The Intermediate Water Depth Limit of the Zakharov Equation and Consequences for Wave Prediction

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ABSTRACT

Finite-amplitude deep-water waves are subject to modulational instability, which eventually can lead to the formation of extreme waves. In shallow water, finite-amplitude surface gravity waves generate a current and deviations from the mean surface elevation. This stabilizes the modulational instability, and as a consequence the process of nonlinear focusing ceases to exist when $kh < 1.363$. This is a well-known property of surface gravity waves. Here it is shown for the first time that the usual starting point, namely the Zakharov equation, for deriving the nonlinear source term in the energy balance equation in wave forecasting models, shares this property as well. Consequences for wave prediction are pointed out.

1. Introduction

Since the beginning of the 1990s, there has been a rapid increase in the understanding of the generation of extreme waves in the open ocean. Different mechanisms have been found to be relevant for the formation of such events [see Kharif and Pelinovsky (2003) for a review]. A number of experimental and theoretical works (Janssen 2003; Onorato et al. 2001, 2004, 2005) have shown that, provided that the spectra are narrow banded and waves are steep, deep-water third-order nonlinearity (four-wave interactions) can lead to focusing of wave energy, even when waves are characterized initially by random phases.

However, some observations of extreme waves have happened at locations close to the coast, where shallow-water effects may become important. For example, the famous Draupner freak wave was observed in water of depth $h_0 = 69$ m, and using the observed dominant frequency and the shallow water dispersion relation, one infers that the dimensionless depth k_0h_0 is just between 1.2 and 1.4 (depending on the choice of the dominant frequency in the spectrum). This prompted a study into the effects of finite depth on the modulational instability. In such conditions finite-amplitude waves generate a wave-induced current, hence for decreasing depth, less and less wave energy is available for nonlinear focusing. As a consequence, the process of nonlinear focusing ceases to exist for sufficiently small water depth, $k_0h_0 < 1.363$. This well-known result was first found by Benjamin (1967) and Whitham (1974) when studying the instability of a uniform, finite-amplitude wave train. Note that in shallow water three-dimensional perturbations or higher-order nonlinearities may lead to modulational instability (Davey and Stewartson 1974; Francius and Kharif 2006; Kristiansen 2005), nevertheless here we will concentrate our attention to four-wave interactions and to long crested waves, leaving the effects of transverse perturbation for future studies.

Here we will establish that the basic evolution equation for surface gravity waves, the Zakharov equation, correctly accounts for the stabilizing effects of wave-induced current and mean sea surface elevation. This holds for intermediate water depth and even for very shallow water when the dynamics of the waves is determined by shallow-water equations such as Boussinesq or Korteweg–de Vries equations (Onorato et al. 2006, unpublished manuscript). An important implica-

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tion for spectral wave modeling in shallow waters is that around $k_0h_0 = 1.363$ there is a considerable reduction of the nonlinear transfer rates. This will be shown explicitly in this work by means of results of Monte Carlo forecasting of the Zakharov equation.

Here we emphasize that our findings are in some way unexpected because the "classical" approach (Herterich and Hasselmann 1980) does not predict the reduction of nonlinear energy transfer for $k_0h_0 = 1.363$. This is an important difference because in shallow water a "typical" saturated wind sea corresponds to a dimensionless depth k_0h_0 of about 1. In that case, in a considerable part of the wave spectrum, the balance is determined by wind input and dissipation predominantly. Moreover, the approach in Herterich and Hasselmann (1980) predicts that the coupling coefficient in the Hasselmann equation shows for $k_0h_0 > 0.7$ a similarity relation between the finite-depth and the infinite-depth cases; this result is not consistent with our findings. Our result is also in contrast with results from Resio et al. (2001), where it was claimed that the Zakharov equation does not include wave-induced currents.

The present paper is organized as follows: We first make a summary of known results on modulational instability theory (we will use the Whitham approach; Whitham 1974); then we will show that the approach based on the Zakharov equation can recover the same results. Last we discuss the consequences for wave predictions in shallow water.

2. Summary of results from the Whitham theory

In shallow water the wave-induced current and mean surface elevation have a stabilizing effect in such a way that for $k_0h_0 < 1.363$ the modulational instability, also known as the Benjamin–Feir (BR) instability (Benjamin and Feir 1967), disappears and there is no self-focusing. This result is understood most easily from Whitham's variational approach (the following description is taken from Whitham (1974, 553–563). The starting point is a modulated wave train of wavenumber k on a wave-induced current β. The nonlinear dispersion relation is found to be

$$\frac{(\omega - k\beta)^2}{gk\tanh kh} = 1 + \frac{9T_0^4 - 10T_0^2 + 9}{4T_0^4}\frac{k^2E}{g}, \qquad (1)$$

with $E = g\eta_0^2/2$ being the wave energy for a single wave train with amplitude η_0. Here, h is the water depth, and the depth factor $T_0 = \tanh(k_0h_0)$ is evaluated at the undisturbed water depth h_0 and wavenumber k_0. This dispersion relation is accompanied by equations for the current β and mean elevation $b = h - h_0$. Whitham finds a particular solution corresponding to set-down in the presence of a wave group, that is,

$$b = -\frac{h_0}{c_S^2 - v_g^2}\frac{S}{h_0}, \qquad (2)$$

with $c_S^2 = gh_0$, $\mathbf{v}_g = \partial\omega/\partial\mathbf{k}$, $\omega_0^2 = gk_0T_0$, $c_0 = \omega_0/k_0$, and S being the radiation stress,

$$S = \left(\frac{2v_g}{c_0} - \frac{1}{2}\right)E, \qquad (3)$$

while the mass transport velocity U becomes

$$U = \beta + \frac{E}{c_0h_0} = \frac{v_g}{h_0}b. \qquad (4)$$

Using Eqs. (2) and (4) in Eq. (1), and linearizing in b, the dispersion relation becomes

$$\omega = \omega_0 + \Omega_2(k_0)\frac{k_0^2E}{c_0}, \qquad (5)$$

and

$$\Omega_2(k_0) = \frac{9T_0^4 - 10T_0^2 + 9}{8T_0^3} - \frac{1}{k_0h_0}\left[\frac{(2v_g - c_0/2)^2}{c_S^2 - v_g^2} + 1\right]. \qquad (6)$$

The stability of a uniform wave train is determined by the sign of the product of the second derivative of ω_0, denoted by ω_0'', and Ω_2 (Whitham 1974). There is instability when $\omega_0''\Omega_2 < 0$. In the present case $\omega_0 = (gk_0T_0)^{1/2}$ and ω_0'' is always negative. The stability of a uniform surface gravity wave train is therefore determined by the sign of the nonlinear term: there is stability for negative Ω_2 and instability in the opposite case.

For large depth $T_0 \to 1$ and the wave-induced current contribution vanishes. In that case $\Omega_2(k_0) \to 1$, which results in the well-known nonlinear dispersion relation for deep-water waves. Clearly, as Ω_2 is positive, a deep-water uniform wave train is unstable, and the nonlinearity leads to focusing of wave energy. For shallow waters, the curly bracketed term in Eq. (6) becomes important. It is positive definite and leads to stabilization of the Benjamin–Feir instability. At $k_0h_0 = 1.363$, Ω_2 vanishes. Hence, for $k_0h_0 < 1.363$ a uniform wave train is stable as Ω_2 is negative.

