\eta\big(\alpha_s(s_1) + \alpha_s(s_2) + \alpha_s(s_3)\big)$ must go to the fixed point of $\operatorname{Pic}(S)$ corresponding to the canonical divisor on S by Proposition 10.3. By Theorem 4.5 it follows that there is a fixed point $u_0 \in \operatorname{Alb}(S)$ such that whenever $(L_{s_1} + L_{s_2} + L_{s_3})$ is a plane section of V:

$$\alpha_s(s_1) + \alpha_s(s_2) + \alpha_s(s_3) = u_0 \text{ in } \operatorname{Alb}(S)\ . \tag{11.9}$$

Since V is simply connected, it follows that the mapping

$$V \longrightarrow \operatorname{Alb}(S)$$

gotten by sending x to $\sum_{i=1}^{6} \alpha_s(s_i)$ where $L_{s_1}, \cdots, L_{s_6}$ are the six lines through x is also a constant mapping. So there is a fixed $u_1 \in \operatorname{Alb}(S)$ such that:

$$\sum \alpha_s(s_i) = u_1 \tag{11.10}$$

whenever $L_{s_1}, \cdots, L_{s_6}$ are the six lines through a point. Fix $s_0 \in S$. Then for $s \in D_{s_0}$:

$$\sigma\big(\alpha_{s_0}(s)\big) = \alpha_{s_0}(t) + \sum_{i=1}^{4} \alpha_{s_0}(t_i) + \text{ (constant)}$$

where $(L_t + L_s + L_{s_0})$ is a plane section of V and $L_{s_0}, L_s, L_{t_1}, \cdots, L_{t_4}$ are the six lines through the point $x = (L_s \cap L_{s_0})$. Applying (11.9) and (11.10):

$$\begin{aligned}\sigma\big(\alpha_{s_0}(s)\big) &= u_0 - \big(\alpha_{s_0}(s_0) + \alpha_{s_0}(s)\big)\\ &\qquad + u_1 - \big(\alpha_{s_0}(s) + \alpha_{s_0}(s)\big) + \text{ (constant)}\ .\\ &= -\ 2\alpha_{s_0}(s) + \text{ (constant)}\ .\end{aligned}$$

Since D_{s_0} is ample, $\alpha_{s_0}(D_{s_0})$ generates $\operatorname{Alb}(S)$ and the lemma follows.

Following [2], the formula (11.9) suggests the definition of an involution:

$$\gamma_s \colon D_s \longrightarrow D_s \tag{11.11}$$

for each $s \in S$ by putting $\gamma_s(s_1) =$ that unique $s_2 \in D_s$ such that $(L_s + L_{s_1} + L_{s_2})$ is a plane section of V. Using (10.6) and the involution γ_s it is clear that D_s is non-singular except at fixpoints of γ_s. Also for generic s there are no fixpoints. Then by (10.10) and the Riemann-Hurwicz formula, the quotient

$$G_s = D_s/\gamma_s \tag{11.12}$$

is a non-singular curve of genus 6. Let ξ_s denote alternatively the quotient mapping $D_s \to G_s$ or the induced homomorphism:

$$\operatorname{Alb}(D_s) \longrightarrow \operatorname{Alb}(G_s)\ .$$

Also let γ_s stand as well for the involution on $\operatorname{Alb}(D_s)$ induced by (11.11). Using (11.9), we have on $\operatorname{Alb}(D_s)$:

(11.13) $$\kappa_s \circ \gamma_s = -\kappa_s \; ; \; \xi_s \circ \gamma_s = \xi_s \; .$$

Let $A = (\text{image}((\text{identity map}) + \gamma_s))$ and $B = (\text{image}((\text{identity map}) - \gamma_s))$. Then by (11.13):

$$A \subseteqq (\ker \kappa_s)$$
$$B \subseteqq (\ker \xi_s) \; .$$

By a dimension argument:

$$A = (\ker \kappa_s)^\circ$$
$$B = (\ker \xi_s)^\circ$$

where $(^\circ)$ denotes as before the component of the identity. Since γ_s respects the standard hermitian form associated to $\mathrm{Alb}(D_s)$, A and B are orthogonal subvarieties of $\mathrm{Alb}(D_s)$. On the other hand, if $\kappa_{s*}: H_1(D_s) \to H_1(S)$ and $\mu_{s*}: H_3(S) \to H_1(D_s)$ are the homology mappings associated to κ_s and μ_s respectively, then $(\gamma \cdot \mu_{s*}(\beta))_{D_s} = (\kappa_{s*}(\gamma) \cdot \beta)_S = 0$ for $\gamma \in \ker \kappa_{s*}$, $\beta \in H_3(S)$. Thus $\mu_s(\mathrm{Pic}(S)) \subseteqq (\ker \kappa_s)^\perp$ and so by dimensions:

(11.14) $$B = (\ker \xi_s)^\circ = \mu_{s*}(\mathrm{Pic}(S)) \; .$$

We next wish to compute the degree of the isogeny $\kappa = \kappa_s|_B: B \to \mathrm{Alb}(S)$. The covering ξ_s induces a non-trivial representation of the fundamental group of G_s in the permutation group $S(2)$, or equivalently, a non-trivial homomorphism

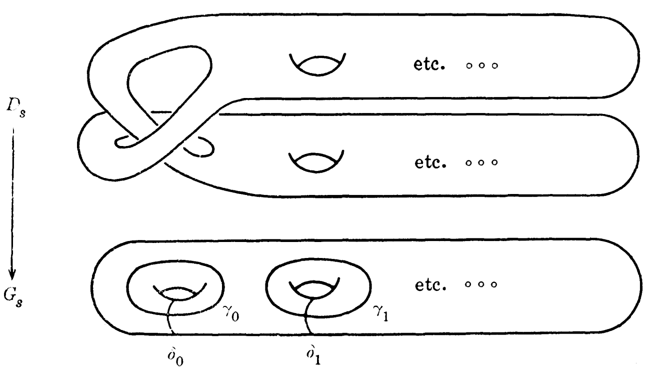
$$\rho: H_1(G_s) \longrightarrow (\mathbf{Z}/2\mathbf{Z}) \; .$$

It follows that there is a basis $\gamma_0, \delta_0, \gamma_1, \delta_1, \cdots, \gamma_5, \delta_5$ for $H_1(G_s)$ such that:

(i) $\rho(\gamma_0) = 1$, $\rho(\gamma_j) = 0$ for $j = 1, \cdots, 5$ and $\rho(\delta_j) = 0$ for $j = 0, \cdots, 5$;

(ii) $(\gamma_j \cdot \gamma_k) = (\delta_j \cdot \delta_k) = 0$, and $(\gamma_j \cdot \delta_k) =$ the Kronecker symbol δ_{jk}.

Such a basis can be represented by cycles in "standard form", that is, each cycle is a connected differentiable submanifold of G_s, any two intersect in at most one point, and all such intersections are transverse. We then have the following picture for the covering ξ_s:



In the obvious way, one constructs a standard basis

$$\{\gamma_0^0, \delta_0^0\} \cup \{\gamma_j^i, \delta_j^i \colon i = 1, 2;\; j = 1, \cdots, 5\}$$

for $H_1(D_s)$ such that:

$$\begin{aligned} &\gamma_s(\gamma_0^0) = \gamma_0^0\,, \quad \gamma_s(\delta_0^0) = \delta_0^0\,, \\ &\gamma_s(\gamma_j^i) = \gamma_j^{i\pm 1}\,, \quad \gamma_s(\delta_j^i) = \delta_j^{i\pm 1} \text{ for } i, j \geqq 1\,; \end{aligned} \tag{11.15}$$

and such that the intersection number of two basis elements in D_s is precisely the same as the intersection number of the basis elements in G_s over which they lie. Then the mapping $\xi_{s*}\colon H_1(D_s) \to H_1(G_s)$ has a kernel which is freely generated by the set

$$\{(\gamma_j^1 - \gamma_j^2)\,,\ (\delta_j^1 - \delta_j^2)\colon\ j = 1, \cdots, 5\}\,. \tag{11.16}$$

Also by (11.13) and the fact that D_s is ample in S, the mapping $\kappa_{s*}\colon H_1(D_s) \to H_1(S)$ takes the set

$$\{\gamma_j^1, \delta_j^1 \colon j = 1, \cdots, 5\} \tag{11.17}$$

onto a basis for $H_1(S)$. By (11.14) and (11.16), however, the abelian variety B is just the subvariety of $\mathrm{Alb}(D_s)$ given by elements of the form

$$\sum \big(a_j(\gamma_j^1 - \gamma_j^2) + b_j(\delta_j^1 - \delta_j^2)\big),\ \ a_j,\ b_j \text{ real.}$$

(Here $H_1(D_s)$ is identified with the lattice U such that $\mathrm{Alb}\,(D_s) = \big(H^{1,0}(D_s)^*/U\big)$.) Therefore the degree of $\kappa\colon B \to \mathrm{Alb}(S)$ must be equal to the order of the group

$$H_1(S)/\kappa_*(\ker \xi_{s*})\,.$$

Using bases (11.16) and (11.17) and the first formula of (11.13), one computes immediately that:

$$\deg(\kappa) = 2^{10}\,. \tag{11.18}$$

THEOREM 11.19. *$\varphi\colon \mathrm{Alb}\,(S) \to J(V)$ is an isomorphism.*

Proof. We have the commutative diagram of isogenies:

$$\begin{tikzcd} & \mathrm{Pic}(S) \arrow[dl, "\mu"'] & \\ B \arrow[dr, "\kappa"'] & & J(V) \arrow[ul, "\lambda"'] \\ & \mathrm{Alb}(S) \arrow[ur, "\varphi"'] & \end{tikzcd}\,.$$

Now the composition $(\kappa\circ\mu\circ\lambda\circ\varphi) = -2$ (identity map) by Lemma 11.8. But $\deg(\kappa) = 2^{10}$. Therefore φ, λ, and μ must all be isomorphisms.

