THE INTERPOLATION OF QUADRATIC NORMS

BY

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In Memory—R. E, Fullerton 1916–1963

The theory of the interpolation of Banach spaces has been widely developed in recent years by a number of authors. It is natural to expect that the interpolation theory for Hilbert spaces should have a particularly simple character; that this is in fact the case is shown in the present study where a complete description of exact quadratic interpolation norms and exact quadratic interpolation methods is given. Since the literature on interpolation theory has been characterized by an expert as impenetrable [11] it has seemed worthwhile to make the exposition as complete and detailed as possible.

Our arguments depend in an essential way on the beautiful theory of monotone matrix functions and Cauchy interpolation problems discovered by Loewner in 1934, and our theory may be regarded as a natural application of Loewner's results.

The description of the exact quadratic interpolation methods, given by our Theorem 2, has already been found by Foias and Lions [6] who establish a corresponding result under somewhat stronger hypotheses. It should be emphasized that our definition of interpolation norms and interpolation methods differs only superficially from that regularly used in the literature [3]. We should also remark that the functions $k(\lambda)$ which give rise to the exact quadratic interpolations are the positive functions, concave of infinite order on the unit interval. This class has been studied by Krauss [7] and also Bendat and Sherman [4].

Let V be a linear space over the complex numbers upon which there is defined a pair of norms $\|x\|_0$ and $\|x\|_1$. We shall usually assume that those norms are compatible, that is to say, that any sequence $\{x_k\}$ in V which is simultaneously Cauchy for both norms,

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and which converges to 0 for one of the norms, necessarily converges to 0 for the other. We introduce the norm $\|x\|_2$ defined by the equation $\|x\|_2^2 = \|x\|_0^2 + \|x\|_1^2$ and the Banach space $\mathcal{H}_2$, the completion of V with respect to that norm. Since the norm of $\mathcal{H}_2$ majorates the initial norms on V, it is evident that these norms may be extended by continuity to the whole of $\mathcal{H}_2$ in a unique way. We write the extended forms in the same way, viz. $\|x\|_0$ and $\|x\|_1$ and we note that for these extended functions the inequalities $\|x\|_0 \leq \|x\|_2$ and $\|x\|_1 \leq \|x\|_2$ are valid. The extended functions are obviously semi-norms on the space $\mathcal{H}_2$ and our hypothesis that the initial norms were compatible is equivalent to the assertion that the extended functions are norms and not merely semi-norms on $\mathcal{H}_2$.

A norm $\|x\|_*$ defined on $\mathcal{H}_2$ will be called an interpolation norm there (relative to the pair of initial norms, of course) if and only if it is compatible with the norm of $\mathcal{H}_2$ and has the property that every linear transformation T of $\mathcal{H}_2$ into itself which is continuous when that space is given the norm $\|x\|_0$ and which is also continuous when that space is given the norm $\|x\|_1$ must also be continuous when the space is given the norm $\|x\|_*$. It is not difficult to show that this is equivalent to the following more formal definition. A norm $\|x\|_*$ on $\mathcal{H}_2$ is an interpolation norm there if and only if

(i) $\|x\|_*$ is compatible with $\|x\|_2$

(ii) there exists a constant C_* such that any linear transformation T on $\mathcal{H}_2$ into itself which satisfies $\|Tx\|_k \leq \|x\|_k$ for all x in $\mathcal{H}_2$ and $k=0, 1$ must satisfy $\|Tx\|_* \leq C_* \|x\|_*$ for all x.

Since the identity is such a linear transformation T, evidently $C_* \geq 1$. It is also clear that the transformations T considered in our definition must be continuous linear transformations of $\mathcal{H}_2$ into itself of bound at most 1. We also note that the inequalities occurring in the definition need only be supposed to hold for elements x belonging to V since V is dense in $\mathcal{H}_2$. The restriction of an interpolation norm to V will be called an interpolation norm on that space.

It would seem more natural not to invoke the space $\mathcal{H}_2$ in the definition of interpolation norms and to consider the family of transformations T mapping V into itself which are continuous both for the norm $\|x\|_0$ as well as the norm $\|x\|_1$, however, the class of transformations so determined may be too restricted for convenient applications of the theory; we therefore take transformations mapping V into $\mathcal{H}_2$.

The interpolation norms are always continuous relative to the norm $\|x\|_2$; there exists a constant M such that $\|x\|_* \leq M\|x\|_2$. This is a consequence of the fact that the space $\mathcal{H}_2$ is complete relative to the norm $\|x\|_2 + \|x\|_*$; this norm is compatible with $\|x\|_2$ and therefore is equivalent to it.

An interpolation norm is called *exact* if the constant C_* of the definition may be taken equal to 1.

Throughout our discussion we will be concerned only with quadratic norms and quadratic interpolation norms. The space $\mathcal{H}_2$ will therefore be a Hilbert space and the quadratic norm $\|x\|_0$ will give rise to a positive, bounded operator in a known way. We will have $\|x\|_0^2 = (Hx, x)_2$ for all x in $\mathcal{H}_2$ and some operator H for which $0 \leqslant H \leqslant I$, where I is the identity operator on $\mathcal{H}_2$. We see immediately that $\|x\|_1^2 = ((I-H)x, x)_2$ and that $0 \leqslant I-H \leqslant I$. The hypothesis that the initial norms were compatible on V, which was equivalent to the hypothesis that the initial norms on $\mathcal{H}_2$ were norms and not semi-norms is clearly equivalent to the hypothesis that the numbers 0 and 1 do not occur as eigenvalues of the operator H. Later in our discussion we will want to remove this hypothesis, to consider incompatible semi-norms on V, and to admit the numbers 0 and 1 as eigenvalues of H, but this is not convenient at the outset.

The specification of H is a complete description of the pair of initial norms. A quadratic interpolation norm similarly corresponds to a positive and bounded operator K where $\|x\|_*^2 = (Kx, x)_2$.

If T is a continuous linear transformation on $\mathcal{H}_2$, the assertion that T is continuous relative to the norm $\|x\|_0$ is the assertion that there exists a positive number t_0 such that for all x in $\mathcal{H}_2$ $\|Tx\|_0^2 \leqslant t_0 \|x\|_0^2$ and this inequality may be written $(HTx, Tx)_2 \leqslant t_0 (Hx, x)_2$. This is equivalent to the operator inequality $T^*HT \leqslant t_0 H$. In a similar way, the continuity of T relative to the other initial norm may be written $T^*(I-H)T \leqslant t_1(I-H)$. Thus a positive and bounded operator K corresponds to a quadratic interpolation norm if and only if for any operator T on $\mathcal{H}_2$ the inequalities $T^*HT \leqslant H$ and $T^*(I-H)T \leqslant I-H$ together imply $T^*KT \leqslant C_* K$. The quadratic interpolation norm will be exact if $C_* = 1$ here.

In the present study we are concerned exclusively with the exact quadratic interpolation norms, hence we seek, associated with any operator H on $\mathcal{H}_2$ for which $0 < H < I$ the class of all positive and bounded operators K for which the hypothesis

$$T^*HT \leqslant H \quad \text{and} \quad T^*(I-H)T \leqslant I-H \tag{1}$$

implies

$$T^*KT \leqslant K. \tag{2}$$

It is clear that the class of such operators forms a convex cone, completely determined by the operator H. Our principal result, Theorem 1 below, asserts that this cone consists precisely of operators of the form $K = k(H)$ where the function $k(\lambda)$ is given by the formula

$$k(\lambda) = \int_0^1 \frac{\lambda(1-\lambda)}{\lambda s + (1-\lambda)(1-s)}\, d\varrho(s), \tag{3}$$

$d\varrho(s)$ being a positive Radon measure on the unit interval $0 \leqslant s \leqslant 1$.