In the opposite case of very small depth, hence $k_0h_0 \ll 1$, one finds that $\Omega_2 = -(9/8)(k_0h_0)^{-3}$ and hence

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a uniform wave train is, as expected, stable. The resulting dispersion relation corresponds exactly with the nonlinear dispersion relation as obtained from the Nonlinear Schrödinger equation in shallow water (Hasimoto and Ono 1972; Mei 1983).

3. The Zakharov equation for arbitrary depth

The results just shown can in principle be of some relevance for forecasting of wind waves in the ocean; it is therefore desirable that the nonlinear source term in the energy balance equation commonly used in wave-forecasting models share the same properties as described above, that is, in the narrowband approximation, the coupling coefficient of the Hasselmann equation should go to zero for $k_0h_0 = 1.363$. None of the previous studies of which we are aware (Resio et al. 2001; Herterich and Hasselmann 1980; Gorman 2003; Zakharov 1999; Lin and Perrie 1997) have noticed such a property. As we will see, the difficulty of obtaining the aforementioned result resides in the fact that in the shallow-water limit the coupling coefficient develops some small denominators that have to be treated properly in order to obtain the correct results.

The nonlinear source term in the energy balance equation is based on the Zakharov equation; therefore that will be our starting point. The Zakharov equation describes the evolution in time of free waves in Fourier space for the so-called complex action density variable $a(\mathbf{k}, t)$. It takes the following form:

$$\frac{\partial a_1}{\partial t} + i\omega_1 a_1 = -i\int d\mathbf{k}_{2,3,4}\, T_{1,2,3,4} a_2^* a_3 a_4 \delta_{1+2-3-4}, \tag{7}$$

where we use the following notation: $a_i = a(\mathbf{k}_i, t)$, $d\mathbf{k}_{2,3,4} = d\mathbf{k}_2 d\mathbf{k}_3 d\mathbf{k}_4$, and $\delta_{1+2-3-4} = \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4)$. Here, the asterisk stands for complex conjugate; $T_{1,2,3,4}$ is the coupling coefficient that will be given below. Bound waves can be recovered from the dynamics of the free waves using the following transformation:

$$A_1 = a_1 + \int d\mathbf{k}_{2,3}\,[A^{(1)}_{1,2,3} a_2 a_3 \delta_{1-2-3} + A^{(2)}_{1,2,3} a_2^* a_3 \delta_{1+2-3} + A^{(3)}_{1,2,3} a_2^* a_3^* \delta_{1+2+3}] + O(a^3). \tag{8}$$

Here, the variable $A_i = A(\mathbf{k}_i, t)$ is still a wave action variable but it includes also bound waves.

In the Hamiltonian description of surface gravity waves (Zakharov 1968), the transformation corresponds to a canonical transformation that is used to remove nonresonant terms in the Hamiltonian. More simply, this transformation, as will be shown in the next section, corresponds to a generalized Stokes expansion that, in the narrowband approximation, results exactly in a Stokes series. Clearly, the theory discussed here is only valid if the second-order integral term in Eq. (8) is much smaller than the first-order term; in the limit of monochromatic and shallow-water waves, this condition corresponds to a small Stokes number (see also Zakharov 1999). The observable variables, that is, the surface elevation $\hat{\eta}(\mathbf{k}, t)$ and the velocity potential $\hat{\psi}(\mathbf{k}, t)$, are related to the variable $A(\mathbf{k}, t)$ in the following way:

$$\hat{\eta}(\mathbf{k}, t) = \sqrt{\frac{\omega}{2g}}[A(\mathbf{k}, t) + A^*(-\mathbf{k}, t)] \quad \text{and}$$

$$\hat{\psi}(\mathbf{k}, t) = -i\sqrt{\frac{g}{2\omega}}[A(\mathbf{k}, t) - A^*(-\mathbf{k}, t)]. \tag{9}$$

For a homogeneous random sea one then finds the following relation between the action density spectrum $\langle A(\mathbf{k}_1, t)A^*(\mathbf{k}_2, t)\rangle = N(\mathbf{k}_1)\delta(\mathbf{k}_1 - \mathbf{k}_2)$ and the surface elevation spectrum $\langle \eta(\mathbf{k}_1, t)\eta^*(\mathbf{k}_2, t)\rangle = F_\eta(\mathbf{k}_1)\delta(\mathbf{k}_1 - \mathbf{k}_1)$:

$$N = \frac{gF_\eta}{\omega}, \tag{10}$$

where $\langle\rangle$ denotes an ensemble average. The Zakharov equation is a very compactly written equation; it contains a lot of interesting physics, and here we would like to explore this for the general case of intermediate depth. We will derive, for the case of a single wave, important relations such as the dispersion relation, the expression for the mean surface elevation and the mean current, and we will compare the results with Whitham (1974). In particular, we would like to check that wave-induced current and mean surface elevation indeed have a damping effect on the Benjamin–Feir instability in such a way that for $k_0h_0 = 1.363$ the instability disappears. In other words, for $k_0h_0 > 1.363$ nonlinearity focuses wave energy, while in the opposite case we have defocusing. Before entering in the discussion it is useful to report the analytical form of the coupling coefficients that will be used in the analysis.

Analytical form of the coupling coefficients

The expressions for the coupling coefficient in the Zakharov equation and in the canonical transformation Eq. (8) are taken from Zakharov (1992) and Krasitskii (1994). The coupling coefficient $T_{1,2,3,4}$ in the Zakharov equation is given by

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$$T_{1,2,3,4} = W_{1,2,3,4} - V^{(-)}_{1,3,1-3}V^{(-)}_{4,2,4-2}\left(\frac{1}{\omega_3 + \omega_{1-3} - \omega_1} + \frac{1}{\omega_2 + \omega_{4-2} - \omega_4}\right)$$
$$- V^{(-)}_{2,3,2-3}V^{(-)}_{4,1,4-1}\left(\frac{1}{\omega_3 + \omega_{2-3} - \omega_2} + \frac{1}{\omega_1 + \omega_{4-1} - \omega_4}\right)$$
$$- V^{(-)}_{1,4,1-4}V^{(-)}_{3,2,3-2}\left(\frac{1}{\omega_4 + \omega_{1-4} - \omega_1} + \frac{1}{\omega_2 + \omega_{3-2} - \omega_3}\right)$$
$$- V^{(-)}_{2,4,2-4}V^{(-)}_{3,1,3-1}\left(\frac{1}{\omega_4 + \omega_{2-4} - \omega_2} + \frac{1}{\omega_1 + \omega_{3-1} - \omega_3}\right)$$
$$- V^{(-)}_{1+2,1,2}V^{(-)}_{3+4,3,4}\left(\frac{1}{\omega_{1+2} - \omega_1 - \omega_2} + \frac{1}{\omega_{3+4} - \omega_3 - \omega_4}\right)$$
$$- V^{(+)}_{-1-2,1,2}V^{(+)}_{-3-4,3,4}\left(\frac{1}{\omega_{1+2} + \omega_1 + \omega_2} + \frac{1}{\omega_{3+4} + \omega_3 + \omega_4}\right), \quad (11)$$

where the coefficients $V^{(\pm)}_{1,2,3}$ are

$$V^{(\pm)}_{1,2,3} = \frac{1}{4\sqrt{2}}\left[(\mathbf{k}_1 \cdot \mathbf{k}_2 \pm q_1q_2)\left(\frac{g\omega_3}{\omega_1\omega_2}\right)^{1/2} + (\mathbf{k}_1 \cdot \mathbf{k}_3 \pm q_1q_3)\left(\frac{g\omega_2}{\omega_1\omega_3}\right)^{1/2} + (\mathbf{k}_2 \cdot \mathbf{k}_3 + q_2q_3)\left(\frac{g\omega_1}{\omega_2\omega_3}\right)^{1/2}\right], \quad (12)$$

with $k_i = |\mathbf{k}_i|$ and $\omega_i = \omega(k_i)$, and where $q_i = \omega_i^2/g$. Here, $W_{1,2,3,4}$ is given by the following analytical expression:

$$W_{1,2,3,4} = U_{-1,-2,3,4} + U_{3,4,-1,-2} - U_{3,-2,-1,4} - U_{-1,3,-2,4} - U_{-1,4,3,-2} - U_{4,-2,3,-1}, \quad (13)$$

with

$$U_{1,2,3,4} = \frac{1}{16}\left(\frac{\omega_3\omega_4}{\omega_1\omega_2}\right)^{1/2}[2(k_1^2q_2 + k_2^2q_1) - q_1q_2(q_{1+3} + q_{2+3} + q_{1+4} + q_{2+4})]. \quad (14)$$