We now wish to complete the list of formulas begun in (9.16) and (9.17). First of all, by (11.4), if $\mathcal{B} = \{\chi, \delta, \eta_1, \cdots, \eta_8\}$ is the basis for $H^1(S)$ used in

§ 9, then the determinant of the matrix

$$\left\{\int_{D_s} \alpha \wedge \beta\right\}_{\alpha\ \beta \in \mathfrak{B}}$$

is 2^{10}. Therefore using (9.10)–(9.12) we can improve on (9.12) (iii), namely we may conclude that:

$$\int_{D_s} \chi \wedge \delta = 2 . \tag{11.20}$$

By Theorem 11.19, under the mapping

$$(\varphi \circ \alpha_s)\colon S \longrightarrow J(V)$$

we can identify $\mathfrak{B}$ with a basis for $H^1(J(V))$. By (9.10), (9.11), (9.12) (ii), (11.20), and (11.4), the polarizing class $\Omega_V = \Omega(\mathcal{J}(V))$ is given on $J(V)$ by:

$$\Omega_V = \left(\chi \wedge \delta + \textstyle\sum_{k=1}^{4} \eta_k \wedge \eta_{4+k}\right) . \tag{11.21}$$

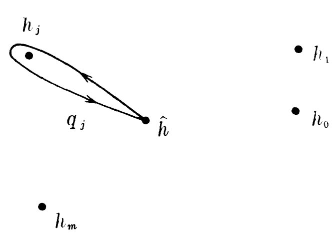
Let us return to the pencil $\{S_h\}_{h \in [R]}$ of divisors on the threefold A_R considered in (7.11) and § 9. Associated to $\{S_h\}_{h \in [R]}$ we have the Lefschetz pencil $\{Y_h\}_{h \in [R]}$ of hyperplane sections of the non-singular cubic fourfold Y. For $P' = ([R] - \{h_0, \cdots, h_m\})$ we have the continuous family of homomorphisms

$$H_1(S_h) \xrightarrow{\varphi_h} H_3(Y_h) \xrightarrow{\lambda_h} H_3(S_h) \quad h \in P' , \tag{11.22}$$

which by Theorem 11.9 and Lemma 11.6 must be isomorphisms. Let

$$T_{V,j}\colon Y_{\hat{h}} \longrightarrow Y_{\hat{h}}$$
$$T_{S,j}\colon S_{\hat{h}} \longrightarrow S_{\hat{h}}$$

be the *Picard-Lefschetz diffeomorphisms* (see [5; pages 42–43]) associated to the path q_j in P' as shown:



According to theorems of Lefschetz (*Analysis Situs*. Paris: Gauthier-Villars, 1950, pages 93 and 106–08), there is a "vanishing cycle" $\delta_j \in H_3(Y_{\hat{h}})$ associated to each $j = 0, \cdots, m$ such that:

(i) $\{\delta_j\}_{j=0,\cdots,m}$ generate $H_3(Y_{\hat{h}})$ (since $H_3(Y) = 0$);

(ii) the vanishing cycles are all conjugate under the action of $\pi_1(P')$ on $H_3(Y_{\hat{h}})$ induced by the Picard-Lefschetz diffeomorphisms (see end of § 5);

(iii) $(T_{V,j})_*(\alpha) = \alpha \pm (\alpha \cdot \delta_j)\delta_j$ for all $\alpha \in H_3(Y_{\hat{h}})$.

Thus the fundamental group $\pi_1(P')$ acts irreducibly on $H_3(Y_{\hat{h}})$ and so by using (11.22), it also acts irreducibly on $H_1(S_{\hat{h}})$ and also

LEMMA 11.23. *$\pi_1(P')$ acts irreducibly on $H^1(S_{\hat{h}}) \otimes \mathbf{C}$.*

Choosing a basepoint $s_0 \in S_Y$ such that $(s_0, h) \in S_h$ for all $h \in [R]$ and letting $D_{s_0,h}$ be the incidence divisor for s_0 in S_h we have clearly that $(T_j)_*(\{D_{s_0,\hat{h}}\}) = \{D_{s_0,\hat{h}}\} \in H_2(S_{\hat{h}})$ for all $j = 0, \cdots, m$. Thus if $\xi_{\hat{h}} \in H^2(S_{\hat{h}})$ is the Poincaré dual of $\{D_{s_0,\hat{h}}\}$ on $S_{\hat{h}}$:

(11.24) $\xi_{\hat{h}}$ is invariant under the action of $\pi_1(P')$.

Since the Picard-Lefschetz diffeomorphisms respect the intersection pairing on $Y_{\hat{h}}$ we have that Ω_V is invariant under the automorphism induced on $J(V_{\hat{h}})$ by T_j for each j so that again using (11.22):

(11.25) $\xi_0 = (\varphi \circ \alpha_s)^*(\Omega_V)$ is invariant under the action of $\pi_1(P')$.

For $S = S_{\hat{h}}$ the form

$$B_1\colon \big(H^1(S) \otimes \mathbf{C}\big) \times \big(H^1(S) \otimes \mathbf{C}\big) \longrightarrow \mathbf{C}$$
$$(\alpha, \alpha') \longmapsto \int_S \alpha \wedge \bar{\alpha}' \wedge \xi_{\hat{h}}$$

is non-degenerate. Let $B_2(\alpha, \alpha') = \int_S \alpha \wedge \bar{\alpha}' \wedge \xi_0$. Since $(\varphi \circ \alpha_s)$ is an immersion (see beginning of § 12), B_2 is also non-degenerate. There is certainly some real number $c \neq 0$ such that $B_1 + cB_2$ is degenerate. Let $A = \{\alpha \in (H^1(S) \otimes \mathbf{C})\colon B_1(\alpha, \alpha') + cB_2(\alpha, \alpha') = 0$ for all $\alpha' \in H^1(S)\}$. By (11.24) and (11.25), A is an invariant subspace of $H^1(S) \otimes \mathbf{C}$ so that by Lemma 11.23

$$B_1 + cB_2 = 0 \ . \tag{11.26}$$

Since B_1 and B_2 take integral values on $H^1(S)$, c is in fact a rational number. By (10.14) therefore

$$\xi_{\hat{h}} = -c\xi_0$$

and

$$5 = (\{D_s\} \cdot \{D_s\}) = \int_{D_s} \xi_{\hat{h}} = -c \int_{D_s} \xi_0 = -c \cdot 10$$

by Lemma 11.5. Thus:

LEMMA 11.27. *On S, the Poincaré dual of $\{D_s\}$ is the class*

$$(1/2)\big((\chi \wedge \delta) + \textstyle\sum_{k=1}^{4} \eta_k \wedge \eta_{4+k}\big) \ .$$

We are now in a position to complete the list of integration formulas on

S begun in § 9. For convenience we list those which we have derived up to this point:

$$\begin{aligned}
&\int_{D_s} \chi \wedge \delta = 2 = \int_{D_s} \eta_k \wedge \eta_{4+k} && \text{for } k = 1, \cdots, 4\,; \\
(11.28) \qquad &\int_{D_s} \chi \wedge \eta_k = 0 = \int_{D_s} \delta \wedge \eta_k && \text{for } k = 1, \cdots, 8\,; \\
&\int_{D_s} \eta_k \wedge \eta_l = 0 && \text{for } l > k\,,\ \ l \neq k + 4\,.
\end{aligned}$$

Also we know that

$$\begin{aligned}
&\int_S \eta_k \wedge \eta_{4+k} \wedge \eta_l \wedge \eta_{4+l} = 1 && \text{for } 1 \leqq k < l \leqq 4\,; \\
&\int_S \eta_p \wedge \eta_q \wedge \eta_r \wedge \eta_s = 0 && \text{for other } p, q, r, s\,; \\
(11.29) \qquad &\int_S \chi \wedge \eta_k \wedge \eta_l \wedge \eta_m = 0 && \text{for all } k, l, m\,; \\
&\int_S \chi \wedge \delta \wedge \eta_k \wedge \eta_l = 0 && \text{for } l > k\,,\ \ l \neq 4 + k\,.
\end{aligned}$$

Lemma 11.27 allows us to improve on (9.17) and conclude:

$$(11.30) \qquad \int_S \chi \wedge \delta \wedge \eta_k \wedge \eta_{4+k} = 1 \qquad \text{for } 1 \leqq k \leqq 4\,.$$

Finally, since $\int_{D_s} \delta \wedge \eta_k = 0$ for all k, Lemmas 9.18 and 11.27 give that:

$$(11.31) \qquad \int_S \delta \wedge \eta_k \wedge \eta_l \wedge \eta_m = 0 \qquad \text{for all } k, l, m\,.$$

12. The Gauss map and the tangent bundle theorem

From (10.14) we have that

$$(12.1) \qquad H^{2,0}(S) \approx H^{1,0}(S) \wedge H^{1,0}(S)\,.$$

Thus the mappings $\alpha_s\colon S \to \mathrm{Alb}(S)$ defined in (4.6) are immersions since by Lemma 10.13 the linear system of canonical divisors on S has no basepoints. Let $\mathrm{Gr}\,(2, T(\mathrm{Alb}(S), 0))$ be the Grassmann variety of two-dimensional subspaces of the tangent space to $\mathrm{Alb}(S)$ at the basepoint 0. Now each isomorphism

$$(12.2) \qquad T(\mathrm{Alb}(S), 0) \approx \mathbf{C}^5$$

induces an associated isomorphism

$$\mathrm{Gr}\,(2, T(\mathrm{Alb}(S), 0)) \approx \mathrm{Gr}\,(2, 5)\,.$$

Also there is a rational mapping

(12.3) $$\mathcal{G}\colon \alpha_s(S) \longrightarrow \operatorname{Gr}(2, T(\operatorname{Alb}(S), 0))$$

defined by assigning to $u \in \alpha_s(S)$ the subspace $T((\alpha_s(S) - u), 0)$ of $T(\operatorname{Alb}(S), 0)$. $\mathcal{G}$ is called the *Gauss map* associated to $\alpha_s(S)$ and the purpose of this chapter is to prove that, for appropriate choice of isomorphism (12.2), there is a commutative diagram:

(12.4)
$$\begin{tikzcd} S \arrow[r, "\alpha_s"] \arrow[d, "\varepsilon"] & \alpha_s(S) \arrow[d, "\mathcal{G}"] \\ \operatorname{Gr}(2,5) \approx \operatorname{Gr}(2, T(\operatorname{Alb}(S), 0)) & \end{tikzcd}$$

where ε is the standard inclusion of S (see (7.1)). Notice that (12.1) and Lemma 10.13 give immediately that if $\xi\colon \operatorname{Gr}(2,5) \to \mathbf{P}_9$ is the Plücker embedding then for any choice in (12.2):

LEMMA 12.5. *There is an automorphism σ of $\mathbf{P}_9$ such that $(\xi\circ\mathcal{G}\circ\alpha_s) = (\sigma\circ\xi\circ\varepsilon)$.*

The stronger fact (12.4) is considerably harder to prove. Since it is the central geometric fact of the paper, we will present two proofs, the first an analytic proof using residues and the second an algebro-geometric proof. The first proof is shorter and more direct, and in the second we will derive some geometric facts which will in any case be useful in § 13.