THEOREM 1. *A norm $\|x\|_*$ on $\mathcal{H}_2$ is an exact quadratic interpolation norm relative to the pair of initial norms determined by H and $I-H$ if and only if there exists a monotone non-decreasing bounded function $\varrho(s)$ on the closed unit interval such that*

$$\|x\|_*^2 = \int_0^1 \int_0^1 \frac{\lambda(1-\lambda)}{\lambda s + (1-\lambda)(1-s)} \, d\varrho(s) \, d(E_\lambda x, x)_2, \tag{4}$$

where $H = \int_0^1 \lambda \, dE_\lambda$.

The proof of the theorem is lengthy; before embarking on it we turn to the subject of exact quadratic interpolation methods. In most applications of interpolation theory, one is presented with two spaces V' and V'' on each of which is defined a pair of initial norms. As before we pass to spaces $\mathcal{H}_2'$ and $\mathcal{H}_2''$ and seek norms $\|x'\|_*$ and $\|x''\|_*$ on each of these spaces which are to be compatible with the norms of those spaces and which are to be such that the linear transformations T from $\mathcal{H}_2'$ to $\mathcal{H}_2''$ which are continuous when both spaces are provided with the norms $\|x'\|_0$ and $\|x''\|_0$ respectively, and which are also continuous when the spaces are provided with the norms $\|x'\|_1$ and $\|x''\|_1$ respectively, must be continuous when they are given the norms $\|x'\|_*$ and $\|x''\|_*$ respectively. An interpolation method is an assignment of such norms. It is necessary to give a somewhat pedantic definition.

We consider the class of triples $[\mathcal{H}_2, \|x\|_0, \|x\|_1]$ where $\mathcal{H}_2$ is a Banach space with norm $\|x\|_2$ and $\|x\|_k$, $k=0, 1$ are norms on $\mathcal{H}_2$ for which $\|x\|_2^2 = \|x\|_0^2 + \|x\|_1^2$; we also consider the class of pairs $[\mathcal{B}, \|x\|_*]$ where $\mathcal{B}$ is a Banach space and $\|x\|_*$ a norm on $\mathcal{B}$ compatible with the norm of $\mathcal{B}$. We consider mappings M of the first class into the second; such mappings may be conveniently written

$$M[\mathcal{H}_2, \|x\|_0, \|x\|_1] = [M(\mathcal{H}_2), \|x\|_M].$$

An interpolation method is such a function M having the following two properties:

(i) $M(\mathcal{H}_2) = \mathcal{H}_2$ for all Banach spaces $\mathcal{H}_2$ and

(ii) there exists a constant C_M such that if

$$M[\mathcal{H}_2', \|x'\|_0, \|x'\|_1] = [\mathcal{H}_2', \|x'\|_M]$$
$$M[\mathcal{H}_2'', \|x''\|_0, \|x''\|_1] = [\mathcal{H}_2'', \|x''\|_M]$$

then every linear transformation T from $\mathcal{H}_2'$ to $\mathcal{H}_2''$ for which $\|Tx'\|_k \leq \|x'\|_k$ for all x' in $\mathcal{H}_2'$ and $k=0, 1$ also satisfies the inequality $\|Tx'\|_M \leq C_M \|x'\|_M$ for all such x'.

The method is exact if C_M may be taken equal to 1 and is quadratic if all of the norms appearing in the definition are quadratic and the spaces are Hilbert spaces. As a consequence of Theorem 1 we obtain the following description of all exact quadratic interpolation methods.

THEOREM 2. *The exact quadratic interpolation methods are in a one-to-one correspondence with the set of functions $k(\lambda)$ of the form* (3)*; if M corresponds to $k(\lambda)$, then the norm $\|x\|_M$ is given by $\|x\|_M^2=(k(H)x, x)_2$ where H represents $\|x\|_0^2$ in the Hilbert space $\mathcal{H}_2$.*

Proof. We first show that the methods described by the theorem are indeed exact quadratic interpolation methods. Suppose $k(\lambda)$ given and $\mathcal{H}_2'$ and $\mathcal{H}_2''$ two Hilbert spaces on each of which is defined a pair of quadratic initial norms. We form the direct sum of those spaces: $\hat{\mathcal{H}}=\mathcal{H}_2'\oplus\mathcal{H}_2''$ and define on it initial norms and the corresponding operators as follows.

$$\|\hat{x}\|_0^2=\|[x'; x'']\|_0^2=\|x'\|_0^2+\|x''\|_0^2=(\hat{H}\hat{x}, \hat{x})=(H'x', x')_2+(H''x'', x'')_2,$$

$$\|\hat{x}\|_1^2=\|x'\|_1^2+\|x''\|_1^2=((I-\hat{H})\hat{x}, \hat{x})=((I-H')x', x')_2+((I-H'')x'', x'')_2.$$

Now $\|\hat{x}\|_*^2=(k(\hat{H})\hat{x}, \hat{x})=(k(H')x', x')_2+(k(H'')x'', x'')_2$ defines an interpolation norm on the space $\mathcal{H}$. Those transformations T mapping the first factor into the second having bound at most 1 for the initial norms correspond then to transformations $\hat{T}$ on $\hat{\mathcal{H}}$ defined by $\hat{T}[x'; x'']=[0; Tx']$, and $\hat{T}$ has bound at most 1 for the interpolation norm. Hence $\|Tx'\|_*\leqslant\|x'\|_*$.

On the other hand, if we suppose that M is an exact quadratic interpolation method, then, since the triples $[\mathcal{H}_2', \|x'\|_0, \|x'\|_1]$ and $[\mathcal{H}_2'', \|x''\|_0, \|x''\|_1]$ occurring in the definition may happen to coincide, the norm $\|x'\|_M$ must be an exact quadratic interpolation norm in the sense of Theorem 1. Hence there exists a function $k(\lambda)$ which a priori depends on M, $\mathcal{H}_2$ and the initial norms such that $\|x\|_M^2=(k(H)x, x)_2$. We have to show that $k(\lambda)$ depends only on M. Corresponding to the spaces $\mathcal{H}_2'$ and $\mathcal{H}_2''$ each of which is provided with a pair of initial norms we have two functions $k'(\lambda)$ and $k''(\lambda)$ so that $\|x'\|_M^2=(k'(H')x', x')_2$ and $\|x''\|_M^2=(k''(H'')x'', x'')_2$. As in the previous argument we pass to the direct sum $\hat{\mathcal{H}}$ and consider the transformation T which embeds $\mathcal{H}_2'$ into $\hat{\mathcal{H}}$: $Tx'=[x'; 0]$. Since T has bound 1 for the initial norms we deduce that $\|[x'; 0]\|_M<\|x'\|_M$. Similarly, considering the projection of $\hat{\mathcal{H}}$ onto $\mathcal{H}_2'$ we deduce the opposite inequality, and it becomes clear that $\|[x'; x'']\|_M^2=\|x'\|_M^2+\|x''\|_M^2$. Finally, since there exists a function $\hat{k}(\lambda)$ of the form (3) defining the norm $\|\hat{x}\|_M$ on $\hat{\mathcal{H}}$ we have

$$(\hat{k}(\hat{H})\hat{x}, \hat{x})_2=(k'(H')x', x')_2+(k''(H'')x'', x'')_2.$$

Since the spaces $\mathcal{H}_2'$ and $\mathcal{H}_2''$ are reducing subspaces of $\hat{\mathcal{H}}$ for $\hat{H}$ we see that we may take

$k'(\lambda)=\hat{k}(\lambda)$ and $k''(\lambda)=\hat{k}(\lambda)$, and if any of the spectra contain an interval the function $\hat{k}(\lambda)$ is uniquely determined, since all functions of the form (3) are analytic in the open unit interval. Thus the exact quadratic interpolation method M is of the type described by the theorem.