The coefficients in the canonical transformation are related to $V^{(\pm)}_{1,2,3}$ as follows:

$$A^{(1)}_{1,2,3} = -\frac{V^{(-)}_{1,2,3}}{\omega_1 - \omega_2 - \omega_3},$$
$$A^{(2)}_{1,2,3} = -2\frac{V^{(-)}_{3,2,1}}{\omega_1 + \omega_2 - \omega_3}, \quad \text{and}$$
$$A^{(3)}_{1,2,3} = -\frac{V^{(+)}_{1,2,3}}{\omega_1 + \omega_2 + \omega_3}. \quad (15)$$

4. Narrowband approximation

a. Mean flow and wave-induced currents

To compare the Zakharov equation with the results of the Whitham theory it is necessary to take the narrowband approximation, that is, we will consider a free wave of the form

$$a_i = \hat{a}_i\delta(\mathbf{k}_i - \mathbf{k}_0), \quad \hat{a}_i = \rho e^{-i\omega_i t}. \quad (16)$$

Substitution of Eq. (16) into the canonical transformation Eq. (8) up to second order in amplitude gives the following expression for the action variable $A(\mathbf{k}, t)$:

$$A_1 = \hat{a}_0\delta(\mathbf{k}_1 - \mathbf{k}_0) - \left[\frac{V^{(-)}_{1,0,0}}{\omega_1 - 2\omega_0}\hat{a}_0^2\delta(\mathbf{k}_1 - 2\mathbf{k}_0) + \frac{V^{(+)}_{1,0,0}}{\omega_1 + 2\omega_0}\hat{a}_0^{*2}\delta(\mathbf{k}_1 + 2\mathbf{k}_0) + 2\int d\mathbf{k}_{2,3}\frac{V^{(-)}_{3,2,1}}{\omega_1 + \omega_2 - \omega_3}|\hat{a}_0|^2\delta(\mathbf{k}_2 - \mathbf{k}_0)\delta(\mathbf{k}_3 - \mathbf{k}_0)\delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3)\right]; \quad (17)$$

hence, as expected, the second-order term generates a second harmonic contribution. The first two terms are easy to deal with because the denominators remain finite. In this case the relevant matrix elements become

$$V^{(\pm)}_{1,0,0} = \frac{1}{4\sqrt{2}}\left[2(\mathbf{k}_1 \cdot \mathbf{k}_0 \pm q_1q_0)\left(\frac{g}{\omega_1}\right)^{1/2} + \frac{k_0^2 + q_0^2}{\omega_0}\sqrt{g\omega_1}\right], \quad \text{at} \quad \mathbf{k}_1 = 2\mathbf{k}_0. \quad (18)$$

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The mean flow contribution is much more awkward because of the apparent singularity caused by the factor $\omega_1 + \omega_2 - \omega_3 = 0$ for $\mathbf{k}_2 \to \mathbf{k}_0$, $\mathbf{k}_3 \to \mathbf{k}_0$, and consequently $\mathbf{k}_1 \to 0$. Strictly speaking, the mean response in the action density diverges and only the mean surface elevation remains finite. In addition, one obtains different answers depending on how the limits are taken. An example of a limit is the one where $\mathbf{k_2} = \mathbf{k}_0$, and $\mathbf{k}_3 = \mathbf{k}_0$, while the limit $\mathbf{k}_1 \to 0$ is only taken afterward. The resulting expression for the mean surface elevation is identical to the one given in Benjamin (1967). The problem with this limit is, however, that by choosing finite $\mathbf{k}_1$ one moves away from the surface determined by the orthonormality condition $\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 = \mathbf{0}$.[1] As suggested by Gorman (2003), we prefer to stick with this condition; therefore, we choose $\mathbf{k}_2$ and $\mathbf{k}_3$ slightly differently in order to satisfy the orthogonality condition. Specifically, we specify

$$\mathbf{k}_3 = \mathbf{k}_0 + \boldsymbol{\epsilon} \quad \text{and} \quad \mathbf{k}_2 = \mathbf{k}_0, \tag{19}$$

where $\boldsymbol{\epsilon}$ is assumed to be small. Because of the orthogonality condition on wavenumbers, the wavelength of wave "1" becomes very long, or,

$$\mathbf{k}_1 = \boldsymbol{\epsilon}. \tag{20}$$

As a consequence the factor $\omega_1 + \omega_2 - \omega_3$ becomes equal to $k_1 c_S - \mathbf{k}_1 \cdot \mathbf{v}_g$, with c_S being the shallow water speed and υ_g being the group velocity.

Hence by choosing the wavenumbers in this fashion we are considering the mean surface elevation and the nonlinear transfer in the limit of a very long wave group! The mean flow response then becomes to lowest significant order

$$\langle A_1 \rangle = -B_0 |\hat{a}_0|^2 \delta(\mathbf{k}_1 - \boldsymbol{\epsilon}), \tag{21}$$

where

$$B_0 = \frac{2V^{(-)}_{0,0,1}}{k_1 c_S - \mathbf{k}_1 \cdot \mathbf{v}_g}, \tag{22}$$

and

$$V^{(-)}_{0,0,1} = \frac{1}{4\sqrt{2}} \left[\frac{k_0^2 - q_0^2}{\omega_0} \sqrt{g\omega_1} + 2\mathbf{k}_0 \cdot \mathbf{k}_1 \left(\frac{g}{\omega_1} \right)^{1/2} \right],$$
$$\text{for} \quad \mathbf{k}_1 \to 0. \tag{23}$$

From now on we will consider only the one-dimensional case; nevertheless we will still use vector notation in order to distinguish magnitude of a vector from the usual vector (which in 1D carries a sign). Hence, we specify the action density as

$$A_1 = \hat{a}_0 \delta(\mathbf{k}_1 - \mathbf{k}_0) - B_2 \hat{a}_0^2 \delta(\mathbf{k}_1 - 2\mathbf{k}_0) - B_{-2} \hat{a}_0^{*2} \delta(\mathbf{k}_1 + 2\mathbf{k}_0) - B_0 |\hat{a}_0|^2 \delta(\mathbf{k}_1 - \boldsymbol{\epsilon}), \tag{24}$$

where B_0 is given by Eq. (22), while

$$B_2 = \frac{V^{(-)}_{1,0,0}}{\omega_1 - 2\omega_0} \quad \text{and} \quad B_{-2} = \frac{V^{(+)}_{1,0,0}}{\omega_1 + 2\omega_0}. \tag{25}$$