Let $A_2^4(V)$ be the vector space of rational four-forms on $\mathbf{P}_4$ with a pole of order two along V. If $E = \mathbf{C}^5$, there is an isomorphism

(12.6) $$\omega\colon E^* \longrightarrow A_2^4(V)$$

defined by putting

$$\omega(H) = (H(y_0, \cdots, y_4)\Omega(y_0, \cdots, y_4))/F(y_0, \cdots, y_4)^2$$

where $F(X_0, \cdots, X_4)$ is the defining polynomial for V and

$$\Omega(y_0, \cdots, y_4) = \sum (-1)^j y_j (dy_0 \wedge \cdots \wedge \widehat{dy_j} \wedge \cdots \wedge dy_4)\ .$$

By [8; page 488], the *Poincaré residue operator* induces an isomorphism:

(12.7) $$\sigma\colon A_2^4(V) \longrightarrow H^{2,1}(V)$$

since $h^{3,0}(V) = 0$. Combining these with the isomorphism

$$\varphi^*\colon H^{2,1}(V) \longrightarrow H^{1,0}(S)$$

(see § 2 and § 11), we get an isomorphism:

(12.8) $$\rho\colon E^* \longrightarrow H^{1,0}(S)\ .$$

Thus for each $s \in S$, ρ gives a linear mapping:

$$\rho_s \colon E^* \longrightarrow T^*(S, s) .$$

On the other hand, $L_s \subseteqq V$ is the "projectification" of a two-dimensional subspace J_s of $\mathbf{C}^5$. Write:

$$J_s^\perp = \{H \in E^* \colon H|_{J_s} \equiv 0\} .$$

If we identify E^* with $\big(T(\mathrm{Alb}(S), 0)\big)^*$ under (12.8) then the commutativity of (12.4) is an immediate corollary of:

PROPOSITION 12.9. *The sequence*

$$0 \longrightarrow J_s^\perp \longrightarrow E^* \xrightarrow{\rho_s} T^*(S, s) \longrightarrow 0$$

is exact for each $s \in S$.

Proof. (12.1) and Proposition 10.3 imply that ρ_s is onto for each $s \in S$. So we must show that if $H|_{J_s} \equiv 0$ then $\rho(H)$ vanishes at s for all s such that

$$L_s \subseteqq [h]$$

where $h \in \mathbf{P}_4^*$ is the element given by $H(X_0, \cdots, X_4) = 0$. We assume that $\big([h] \cap V\big)$ is non-singular, since it will suffice to treat this case, the generic one. Now as in [8; equation (10.8)], there is a residue isomorphism

$$R \colon A_2^4(V) \longrightarrow H^1(V; \hat{\Omega}_V^2) \tag{12.10}$$

where $\hat{\Omega}_V^2$ is the sheaf of closed holomorphic two-forms on V. For generic H we will explicity construct the mapping

$$(R \circ \omega) \colon E^* \longrightarrow H^1(V; \hat{\Omega}_V^2)$$

and then relate it geometrically to the mapping ρ of (12.8). Let $z_1, \cdots, z_4$ be holomorphic local coordinates in $\mathbf{\Gamma}_4$ around a point of $\big(V \cap [h]\big)$ such that:

(i) $[h]$ is given by $z_1 = 0$;

(ii) V is given by $z_4 = 0$.

Locally $\omega(H) = \big(z_1 f(z) dz_1 \wedge \cdots \wedge dz_4/z_4^2\big)$ where f is holomorphic. If we choose $g(z)$ such that

$$\partial g/\partial z_3 = \partial f/\partial z_4 ,$$

then locally

$$\omega(H) = d\Big(\big(z_1 g(z) dz_1 \wedge dz_2 \wedge dz_4 - z_1 f(z) dz_1 \wedge dz_2 \wedge dz_3\big)/z_4\Big) .$$

Next choose a finite covering $\{U_k\}$ of a neighborhood of V in $\mathbf{P}_4$ such that on each U_k:

$\omega(H) = d\eta_k$ where η_k is holomorphic on $(U_k - V)$, vanishes on $\big([h] \cap U_k\big)$ and has first order pole along $(V \cap U_k)$.

If we define $\psi_{jk} = (\eta_j - \eta_k)$ on $(U_j \cap U_k)$, then each ψ_{jk} has the following properties:

(i) $d\psi_{jk} = 0$;

(ii) ψ_{jk} vanishes on $([h] \cap U_{jk})$;

(iii) ψ_{jk} is either identically zero on a component U'_{jk} of U_{jk} or else ψ_{jk} has a first order pole along $(V \cap U'_{jk})$.

Any meromorphic three-form ψ satisfying (i)–(iii) may be expressed (on the corresponding U) in the form:

$$\psi = \big((z_1\alpha \wedge dz_4)/z_4\big) + (z_1\beta/z_4)$$

where α and β do not involve dz_4. Since $d\psi = 0$, we must have in fact that $z_1\beta/z_4$ is *holomorphic* on U. Then the *Poincaré residue* ([8; § 10]) of ψ is given by:

$$R(\psi) = z_1\alpha\,|_{(V \cap U)}$$

which is therefore a closed holomorphic two-form on $(V \cap U)$ which vanishes on $([h] \cap V \cap U)$. The element

$$(R\circ\omega)(H) \in H^1(V;\, \hat{\Omega}^2_V)$$

is then represented by the cocycle:

(12.11) $$\{R(\psi_{jk})\}\ .$$

The important thing about this cocycle is that it vanishes along $([h] \cap V)$. We can now finish the proof of Proposition 12.9 by proving the following:

(12.12) For generic $H \in E^*$, if $[h]$ is the hyperplane defined by H and $L_s \subseteqq [h]$, then for each $\tau \in T(S, s)$ the contraction $\langle \tau, \rho_s(H)\rangle = 0$.

From [15; page 150], there is a natural isomorphism for each $s \in S$:

(12.13) $$\eta\colon T(S, s) \longrightarrow H^0\big(L_s;\, \mathcal{O}(N(V, L_s))\big)$$

where $N(V, L_s)$ denotes as before the normal bundle. There is a contraction

(12.14) $$H^0\big(L_s;\, \mathcal{O}(N(V, L_s))\big) \times H^1(V;\, \hat{\Omega}^2_V) \longrightarrow H^1(L_s;\, \Omega^1_{L_s})$$

defined as follows. If $\eta_0 \in H^0\big(L_s;\, \mathcal{O}(N(V, L_s))\big)$, choose on $(U_{jk} \cap L_s)$ a representative

$$\mu_{jk} \in H^0\big((U_{jk} \cap L_s);\, \mathcal{O}(T(V)\,|_{(U_{jk}\cap L_s)})\big)$$

for $\eta_0\,|_{(U_{jk}\cap L_s)}$. If $\omega = \{\omega_{jk}\} \in H^1(V;\hat{\Omega}^2_V)$, let ξ_{jk} be the contraction

$$\langle \mu_{jk},\, \omega_{jk}\,|_{U_{jk}\cap L_s}\rangle\ .$$

Then ξ_{jk} is an element of $H^0(U_{jk} \cap L_s;\, \Omega^1_V\,|_{U_{jk}\cap L_s})$ and so gives an element $\xi^*_{jk} \in H^0(U_{jk\cap L_s};\, \Omega^1_{(U_{jk}\cap L_s)})$ (where Ω^1 denotes the sheaf of holomorphic one-forms). Then $\{\xi^*_{jk}\} \in H^1(L_s;\, \Omega^1_s)$ depends only on η_0 and ω. Putting $\langle \eta_0, \omega\rangle =$

$\{\xi_{jk}^*\}$, we have the pairing (12.14). Furthermore, for suitable choice of isomorphism

$$H^1(L_s;\, \Omega^1_{L_s}) \approx \mathbf{C}\ ,$$

the mapping (12.13) of Kodaira satisfies at s the identity:

$$\langle \tau,\, (\varphi^* \circ \sigma)(\Omega)\rangle = \langle \eta(\tau),\, R(\Omega)\rangle \tag{12.15}$$

for $\tau \in T(S, s)$ and $\Omega \in A^4_2(V)$. (See (12.7), [8], and [15].) Finally if $\Omega = \omega(H)$ for generic $H \in E^*$ and if $H|_{J_s} \equiv 0$, then using representative (12.11) for $R(\omega(H))$ the right hand side of (12.15) trivially vanishes at s for all $\tau \in T(S, s)$. Thus

$$\langle \tau,\, (\varphi^* \circ \sigma \circ \omega)(H)\rangle = 0$$

which proves (12.12) and hence the proposition.

