THE INTRINSIC THEORY OF THIN SHELLS AND PLATES*

PART I.—GENERAL THEORY

BY

WEI-ZANG CHIEN

Department of Applied Mathematics, University of Toronto

1. Introduction. The development of the theory of thin shells and plates by many authors [1, 2, 3, 4] can be summarized under the following three main heads:

(1) All theories are based upon certain simplifying unproved assumptions. For example: (a) the thickness remains unchanged during the deformation, (b) the middle surface in the unstrained state deforms into the middle surface in the strained state, (c) the normals of the unstrained middle surface deform into the normals of the strained middle surface.

(2) All theories involve the use of displacement to describe the state of deformation. This plan works well in the theory of small deflection, but presents considerable difficulty in the case of large deflection.

(3) The various approximations used in the theory of thin shells and plates are confusing. If one attempts to give a complete picture of the theory, one must be able to introduce a systematic method of approximation, which not only clears away the confusion of various approximations, but also leads to a complete classification of all thin shell and plate problems.

The purpose of this paper is to give a systematic treatment of the general problem of the thin shell, which includes the problem of the thin plate as a special case. The work is based on the usual equations of elasticity for a finite body, supposed to be homogeneous and isotropic. The final equations of Part I are the three equations of equilibrium, (6.34), (6.35), and the three equations of compatibility, (6.43), (6.44), for the six unknowns, $p_{\alpha\beta}$, $q_{\alpha\beta}$, which represent extension and change of curvature of the middle surface. When these quantities are found, the strain and stress throughout the shell or plate can be calculated. The displacement does not appear explicitly in the argument. Since we deal rather with stress, strain and curvature (all tensors), the tensor notation proves much more convenient than any other.

In Part II and III, we shall discuss the various approximate forms of the equations arising from consideration of the thinness of the shell or plate and the smallness (or vanishing) of its curvature. The strain is, of course, always supposed to be small. We obtain a complete classification of all shell and plate problems. There are found to be twelve types of plate problems, and thirty-five types of shell problems. To each type, there corresponds a set of six equations which are simplifications of the equations (6.34), (6.35), (6.43), (6.44) of Part I, with certain terms dropped on account of smallness and the uncalculated residual terms omitted. The equations obtained include all the familiar equations in the field of small deflection and the few equations

* Received May 13, 1943. This paper is part of a thesis submitted in conformity with the requirements for the degree of doctor of philosophy in the University of Toronto, 1942.

The work was carried out under the support of a scholarship granted by the Board of Trustees of the Sino-British Fund.

already known for large deflection. The new results for finite deflection may prove particularly interesting.

The author is much indebted to Prof. J. L. Synge who not merely has directed this work but has actively participated in it.

2. Reduction of force system to reference surface; macroscopic equations of equilibrium. We shall start in this section by reviewing some main results in a previous paper [5].

All theories of thin shells and plates involve the use of a reference surface. Usually the middle surface is taken without explicit distinction between the middle surface in the unstrained state and the middle surface in the strained state. In the present methodical treatment we shall use a general reference surface in the material in sections 2–5. In section 6 and later parts we shall use the surface in the strained state formed by the particles of the middle surface in the unstrained state. To the order of approximation used there, this is indistinguishable from the middle surface in the strained state. (This is generally assumed to be the case; cf. E. Reissner [4].)

The following coordinate system will be used: x^0 at any point A inside the material of the shell or plate is the perpendicular distance of A from the reference surface S_0, and x^α at A are the values of any Gaussian coordinates on S_0 corresponding to the foot of the perpendicular dropped from A. (Throughout the paper, Latin indices have the range 0, 1, 2 and Greek indices the range 1, 2; summation over either of these ranges will be signified by the repetition of an index.) This may be called a normal space coordinate system with respect to S_0.

Let us denote the line element in space by $ds^2=g_{ij}dx^i dx^j$, where g_{ij} is the fundamental tensor. Furthermore let $g_{[0]ij}$ be the values of g_{ij} at S_0, then we have in usual tensor notation

$$g_{00} = g_{[0]00} = g^{00} = g^{00}_{[0]} = 1, \qquad g_{0\alpha} = g_{[0]0\alpha} = g^{0\beta} = g^{0\beta}_{[0]} = 0, \tag{2.1}$$

$$g_{[0]\alpha\beta} = a_{\alpha\beta}, \qquad g^{\alpha\beta}_{[0]} = a^{\alpha\beta}, \qquad \text{det.}\,(g_{[0]ij}) = g_{[0]} = \text{det.}\,(a_{\alpha\beta}) = a, \tag{2.2}$$

where $a_{\alpha\beta}dx^\alpha dx^\beta$ is the metric on S_0.