The surface elevation now becomes

$$\eta = \int d\mathbf{k}\, \hat{\eta} e^{i\mathbf{k}\cdot\mathbf{x}} = \int d\mathbf{k} \left(\frac{\omega}{2g} \right)^{1/2} A(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} + \text{ c.c.}, \tag{26}$$

where c.c. is the complex conjugate. Substituting Eq. (24) into Eq. (26) and introducing the surface elevation amplitude $\eta_0 = (2\omega_0/g)^{1/2} \rho$ gives

$$\eta = \eta_0 \cos\theta - \frac{\eta_0^2}{2\omega_0} \left[\frac{g\omega(\boldsymbol{\epsilon})}{2} \right]^{1/2} [B_0(+\boldsymbol{\epsilon}) + B_0(-\boldsymbol{\epsilon})] - \frac{\eta_0^2}{\omega_0} \left[\frac{g\omega(2\mathbf{k}_0)}{2} \right]^{1/2} [B_2(2\mathbf{k}_0) + B_{-2}(-2\mathbf{k}_0)] \cos 2\theta, \tag{27}$$

where $\theta = \mathbf{k}_0 \cdot \mathbf{x} - \omega_0 t$.

Using the expression for B_0, B_2, and B_{-2} and taking the limit of vanishing ϵ, one finds explicitly that

$$\eta = k_0 \eta_0^2 \Delta + \eta_0 \cos\theta + k_0 \eta_0^2 P \cos 2\theta, \tag{28}$$

with

$$P = \frac{1}{4T_0} \left(\frac{3}{T_0^2} - 1 \right), \quad T_0 = \tanh(k_0 h_0),$$

while

$$\Delta = -\frac{1}{4} \frac{c_S^2}{c_S^2 - \upsilon_g^2} \left[\frac{2(1 - T_0^2)}{T_0} + \frac{1}{k_0 h_0} \right].$$

Both expressions agree with Whitham [1974, his Eqs. (13.123) and (16.99)]. A remarkable property of the mean surface elevation is that, in contrast perhaps to one's expectation, it does not vanish exponentially for large $k_0 h_0$, but it only slowly vanishes like $1/k_0 h_0$.

In the appendix a similar calculation is performed for the mean flow according to the Zakharov equation. To this end Whitham introduced in a natural way the wave-induced mass transport velocity u_w. With wave variance $E = g\eta_0^2/2$ one has

$$u_w = \frac{E}{c_0 h_0} = \frac{1}{2} \frac{g k_0}{\omega_0 h_0} \eta_0^2, \tag{29}$$

[1] The presence of the delta function with argument $\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3$ is a direct consequence of the orthonormality of the Fourier modes used in the expansion.

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and Whitham (1974) finds that the average mass transport velocity U, defined as the sum of the wave-induced transport and the mean circulation velocity $\beta = \partial\langle\phi\rangle/\partial x$, or, $U = \beta + u_w$, obeys the relation $U = v_g b/h_0$, where $b = k_0\eta_0^2\Delta$ is the mean surface elevation. We have determined the mean of the velocity potential, $\langle\phi\rangle$, from Zakharov's Hamiltonian approach in the appendix with exactly the same results.

b. Nonlinear dispersion relation

To obtain the dispersion relation for a single wave in shallow water we require $T_{0,0,0,0}$. Inspecting the general expression for T in Eq. (11), it is evident that once more the limit of equal wavenumbers is awkward, since the first four terms show apparent singularities. We will treat this limit in a similar fashion as in the case of the mean surface elevation. The task is, however, simplified by the abundance of symmetries of T. Let us start with the first singular term, and we perturb all wavenumbers slightly, respecting the resonance condition in the Zakharov equation, hence

$$\mathbf{k}_1 = \mathbf{k}_0 + \boldsymbol{\epsilon}_1, \quad \mathbf{k}_2 = \mathbf{k}_0 + \boldsymbol{\epsilon}_2, \\ \mathbf{k}_3 = \mathbf{k}_0 + \boldsymbol{\epsilon}_3, \quad \text{and} \quad \mathbf{k}_4 = \mathbf{k}_0 + \boldsymbol{\epsilon}_4, \tag{30}$$

in such a way that $\boldsymbol{\epsilon}_1 + \boldsymbol{\epsilon}_2 = \boldsymbol{\epsilon}_3 + \boldsymbol{\epsilon}_4$. The first term in parentheses in Eq. (11), denoted by $f(\mathbf{d})$, becomes in lowest significant order

$$f(\mathbf{d}) = -\frac{|\mathbf{d}|}{16(|\mathbf{d}|c_S - \mathbf{d}\cdot\mathbf{v}_g)} \times \left[k_0^2(1 - T_0^2)\frac{gc_S}{\omega_0} + \frac{2\mathbf{k}_0\cdot\mathbf{d}}{|\mathbf{d}|}\left(\frac{g}{c_S}\right)^{1/2}\right]^2, \tag{31}$$

where $\mathbf{d} = \boldsymbol{\epsilon}_1 - \boldsymbol{\epsilon}_3$. The last singular term in Eq. (11) can be obtained from the first one by interchanging the indices 1, 3 and 4, 2. As a result, this last term equals $f(-\mathbf{d})$. Combining the two terms we have that their sum equals $f(\mathbf{d}) + f(-\mathbf{d})$ and is therefore independent of the sign of the difference vector $\mathbf{d}$.

The second and third singular terms give a similar contribution, and upon taking the limit of vanishing distance $\mathbf{d}$ one finds that the singular terms amount to

$$-\frac{1}{4}\frac{k_0^3c_S^2}{c_S^2 - v_g^2}\left[\frac{(1 - T_0^2)^2}{T_0} + \frac{4gv_g}{\omega_0 c_S^2}(1 - T_0^2) + \frac{4}{k_0h_0}\right]. \tag{32}$$

Making use of the dispersion relation and the expression for the group speed, (32) becomes

$$-\frac{k_0^3}{k_0h_0}\left[\frac{(2v_g - c_0/2)^2}{c_S^2 - v_g^2} + 1\right], \tag{33}$$

where $c_0 = \omega_0/k_0$ is the linear phase speed.

The regular terms in $T_{0,0,0,0}$ can be obtained in an elaborate, but straightforward, manner, and the final result becomes

$$T_{0,0,0,0}/k_0^3 = \frac{9T_0^4 - 10T_0^2 + 9}{8T_0^3} - \frac{1}{k_0h_0}\left[\frac{(2v_g - c_0/2)^2}{c_S^2 - v_g^2} + 1\right]. \tag{34}$$

The dispersion relation now follows in a straightforward manner from the Zakharov equation by substitution of Eq. (16) into Eq. (7). The resulting evolution equation is

$$\frac{d\hat{a}}{dt} + i\omega_0\,\hat{a} = -iT_{0,0,0,0}|\hat{a}|^2\hat{a}. \tag{35}$$

Solving this with the *ansatz* $\hat{a} = \hat{a}_0e^{-i\Omega t}$, the result is

$$\Omega = \omega_0 + T_{0,0,0,0}|\hat{a}_0|^2. \tag{36}$$

To be able to compare with results obtained by Whitham (1974) the energy E of a wave train is introduced. In terms of the action variable $\hat{a}$ the energy E becomes

$$E = \omega_0|\hat{a}|^2. \tag{37}$$

Writing the dispersion relation Eq. (36) as

$$\Omega = \omega_0(k_0) + \Omega_2(k_0)\frac{k_0^2E}{c_0}, \tag{38}$$

one finds for Ω_2,

$$\Omega_2 = T_{0,0,0,0}/k_0^3, \tag{39}$$

where $T_{0,0,0,0}$ is given by Eq. (34).