This then is the analytic proof of the commutativity of (12.14). Before turning to an algebro-geometric discussion, we remark that the proof of Proposition 12.9 can be generalized to give the following:

Let V be a smooth hypersurface of dimension $(2n + 1)$ in a smooth projective variety X. Suppose that $\{Z_s\}_{s \in S}$ is an *algebraic family of algebraic n-cycles* on V. As above we have mappings

$$\begin{array}{ccc} A^{2n+2}_{n+1}(V) & \xrightarrow{\quad R \quad} & H^n(V; \hat{\Omega}^{n+1}_V) \\ \big\downarrow{\scriptstyle \sigma} & & \\ H^{2n+1,0}(V) \oplus \cdots \oplus H^{n+1,n}(V) & \xrightarrow{\ \varphi^*\ } & H^{1,0}(S)\ . \end{array}$$

Then if $\Omega \in A^{2n+2}_{n+1}(V)$ and $R(\Omega)$ is zero on Z_s,

$$(\varphi^* \circ \sigma)(\Omega)\,|_s = 0\ .$$

We now return again to our point of departure just after Lemma 12.5. We proceed by a different route, deriving a series of geometric lemmas.

LEMMA 12.16. *Let* $(S \times S)^0 = ((S \times S) - (\text{diagonal}))$. *For* $(s, t) \in (S \times S)^0$, *define*:

$$\Phi(s, t) = \begin{cases} ([L_s] \cap [L_t]),\ \textit{the hyperplane spanned} \\ \qquad \textit{by } L_s \cup L_t,\ \textit{if}\ (L_s \cap L_t) = \varnothing\ ; \\ \mathfrak{D}(x)\ \ \textit{if}\ \{x\} = (L_s \cap L_t)\ \ (\text{see § 5})\ . \end{cases}$$

Then $\Phi\colon (S \times S)^0 \to \mathbf{P}^*_4$ *is analytic.*

Proof. If suffices to check that if C is a non-singular curve in $(S \times S)^0$ and (s_0, t_0) is an isolated point of

$$C \cap \{(s, t)\colon (L_s \cap L_t) \neq \varnothing\}$$

and $L_{s_0} \cap L_{t_0} = \{x\}$, then

$$\operatorname{limit}_{(s,t)\in C,(s,t)\to(s_0,t_0)} \Phi(s, t) = \mathfrak{D}(x) .$$

Let $h_0 = \operatorname{limit} \Phi(s, t)$. Following [18; page 2], let y be any point in $[h_0]$ such that y does not lie in the plane spanned by L_{s_0} and L_{t_0}. Then if L_0 is a line in $\mathbf{P}_4$ which meets $[h_0]$ at y, and if (s, t) is near (s_0, t_0), there is a unique line $L_{(s,t)}$ meeting L_0, L_s and L_t. If $y \notin V$, then as (s, t) approaches (s_0, t_0), the two points given by $(L_s \cap L_{(s,t)})$ and $(L_t \cap L_{(s,t)})$ must both approach the point $x \in (L_{s_0} \cap L_{t_0})$. Thus a generic line in $[h_0]$ which passes through x has multiple contact with V at x, and so x must be a singular point of $([h] \cap V)$. So $h_0 = \mathfrak{D}(x)$ and the lemma is proved.

LEMMA 12.17. *Let* $\nu: E \to S \times S$ *be an analytic mapping of the unit disc into* $S \times S$ *such that for* $z \in E$, $z \neq 0$, $\nu(z) \notin$ *(diagonal of* $S \times S$*) but* $\nu(0) = (s_0, s_0)$. *Then* $\operatorname{limit}_{z\to 0} \Phi(\nu(z)) \in \mathfrak{D}(L_{s_0})$.

Proof. Let $h_0 = \operatorname{limit} \Phi(\nu(z))$. If the lemma is false, $(V \cap [h_0])$ is non-singular in a neighborhood of L_{s_0}. So there is a tubular neighborhood U of L_{s_0} in $\mathbf{P}_4$ such that, if z is near 0 and $(s, t) = \nu(z)$, U contains $(L_s \cup L_t)$ and $(U \cap V \cap [\Phi(s, t)])$ is diffeomorphic to $(U \cap V \cap [h_0])$. But in $(V \cap [\Phi(s, t)])$, $(\{L_s\}\cdot\{L_s\}) = -1$ and $(\{L_s\}\cdot\{L_t\}) \geqq 0$, so that it is impossible that both L_s and L_t go to L_{s_0} in $(V \cap [h])$. This gives the lemma.

Let $I \subseteqq (S \times S)$ denote as in (2.4) the incidence divisor for the family $\{L_s\}$. For $(s, s') \in I$, in order that (s, s') be a singular point of I it is necessary that s be a singular point of $D_{s'}$ *and* s' be a singular point of D_s. By condition (10.6), this can only happen if $i = s'$ and $(V\cdot[\mathfrak{D}(L_s)]) = 3L_s$. (Recall that $[\mathfrak{D}(L_s)]$ is the plane tangent to V along L_s.) By Lemma 10.15 therefore:

LEMMA 12.18. *I is non-singular except possibly at a finite set of points lying on the diagonal of* $S \times S$.

Let γ_s be as in (11.11) and define:

$$I' = \{(s, s') \in I\colon \gamma_{s'}(s) \neq s\} \; ; \qquad (12.19)$$
$$\pi\colon I' \to S \quad \text{given by} \quad \pi(s, s') = s \; .$$

Thus if L_s is of first type, $\pi^{-1}(s) = (\{s\} \times D_s)$. If L_s is of second type, $\pi^{-1}(s) = (\{s\} \times (D_s - \{t\}))$ where $(2L_s + L_t)$ is a plane section of V. Also, if $(s, s') \in I'$, s is a non-singular point of $D_{s'}$.

LEMMA 12.20. *If* $(s, s') \in I'$, $\operatorname{limit}_{t\in D_{s'},t\to s} \Phi(s, t) = \mathfrak{D}(L_s \cap L_{\gamma_{s'}(s)})$.

Proof. Let $s_1 = \gamma_{s'}(s)$. Then $s_1 \neq s$. If $s_1 \neq s'$, then $(L_t \cap L_{s_1}) = \varnothing$ for all $t \in D_{s'}$, t near s, since V contains no planes. By Lemma 12.16,

$\text{limit}_{t\in D_{s'},t\to s}\,\Phi(s_1, t) = (L_{s_1} \cap L_s)$. But for t near s, $\Phi(s_1, t) = \Phi(s, t)$, and so we are done in case $s_1 \neq s'$. If $s_1 = s'$, let K be the plane spanned by L_s and $L_{s'}$. K is tangent to V along $L_{s'}$ so that $K = [\mathfrak{D}(L_{s'})]$. Also $K \subseteqq [\Phi(s, t)]$ for $t \in D_{s'}$, t near s. Thus $\text{limit}\,\Phi(s, t) \in \mathfrak{D}(L_{s'})$. But by Lemma 12.17, $\text{limit}\,\Phi(s, t) \in \mathfrak{D}(L_s)$, and $(\mathfrak{D}(L_s) \cap \mathfrak{D}(L_{s'})) = \mathfrak{D}(L_s \cap L_{s'})$ since, if $x \in L_s$, $y \in L_{s'}$, and $\mathfrak{D}(x) = \mathfrak{D}(y)$, then the line through x and y lies in V and so must be either L_s or $L_{s'}$. The lemma is therefore proved.

We define two morphisms on I':

(12.21)
$$\begin{aligned}\rho\colon I' &\longrightarrow V\\ (s, s') &\longmapsto (L_s \cap L_{\gamma_{s'}(s)})\ .\end{aligned}$$

(12.22) Let $\tau\colon \mathbf{P}(S) \to S$ be the projective line bundle whose fibre at s is $\mathrm{Gr}(1, T(S, s))$ and put:

$$\begin{aligned}\chi\colon I' &\longrightarrow \mathbf{P}(S)\\ (s, s') &\longmapsto \{T(D_{s'}, s)\}\ .\end{aligned}$$

LEMMA 12.23. *Suppose L_s is a line of first type. Then for (s, s_1) and $(s, s_2) \in I'$: $\rho(s, s_1) = \rho(s, s_2)$ if and only if $\chi(s, s_1) = \chi(s, s_2)$.*

Proof. If C is a non-singular curve in S and $s \in C$ then by Lemma 10.7, $(L_t \cap L_s) = \varnothing$ for $t \in C$, t near s. Furthermore if we put the equation for V into normal form with respect to L_s as in (6.9) and let $B(\alpha_0, \alpha_1)$ be the curves in $\mathrm{Gr}(2, 5)$ defined as in the discussion preceding Lemma 6.18 then there is a unique $(\alpha_0, \alpha_1) \in \mathbf{P}_1$ such that C and $B(\alpha_0, \alpha_1)$ are tangent at s. For that particular (α_0, α_1):

$$\text{limit}_{t\in C,t\to s}\,\Phi(s, t) = \text{limit}_{t\in B(\alpha_0,\alpha_1),t\to s}\,\Phi(s, t)$$

(where $\Phi(s, t)$ is the hyperplane spanned by L_s and L_t). Using Lemma 6.18 and the fact that for all $t \in B(\alpha_0, \alpha_1)$, $\Phi(s, t) = h(\alpha_0, \alpha_1)$, we have that:

$$\chi(s, s_1) = \chi(s, s_2) \text{ if and only if } \mathfrak{D}(\rho(s, s_1)) = \mathfrak{D}(\rho(s, s_2))\ .$$

Since $\mathfrak{D}|_{L_s}$ is injective, we have the lemma.

LEMMA 12.24. *Suppose that L_s is a line of second type. If (s, s_1) and $(s, s_2) \in I'$, then $\mathfrak{D}(\rho(s, s_1)) = \mathfrak{D}(\rho(s, s_2))$ if and only if*

$$\mathfrak{D}(L_s \cap L_{s_1}) = \mathfrak{D}(L_s \cap L_{s_2})\ .$$

(Notice that if, for example, $s = s_1$ then $(L_{s_1} \cap L_s)$ should be interpreted as $\text{limit}_{t\in D_{s_1},t\to s}\,(L_t \cap L_s)$ which exists since D_s is non-singular at s.)