We shall now consider forces in the shell or plate. Let C be a curve on S_0 and A_0 a point on C. Let $n^\alpha_{[0]}$ be the unit vector in S_0 normal to C at A_0, indicating the positive side; let $\Lambda^\alpha_{[0]}$ be an arbitrary unit vector in S_0 at A_0. We consider the system of forces acting across an element of area standing on the element ds_0 of C at A_0, and terminated by the surfaces of the shell or plate. We replace the forces acting on the element by a statically equipollent system acting at A_0. This leads to the following invariants, which in fact define the macroscopic tensors, $T^{\alpha 0}$ (shearing stress tensor), $T^{\alpha\beta}$ (membrane stress tensor), $L^{\alpha\beta}$ (bending moment tensor):

$$\begin{aligned} T^{\alpha 0} n_{[0]\alpha} ds_0 &= \text{shearing force normal to } S_0 \text{ across the element } ds_0, \\ T^{\alpha\beta} n_{[0]\alpha} \Lambda_{[0]\beta} ds_0 &= \text{component in the direction of } \Lambda^\alpha_{[0]} \text{ of membrane stress across} \\ &\quad\ \text{the element } ds_0, \\ L^{\alpha\beta} n_{[0]\alpha} \Lambda_{[0]\beta} ds_0 &= \text{component in the direction of } \Lambda^\alpha_{[0]} \text{ of the bending moment} \\ &\quad\ \text{across the element } ds_0. \end{aligned} \tag{2.3}$$

Similarly, let dS_0 be an element of area of S_0 at A_0. Consider the external forces acting on the volume element consisting the normals to S_0 standing on the element

dS_0 and terminated by the surfaces of the shell or plate. We replace them by a statically equipollent system acting at A_0. This leads to the following invariants, which give the definitions of the external force and external moment tensors, F^i, M^α:

$$\begin{aligned} F^0 dS_0 &= \text{normal component of the external force on } dS_0, \\ F^\alpha \Lambda_{[0]\alpha} dS_0 &= \text{component in the direction of } \Lambda^\alpha_{[0]} \text{ of the external force on } dS_0, \\ M^\alpha \Lambda_{[0]\alpha} dS_0 &= \text{component in the direction of } \Lambda^\alpha_{[0]} \text{ of the external moment on } dS_0. \end{aligned} \tag{2.4}$$

Then from purely statical considerations, we obtain the following six equations of statical equilibrium ([5], p. 110):

$$\begin{aligned} &\text{(a)} \quad T^{\alpha 0}{}_{|\alpha} - \tfrac{1}{2} b_{\alpha\beta} T^{\alpha\beta} + F^0 = 0, \\ &\text{(b)} \quad T^{\beta\alpha}{}_{|\beta} + \tfrac{1}{2} a^{\alpha\beta} b_{\beta\gamma} T^{\gamma 0} + F^\alpha = 0, \\ &\text{(c)} \quad L^{\beta\alpha}{}_{|\beta} + a^{\alpha\beta} \eta_{[0]\beta\gamma} T^{\gamma 0} + M^\alpha = 0, \\ &\text{(d)} \quad \eta_{[0]\alpha\beta} T^{\alpha\beta} - \tfrac{1}{2} b_{\alpha\beta} L^{\alpha\beta} = 0. \end{aligned} \tag{2.5}$$

The symbols have the following meanings:

$$\begin{aligned} T^{\alpha 0}{}_{|\alpha} &= T^{\alpha 0}{}_{,\alpha} + \left\{ {\alpha \atop \pi\alpha} \right\}_a T^{\pi 0}, \\ T^{\alpha\beta}{}_{|\alpha} &= T^{\alpha\beta}{}_{,\alpha} + \left\{ {\alpha \atop \pi\alpha} \right\}_a T^{\pi\beta} + \left\{ {\beta \atop \pi\alpha} \right\}_a T^{\alpha\pi}, \\ b_{\alpha\beta} &= (g_{\alpha\beta,0})_{x^0=0}, \\ \eta_{[0]\alpha\beta} &= a^{1/2} \epsilon_{\alpha\beta}, \qquad \epsilon_{11} = \epsilon_{22} = 0, \qquad \epsilon_{12} = -\epsilon_{21} = 1. \end{aligned} \tag{2.6}$$