Whitham (1974) derived the nonlinear dispersion relation for shallow-water waves using a variational approach and his result [his Eq. (16.103); see also Eq. (6)] is in exact agreement with the present result displayed in Eq. (39). Combining Whitham's analysis and our work, it appears that the singular terms in $T_{1,2,3,4}$ result from the wave-induced changes in mean sea surface level and the wave-induced mean flow. These changes have a stabilizing effect on the Benjamin–Feir instability as k_0h_0 decreases from the deep-water limit. The critical value for stability is determined by the value of k_0h_0 for which $\Omega_2 = 0$. This value is found numerically to be $k_0h_0 = 1.363$. For $k_0h_0 > 1.363$, modulations grow, while instability is absent in the opposite case.

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This threshold for instability was deduced by Whitham [see his account in Whitham (1974)] from the variational approach and by Benjamin (1967) by means of a Fourier mode analysis. There are important implications for the probability distribution function (PDF) of the surface elevation. For $k_0h_0 > 1.363$, nonlinearities result in focusing of wave energy, and hence the kurtosis of the PDF is positive, reflecting the increased probability of extreme waves. In the opposite case nonlinearities result in defocusing, and hence the kurtosis of the PDF is negative (Janssen 2003). It should be mentioned that in this work [as in Janssen (2003)] we are only considering the deviation from Gaussian statistics as a result of nonlinear interactions among free waves, while the effect of bound modes on the kurtosis will be considered in a separate paper.

Note that in Resio et al. (2001) this problem has been considered before. These authors find that the Zakharov equation does not include wave-induced currents, and their $T_{0,0,0,0}$ is given by the first term on the right-hand side in Eq. (34). We can reproduce their result by numerically taking the limit in such a way that the condition $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4$ is not satisfied. We argue that it is essential to satisfy the wavenumber condition because this condition follows from a basic property of the basis functions, namely the orthonormality property (see Gorman 2003). Last, note that Lin and Perrie (1997) also report the nonlinear correction to the dispersion relation, but, nevertheless only the first term in the right-hand side of Eq. (34) is included, because in their perturbation analysis effects of the wave-induced current and mean surface elevation were not taken into account.

We mention that in a different context, namely for deep water, Stiassnie (1984) obtained the higher-order nonlinear Schrödinger equation from the Zakharov equation. Therefore, he has noticed that the Zakharov equation is able to represent the effects of the wave-induced mean current or setup (or -down) of the free surface. While in intermediate water depth, the contribution to the mean flow appears of leading order in spectral width; in infinite water depth, one has to go to higher order.

c. Modulational instability and effects of finite depth

In the context of the Zakharov Eq. (7) the stability of a weakly nonlinear wave train was studied by Zakharov (1968) and Crawford et al. (1981). To test the stability of a uniform wave train with wavenumber k_0 and complex amplitude A_0, it is perturbed by a pair of sidebands with wavenumbers $k_\pm = k_0 \pm K$ (where K is the modulation wavenumber) and amplitudes $A_\pm$, for example,

$$a = A_0\delta(k - k_0) + A_+\delta(k - k_+) + A_-\delta(k - k_-).$$

This expression for the action variable is substituted in the Zakharov equation, and it is assumed that the sideband amplitudes are small when compared with the amplitude A_0 of the carrier wave; therefore the square of small quantities may be neglected. The resulting evolution equations for the complex amplitudes may then be solved exactly, and for the amplitude of the carrier wave one finds that

$$A_0 = a_0e^{-i\omega_2 t},$$

where ω_2 denotes the correction of the dispersion relation due to nonlinearity. It is given by

$$\omega_2 = T_{0,0,0,0}|a_0|^2,$$

and agrees with the nonlinear part of the dispersion relation Eq. (36). The equations for the amplitudes of the sidebands are linear and can therefore be solved in terms of exponential functions, involving an as-yet-unknown oscillation frequency Ω. A nontrivial solution is then found provided that Ω satisfies the dispersion relation (Crawford et al. 1981)

$$\Omega = (T_{+,+} - T_{-,-})\,a_0^2 \pm \left\{-T_{+,-}T_{-,+}\,a_0^4 + \left[-\frac{1}{2}\Delta\omega + a_0^2(T_{+,+} + T_{-,-} - T_{0,0,0,0})\right]^2\right\}^{(1/2)}, \tag{40}$$

where $T_{\pm,\pm} = T(k_\pm, k_0, k_0, k_\pm)$ and $T_{\pm,\mp} = T(k_\pm, k_\mp, k_0, k_0)$, while $\Delta\omega = 2\omega(k_0) - \omega(k_+) - \omega(k_-)$ is a frequency mismatch. We have instability provided that the term under the square root is negative.

In Fig. 1 the normalized growth rate $\Im(\Omega)/(½)\omega_0 s^2$ (where s equals the steepness $k_0\eta_0$), obtained from Eq. (40), is plotted as function of the normalized sideband wavenumber $\Delta = K/2k_0s$ at different values of the dimensionless depth k_0h_0.

In agreement with one's expectations, the growth rate is reduced for decreasing values of the dimensionless depth k_0h_0, and for $k_0h_0 < 1.363$ the instability disappears. However, this result can only be obtained when the relevant nonlinear transfer coefficients T in Eq. (40) are evaluated by perturbing the wavenumbers according to Eq. (30).

5. Consequences for wave prediction in shallow water

The threshold for instability at $k_0h_0 = 1.363$ has important consequences for wave modeling in shallow wa-

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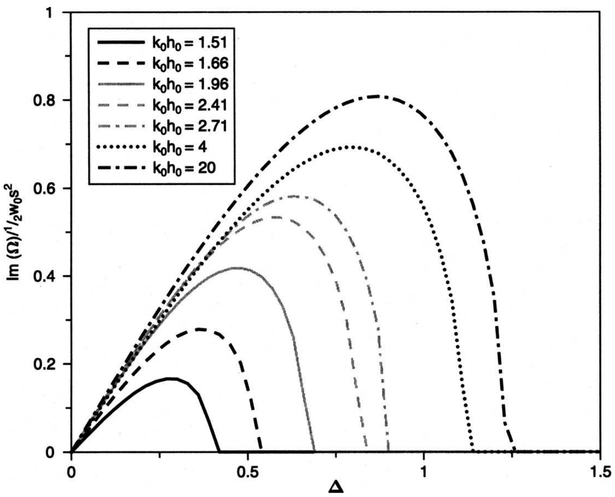

FIG. 1. Two-dimensional normalized growth rate as function of the normalized perturbation wavenumber for different values of the dimensionless water depth k_0h_0.