Proof. Suppose $(s, s') \in I'$ and $s \neq s'$. By Lemma 12.17, if $h = \text{limit}_{t\in D_{s'},t\to s}\,\Phi(s, t)$ then $[\mathfrak{D}(L_s)] \subseteqq [h]$. Clearly $L_{s'} \subseteqq [h]$ also. Since $L_{s'} \not\subseteqq$

$[\mathfrak{D}(L_s)]$ by the definition of I', $h = \mathfrak{D}(L_s \cap L_{s'})$, the tangent hyperplane to V at $(L_{s'} \cap [\mathfrak{D}(L_s)])$. By Lemma 12.20 therefore:

$$\mathfrak{D}(L_s \cap L_{s'}) = \mathfrak{D}(L_s \cap L_{\gamma_{s'}(s)}). \tag{12.25}$$

By continuity, (12.25) continues to hold if $s = s'$. The lemma now follows directly from Lemma 12.20.

If L_s is of second type, we see by (6.15) that the tangent directions to S at s are given by the Schubert varieties:

$$E_x = \{t \in \mathrm{Gr}(2, 5)\colon x \in L_t \subseteqq [\mathfrak{D}(L_s)]\} \quad \text{for } x \in L_s\,. \tag{12.26}$$

LEMMA 12.27. *Let L_s be a line of second type and let $(s, s') \in I$. Then if $x = (L_s \cap L_{s'})$, $D_{s'}$ and E_x are tangent at s.*

Proof. Let $\pi_G\colon M \to \mathrm{Gr}(2, 5)$ be as in the proof of Lemma 10.18. Then if $D_{s'}$ and E_y are tangent at s, we have as in that proof: $\pi_{\mathbf{P}_4}\colon \pi_G^{-1}(D_{s'}) \to \mathbf{P}_4$ is not of maximal rank at (s, y). On the other hand, if $y' \in L_s$, $y' \neq y$, then $\pi_{\mathbf{P}_4}\colon \pi_G^{-1}(E_y) \to \mathbf{P}_4$ is clearly of maximal rank at (s, y') and so $\pi_{\mathbf{P}_4}\colon \pi_G^{-1}(D_{s'}) \to \mathbf{P}_4$ must also be of maximal rank at (s, y') since $\pi_G^{-1}(E_y)$ and $\pi_G^{-1}(D_{s'})$ are tangent along $(\{s\} \times L_s)$. But if $s \neq s'$ and $\{x\} = (L_s \cap L_{s'})$, $\pi_{\mathbf{P}_4}\colon \pi_G^{-1}(D_{s'}) \to \mathbf{P}_4$ cannot be of maximal rank at (s, x) since non-degeneracy at (s, x) would imply that:

$$(\pi_{\mathbf{P}_4})_*\big(T(\pi_G^{-1}(D_{s'}), (s, x))\big) \subseteqq T([\mathfrak{D}(L_s)], x)$$

which in turn would imply that $L_{s'} \subseteqq [\mathfrak{D}(L_s)]$. This cannot be since $(s, s') \in I'$. The lemma is therefore proved if $s \neq s'$. The case $s = s'$ follows by a continuity argument.

COROLLARY 12.28. *If L_s is of second type and (s, s_1) and $(s, s_2) \in I'$, $\chi(s, s_1) = \chi(s, s_2)$ if and only if $(L_s \cap L_{s_1}) = (L_s \cap L_{s_2})$. Also $\mathfrak{D}(\rho(s, s_1)) = \mathfrak{D}(\rho(s, s_2))$ if and only if $(L_s \cap L_{s_1})$ and $(L_s \cap L_{s_2})$ go to the same point under $\mathfrak{D}\colon L_s \to \mathbf{P}_4^*$.*

By Lemma 12.23 and Corollary 12.28, there is a unique rational map

$$\theta^*\colon \mathbf{P}(S) \longrightarrow V^* = \mathfrak{D}(V)$$

such that the diagram

$$\begin{array}{ccc} I' & \xrightarrow{\rho} & V \\ \big\downarrow{\scriptstyle\chi} & & \big\downarrow{\scriptstyle\mathfrak{D}} \\ \mathbf{P}(S) & \xrightarrow{\theta^*} & V^* \end{array}$$

is commutative. If L_s is of first type, then by Lemma 12.23 and the injectivity of $\mathfrak{D}|_{L_s}$, we have that

$$\theta^*\colon \tau^{-1}(s) \longrightarrow \mathfrak{D}(L_s) \tag{12.29}$$

is an isomorphism. If L_s is of second type then by Corollary 12.28, θ^* is defined at a generic point of $\tau^{-1}(s)$ and

$$\theta^*: \tau^{-1}(s) \longrightarrow \mathfrak{D}(L_s)$$

is generically two-to-one. Since $\mathfrak{D}$: $V \to V^*$ is generically injective, there is a unique lifting θ: $\mathbf{P}(S) \to V$ of θ^*. By Zariski's Main Theorem ([10; pages 43–48]) and the fact that $\mathfrak{D}$: V $\to V^*$ is finite-to-one, θ is defined at each point at which θ^* is defined. Since $\mathfrak{D}$ is generically injective, $\theta(\tau^{-1}(s)) \subseteqq \bar{L}_s$. If L_s is of first type, $\theta^* |_{\tau^{-1}(s)}$ and $\mathfrak{D}|_{L_s}$ are isomorphisms so that

$$\theta: \tau^{-1}(s) \longrightarrow L_s \tag{12.30}$$

is an isomorphism. If L_s is of second type, $\mathfrak{D}|_{L_s}$ and $\theta^*|_{\tau^{-1}(s)}$ are both generically two-to-one so that the mapping (12.30) is an injection.

PROPOSITION 12.31. *The mapping* θ_S: $\mathbf{P}(S) \to T_V$ *defined by* $\theta_S(b) = (\tau(b), \theta(b))$ *is an isomorphism of projective line bundles over S.*

Proof. By the preceding discussion θ_S is an injective, fibre-preserving map which is defined at a generic point of each fibre of τ: $\mathbf{P}(S) \to S$. Using the fact that any such map is determined in the neighborhood of a fibre by its values on three distinct local sections of τ: $\mathbf{P}(S) \to S$, the proposition follows.

We are now in a position to derive a proof of the commutativity of (12.4). Using the immersion

$$\alpha_{s_0}: S \longrightarrow \mathrm{Alb}(S)$$

we define a morphism

$$\theta_1: \mathbf{P}(S) \longrightarrow \mathrm{Gr}\,(1,\ T(\mathrm{Alb}(S),\ 0))$$

by associating to $b \in \mathbf{P}(S)$ the subspace

$$T((\alpha_{s_0}(C) - \alpha_{s_0}(s)),\ 0)$$

where C is any curve which passes through $s = \tau(b)$ with direction b. By Proposition 12.31, θ: $\mathbf{P}(S) \to V$ is a morphism and we assert:

LEMMA 12.32. *For* b, $b' \in \mathbf{P}(S)$, *if* $\theta(b) = \theta(b')$ *then* $\theta_1(b) = \theta_1(b')$.

Proof. It suffices to check this for generic b and b'. Let $\tau(b) = s$, $\tau(b') = s'$. If $s = s'$ the lemma is trivial. If $s \neq s'$ and $x = \theta(b) = \theta(b')$, then $x \in \theta(\tau^{-1}(s)) = L_s$ and $x \in \theta(\tau^{-1}(s')) = L_{s'}$. Let $t = \gamma_s(s') = \gamma_{s'}(s)$. Then $x = \rho(s, t) = (\theta \circ \chi)(s, t) = \theta(\{T(D_t, s)\})$. But $x = \theta(b)$ and $\theta|_{\tau^{-1}(s)}$ is injective. Thus $b = \{T(D_t, s)\}$. Similarly $b' = \{T(D_t, s')\}$. Now by (11.13), $(\alpha_{s_0}|_{D_t} \circ \gamma_t) = -\alpha_{s_0}|_{D_t} +$ (constant). So $\theta_1(\{T(D_t, s)\}) = \theta_1(\{T(D_t, s')\})$ and the lemma is proved.

By Lemma 12.32, there is an induced mapping

$$\lambda\colon V \longrightarrow \mathrm{Gr}(1,\ T(\mathrm{Alb}(S),\ 0))$$

such that $(\lambda\circ\theta) = \theta_1$. We then have the commutative diagram

(12.33)
$$\begin{tikzcd}
T_V \arrow[d, "\pi_V"'] & \mathbf{P}(S) \arrow[l, "\theta_S"'] \arrow[dl, "\theta"'] \arrow[d, "\theta_1"] \\
V \arrow[r, "\lambda"] \arrow[d, phantom, sloped, "\subseteq"] & \mathrm{Gr}(1,\ T(\mathrm{Alb}(S),\ 0)) \arrow[d, phantom, sloped, "\simeq"] \\
\mathbf{P}_4 & \mathbf{P}_4
\end{tikzcd}$$

By its definition, θ_1 induces a mapping of S into $\mathrm{Gr}(2,\ T(\mathrm{Alb}(S),\ 0))$ and using (12.3):

(12.34) The mapping $S \to \mathrm{Gr}(2,\ T(\mathrm{Alb}(S),\ 0))$ is precisely the composition $(\mathcal{G}\circ\alpha_s)$.

By Lemmas 12.5 and 10.2:

(12.35) $\theta_1(\mathbf{P}(S))$ is contained in no hyperplane of $\mathrm{Gr}(1,\ T(\mathrm{Alb}(S),\ 0))$.

If H is a generic hyperplane $\subseteqq \mathrm{Gr}(1,\ T(\mathrm{Alb}(S),\ 0))$:

$$0 < (\{L_s\}\cdot\{\lambda^{-1}(H)\})_V \leqq (\{\tau^{-1}(s)\}\cdot\{\theta_1^{-1}(H)\})_{\mathbf{P}(s)} = 1 .$$

Therefore if V_h denotes as before a hyperplane section of V:

(12.36) λ is induced by sections of the bundle $L(V_h)$.