As we remarked in the introduction, a form of Theorem 2 has been established by Foias and Lions [6]; in our terminology these authors suppose that an exact quadratic interpolation method M is defined by a function $m(\lambda)$ so that $\|x\|_M^2=(m(H)x, x)_2$ and deduce that $m(\lambda)$ is necessarily of the form (3); conversely, they show that such functions define exact quadratic interpolation methods.

Virtually all of the rest of the paper consists in the proof of Theorem 1.

Suppose, now, that K is a positive operator which gives rise to an exact interpolation on the space $\mathcal{H}_2$. Thus the inequalities (1) imply (2). In particular, if E is a projection which commutes with the operator H, from the evident inequalities $EHE\leqslant H$ and $E(I-H)E\leqslant I-H$ we deduce that $EKE\leqslant K$. We invoke next the following elementary lemma.

LEMMA 1. *If E is a projection in Hilbert space and K a positive and bounded operator, the inequality $EKE\leqslant K$ implies that E commutes with K.*

Proof. Choose x in the range of E and y in its null space to form $u=x+ty$ where t is any complex number. Now

$$(EKEu, u) = (KEu, Eu) = (Kx, x)\leqslant (Ku, u) = (Kx, x)+|t|^2(Ky, y)+2\mathrm{Re}[\bar{t}(Kx, y)].$$

Since t is arbitrary, evidently $(Kx, y)=0$, whence $(I-E)KE=0$ or $EKE=KE$. Taking adjoints, we find that $EKE=EK=KE$.

It follows from our lemma, then, that any subspace $\mathfrak{M}$ of $\mathcal{H}_2$ which is a reducing subspace for H must also be one for K. If $\mathfrak{M}$ is considered as a space in itself, the restriction of H to $\mathfrak{M}$ gives rise to the restriction of the pair of initial norms, and the restriction of K corresponds to the restriction to $\mathfrak{M}$ of the interpolation norm. Since any continuous linear transformation T of $\mathfrak{M}$ into itself can be extended to a continuous linear transformation $\hat{T}$ of $\mathcal{H}_2$ into itself in such a way that the bounds of T for the norms $\|x\|_0$ and $\|x\|_1$ are not increased (we have only to set $T=0$ on the orthogonal complement of $\mathfrak{M}$), it follows that the restricted interpolation norm is in fact an interpolation norm on $\mathfrak{M}$ relative to the restricted initial norms.

When the space $\mathcal{H}_2$ is separable, the fact that K commutes with every projection commuting with H implies that K is a function of H [10]. We have $K=k(H)=\int k(\lambda)dE_\lambda$ if $H=\int \lambda dE_\lambda$, the function $k(\lambda)$ being measurable with respect to the projection valued measure dE_λ. The function $k(\lambda)$ is determined, of course, only up to a set of spectral measure 0; in particular it is wholly undetermined on the complement of the spectrum of H. Since K is a bounded and positive operator, we may always suppose that $0 \leqslant k(\lambda) \leqslant \|K\|$. We shall show presently that there exists a canonical choice for $k(\lambda)$ which makes it a continuous function on the spectrum of H.

These considerations make it clear that the hypothesis of separability of $\mathcal{H}_2$ which we introduced provisionally above makes no difference. For if K appears as a continuous function of H on every separable reducing subspace we have only to verify that the same function occurs for every such subspace as follows. We choose $\mathcal{M}_0$, a separable reducing subspace of H such that the spectrum of the restriction of H to $\mathcal{M}_0$ coincides with the spectrum of H relative to $\mathcal{H}_2$. On $\mathcal{M}_0$ we have $K = k(H)$ where $k(\lambda)$ is a continuous function on the spectrum. If $\mathcal{M}_1$ be any other separable reducing subspace, so also is the direct sum $\mathcal{M}_0 \oplus \mathcal{M}_1$ upon which K appears as a continuous function of H, necessarily the function $k(\lambda)$ above. Thus $K = k(H)$ on any separable reducing subspace, hence everywhere. There is also no loss of generality in our assuming the space separable in the sequel.

Throughout our arguments we shall make frequent use of this device for the study of K and $k(\lambda)$; we descend to a reducing subspace for H and study K as a function of H on the subspace.

LEMMA 2. *$K = k(H)$ where $k(\lambda)$ is continuous and concave on the spectrum of H.*

Proof. We first consider the case where the spectrum of H does not contain either 0 or 1; there exists therefore a small positive η such that the spectrum is contained in the interval $(\eta, 1-\eta)$. Choose λ_0 in the spectrum and a positive ε for which $\varepsilon < \eta/2$. Let $\mathcal{M}$ be the reducing subspace of H associated with the interval $(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$ and let T be any unitary transformation of $\mathcal{M}$ into itself. For x in $\mathcal{M}$ then

$$\|Tx\|_0^2 \leqslant (\lambda_0 + \varepsilon)\,\|Tx\|_2^2 = (\lambda_0 + \varepsilon)\,\|x\|_2^2 \leqslant \frac{\lambda_0 + \varepsilon}{\lambda_0 - \varepsilon}\,\|x\|_0^2 = \left(1 + \frac{2\varepsilon}{\lambda_0 - \varepsilon}\right)\|x\|_0^2,$$

and since $\lambda_0 - \varepsilon$ is at least $\eta/2$ we have $\|Tx\|_0^2 \leqslant (1 + 4\varepsilon/\eta)\,\|x\|_0^2$. Similarly

$$\|Tx\|_1^2 \leqslant (1 + 4\varepsilon/\eta)\,\|x\|_1^2$$

and hence $\|Tx\|_*^2 \leqslant (1 + 4\varepsilon/\eta)\,\|x\|_*^2$.

Let M be the essential supremum of $k(\lambda)$ over the interval $(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$, that is to say, the supremum of all numbers t such that the set $k(\lambda) \geqslant t$ has positive spectral measure; similarly m is the essential infimum of that function. Both numbers are finite and positive

since $k(\lambda)$ gives rise on $\mathcal{M}$ to a norm equivalent to that of $\mathcal{H}_2$. For a small ε' we can therefore find a set E_1 with positive spectral measure supporting a normalized element x_1 in $\mathcal{M}$ such that $k(\lambda) > M - \varepsilon'$ on E_1; there is also another measurable set E_2 supporting a normalized x_2 where $k(\lambda) < m + \varepsilon'$ on E_2. Evidently there exists a unitary transformation T of $\mathcal{M}$ into itself which carries x_2 into x_1.
Accordingly

$$M - \varepsilon' = (M - \varepsilon')\,\|x_1\|_2^2 \leqslant \|x_1\|_*^2 \leqslant (1 + 4\,\varepsilon/\eta)\,\|x_2\|_*^2 \leqslant (1 + 4\,\varepsilon/\eta)\,(m + \varepsilon')\,\|x_2\|_2^2$$
$$= (1 + 4\,\varepsilon/\eta)\,(m + \varepsilon'),$$

and since ε' is arbitrarily small, $M/m < 1 + 4\varepsilon/\eta$, whence

$$M - m \leqslant 4m\,\varepsilon/\eta \leqslant 4\|K\|\varepsilon/\eta.$$

For such small ε we next define on the spectrum the functions

$$M_\varepsilon(\lambda) = \text{essential sup } k(\lambda) \quad \text{over} \quad (\lambda - \varepsilon, \lambda + \varepsilon)$$

as well as
$$m_\varepsilon(\lambda) = \text{essential inf } k(\lambda) \quad \text{over} \quad (\lambda - \varepsilon, \lambda + \varepsilon).$$