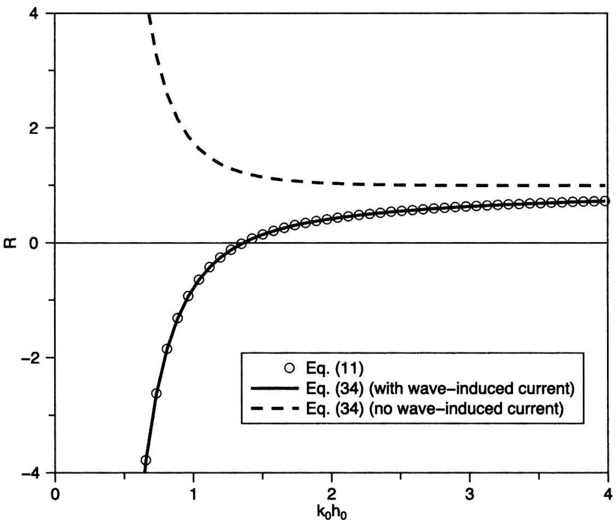

FIG. 2. Depth dependence of the numerical [Eq. (11)] and analytical narrowband approximation [Eq. (34)] of the nonlinear transfer coefficient normalized with the deep-water value. The effect of the wave-induced current and mean surface elevation is shown as well.

ters of intermediate depth. The reason is that for these dimensionless depths there is a considerable reduction of the nonlinear transfer, and hence the shape of the wave spectrum is only determined by the balance of wind input and dissipation. To illustrate the stabilizing effect of the wave-induced current and mean surface elevation we have plotted in Fig. 2 the narrowband transfer coefficient $R = T_{0,0,0,0}/k_0^3$ as function of dimensionless depth using Eq. (34) with and without the wave-induced current effects. Including wave-induced effects shows that indeed the transfer coefficient changes sign at $k_0h_0 = 1.363$ while the transfer only approaches very slowly the deep-water value [in agreement with the fact that the mean surface elevation slowly vanishes like $1/(k_0h_0)$ and not exponentially]. In Fig. 2 we have also plotted the narrowband approximation of the nonlinear transfer using the complete expression in Eq. (11). To take the limit numerically we again perturbed the relevant wavenumbers according to Eq. (30). The agreement with the analytical result Eq. (34) is satisfactory. Hence, the Zakharov equation contains the effects of wave-induced current and mean surface elevation.

These wave-induced effects have an even more pronounced effect in the kinetic equation for the action density, as the nonlinear transfer coefficient is squared. Recall that according to Janssen (2003) the corresponding kinetic equation becomes

$$\frac{\partial}{\partial t}N_4 = 4\int d\mathbf{k}_{1,2,3}T^2_{1,2,3,4}\delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4)R_i(\Delta\omega, t) \times [N_1N_2(N_3 + N_4) - N_3N_4(N_1 + N_2)], \quad (41)$$

where $R_i(\Delta\omega, t) = \sin(\Delta\omega t)/\Delta\omega$ and $\Delta\omega = \omega_1 + \omega_2 - \omega_3 - \omega_4$. In Fig. 3 we have plotted $R^2 = T^2_{0,0,0,0}/k_0^6$ as a function of dimensionless depth and compared it with the case in the absence of wave-induced effects. Clearly, in a wide range of dimensionless depth around $k_0h_0 = 1.363$ the nonlinear transfer is small. Although the narrowband approximation to the nonlinear transfer only has a very restricted validity, it nevertheless

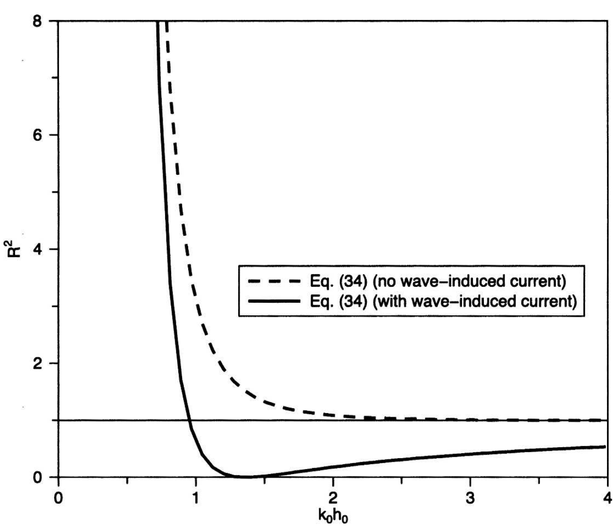

FIG. 3. Depth dependence of the square of the nonlinear transfer coefficient in the narrowband approximation [Eq. (34)] in comparison with the case in which wave-induced effects are removed.

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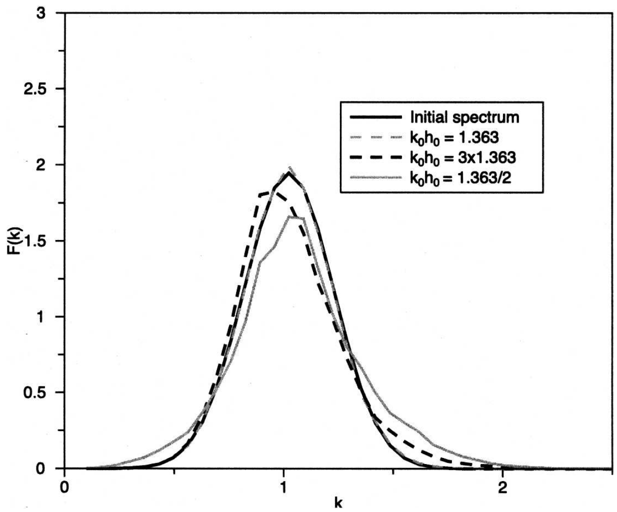

FIG. 4. Spectral evolution of shallow-water case (k_0h_0 = 1.363/2), an intermediate-depth case (k_0h_0 = 1.363), and a deep-water case ($k_0h_0 = 3 \times 1.363$), showing upshifting and downshifting of the spectrum, respectively, caused by nonlinear interactions. The BFI = 1. The spectra have been scaled in such a manner that the total surface is 1.

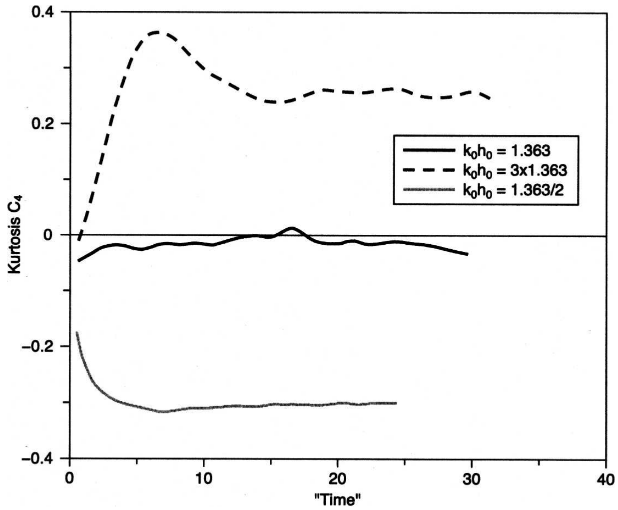

FIG. 5. Time evolution of kurtosis for BFI = 1. For deep-water ($k_0h_0 = 3 \times 1.363$) nonlinearly focused waves, there results positive kurtosis while for shallow water ($k_0h_0 < 1.363$) we have defocusing giving a negative kurtosis.

indicates that the wave-induced effects should have a dramatic impact on the downshifting of the peak of the wave spectrum in shallow water. When the peak wavenumber of the spectrum approaches the threshold value $k_{thr} = 1.363/h_0$, one would expect that the downshift of the spectrum is reduced.