But (12.35) and (12.36) taken together imply that λ is induced by the *entire* five-dimensional vector space of sections of $L(V_h)$. (See Lemma (A. 1) of Appendix.) So (12.33) can be completed by an appropriate isomorphism between the two copies of $\mathbf{P}_4$ appearing in the diagram. Thus by (12.34):

THEOREM 12.37. *For proper choice of isomorphism* (12.2), *the following diagram is commutative*:

$$\begin{tikzcd}
S \arrow[r, "\alpha_s"] \arrow[d, "\varepsilon"] & \alpha_s(S) \arrow[d, "\mathcal{G}"] \\
\mathrm{Gr}(2,\ 5) \approx & \mathrm{Gr}\,(2,\ T(\mathrm{Alb}(S),\ 0)) .
\end{tikzcd}$$

Since $\alpha_s\colon S \to \mathrm{Alb}\,(S)$ is an immersion, $(G\circ\alpha_s)$ induces the tangent bundle on S so that:

COROLLARY 12.38. *Let $U(S)$ be the two-dimensional vector bundle on S induced by the inclusion*

$$\varepsilon\colon S \longrightarrow \mathrm{Gr}(2,\ 5) .$$

Then $U(S)$ is isomorphic to the tangent bundle $T(S)$.

Lastly since $V \approx \bigcup\{L_t \colon t \in \mathcal{G}(\alpha_s(S)) \subseteqq \mathrm{Gr}(2, 5)\}$, we have

COROLLARY 12.39. *The cubic threefold V is determined by its surface of lines S.*

13. The "double-six", Torelli, and irrationality theorems

As in § 3, let $\Omega_V = \Omega(\mathcal{J}(V))$ denote the polarizing class of the principally polarized abelian variety $\mathcal{J}(V)$. Let $\alpha_s \colon S \longrightarrow \mathrm{Alb}(S)$ and $\varphi \colon \mathrm{Alb}(S) \xrightarrow{\approx} J(V)$ be as in previous chapters. Referring to Definition 3.15 and Lemma 11.5, we have the following:

PROPOSITION 13.1. *$(\varphi \circ \alpha_s)_*(\{S\})$ has as its Poincaré dual on $J(V)$ the class $(\Omega_V^3/3\,!)$.*

Proof. Let $\xi_1, \cdots, \xi_{10}$ be any basis for $H^1(J(V))$ such that

$$\Omega_V = \sum \xi_k \wedge \xi_{5+k}\ .$$

If we continue to denote by ξ_k the pull-back of ξ_k to S under $(\varphi \circ \alpha_s)^*$, then from Lemma 11.5 we have:

$$\begin{aligned} &\int_{D_s} \xi_k \wedge \xi_{5+k} = 2 && \text{for } k = 1, \cdots, 5\ ; \\ &\int_{D_s} \xi_k \wedge \xi_l = 0 && \text{for other } l > k\ . \end{aligned} \tag{13.2}$$

To prove the proposition, it suffices to prove that

$$\int_S \xi_k \wedge \xi_{5+k} \wedge \xi_l \wedge \xi_{5+l} = 1 \qquad \text{for } k \neq l\ ,$$

$$\int_S \xi_p \wedge \xi_q \wedge \xi_r \wedge \xi_s = 0 \qquad \text{when } \{p, q, r, s\} \neq \{k, 5 + k, l, 5 + l\}\ .$$

Since this is strictly a numerical result, it suffices to prove the result for $S = S_{\hat{h}}$ with $S_{\hat{h}}$ as in § 9. By (11.21), we may take as our basis for $H^1(S)$ the set $\{\chi, \delta, \eta_1, \cdots, \eta_8\}$ of § 9. Then the proposition is proved by the formulas (11.29)–(11.31).

(Thus $\mathcal{J}(V)$ is a principally polarized abelian variety of level 2 (see § 3).)

Define the morphism

$$\Psi \colon S \times S \longrightarrow J(V) \tag{13.3}$$

by $\Psi(s_1, s_2) = \varphi(\alpha_s(s_1) - \alpha_s(s_2))$. By Theorem 12.37, Ψ is an immersion at each point (s_1, s_2) such that $(L_{s_1} \cap L_{s_2}) = \varnothing$. Thus the image $\Psi(S \times S)$ in $J(V)$ (counted with multiplicity one) is an effective divisor on $J(V)$ which we denote by θ_S. Notice that θ_S is even (that is, $\theta_S = -\theta_S$) and independent of the choice of basepoint s for the mapping α_s. Notice also that Ψ(diagonal of $S \times S) = 0 \in J(V)$. Take a generic $(s_1, s_1') \in (S \times S)$. Then $(L_{s_1} \cap L_{s_1}') = \emptyset$ and there are five lines $t_2, \cdots, t_6 \in S$ such that L_{t_j} is incident to both L_{s_1} and L_s for $j = 2, \cdots, 6$. Referring to (11.11), define:

$$s_j = \gamma_{t_j}(s_1') \quad j = 2, \cdots, 6\,,$$
$$s_j' = \gamma_{t_j}(s_1) \quad j = 2, \cdots, 6\,.$$

By (11.9), $\Psi(s_j, s_j') = \Psi(s_k, s_k')$ for $j, k = 1, 2, \cdots, 6$. The pair $\big((s_1, s_2, \cdots, s_6),$ $(s_1', s_2', \cdots, s_6')\big)$ is known classically as a "double-six" of lines on a cubic surface (see [18; page 7]). Since $\dim \Psi(S \times S) = 4$, Ψ is generically finite-to-one and, by what we have seen, Ψ is generically at least six-to-one. But by Proposition 13.1, $\Psi_*(\{S \times S\})$ has as its Poincaré dual on $J(V)$ the class $6\Omega_V$. Since Ω_V is not divisible in $H^2(J(V))$, we have:

THEOREM 13.4. *$\{\theta_S\}$ has as its Poincaré dual on $J(V)$ the polarizing class Ω_V.*

As in (12.3) we can define a Gauss map

$$\mathcal{G}\colon \theta_S \longrightarrow \mathrm{Gr}\,(4,\ T(J(V), 0)) \tag{13.5}$$

by assigning to $u \in \theta_S$ the subspace $T(\theta_S - u, 0)$ of $T(J(V), 0)$. Then Theorem 12.37 implies that for appropriate choice of isomorphism (12.2) we have the following commutative diagram of rational mappings:

$$\begin{array}{ccc} S \times S & \xrightarrow{\ \Psi\ } & \theta_S \subsetneqq J(V) \\ \Big\downarrow{\scriptstyle \Phi} & & \Big\downarrow{\scriptstyle \mathcal{G}} \\ \mathbf{P}_4^* & \approx & \mathrm{Gr}\,(4,\ T(J(V), 0)) \end{array} \tag{13.6}$$

where Φ is as in Lemma 12.16. Let $V^* = \mathfrak{D}(V)$, the dual variety to V in $\mathbf{P}_4^*$ (see § 5). As in [1; § 7], let N denote the closure of the graph of Φ in $(S \times S) \times \mathbf{P}_4^*$ and define

$$E_h = \pi_{S\times S}\big(N \cap \big((S \times S) \times \{h\}\big)\big) \tag{13.7}$$

for $h \in \mathbf{P}_4^*$. If $h \notin V^*$, then by Lemma 12.17, $\big(E_h \cap (\text{diagonal of } S \times S)\big) = \emptyset$. Then E_h contains $n_0 = \left(2 \times \binom{27}{2}\right)$ points since $\big(V \cap [h]\big)$ is a non-singular cubic surface. For $h \in V^*$, either E_h is infinite or E_h contains a number of points no larger than n_0. But clearly the set of points h such that E_h is infinite has dimension $\leqq 2$. Also for generic $h \in V^*$, $\big(V \cap [h]\big)$ has only one ordinary double point by Proposition 5.16. Thus for generic $h \in V^*$, E_h has at least $\left(2 \times \binom{22}{2}\right)$ points. And E_h for generic $h \in V^*$ must have less than n_0 points since otherwise there would exist a non-trivial connected covering space of $(\mathbf{P}_4^* - (\text{set of codimension} \geqq 2))$ which is impossible. By [1; § 7], for any rational map $f\colon X \to \mathbf{P}_d$ from a normal irreducible projective variety of dimension d onto $\mathbf{P}_d$, there is a divisor D in $\mathbf{P}_d$ uniquely determined by the condition that it is the largest effective divisor such that for x generic in D, $f^{-1}(x)$ has fewer than n elements, where n is the cardinality of $f^{-1}(y)$ for generic $y \in \mathbf{P}_d$. D is called the *branch locus* of f and will be denoted by $b(f)$. Then we have:

LEMMA 13.8. $b(\Phi) = V^* \subseteqq \mathbf{P}_4^*$.