Evidently
$$0 \leqslant M_\varepsilon(\lambda) - m_\varepsilon(\lambda) \leqslant 4\|K\|\varepsilon/\eta.$$

As ε approaches 0, the functions $M_\varepsilon(\lambda)$ diminish monotonically, converging uniformly to a function $k^*(\lambda)$ on the spectrum which is also the uniform limit of the monotone increasing family $m_\varepsilon(\lambda)$. It is easy to see that $k^*(\lambda)$ is continuous on the spectrum, and since the inequality $m_\varepsilon(\lambda) \leqslant k(\lambda) \leqslant M_\varepsilon(\lambda)$ holds almost everywhere for the spectral measure dE_λ, for all small ε, the function $k^*(\lambda)$ is then equivalent to $k(\lambda)$ for that measure. This canonical determination of $k(\lambda)$ is evidently unique. We now show that it is a concave function on the spectrum of H. Let f, g be a pair of elements of $\mathcal{H}_2$ and let T be the linear transformation defined by $Tx = (x, g)_2 f$. Since K has a bounded inverse, we may write T in the form $Tx = (x, K^{-1}g)_* f$, whence

$$\|Tx\|_*^2 = |(x, K^{-1}g)_*|^2\,\|f\|_*^2 = |(x, K^{-1}g)_*|^2\,(Kf, f)_2.$$

From the fact that equality can be attained in the Schwartz inequality we obtain the bound of T when the space is given the interpolation norm: $\|T\|_*^2 = (K^{-1}g, g)_2 (Kf, f)_2$ and by the same calculation we obtain the bounds of T relative to either of the initial norms:

$$\|T\|_0^2 = (H^{-1}g, g)_2 (Hf, f)_2 \quad \text{and} \quad \|T\|_1^2 = ((I - H)^{-1}g, g)_2((I - H)f, f)_2.$$

Since K corresponds to an exact interpolation, we find that for all f and g

$$(K^{-1}g, g)_2(Kf, f)_2 \leqslant \max\,[(H^{-1}g, g)_2(Hf, f)_2,\ ((I-H)^{-1}g, g)_2((I-H)f, f)_2].$$

Consider three points in the spectrum of H: $0<\lambda_1<\lambda_2<\lambda_3<1$ and three elements e_1, e_2 and e_3 each normalized in $\mathcal{H}_2$ and belonging to the spectral subspaces for H associated, respectively, with the intervals $(\lambda_1-\varepsilon, \lambda_1+\varepsilon)$, $(\lambda_2-\varepsilon, \lambda_2+\varepsilon)$ and $(\lambda_3-\varepsilon, \lambda_3+\varepsilon)$ where the positive ε is so small that these intervals are disjoint and contained in the interval $(\eta/2, 1-\eta/2)$. We choose $g=e_2$ and $f=\alpha e_1+\beta e_3$ where the positive α and β are so chosen that $\lambda_2=\alpha^2\lambda_1+\beta^2\lambda_3$, and $1-\lambda_2=\alpha^2(1-\lambda_1)+\beta^2(1-\lambda_3)$. Now $(H^{-1}g, g)_2\leqslant(\lambda_2-\varepsilon)^{-1}$ and $(Hf, f)_2\leqslant\alpha^2(\lambda_1+\varepsilon)+\beta^2(\lambda_3+\varepsilon)=\lambda_2+\varepsilon$. Thus

$$(H^{-1}g, g)_2\,(Hf, f)_2\leqslant 1+\frac{2\,\varepsilon}{\lambda_2-\varepsilon}\quad\text{and similarly }((I-H)^{-1}g, g)_2\,((I-H)\,f, f)_2\leqslant 1+\frac{2\,\varepsilon}{(1-\lambda_2)-\varepsilon}$$

and therefore $(K^{-1}g, g)_2(Kf, f)_2\leqslant 1+O(\varepsilon)$.

From the uniform continuity of $k(\lambda)$ on the closed spectrum of H it follows that there exists a positive $\omega=\omega(\varepsilon)$ which diminishes to 0 as ε does and such that $(K^{-1}g, g)_2\geqslant(k(\lambda_2)+\omega)^{-1}$ and $(Kf, f)_2\geqslant\alpha^2k(\lambda_1)+\beta^2k(\lambda_3)-\omega$. Accordingly

$$\alpha^2k(\lambda_1)+\beta^2k(\lambda_3)\leqslant[1+O(\varepsilon)][k(\lambda_2)+\omega]+\omega,$$

where $a^2\lambda_1+\beta^2\lambda_3=\lambda_2$. As ε approaches 0 we obtain the concavity of $k(\lambda)$. If $[a, b]$ is the smallest closed interval containing the spectrum of H, it is evident that we can obtain a concave positive function on $[a, b]$ which coincides with $k(\lambda)$ on the spectrum by defining the extended function so that it is linear on the complementary intervals. This function is clearly Lipschitzian.

There remains the case when the spectrum of H is not contained in an interval of the form $(\eta, 1-\eta)$. However, from the foregoing argument, if we consider the interpolation norm on the reducing subspace for H associated with the spectral interval $(\eta, 1-\eta)$, we see that $k(\lambda)$ has a canonical determination as a continuous and concave function on that interval. Thus $k(\lambda)$ may be defined at one or more of the end points of the unit interval, neither of which is an eigenvalue for H, in such a way that $k(\lambda)$ is concave and continuous on the whole spectrum. The proof is complete.

LEMMA 3. *If H' and H'' are two operators with the same spectrum, and $k(\lambda)$ is a continuous function on the spectrum such that $K'=k(H')$ is an exact interpolation operator for H', then $K''=k(H'')$ gives rise to an exact interpolation for H''.*

For the proof of Lemma 3 it will be necessary to make use of the following elementary lemma.

LEMMA 4. *If A and B are two positive operators with spectrum in the interval $(\eta, 1-\eta)$ where $0<\eta<\frac{1}{2}$, and if $\|A-B\|<\varepsilon$ and $\lambda\geqslant 1$ then for transformations T of bound 1 the inequality $T^*AT\leqslant\lambda A$ implies $T^*BT\leqslant\lambda(1+2\,\varepsilon/\eta)\,B$.*

The proof is quite straightforward:

$$T^*BT = T^*(B-A)\,T+T^*AT\leqslant\lambda A+\varepsilon I=\lambda B+\lambda(A-B)+\varepsilon I\leqslant\lambda B+\lambda 2\varepsilon I\leqslant\lambda B+\lambda 2\varepsilon/\eta B.$$

Proof of Lemma 3. As before, we first suppose that the spectrum of H' and H'' is bounded away from the end points of the unit interval, and therefore contained in an interval of the form $(\eta, 1-\eta)$. Since the function $k(\lambda)$ may be taken to be a Lipschitzian function on the whole interval, we may suppose that its Lipschitz constant is at most 1, because $k(\lambda)$ gives rise to an exact interpolation norm if and only if $Ck(\lambda)$ does, where C is any positive constant.