To test this conjecture we simulated the evolution of the wave spectrum by performing Monte Carlo simulations with Eq. (7) (Janssen 2003) for a number of cases, namely $k_0h_0 = 1.363/2$, $k_0h_0 = 1.363$, and $k_0h_0 = 3 \times 1.363$. The size of the ensemble is 500, while the Benjamin–Feir Index equals 1. Results for the spectrum and the nonlinear transfer are displayed in Fig. 4. Clearly, the "deep" water simulation shows the expected downshift of the spectrum, while the intermediate water depth case ($k_0h_0 = 1.363$) shows no change of the spectrum at all, while the shallow-water case shows signs of an upshifting of the peak of the spectrum. Evidently, a simple scaling of the deep-water nonlinear transfer for shallow-water cases (as is common practice in wave modeling) does not seem to be a realistic option.

From the numerical simulations we have also obtained the time evolution of the kurtosis C_4, defined as $C_4 = \langle\eta^4\rangle/3\langle\eta^2\rangle^2 - 1$. These results are plotted in Fig. 5 and are in agreement with our expectations. For deep water we find a positive kurtosis [in agreement with Janssen (2003)]; hence there is an increased probability for extreme events. In shallow water, on the other hand, kurtosis is found to be negative, and thus it is less likely than normal to find extreme waves.

These simulations have been repeated with the kinetic Eq. (41), and for $k_0h_0 \geq 1.363$ a good agreement with the Monte Carlo Simulations is found. For $k_0h_0 \ll 1.363$ however we see from Fig. 2 that in shallow water the nonlinear transfer coefficient increases very rapidly with decreasing dimensionless depth, so we very quickly end up with a strongly nonlinear case. In such circumstances the range of validity of the kinetic equation is much restricted (Zakharov 1999).

Referring to Janssen (2003) where the properties of the Zakharov equation were discussed, it was argued that one is basically studying the balance between dispersion and nonlinearity (see also Onorato et al. 2001). Thus, balancing the nonlinear term and the dispersive term in the narrowband version of Eq. (7) therefore gives the dimensionless number (see also Onorato et al. 2006)

$$-\frac{v_g^2}{c^2}\frac{gT_{0,0,0,0}}{\omega_0}\frac{1}{k_0^4\omega_0''}\frac{s^2}{\sigma_\omega'^2}. \tag{42}$$

Since our interest is in the dynamics of a continuous spectrum of waves the slope parameter s and the relative width σ'_ω of the frequency spectrum relate to spectral properties, hence $s = (k_0^2\langle\eta^2\rangle)^{1/2}$, with $\langle\eta^2\rangle$ the average surface elevation variance, and $\sigma'_\omega = \sigma_\omega/\omega_0$. For positive sign of the dimensionless parameter in Eq. (42) there is focusing (modulational instability), while in the opposite case there is defocusing, of the weakly nonlinear wave train.

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Using the dispersion relation for deep-water gravity waves and the deep-water expression for the nonlinear interaction coefficient, $T_{0,0,0,0} = k_0^3$, the BF Index [which is basically the square root of Eq. (42)] becomes

$$\mathrm{BFI} = \frac{s\sqrt{2}}{\sigma'_\omega}. \tag{43}$$

However, in the general shallow-water case the appropriate dimensionless parameter (denoted by B_S in order to avoid confusion with the deep-water case) becomes

$$B_S = \frac{s\sqrt{2}}{\sigma'_\omega}\frac{v_g}{c_0}(g|T_{0,0,0,0}|/k_0^4\omega_0|\omega_0''|)^{1/2}. \tag{44}$$

For $k_0h_0 < 1.363$ the factor $T_{0,0,0,0}$ increases rapidly with decreasing depth and the B_S parameter becomes quickly much larger than 1. In other words, one then deals with a strongly nonlinear problem. The present form of the kinetic equation has been obtained for weakly nonlinear waves only (i.e., $B_S < 1$). This condition imposes serious restrictions on, in particular, the steepness of the waves, and for $k_0h_0 < 1.363$ the condition $B_S < 1$ was not satisfied in the present Monte Carlo simulations of the Zakharov equation.

6. Conclusions

The threshold value for instability $k_0h_0 = 1.363$ plays an important role in understanding the generation of freak waves and in understanding the spectral evolution in shallow water. A simple scaling of the deep-water nonlinear transfer for shallow-water cases does not seem to exist. Nonlinear energy transfer in intermediate water depths has been studied before. Herterich and Hasselmann (1980) also mentioned that the shallow-water ($kh < 0.7$) energy transfer cannot be scaled using the deep-water transfer, but according to these authors this occurs for much smaller values with respect to 1.363 as found in this work. For very small values of k_0h_0 the perturbation approach breaks down and Herterich and Hasselmann (1980) do not discuss this problem any further. However, there is a large body of literature from the 1960s pointing out that the narrow-band approximation to the nonlinear transfer vanishes at $k_0h_0 = 1.363$. For these values of dimensionless depth the perturbation approach is appropriate, and therefore one should deal with the nonscaling behavior of the shallow-water nonlinear energy transfer. In addition, wind waves at these dimensionless depths are a common feature near oil rigs and buoys; hence use of a more appropriate scaling factor, for example, the one from Eq. (34), should be investigated.

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APPENDIX

Evaluation of the Potential for a Single Wave

Let $\hat{\phi}$ be the Fourier transform of the velocity potential ϕ and let $\hat{\psi}$ be the Fourier transform of the value of the potential at the surface. To second order in amplitude one then finds the following relation between $\hat{\phi}$ and $\hat{\psi}$:

$$\hat{\phi}_1 = \tanh(k_1h_0)\left(\hat{\psi}_1 - \int d\mathbf{k}_{2,3}\, q_2\hat{\psi}_2\hat{\eta}_3\delta_{1-2-3}\right). \tag{A1}$$

The velocity potential is then given by

$$\phi(x) = \int d\mathbf{k}\hat{\phi}(\mathbf{k})\frac{\cosh[k(z+h_0)]}{\sinh(kh_0)}e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{A2}$$

and using Eq. (A1) the potential at $z = 0$, the mean surface, becomes

$$\phi(x) = \int d\mathbf{k}\hat{\psi}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} - \int d\mathbf{k}_{1,2,3}\, e^{i\mathbf{k}_1\cdot\mathbf{x}}\, q_2\hat{\psi}_2\hat{\eta}_3\delta_{1-2-3}. \tag{A3}$$

Note that $\hat{\psi}$ and $\hat{\eta}$ are given in terms of the action variable $A(\mathbf{k})$ by Eq. (9), while for a single wave the action variable is given by Eq. (24). There are two contributions to $\phi(x)$, which are denoted by $\mathcal{A}$ and $\mathcal{B}$.

The first one, $\mathcal{A}$, is given by

$$\mathcal{A} = \int d\mathbf{k}\hat{\psi}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} = -i\int d\mathbf{k}\sqrt{\frac{g}{2\omega}}A(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} + \text{ c.c.} \tag{A4}$$

Making use of Eq. (24) and the introduction of the amplitude η_0, one finds in the limit of small ϵ

$$\mathcal{A} = \frac{g\eta_0}{\omega_0}\sin\theta - \frac{g\eta_0^2}{\omega_0}\sqrt{\frac{g}{2\omega_2}}[B_2(2\mathbf{k}_0) - B_{-2}(-2\mathbf{k}_0)]\sin 2\theta - \frac{1}{2}\frac{g\eta_0^2}{\omega_0}\sqrt{\frac{g}{2\omega(\epsilon)}}[B_0(\epsilon) - B_0(-\epsilon)]\sin\epsilon\cdot\mathbf{x}, \tag{A5}$$

thus giving a linear oscillation, a second harmonic, and a mean flow contribution.