Let $\tilde{\theta}_S$ be the normalization of θ_S. Then there is induced the following commutative diagram:

$$\begin{tikzcd}(S \times S) \arrow[r, "\tilde{\Psi}"] \arrow[d, "\Phi"] & \tilde{\theta}_S \arrow[d, "\beta"] \\ \mathbf{P}_4^* & \theta_S \arrow[l, "G"'] \end{tikzcd} \tag{13.9}$$

where $\tilde{\Psi}$ and β are morphisms. Since $\tilde{\Psi}$ is finite-to-one except over a set of codimension $\geqq 2$:

$$b(G \circ \beta) \subseteqq b(\Phi) \; .$$

But $b(\Phi) = V^*$ is irreducible and $b(\mathcal{G} \circ \beta) \neq \varnothing$ since $\tilde{\Psi}$ is generically six-to-one, Φ is generically n_0-to-one and $(\mathbf{P}_4^* - (\text{set of codimension} \geqq 2))$ has no non-trivial connected covering spaces. Thus we have:

$$b(\mathcal{G} \circ \beta) = b(\Phi) = V^* \; . \tag{13.10}$$

THEOREM 13.11. *Let V_1 and V_2 be two non-singular cubic threefolds. If $\mathcal{J}(V_1) \approx \mathcal{J}(V_2)$ then $V_1 \approx V_2$.*

Proof. If $\mathcal{J}(V_1) \approx \mathcal{J}(V_2)$ and if $S_j = S_{V_j}$ for $j = 1, 2$ then there is an isomorphism $\nu\colon J(V_1) \to J(V_2)$ such that $\nu(\theta_{S_1})$ and θ_{S_2} both give the theta-divisor of $\mathcal{J}(V)$ by Theorem 13.4. Thus there exists $u \in J(V_2)$ such that:

$$\nu(\theta_{S_1}) = \theta_{S_2} + u \; .$$

Then there is a commutative diagram

$$\begin{tikzcd}\tilde{\theta}_{S_1} \arrow[r, phantom, "\approx"] \arrow[d, "(\mathcal{G} \circ \beta_1)"'] & \tilde{\theta}_{S_2} \arrow[d, "(\mathcal{G} \circ \beta_2)"] \\ \mathbf{P}_4^* \arrow[r, phantom, "\approx"] & \mathbf{P}_4^* \; . \end{tikzcd}$$

Using 13.10 therefore, $V_1^* \approx V_2^*$. Since for $j = 1, 2$

$$\mathcal{D}_{V_j}\colon V_j \longrightarrow V_j^*$$

is finite-to-one and generically one-to-one, the theorem now follows easily from Zariski's Main Theorem.

Notice that the method of proof of Theorem 13.11 is analogous to the method of Andreotti in [1]. Our final theorem also derives from the techniques and results of that same work.

THEOREM 13.12. *Let V be a non-singular cubic threefold. Then V is not birationally equivalent to* $\mathbf{P}_3$.

Proof. By Corollary 3.26, it suffices to show that $\mathcal{J}(V)$ is not isomorphic to $\mathcal{J}(C)$ for some non-singular curve C. Suppose that such a curve C did exist. Let

$$\kappa\colon C \longrightarrow \mathbf{P}_4$$

be the morphism induced by sections of the canonical bundle of C. Following [1], if C is not hyperelliptic let $C^* = \{h \in \Gamma_4^*\colon \kappa(C)$ and $[h]$ are tangent at some point$\}$, and if C is hyperelliptic, put

$$\begin{aligned} C^* = \{h \in \mathbf{P}_4^*\colon &\text{ either } \kappa(C) \text{ and } [h] \text{ are somewhere tangent, or} \\ &[h] \text{ contains one of the 12 branch points of the ramified} \\ &\text{two-sheeted covering } C \to \kappa(C)\}\ . \end{aligned}$$

If θ_C is the theta-divisor for the principally polarized abelian variety $\mathcal{J}(C)$, then we have a birational morphism

$$\gamma\colon C^{(4)} \longrightarrow \theta_C \subseteqq J(C)$$

by Riemann's theorem. Let $\mathcal{G}_C$ denote the Gauss map

$$\mathcal{G}_C\colon \theta_C \longrightarrow \mathbf{P}_4^*\ .$$

By [1; pages 820-21], $b(\mathcal{G}\circ\gamma) = C^*$. Since $\mathcal{J}(V) \approx \mathcal{J}(C)$, it follows that we can identify $J(V)$ and $J(C)$ so that $\theta_S = \theta_C + u$. We then have the commutative diagram:

$$\begin{tikzcd}
S \times S \arrow[r, "\Psi"] \arrow[drr, "\Phi"'] & \theta_S \arrow[dr, "\mathcal{G}"] & \approx & \theta_C \arrow[dl, "\mathcal{G}_C"'] & C^{(4)} \arrow[l, "\gamma"'] \arrow[dll] \\
& & \mathbf{P}_4^* & &
\end{tikzcd}$$

Putting $\delta = (\gamma^{-1}\circ\Psi)$, we have:

$$V^* = b(\Phi) = b\big((\mathcal{G}_C\circ\gamma)\circ\delta\big) \supseteqq b(\mathcal{G}_C\circ\gamma) = C^*\ .$$

However C^* as defined above contains a linear subspace of dimension 2 in $\mathbf{P}_4^*$ (in fact it contains an infinite number of such subspaces), whereas by Corollary 6.2, V^* contains no such subspace. This gives the desired contradiction to the assumption that $\mathcal{J}(V) \approx \mathcal{J}(C)$ and so the theorem is proved.

APPENDICES

A. Equivalence relations on the algebraic one-cycles lying on a cubic threefold

Let $V \subseteqq \mathbf{P}_4$ be a non-singular cubic three-fold. There is a free abelian group, called the group of algebraic one-cycles, which has the set of irreducible algebraic curves lying on V as a distinguished basis. We denote this group by $C(V)$. Assigning to each irreducible curve its degree as an algebraic subvariety of $\mathbf{P}_4$, we can define a homomorphism:

$$\deg\colon C(V) \longrightarrow \mathbf{Z} .$$

The kernel of this map, which we denote by $H(V)$, is the group of algebraic one-cycles homologous to zero. Before discussing some relations between certain subgroups of $H(V)$, we will need to prove a lemma originally due to Fano:

LEMMA A.1. *Let W be an effective divisor on V. Then there is an effective divisor Y on $\mathbf{P}_4$ such that:*

$$W = (Y \cdot V) .$$

Proof. By the Lefschetz theorem applied to $V \subseteqq \mathbf{P}_4$, the natural mapping $H^2(\mathbf{P}_4) \rightarrow H^2(V)$ is an isomorphism so that there is some divisor Y_1 on $\mathbf{P}_4$ such that W is homologous, and therefore linearly equivalent, to $(Y_1 \cdot V)$. Also, if H is a hyperplane on $\mathbf{P}_4$, the sheaf sequence:

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}_4}\big((k-3)H\big) \longrightarrow \mathcal{O}_{\mathbf{P}_4}(kH) \longrightarrow \mathcal{O}_V\big(k(H \cdot V)\big) \longrightarrow 0$$

is exact. If $k > 0$:

$$H^1\big(\mathbf{P}_4;\, \mathcal{O}((k-3)H)\big) \approx H^3\big(\mathbf{P}_4;\, \mathcal{O}((-k-2)H)\big) = 0$$

(see [13; § 15 and 18]). Thus we obtain that the mapping

$$H^0\big(\mathbf{P}_4;\, \mathcal{O}(kH)\big) \longrightarrow H^0\big(V;\, \mathcal{O}(k(V \cdot H))\big)$$

is a surjection. Since $Y_1 \sim kH$ for some $k > 0$, the lemma follows.

Now let C be an irreducible curve on V and let S and T be as in (7.1). By desingularizing the components of $\pi_V^{-}(C) \subseteqq T$ and discarding any components which arise from Eckardt points of V, we obtain a non-singular curve C_1 and a mapping

$$\psi\colon C_1 \longrightarrow T$$

such that $(\pi_V \circ \psi)\colon C_1 \rightarrow C$ is finite-to-one and generically six-to-one. (Note that ψ is not necessarily generically injective since some components of $\pi_V^{-1}(C)$ may have to be counted with multiplicity > 1.) Next we define a $\mathbf{P}_1$-bundle over C_1:

$$T_1 = \{(u, x) \in C_1 \times V: x \in L_{\pi_S(\psi(u))}\} .$$

Now we can construct sections of the bundle $\pi_1: T_1 \to C_1$ as follows:

(i) define $\tau: C_1 \to T_1$ by $\tau(u) = (u, (\pi_V \circ \psi)(u))$;

(ii) for a generic hyperplane $H \subseteq \mathbf{P}_4$ define $\sigma: C_1 \to T_1$ by $\sigma(u) = (u, (L_{\pi_S(\psi(u))} \cdot H))$.

Using Lemma A.1, we have that there exists a divisor $Y \subseteq \mathbf{P}_4$ such that (counting multiplicities):

$$(Y \cdot V) = \pi_V(T_1) .$$

Therefore (again keeping track of multiplicities):

$$(Y \cdot V \cdot H) = \pi_V(\sigma(C_1)) .$$

But in the surface T_1, $\sigma(C_1)$ is homologous, and in fact linearly equivalent, to $(\tau(C_1) + (\text{sum of fibres of } \pi_1))$. Thus projecting onto V by $\pi_V: T_1 \to V$, we have

LEMMA A.2. *Let C be an irreducible curve on V. Then there exists a divisor Y on $\mathbf{P}_4$ such that $6C$ is rationally equivalent to* $((H \cdot Y \cdot V) + (\text{sum of lines}))$ *on V.*

Let $R(V)$ be the subgroup of $H(V)$ consisting of all algebraic one-cycles *rationally equivalent* to zero, and $A(V)$ the subgroup of all algebraic one-cycles *algebraically equivalent* to zero. Then

$$R(V) \subseteq A(V) \subseteq H(V) ,$$

and dividing by $R(V)$ we get an inclusion of quotient groups:

$$\mathcal{A}(V) \subseteq \mathcal{H}(V) .$$

If we let $\mathcal{L}(V)$ be the subgroup of $\mathcal{A}(V)$ generated by elements of the form

$$\sum n_j L_{s_j} \qquad (s_j \in S, \sum n_j = 0)$$

then by Lemma A.2:

PROPOSITION A.3. $6\mathcal{H}(V) \subseteq \mathcal{L}(V)$.

B. Unirationality

Recall that a threefold V is unirational if there is a generically finite-to-one rational mapping

$$f: \mathbf{P}_3 \longrightarrow V .$$

It was evidently known to Max Noether that the cubic threefold is unirational. We will give a construction for this which was pointed out to us by J. Fogarty.

Let L_0 be a line in V and let $[\mathfrak{D}(x)]$ denote the tangent hyperplane to V

at the point $x \in L_0$. Then there is a $\mathbf{P}_2$-bundle

$$B \longrightarrow L_0$$

with fibre $= \{t \in \mathrm{Gr}(2, 5)\colon x \in L_t \subseteqq [\mathcal{D}(x)]\}$. Evidently B is a rational threefold.