Let $g(\lambda)$ be a monotone increasing function on the unit interval which assumes only a finite number of values, and for which uniformly $|g(\lambda)-\lambda|<\varepsilon$ on the spectrum of H' where $\varepsilon<\eta/2$. We also require that the finite set of numbers which $g(\lambda)$ assumes be points of the spectrum of H'. We form the operators $G'=g(H')$ and $G''=g(H'')$ and note that $\|H''-G''\|<\varepsilon$ and $\|H'-G'\|<\varepsilon$. The operators G' and G'' have the same spectrum, and the function $k(\lambda)$ is unambiguously defined on that spectrum; we form therefore $K_1=k(G')$ and $K_2=k(G'')$. Now $K_1-K'=k(G')-k(H')$, and the Lipschitz constant of $k(\lambda)$ being 1, we infer that $\|K_1-K'\|<\varepsilon$ as well as $\|K_2-K''\|<\varepsilon$. We may take η a lower bound for $k(\lambda)$ on the spectrum. The hypothesis $T^*G'T\leqslant G'$ and $T^*(I-G')\,T\leqslant(I-G')$ for operators T of bound 1, implies, by virtue of Lemma 4, the inequalities $T^*H'T\leqslant(1+2\varepsilon/\eta)H'$ and $T^*(I-H')\,T\leqslant(1+2\varepsilon/\eta)(I-H')$ and because K' gives rise to an interpolation norm for H' we then have $T^*K'T\leqslant(1+2\varepsilon/\eta)K'$, whence by Lemma 4, $T^*K_1T\leqslant(1+2\varepsilon/\eta)^2K_1$. Since there is a unitary equivalence between G' and G'' and K_1 and K_2, at least when the spectrum of H' has no isolated points, it follows that the hypothesis $T^*G''T\leqslant G''$ and $T^*(I-G'')\,T\leqslant(I-G'')$ implies that $T^*K_2T\leqslant(1+2\varepsilon/\eta)^2K_2$. Accordingly, $T^*H''T\leqslant H''$ and $T^*(I-H'')\,T\leqslant I-H''$ together imply $T^*G''T\leqslant(1+2\varepsilon/\eta)G''$ and $T^*(I-G'')\,T\leqslant(1+2\varepsilon/\eta)$ $(I-G'')$ and therefore $T^*K_2T\leqslant(1+2\varepsilon/\eta)^3K_2$. Because of Lemma 4, finally, $T^*K''T\leqslant$ $(1+2\varepsilon/\eta)^4K''$. Since the ε was arbitrary, $T^*K''T\leqslant K''$ is then a consequence of the inequalities. The lemma is therefore proved when the spectrum of H' has no isolated points and is bounded away from the end points of the interval.

In this argument we have supposed that there were no isolated points in the spectrum of H' in order that the operators G' and G'' should be unitarily equivalent. Even if there were such isolated points, they would occur as eigenvalues of H' and H'' and if these eigenvalues had the same multiplicity that unitary equivalence would still exist and our previous argument would hold. We have now to show that the multiplicity of these eigenvalues is immaterial. Obviously there is a reducing subspace for H' such that the part of H' in that subspace has the same spectrum as that of H' on $\mathcal{H}_2$ and such that the isolated points of it occur as eigenvalues of unit multiplicity. On that subspace $k(\lambda)$ gives rise to an exact quadratic interpolation, and there is no loss of generality if we suppose that these isolated points occur as eigenvalues of unit multiplicity for H'.

If $K''=k(H'')$ does not give rise to an exact interpolation for H'' there exists a linear transformation T of bound 1 in $\mathcal{H}_2$ such that $T^*H''T \leqslant H''$ and $T^*(I-H'')T \leqslant I-H''$ and a normalized x in $\mathcal{H}_2$ for which $(K''Tx, Tx)_2 > (K''x, x)_2$. We write the expansion of x and Tx as follows:

$$x = \sum a_i f_i + f^*,$$
$$Tx = \sum b_i f'_i + f'^*.$$

Here the elements f^*, f'^* are orthogonal to the linear span of the eigenspaces $\mathfrak{M}_i$ associated with isolated eigenvalues of H''; the elements f_i and f'_i both belong to $\mathfrak{M}_i$ and are normalized. We take U as a unitary transformation on $\mathcal{H}_2$ which reduces to the identity on the orthogonal complement of the $\mathfrak{M}_i$ and which leaves each $\mathfrak{M}_i$ invariant, but carries f'_i into f_i. Evidently U commutes with H'', and if we form $S=UT$ we have $S^*H''S \leqslant H''$ as well as $S^*(I-H'')S \leqslant I-H''$ and also $(K''Sx, Sx)_2 > (K''x, x)_2$. Passing finally to the reducing subspace of H'' determined by the orthogonal complement of the $\mathfrak{M}_i$ as well as the span of the sequence $\{f_i\}$ we see that the part of H'' in that subspace cannot admit $k(\lambda)$ as an exact interpolation function, although the isolated points of its spectrum are eigenvalues of unit multiplicity. This is a contradiction.

The proof of the lemma will be complete if we consider finally the case when the spectrum of H' is not bounded away from the end points of the unit interval. We argue as before: if $K''=k(H'')$ does not give rise to an exact quadratic interpolation for H'' there exists T for which $T^*H''T \leqslant H''$ as well as $T^*(I-H'')T \leqslant I-H''$ and a normalized x in $\mathcal{H}_2$ such that $(K''Tx, Tx)_2 > (K''x, x)_2$. The latter inequality may be written in integral form

$$\int_0^1 k(\lambda)\, d(E''_\lambda Tx, Tx)_2 > \int_0^1 k(\lambda)\, d(E''_\lambda x, x)_2.$$

Since the end points carry no positive mass for these measures, there evidently exists a small positive η for which

$$\int_\eta^{1-\eta} k(\lambda)\, d(E''_\lambda Tx, Tx)_2 > \int_\eta^{1-\eta} k(\lambda)\, d(E''_\lambda x, x)_2.$$

Accordingly, if $\mathcal{M}=P''_\eta(\mathcal{H}_2)$ where $P''_\eta=E''_{1-\eta}-E''_\eta$ the function $k(\lambda)$ cannot give rise to an exact quadratic interpolation associated with the part of H'' in $\mathcal{M}$; this, in turn contradicts the part of the lemma which has been proved, since $k(\lambda)$ does give rise to such an interpolation for the part of H' in the range $P'_\eta(\mathcal{H}_2)$ where $P'_\eta=E'_{1-\eta}-E'_\eta$. The proof is complete.

In view of the previous lemmas we are able to formulate our problem as follows. Associated with every closed subset F of the unit interval which does not contain the end points of that interval as isolated points, there exists a convex cone C_F of non-negative functions continuous on F such that if $k(\lambda)$ belongs to C_F and H is any positive operator, the spectrum of which is a subset of F, then $K=k(H)$ gives rise to an exact quadratic interpolation for H. Moreover, every function giving rise to such an interpolation is contained in the cone C_F where F is the spectrum of H. If F' is a subset of F'', the restrictions to F' of functions in $C_{F''}$ belong to $C_{F'}$. Our purpose is to show that for all F the functions in C_F are precisely the restrictions to F of functions of the form (3); this is the content of Theorem 1. Half of the theorem is established by the next lemma.