The Christoffel symbols are calculated for $a_{\alpha\beta}$. The quantities $(1/2)b_{\alpha\beta}$ are the coefficients of the second fundamental form of S_0; they vanish if S_0 is a plane. The radius of curvature R in the direction of a unit vector $\mu^\alpha_{[0]}$ (counted positive when S_0 is convex in the sense of x^0 increasing) is given by

$$2/R = b_{\alpha\beta} \mu^\alpha_{[0]} \mu^\beta_{[0]}. \tag{2.7}$$

By eliminating $T^{\alpha 0}$ from (2.5) we get a set of three equations,

$$T^{\pi\alpha}{}_{|\pi} + (1/2) a^{\alpha\beta} b_{\beta\gamma} \eta^{\gamma\lambda}_{[0]} a_{\lambda\delta} L^{\pi\delta}{}_{|\pi} + F^\alpha + \tfrac{1}{2} a^{\alpha\beta} b_{\beta\gamma} \eta^{\gamma\delta}_{[0]} a_{\delta\lambda} M^\lambda = 0, \tag{2.8a}$$

$$\eta^{\delta\pi}_{[0]} a_{\pi\gamma} L^{\lambda\gamma}{}_{|\lambda\delta} - \tfrac{1}{2} b_{\pi\lambda} T^{\pi\lambda} + F^0 + a_{\pi\lambda} \eta^{\gamma\pi}_{[0]} M^\lambda{}_{|\gamma} = 0. \tag{2.8b}$$

These equations, rather than the equations (2.5), are fundamental in the later theory. It should be noted that in the case of repeated covariant differentiation with respect to $a_{\alpha\beta}$, the order of the operations cannot be changed unless the total curvature K of the reference surface S_0 (cf. Eq. (3.13)) is equal to zero.

The above equations are valid for shells and plates of finite thickness. When we come to deal with approximations based on the smallness of certain quantities, we must of course consider only the magnitudes of dimensionless quantities. It is best therefore to work with dimensionless quantities throughout. Let us introduce a standard length L, a lateral dimension of the shell or plate (e.g. the diameter in the case of a circular plate). By ds we shall understand the distance between two adjacent points, divided by L; this dimensionless ds may be called the reduced distance. Similarly all coordinates x^i are supposed to be in reduced or dimensionless form. Consequently the fundamental tensor g_{ij} is dimensionless, and all tensor operations (such as raising or lowering suffixes or covariant differentiation) are dimensionless operations. We also reduce stress to dimensionless form by dividing by Young's modulus E, and body forces by dividing by E/L.

All the relations written above hold equally well for reduced or dimensionless quantities. We shall carry through the work with these quantities; if we wish to translate conclusions into ordinary dimensional form, we have simply to multiply by that combination of the form $L^m E^n$ which restores the required dimensionality. Young's modulus will not appear explicitly in our work, since it is eliminated from the stress-strain relations by the process of reduction.

We also note that a thin shell or plate is defined as one whose thickness is small in comparison with a lateral dimension L. Customarily, a thin shell is defined as one whose thickness is small compared with the radius of curvature; this is unsatisfactory in the limiting case of a plate, and also in the case of a shell whose thickness is small in comparison with the radius of curvature but of the same order as the lateral dimension L.

3. Representation of $T^{\alpha 0}$, $T^{\alpha\beta}$, $L^{\alpha\beta}$, F^i, M^α as power series in the thickness. Let C be a curve on the reference surface S_0, and Σ the surface formed by erecting normals to S_0 along C. Let ds_0 be an element of C, and $d\Sigma$ the strip formed by the normals on ds_0 (Fig. 1). Let ds be the length of the arc of intersection of $d\Sigma$ and the surface x^0 = constant, passing through any point A in $d\Sigma$. Let n_α be the unit vector normal to $d\Sigma$ at A, and $n_{[0]\alpha}$ the unit vector normal to $d\Sigma$ at the reference surface S_0.

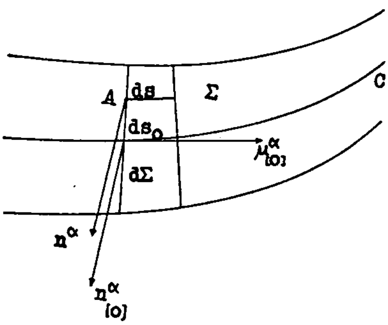

FIG. 1.