The second one, $\mathcal{B}$, reads

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$$\mathcal{B} = -\int \mathrm{d}\mathbf{k}_{1,2,3}\, e^{i\mathbf{k}_1\cdot\mathbf{x}} q_2 \hat{\psi}_2 \hat{\eta}_3 \delta_{1-2-3}. \tag{A6}$$

This is already quadratic in amplitude so only the linear representation of $\hat{\psi}$ and $\hat{\eta}$ is required. As a result one finds that

$$\mathcal{B} = -\frac{1}{2}\frac{g\eta_0^2}{\omega_0} q_0 \sin 2\theta. \tag{A7}$$

Combining the two one finds for the potential of a single wave that

$$\phi(x) = \frac{g\eta_0}{\omega_0}\sin\theta - \frac{g\eta_0^2}{\omega_0}\sin 2\theta\left\{\sqrt{\frac{g}{2\omega_2}}[B_2(2\mathbf{k}_0) - B_{-2}(-2\mathbf{k}_0)] + \frac{1}{2}q_0\right\} - \frac{1}{2}\frac{g\eta_0^2}{\omega_0}\sqrt{\frac{g}{2\omega(\boldsymbol{\epsilon})}}[B_0(\boldsymbol{\epsilon}) - B_0(-\boldsymbol{\epsilon})]\sin\boldsymbol{\epsilon}\cdot\mathbf{x}. \tag{A8}$$

Making use of the expressions for B_0, B_2, and B_{-2}, one finds in the limit of small $\boldsymbol{\epsilon}$ that

$$\phi(x) = \beta x + \frac{g\eta_0}{\omega_0}\sin\theta + \nu\eta_0^2\sin 2\theta, \tag{A9}$$

where

$$\nu = \frac{3}{8}\frac{\omega_0}{T_0^4}(1 - T_0^4), \tag{A10}$$

while

$$\beta = -\frac{1}{4}\frac{k_0\eta_0^2}{c_S^2 - v_g^2}\left[\frac{gv_g}{T_0}(1 - T_0^2) + \frac{2g^2}{\omega_0}\right]. \tag{A11}$$

Note that Eq. (A10) is in complete agreement with Whitham [1974, his Eq. (13.123) for $z = 0$]. From the variational approach Whitham finds that the mass flux involves the normal contribution of the current β and a contribution by the waves. This therefore suggests introducing the mass transport velocity U as

$$U = \beta + u_w, \tag{A12}$$

where, with $E = (1/2)g\eta_0{}^2$, as expected

$$u_w = \frac{E}{c_0 h_0} = \frac{1}{2}\frac{gk_0}{\omega_0 h_0}\eta_0^2. \tag{A13}$$

Whitham deduces the following relation between U and the mean elevation $b = k_0\eta_0^2\Delta$:

$$U = \frac{v_g}{h_0} b. \tag{A14}$$

It is concluded that the narrowband version of the Zakharov equation is in complete accord with the results of Whitham's variational approach.

REFERENCES

Benjamin, T. B., 1967: Instability of periodic wavetrains in nonlinear dispersive systems. *Proc. Roy. Soc. London,* **A299,** 59–75.

——, and J. E. Feir, 1967: The desintegration of wavetrains on deep water. Part 1: Theory. *J. Fluid Mech.,* **27,** 417–430.

Crawford, D. R., B. M. Lake, P. G. Saffman, and H. C. Yuen, 1981: Stability of weakly nonlinear deep-water waves in two and three dimensions. *J. Fluid Mech.,* **105,** 177–191.

Davey, A., and K. Stewartson, 1974: On three-dimensional packets of surface waves. *Proc. Roy. Soc. London,* **338,** 101–110.

Francius, M., and C. Kharif, 2006: Three dimensional instabilities of periodic gravity waves in shallow water. *J. Fluid Mech.,* **561,** 417–437.

Gorman, R. M., 2003: The treatment of discontinuities in computing the nonlinear energy transfer for finite-depth gravity wave spectra. *J. Atmos. Oceanic Technol.,* **20,** 206–216.

Hasimoto, H., and H. Ono, 1972: Nonlinear modulation of gravity waves. *J. Phys. Soc. Japan,* **33,** 805–811.

Herterich, K., and K. Hasselmann, 1980: A similarity relation for the nonlinear energy transfer in a finite-depth gravity-wave spectrum. *J. Fluid Mech.,* **97,** 215–224.

Janssen, P. A. E. M., 2003: Nonlinear four-wave interactions and freak waves. *J. Phys. Oceanogr.,* **33,** 863–884.

Kharif, C., and E. Pelinovsky, 2003: Physical mechanisms of the rogue wave phenomenon. *Eur. J. Mech. B,* **22,** 603–634.

Krasitskii, V. P., 1994: On reduced equations in the Hamiltonian theory of weakly nonlinear surface waves. *J. Fluid Mech.,* **272,** 1–20.

Kristiansen, O., D. Fructus, D. Clamond, and J. Grue, 2005: Simulations of crescent water wave patterns on finite depth. *Phys. Fluids,* **17,** doi:10.1063/1.1920351.

Lin, R. Q., and W. Perrie, 1997: A new coastal wave model. Part III: Nonlinear wave–wave interaction. *J. Phys. Oceanogr.,* **27,** 1813–1826.

Mei, C. C., 1983: *The Applied Dynamics of Ocean Surface Waves*. John Wiley and Sons, 740 pp.

Onorato, M., A. R. Osborne, M. Serio, and S. Bertone, 2001: Freak waves in random oceanic sea states. *Phys. Rev. Lett.,* **86,** 5831–5834.

——, ——, ——, C. Brandini, and C. T. Stansberg, 2004: Observation of strongly non-Gaussian statistics for random sea surface gravity waves in wave flume experiments. *Phys. Rev. E,* **70,** doi:10.1103/PhysRevE.70.067302.

——, ——, ——, and L. Cavaleri, 2005: Modulational instability and non-Gaussian statistics in experimental random water-wave trains. *Phys. Fluids,* **17,** doi:10.1063/1.1946769.

——, ——, ——, ——, C. Brandini, and C. T. Stansberg, 2006: Extreme waves, modulational instability and second order theory: Wave flume experiments on irregular waves. *Eur. J. Mech. B,* **25,** 586–601.

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Resio, D. T., J. H. Pihl, B. A. Tracey, and C. L. Vincent, 2001: Nonlinear energy fluxes and the finite depth equilibrium range in wave spectra. *J. Geophys. Res.,* **106,** 6985–7000.

Stiassnie, M., 1984: Note on the modified nonlinear Schrödinger equation for deep water waves. *Wave Motion,* **6,** 431–433.

Whitham, G. B., 1974: *Linear and Nonlinear Waves*. John Wiley and Sons, 636 pp.

Zakharov, V. E., 1968: Stability of periodic waves of finite amplitude on the surface of a deep fluid. *J. Appl. Mech. Technol. Phys.,* **9,** 190–194.

——, 1992: Inverse and direct cascade in the wind-driven surface wave turbulence and wave-breaking. *Breaking Waves: IUTAM Symposium, Sydney, Australia, 1991,* M. L. Banner and R. H. J. Grimshaw, Eds., Springer-Verlag, 69–91.

——, 1999: Statistical theory of gravity and capillary waves on the surface of a finite-depth fluid. *Eur. J. Mech. B,* **18,** 327–344.

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