LEMMA 5. *If $k(\lambda)$ is of the form* (3) *and F arbitrary, the restriction of $k(\lambda)$ to F belongs to C_F.*

Proof. If there exists an operator H for which $K=k(H)$ is not an exact quadratic interpolation then there exists a T of bound 1 on $\mathcal{H}_2$ for which $T^*HT\leqslant H$ and $T^*(I-H)T\leqslant I-H$ and a normalized x in $\mathcal{H}_2$ for which $(KTx, Tx)_2>(Kx, x)_2$. Since the property of being an exact interpolation function depends only on the spectrum of H, we may suppose that H has a complete set of eigenvectors, the corresponding eigenvalues forming, of course, a dense subset of the spectrum. The inequality above may then be written in integral form:

$$\sum_1^\infty k(\lambda_i)\,|(Tx, f_i)_2|^2>\sum_1^\infty k(\lambda_i)\,|(x, f_i)_2|^2=(Kx, x)_2,$$

and for sufficiently large N we have

$$\sum_1^N k(\lambda_i)\,|(Tx, f_i)_2|^2>(Kx, x)_2.$$

If we take P as the projection on the first N eigenvectors this may be written

$$(KPTx, PTx)_2>(Kx, x)_2.$$

Set $y=Px$ and pass to the transformation $S=PTP$ on the space $\mathcal{M}=P(\mathcal{H}_2)$; since $\mathcal{M}$ is a reducing subspace for H we have $S^*HS\leqslant H$ and $S^*(I-H)S\leqslant I-H$. Moreover, $PTx=Sy+PT(I-P)x$, and with increasing N the second term converges to 0. Thus $(KSy, Sy)_2$

converges to $(KPTx, PTx)_2 > (Kx, x)_2 \geqslant (Ky, y)_2$ and so for large enough N $(KSy, Sy)_2 > (Ky, y)_2$. It follows that $k(\lambda)$ does not give rise to an exact quadratic interpolation for the finite dimensional space $\mathcal{M}$. Thus it will be necessary to show only that functions $k(\lambda)$ of the form (3) give rise to exact quadratic interpolations for finite dimensional spaces, i.e., that such functions belong to C_F for finite sets F.

The class of functions of the form (3) is obviously a convex cone, and therefore we have only to show that the suitably normalized extreme points, the functions

$$k_s(\lambda) = \frac{\lambda(1-\lambda)}{\lambda s + (1-\lambda)(1-s)}, \quad 0 \leqslant s \leqslant 1,$$

are exact interpolation functions on any finite set F. Now it is trivial that for $s=0$ or $s=1$ the resulting functions, $k_0(\lambda)=\lambda$ and $k_1(\lambda)=1-\lambda$ belong to C_F; we therefore suppose $0<s<1$.

On the finite dimensional space $\mathcal{H}_2$ the operators $I, H, I-H, K_s = k_s(H)$ give rise to four different, but equivalent quadratic norms, viz. $\|x\|_2, \|x\|_0, \|x\|_1, \|x\|_s$. These norms then give rise to four equivalent norms on the finite-dimensional operator space which we denote with the same subscripts: $\|T\|_2, \|T\|_0, \|T\|_1$ and $\|T\|_s$. We have only to show the exactness of the interpolation, namely the inequality

$$\|T\|_s \leqslant \max(\|T\|_0, \|T\|_1)$$

and since all three functions are norms, a fortiori continuous, it is enough to show this for a dense subset of the operator space, viz. the operators T which have inverses.

Now it is easy to show that on a finite dimensional Hilbert space the operator inequality $0<A\leqslant B$ is completely equivalent to the inequality $0<B^{-1}\leqslant A^{-1}$; our hypothesis (1) therefore reads

$$H^{-1} \leqslant T^{-1}H^{-1}T^{*-1}$$

and

$$(I-H)^{-1} \leqslant T^{-1}(I-H)^{-1}T^{*-1}$$

and we want to prove

$$K_s^{-1} \leqslant T^{-1}K_s^{-1}T^{*-1}.$$

From the form of k_s, however, $K_s^{-1} = s(I-H)^{-1} + (1-s)H^{-1}$, that is to say, K_s^{-1} is a convex combination of H^{-1} and $(I-H)^{-1}$ and the proof of Lemma 5 is complete.

The proof of the necessity in Theorem 1 is a good deal harder than the proof of the sufficiency; before embarking on it we make a few remarks serving to simplify the problem. First of all, it is enough to prove the theorem for finite sets F, since if $k(\lambda)$ belongs to C_F for some infinite F, and if $\{F_n\}$ is an increasing sequence of finite subsets of F, the union of which is dense in F, the restriction of $k(\lambda)$ to F_n is of the form (3) when the theorem is true for finite sets. Accordingly, there exists a sequence $k_n(\lambda)$ of functions of the form (3) converging on F to $k(\lambda)$ and which are uniformly bounded on the closed interval $0 \leqslant \lambda \leqslant 1$. It is therefore clear that $k(\lambda)$ is the restriction to F of a function of the form (3).

Hitherto we have been studious to avoid semi-norms and to speak only of norms; thus we always had the hypothesis that the spectrum of H did not contain the points 0 or 1 as eigenvalues. In general, that hypothesis was not necessary for our considerations and we shall return to this point later. At the moment we content ourselves with the following remark: if $k(\lambda)$ is an interpolation function associated with a positive definite operator H on the Hilbert space $\mathcal{H}_2$, and if we extend $k(\lambda)$ so that $k(0)=0$, and if, moreover, we extend the space $\mathcal{H}_2$ by taking its direct sum with some further Hilbert space $\mathcal{M}$ to obtain $\hat{\mathcal{H}}=\mathcal{H}_2 \oplus \mathcal{M}$ and consider the operator $\hat{H}$ on $\hat{\mathcal{H}}$ which coincides with H on $\mathcal{H}_2$ and which vanishes on $\mathcal{M}$, then the extended $\hat{k}(\lambda)$ gives rise to an interpolation seminorm for the operator $\hat{H}$. Indeed, if we write the generic element of $\hat{\mathcal{H}}$ in the form $f=[x; z]$ where x is in $\mathcal{H}_2$ and z in $\mathcal{M}$ then

$$\|f\|_0^2 = \|x\|_0^2 = (\hat{H}f, f),$$

$$\|f\|_1^2 = \|x\|_1^2 + \|z\|^2 = ((I-\hat{H})f, f),$$

and the one initial form is a semi-norm on $\hat{\mathcal{H}}$ while the other is a norm. We have $\|f\|_*^2 = (\hat{k}(\hat{H})f, f) = (k(H)x, x)_2 = \|x\|_*^2$ which is surely a seminorm on $\hat{\mathcal{H}}$ compatible with the norm of the space. If T is a linear transformation of $\hat{\mathcal{H}}$ into itself which is continuous for the semi-norm $\|f\|_0$ then evidently $\|f\|_0=0$ implies $\|Tf\|_0=0$, i.e., the subspace $\mathcal{M}$ is invariant under T. If then E is the projection in $\hat{\mathcal{H}}$ onto $\mathcal{H}_2$ we have $\|Tf\|_* = \|ETEf\|_*$ for all f. But ETE maps $\mathcal{H}_2$ into itself, accordingly, if T has bound 1 relative to each of the initial semi-norms, so also does ETE, and therefore the restriction of ETE to $\mathcal{H}_2$, and since $\|x\|_*$ is an exact interpolation norm on $\mathcal{H}_2$ we have $\|Tf\|_* \leqslant \|f\|_*$.

In a similar way the point $\lambda=1$ may be adjoined to F and the function $k(\lambda)$ extended by the definition $k(1)=0$. By a slight modification of the argument used in the proof of Lemma 2 we can show that the extended function is also concave. This fact is sufficient to prove Theorem 1 in the special case when F contains only two points; the linear function which coincides with $k(\lambda)$ on such an F is non-negative throughout the unit interval and is therefore of the form (3), the mass $d\varrho(s)$ being concentrated at the points $s=0$ and $s=1$. Since the theorem is a triviality when F is a one-point set, we will suppose in the sequel that F consists of at least three points.

Let F^* be the set obtained from F by reflecting it through the point $\lambda=\frac{1}{2}$. If $k(\lambda)$ belongs to C_F, the function $k^*(\lambda)$ defined on F^* by $k^*(\lambda)=k(1-\lambda)$ is evidently in C_{F^*}, since all that has been done is to interchange of H and $I-H$. Less obvious is the following lemma, which we prove only for finite sets F in the open unit interval, although it is true in general.