We can define a rational map by the rule

$$\big(f(x, t) + 2x\big) = (L_t \cdot V)$$

for $(x, t) \in B$. Then $f\colon B \rightarrow V$ is defined except along a curve and the set of adherence points to f along that curve lies in

$$W = \bigcup \{L_s\colon s \in S \text{ and } (L_s \cap L_0) \neq \varnothing\} .$$

For $y \in (V - W)$, let K be the plane through y and L_0. Then $f^{-1}(y)$ is the set of points $(x, t) \in B$ such that:

(i) $y \in L_t$;

(ii) x is a singular point of $(K \cdot V)$.

Since $(K \cdot V) = L_0 +$ (quadric curve), f is generically two-to-one.

C. Mumford's theory of Prym varieties and a comment on moduli

D. Mumford has developed a theory of "conic bundles" and associated Prym varieties. His theory gives also a proof that if V is a non-singular cubic threefold then $J(V)$ is not the Jacobian variety of a curve. Furthermore, it sheds some light on the singularities of the polarizing divisor θ_V. We shall briefly describe his results here.

Let L be a line lying in V and let

$$\pi_L\colon \mathbf{P}_4 \longrightarrow \mathbf{P}_2$$

be a generic projection centered along L. If V_L is the variety obtained by blowing up V along L, then π_L induces a morphism

$$\pi\colon V_L \longrightarrow \mathbf{P}_2 . \tag{C.1}$$

The fibres of π are given by conic curves on V which are coplanar with L. By [18; pages 3–5], these fibres are non-singular except along a plane curve G_L of degree five along which the fibre becomes the sum of two lines. For generic L, we have seen in § 10 that G_L is the quotient of the incidence divisor D_L under a fix-point-free involution γ_L.

The situation (C.1) is called a conic bundle. Note that $V_L \rightarrow V$ gives an isomorphism on the principally polarized intermediate Jacobians:

$$\mathcal{J}(V) \xrightarrow{\approx} \mathcal{J}(V_L) . \tag{C.2}$$

If we put $\mathcal{J}(D_L) = (W, U, \mathcal{H})$, then γ_L induces an involution on W which leaves $\mathcal{H}$ invariant and also takes U onto itself. This gives a decomposition

$$W_1 \oplus W_{-1}$$

into eigenspaces corresponing to the eigenvalues $+1$ and -1 of the involution. $\operatorname{Im}\mathcal{H}$ is not unimodular on the lattice

$$U_{-1} = (U \cap W_{-1})$$

but it is divisible as an integral-valued bilinear form and $\operatorname{Im}\big((1/2)\mathcal{H}\big)$ is integral-valued and unimodular. The resulting principally polarized abelian variety

$$\big(W_{-1},\ U_{-1},\ (1/2)\mathcal{H}\big)$$

is called the *Prym variety* associated to (D_L, γ_L), which we shall denote by $\mathcal{P}(D_L, \gamma_L)$.

Mumford has proved that for conic bundles with singular fibres which are always the union of two *distinct* linear components, there is an isomorphism between the intermediate Jacobian variety of the conic bundle and the Prym variety associated to the curve with fix-point-free involution given by the components of the singular fibres. Thus in our situation:

$$\mathcal{J}(V_L) \approx \mathcal{P}(D_L, \gamma_L)\ . \tag{C.3}$$

The question then arises as to which Prym varieties can be Jacobian varieties of curves. If $(W, U, \mathcal{H})$ is a principally polarized abelian variety and θ is the corresponding theta-divisor on (W/U), then it is known that a necessary condition for $(W, U, \mathcal{H})$ to be the Jacobian of a curve is that:

$$\dim(\theta_{\text{sing}}) \geqq (\dim W) - 4 \tag{C.4}$$

where θ_{sing} is the singular locus of the subvariety θ of (W/U).

Let D be a non-singular irreducible algebraic curve, γ a fixed-point-free involution, and G the quotient curve. If the associated Prym variety $\mathcal{P}(D, \gamma)$ has theta-divisor θ_γ, then Mumford has shown that:

$$\dim\big((\theta_\gamma)_{\text{sing}}\big) < (g - 5) \tag{C.5}$$

where $g =$ genus of G except in the following case:

(i) G is hyperelliptic;

(ii) G is a three-sheeted covering of $\mathbf{P}_1$;

(iii) G is a double covering of an elliptic curve;

(iv) G is a curve of genus 5 having some vanishing theta-nulls;

(v) G is a plane quintic.

By (C.3), it is case (v) which interests us here. In this case the unramified coverings $D \rightarrow G$ fall into two classes depending on the parity of the dimension of a certain linear system on G. To see what this is, let L_1 be the line bundle on G which is constructed from the representation

$$\pi_1(G) \longrightarrow \{1, -1\} \subseteq \mathbf{C}^*$$

associated to the covering $D \to G$ and let L_2 be the ample bundle on $G \subseteq \mathbf{P}_2$ gotten by restricting the bundle whose divisor is a line on $\mathbf{P}_2$. Then there are two cases:

(i) if $\dim H^0(G; \mathcal{O}(L_1 \otimes L_2))$ is even, $\dim ((\theta_\gamma)_{\text{sing}}) = 1$ so that we cannot rule out the possibility that the Prym variety is the Jacobian of a curve;

(ii) if $\dim H^0(G; \mathcal{O}(L_1 \otimes L_2))$ is odd, then the singular set of θ_γ contains exactly one point. Now for the case of the cubic threefold, (D_L, γ_L) falls into this last case, so that by (C.3) and (C.4) the intermediate Jacobian cannot be the Jacobian of a curve. Also, by § 13, the singular point of $\theta_\gamma = \theta_S \subseteq J(V)$ must just be the image of the diagonal of $S \times S$ under the difference map $(S \times S) \to \theta_S \subseteq J(V)$.

Finally, we should like to give a comment about moduli of the set of cubic threefolds. It is easily seen that the isomorphism class of a cubic threefold V depends on ten parameters (see [8; pages 493–94]). If a plane quintic G_L arises as in (C.1), then the ramification locus of the Gauss mapping applied to the theta-divisor of $\mathcal{P}(D_L, \gamma_L)$ allows us to reconstruct V, and for fixed V the family $\{G_L\}$ depends on two parameters. Since the family of all plane quintics also depends on twelve parameters, this suggests that one may be able to take a generic plane quintic G and an unramified double covering $D \to G$ such that $\dim H^0(G; \mathcal{O}(L_1 \otimes L_2))$ is odd and recover a cubic threefold from the ramification locus of the Gauss map associated to the theta-divisor on the Prym variety.

COLUMBIA UNIVERSITY
PRINCETON UNIVERSITY

BIBLIOGRAPHY

[1] A. ANDREOTTI, *On a theorem of Torelli*, Amer. J. Math. **80** (1958), 801-828.

[2] E. BOMBIERI and H.P.F. SWINNERTON-DYER, *On the local zeta function of a cubic threefold*, Ann. Sc. Norm. Sup. Pisa, **21** (1967), 1-29.

[3] S. CHERN, Complex Manifolds without Potential Theory, Princeton, D. Van Nostrand Company, 1967.

[4] ———, *On the characteristic classes of complex sphere bundles and algebraic varieties*, Amer. J. Math. **75** (1953), 565-596.

[5] CLEMENS, GRIFFITHS, et al., Seminar on Degeneration of Algebraic Varieties (mimeo notes) Institute for Advanced Study, Princeton, New Jersey, Fall Term, 1969-70.

[6] G. FANO, *Sul sistema ∞^2 di rette contenuto in una varietà cubica generale dello spazio a quattro dimensioni*, Atti R. Acc. Sc. Torino, **XXXIX** (1904), 778-792.

[7] F. GHERARDELLI, *Un 'osservazione sulla varieta' cubica di* $\mathbf{P}^4$, Rend. Seminario Matematico di Milano (1967).

[8] P. GRIFFITHS, *On the periods of certain rational integrals*: I *and* II. Ann. of Math. **90** (1969) 460-541.

[9] ———, Some transcendental methods in the study of algebraic cycles, Several Complex Variables II—Maryland 1970 (Lecture notes in mathematics, no. 185), New York: Springer-Verlag, 1971.

[10] A. GROTHENDIECK, and J. DIEUDONNE, E.G.A. IV (3e partie), Pub. Math. I.H.E.S. (no. 28), 1966.

[11] R. HARTSHORNE, Ample Subvarieties of Algebraic Varieties (Lecture notes in mathematics, no. 156), New York. Springer-Verlag, 1970.

[12] H. HIRONAKA, *Resolution of singularities of an algebraic variety over a field of characteristic zero* I-II, Ann. of Math. **79** (1964), 109-326.

[13] F. HIRZEBRUCH, Topological Methods in Algebraic Geometry, New York, Springer-Verlag, 1966.

[14] W. HOYT, *On products and algebraic families of Jacobian varieties*, Ann. of Math. **77** (1963), 415-423.

[15] K. KODAIRA, *A theorem of completeness of characteristic systems for analytic families of compact submanifolds of complex manifolds*, Ann. of Math. **75** (1962), 146-162.

[16] T. MATSUSAKA, *On a characterization of a Jacobian variety*, Memoirs Coll. Sci. Kyoto, ser. A, **XXXII**, no. 1 (1959), 1-19.

[17] D. MUMFORD, Abelian Varieties, (Tata Institute of Fundamental Research Studies in Mathematics), London: Oxford Univ. Press, 1970.

[18] B. SEGRE, The Non-singular Cubic Surfaces, Oxford: Clarendon Press 1942.

[19] J. A. TODD, *On the topology of certain algebraic threefold loci*, Proc. Edinburgh Math. Soc. (II), **4** (1935), 175-184.

[20] A. WALLACE, Homology Theory on Algebraic Varieties, New York. Pergamon Press, 1958.

[21] O. ZARISKI, Algebraic Surfaces, New York, Chelsea Publishing Company, 1948.

(Received July 9, 1971)