LEMMA 6. *If $k(\lambda)$ belongs to C_F, where F is a finite subset of $0<\lambda<1$, then the function $\check{k}(\lambda)=\lambda(1-\lambda)/k(1-\lambda)$ belongs to C_{F^*}.*

Proof. In view of the previous remark, it is enough to show that the function $g(\lambda)=\lambda(1-\lambda)/k(\lambda)$ belongs to C_F, and this is equivalent to showing that the operator $G=K^{-1}H(I-H)$ gives rise to an exact quadratic interpolation. We may suppose that the Hilbert space $\mathcal{H}_2$ is finite dimensional. Since F is the spectrum of H and is bounded away from 0 and 1 it follows that the operators H, $I-H$, K and G are all positive commuting operators with inverses. We let S be the positive square root of $H(I-H)$ and note that $SK^{-1}S=G$ as well as $S^{-1}HS^{-1}=(I-H)^{-1}$ and $S^{-1}(I-H)S^{-1}=H^{-1}$. In order to show that G corresponds to an interpolation norm we must show that the inequalities $T^*HT\leqslant H$ and $T^*(I-H)T\leqslant I-H$ imply $T^*GT\leqslant G$, and it is sufficient to show this for operators T which have inverses. In the calculation which follows we make use of the fact that $0<A\leqslant B$ implies $B^{-1}\leqslant A^{-1}$ as well as $T^*AT\leqslant T^*BT$ for any T.

From $T^*(I-H)T\leqslant I-H$ we have

$$\begin{aligned}
&(I-H)^{-1}\leqslant T^{-1}(I-H)^{-1}T^{*-1},\\
&T(I-H)^{-1}T^*\leqslant (I-H)^{-1},\\
&TS^{-1}HS^{-1}T^*\leqslant S^{-1}HS^{-1},\\
&STS^{-1}HS^{-1}T^*S\leqslant H,\\
&M^*HM\leqslant H \text{ where } M=S^{-1}T^*S.
\end{aligned}$$

In a similar way, from $T^*HT\leqslant H$ we obtain $M^*(I-H)M\leqslant I-H$, whence, since K gives rise to an interpolation norm, $M^*KM\leqslant K$, and therefore

$$\begin{aligned}
&K^{-1}\leqslant M^{-1}K^{-1}M^{*-1},\\
&MK^{-1}M^*\leqslant K^{-1},\\
&S^{-1}T^*SK^{-1}STS^{-1}\leqslant K^{-1},\\
&T^*SK^{-1}ST\leqslant SK^{-1}S=G,\\
&T^*GT\leqslant G, \text{ as required.}
\end{aligned}$$

It is convenient to change variables by the mapping $Z(\lambda)=\lambda/(1-\lambda)$ carrying the unit interval into the right half-axis. To the function $k(\lambda)$ in C_F we associate a function $\phi(z)$ defined on $Z(F)$ by the equation

$$\phi(z) = (z+1)\, k\left(\frac{z}{z+1}\right).$$

For the extreme points of the class of functions $k(\lambda)$ of the form (3) we obtain the correspondences: $k_0(\lambda)=\lambda$ and $\phi_0(z)=z$; $k_1(\lambda)=1-\lambda$ and $\phi_1(z)=1$; while for

$$k_s(\lambda)=\frac{\lambda(1-\lambda)}{\lambda s+(1-\lambda)(1-s)}, \quad 0<s<1,$$

we obtain $$\phi_s(z)=\frac{m}{\omega-z}-\frac{m}{\omega} \quad \text{where } m=\frac{1-s}{s^2}>0 \text{ and}$$

$\omega=(s-1)/s<0$. The functions of the form (3) therefore correspond to functions $\phi(z)$, positive and regular on the right half-axis, which have positive imaginary part in the upper half-plane. This class of functions, studied at length in [2] and [9] is denoted by the letter P' in [2] where the letter P is reserved for the general class of functions, analytic in the upper half-plane with positive imaginary part. Since the most general function in the class P' may be written

$$\phi(z)=\alpha z+\beta_0+\int_{-\infty}^{0}\left[\frac{1}{\omega-z}-\frac{1}{\omega}\right] d\mu(\omega), \tag{5}$$

where $\alpha\geqslant 0$, $\beta_0\geqslant 0$ and $d\mu$ is a positive measure for which $\int (d\mu(\omega))/(\omega(\omega-1))$ is finite, we see that the functions of the form (3) correspond precisely to the functions in P'. Our object, then, is to show that $k(\lambda)$ in C_F corresponds to a function $\phi(z)$ defined on $Z(F)$ which is the restriction to that set of a function in the class P'.

Let n be an integer $\geqslant 1$ and suppose that $Z(F)$ contains the following set of $2n$ positive numbers

$$\xi_1<\eta_1<\xi_2<\eta_2<\dots<\xi_n<\eta_n. \tag{6}$$

Let V be an n-dimensional Hilbert space. It is easy to show that there exists a self-adjoint operator A on V having the numbers $\{\xi_i\}$ for its spectrum, and a one-dimensional projection E such that the operator $B=A+cE$ for an appropriate positive c has the numbers $\{\eta_i\}$ for its spectrum. Evidently $A\leqslant B$. Next we form the direct sum of V with itself to obtain $W=V\oplus V$ with generic element $f=[x;y]$. Naturally $\|f\|_W^2=\|x\|_V^2+\|y\|_V^2$, and we let L denote the operator defined by $L[x;y]=[Ax;By]$. On W we introduce the initial norms

$$\|f\|_0^2=(Ax,x)_V+(By,y)_V=(Lf,f)_W$$

and $$\|f\|_1^2=\|x\|_V^2+\|y\|_V^2=\|f\|_W^2.$$

Accordingly $$\|f\|_2^2=((L+I)f,f)_W,$$

and therefore $$\|f\|_0^2=(Hf,f)_2=((L+I)\,Hf,f)_W=(Lf,f)_W,$$

whence $H=(L+I)^{-1}L$. Since the spectrum of L consists of the $2n$ numbers (6), the spectrum of H is a subset of F and we may use the function $k(\lambda)$ in C_F to obtain an interpolation norm. We obtain

$$\|f\|_*^2=(Kf,f)_2=((L+I)k(H)f,f)_W=(\phi(L)f,f)_W,$$

and since the factor spaces making up W are invariant under L

$$\|f\|_*^2=(\phi(A)x,x)_V+(\phi(B)y,y)_V.$$

Now for the transformation T defined on W by $T[x;y]=[y;0]$ we have

$$\|[y;0]\|_1^2\leqslant\|y\|_V^2+\|x\|_V^2=\|[x;y]\|_1^2,$$

$$\|[y;0]\|_0^2=(Ay,y)_V\leqslant(Ax,x)_V+(By,y)_V=\|[x;y]\|_0^2,$$

since $A\leqslant B$. Hence for the interpolation norm

$$\|[y;0]\|_*^2=(\phi(A)y,y)_V\leqslant(\phi(A)x,x)_V+(\phi(B)y,y)_V=\|[x;y]\|_*^2.$$

In particular, for elements with $x=0$, we have $(\phi(A)y,y)_V\leqslant(\phi(B)y,y)_V$ for every y in V. From a theorem of Loewner [9] then the determinant

$$\det\left|\frac{\phi(\xi_i)-\phi(\eta_j)}{\xi_i-\eta_j}\right| \tag{7}$$

is non-negative.

Let $Z=Z(F)$ consist of the points

$$z_1<z_2<z_3<\ldots<z_l,\quad z_1>0, \tag{8}$$

and let $P(Z)$ denote the convex cone of all real functions $f(z)$ defined on Z which are the restrictions to that set of functions in the class P, real and regular on an interval containing the closed interval $z_1\leqslant z\leqslant z_l$; similarly let $P'(Z)$ denote the smaller cone of restrictions to Z of functions in the class P'. For any real $f(z)$ defined on Z and any subset S of Z consisting of an even number of points we write S in the form (6) and compute the corresponding determinant of the type (7); this determinant is called the Loewner determinant of f associated with S. Evidently there are as many Loewner determinants as there are subsets of S of even cardinality. In another publication on the Loewner theory [5] the author has established the following theorem, which does not require the hypothesis $z_1>0$.

THEOREM A. *A real function $f(z)$ defined on Z belongs to $P(Z)$ if and only if*

I. *Every Loewner determinant of f is non-negative and*

II. *If a Loewner determinant vanishes, so also do all other Loewner determinants of the same or higher order.*

We have already seen that if $\phi(z)$ is defined on $Z=Z(F)$ by $\phi(z)=(z+1)k(z/(z+1))$ where $k(\lambda)$ belongs to C_F then the Loewner determinants of $\phi(z)$ are non-negative, and this remains true when 0 is adjoined to Z and ϕ extended by the definition $\phi(0)=0$.

If we consider next F^*, the reflection of F through $\lambda=\frac{1}{2}$, we find that $Z(F^*)$ is the set Z^* of reciprocals of numbers in $Z=Z(F)$. The function $k^*(\lambda)$ defined on F^* by $k^*(\lambda)=k(1-\lambda)$ corresponds to $\phi^*(z)$ defined on Z^* by the equation $\phi^*(z)=z\phi(1/z)$, and the Loewner determinants of this function are non-negative, even when 0 is adjoined to Z^* and ϕ^* extended by the definition $\phi^*(0)=0$. In a similar way we can consider $\check{k}(\lambda)=\lambda(1-\lambda)/(k(1-\lambda))$ defined on F^* which corresponds to $\check{\phi}(z)$ defined on Z^* by the equation $\check{\phi}(z)=[\phi(1/z)]^{-1}$; if we set $\check{\phi}(0)=0$ we obtain a function all of whose Loewner determinants are non-negative.

The proof of Theorem 1 will then be almost complete if we invoke another theorem from the Loewner theory.

THEOREM B. *If Z is of the form (8), $z_1>0$ and Z^* the set of reciprocals of numbers in Z, then a real function $f(z)$ defined on Z belongs to $P'(Z)$ if the following three conditions are satisfied.*

III. *When f is defined at $z=0$ by $f(0)=0$, the extended function belongs to $P(Z\cup 0)$.*

IV. *The function $f^*(z)$ defined on $Z^*\cup 0$ by $f^*(0)=0$ and $f^*(z)=zf(1/z)$ belongs to $P(Z^*\cup 0)$.*

V. *The function $\check{f}(z)$ defined on $Z^*\cup 0$ by $\check{f}(0)=0$ and $\check{f}(z)=[f(1/z)]^{-1}$ belongs to $P(Z^*\cup 0)$.*

We remark that the function $\sqrt{\lambda(1-\lambda)}$ is of the form (3) and hence, in view of Lemma 5, belongs to every C_F; under the mapping $Z(\lambda)$ it corresponds to the function $\sqrt{z}$ in P'. It is well known that no Loewner determinant of $\sqrt{z}$ vanishes. If $k(\lambda)$ belongs to C_F, so also does $k(\lambda)+\varepsilon\sqrt{\lambda(1-\lambda)}$ for small positive ε, as well as the two associated functions $k^*(\lambda)+\varepsilon\sqrt{\lambda(1-\lambda)}$ and $\check{k}(\lambda)+\varepsilon\sqrt{\lambda(1-\lambda)}$. From these, we obtain the corresponding functions defined on Z and Z^*, viz. $\phi(z)+\varepsilon\sqrt{z}$, $\phi^*(z)+\varepsilon\sqrt{z}$ and $\check{\phi}(z)+\varepsilon\sqrt{z}$. For all three functions the Loewner determinants are non-negative, and for all sufficiently small positive ε those determinants are even strictly positive, since those determinants, of which there are only a finite number in all, are polynomials in ε. If one of the polynomials vanished identically, it would follow that the coefficient of the highest power of ε also did, and such a coefficient is a Loewner determinant of $\sqrt{z}$, none of whose Loewner determinants vanishes. Thus, for sufficiently small positive ε $k(\lambda)+\varepsilon\sqrt{\lambda(1-\lambda)}$ is of the form (3), and hence $k(\lambda)$ is too. This completes the proof of Theorem 1.

In the proof of the theorem we were compelled to consider, at least in the case of a finite-dimensional space, the possibility of the exact quadratic interpolation of semi-norms. It is worth-while to review our argument with this in mind. Had we begun with a pair of possibly not compatible norms on V, the passage to $\mathcal{H}_2$ would have given rise to a pair of semi-norms on that space, the null spaces of which would intersect only in the zero vector. The exact quadratic interpolations would have to give rise to semi-norms $\|x\|_*$ rather than norms, in general, on $\mathcal{H}_2$. The content of the lemmas which we established would still be valid, except the assertion about the continuity of $k(\lambda)$ on the spectrum; Lemma 2 would guarantee only that $k(\lambda)$ was continuous at every point of the spectrum which was not 0 or 1, and that $k(\lambda)$ was concave. Since the class of interpolation functions is convex, we see that the functions we must consider in this more general case are those which, on the open interval, are of the form (3) and which are so defined at the end points that they are non-negative and lower semi-continuous. We may therefore state a slight generalization of Theorem 1, and the corresponding obvious generalization for interpolation methods.

THEOREM 1′. *A semi-norm $\|x\|_*$ on $\mathcal{H}_2$ is an exact quadratic interpolation semi-norm relative to the pair of initial semi-norms determined by H and $I-H$ if and only if there exists a non-negative, lower semi-continuous function $k(\lambda)$ on the closed unit interval which, on the open interval, coincides with a function of the form* (3) *such that for all x in $\mathcal{H}_2$*

$$\|x\|_*^2=\int_0^1 k(\lambda)\,d(E_\lambda x, x)_2,$$

where $H=\int_0^1 \lambda\, dE_\lambda$.

THEOREM 2′. *The exact quadratic interpolation methods for semi-norms are given by functions $k(\lambda)$ of the type of Theorem* 1′.

Let us remark that various interpolation methods based on the three-lines theorem [1, 8, 12] in the case of quadratic norms correspond in our notation to interpolations described by the functions

$$k_\alpha(\lambda)=\lambda^{1-\alpha}(1-\lambda)^\alpha, \quad 0\leqslant\alpha\leqslant 1.$$

The functions $\phi(z)$ of the form (5) introduced in the proof of Theorem 1 often appear more naturally in the applications of the interpolation theorem than the functions $k(\lambda)$.

Thus, if μ is a measure on some measure space X and $w(x)$ a non-negative measurable function and V an appropriate space of measurable functions, the initial norms

$$\|u\|_0^2 = \int_X |u(x)|^2 \, d\mu(x) \quad \text{and} \quad \|u\|_1^2 = \int_X |u(x)|^2 \, w(x) \, d\mu(x)$$

give rise to the family of interpolation norms

$$\|u\|_*^2 = \int_X |u(x)|^2 \, \phi(w(x)) \, d\mu(x),$$

where $\phi(z)$ belongs to the class P', i.e. is of the form (5).

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