Here we note that there are two distinct classes of quantities: (1) those defined only on the reference surface, such as $a_{\alpha\beta}$, $b_{\alpha\beta}$, $T^{\alpha 0}$, $T^{\alpha\beta}$, $L^{\alpha\beta}$, M^α, F^i, $E^{ij}_{[m]}$ (normal derivatives of the stress tensor on S_0); (2) those defined at any point in the material of the shell, such as g_{ij}, E_{ij} (stress tensor), λ_i (parallel vector field), n_α. For all space tensors, the change of indices will be effected by applying g_{ij} or g^{ij}; and for all surface tensors, the change of indices will be effected by applying $g_{[0]ij}$ and $g^{ij}_{[0]}$ [defined as in (2.1), (2.2)].

By the definition (2.3), the shearing stress tensor and the membrane stress tensor are calculated by the invariant formula:

$$T^{\alpha i} n_{[0]\alpha} \lambda_{[0]i} ds_0 = \int_h (E^{\alpha i} n_\alpha \lambda_i ds) dx^0, \tag{3.1}$$

where λ_i is any parallel field of unit vectors, $\lambda_{[0]i}$ the same vector field at the reference surface S_0, and $E^{\alpha i}$ part of the stress tensor E^{ij}. The symbol h under the sign of integration indicates here (and throughout the paper) that the integral runs from $x^0 = -h_{(-)}$ to $x^0 = +h_{(+)}$, where both $h_{(+)}$ and $h_{(-)}$ are positive functions of x^α (for thin shells or plates, they are assumed to be small). Furthermore, the bending moment tensor is caculated from the invariant formula:

$$L^{\alpha\beta}n_{[0]\alpha}\lambda_{[0]\beta}ds_0 = \int_h (g_{\pi\gamma}\eta^{\gamma\beta}E^{\alpha\pi}\lambda_\beta n_\alpha ds)x^0dx^0, \quad (3.2)$$

where

$$\eta^{\alpha\beta} = g^{-1/2}\epsilon^{\alpha\beta}, \qquad g = \det.\,(g_{\alpha\beta}),$$
$$\epsilon^{11} = \epsilon^{22} = 0, \qquad \epsilon^{12} = -\epsilon^{21} = 1. \quad (3.3)$$

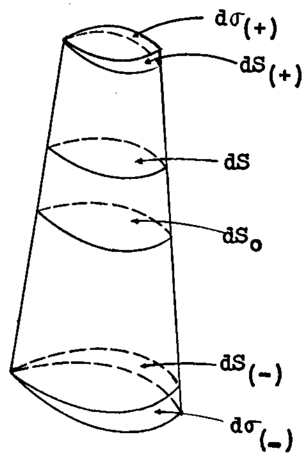

FIG. 2.

For the external force system, assume that $X^i\lambda_i$ is the component of the body for per unit volume in the arbitrary direction of λ_i at any point in the shell, and $Z^i_{(+)}\lambda_{(+)i}$ and $Z^i_{(-)}\lambda_{(-)i}$ the components of the given loads per unit are applied to the upper and lower surfaces respectively. Let us consider (Fig. 2) a portion of the shell or plate obtained by drawing normals to its reference surface over a surface element dS_0. The portions of the boundary surfaces of the shell or plate cut out by these normals are $d\sigma_{(+)}$ and $d\sigma_{(-)}$. Let dS be the corresponding element drawn at constant normal distance from the reference surface S_0. Then the external force and moment components are calculated from the invariant formulae:

$$F^i\lambda_{[0]i}dS_0 = \int_h (X^i\lambda_i dS)dx^0 + Z^i_{(+)}\lambda_{(+)i}d\sigma_{(+)} + Z^i_{(-)}\lambda_{(-)i}d\sigma_{(-)}, \quad (3.4)$$

$$M_\alpha\lambda^\alpha_{[0]}dS_0 = \int_h (\eta_{\beta\alpha}\lambda^\alpha X^\beta dS)x^0dx^0 + \eta_{(+)\beta\alpha}\lambda^\alpha_{(+)}Z^\beta_{(+)}h_{(+)}d\sigma_{(+)}$$
$$- \eta_{(-)\beta\alpha}\lambda^\alpha_{(-)}Z^\beta_{(-)}h_{(-)}d\sigma_{(-)}, \quad (3.5)$$

where $\eta_{(+)\alpha\beta}$, $\eta_{(-)\alpha\beta}$ are the values of $\eta_{\alpha\beta}$ at the upper and lower surfaces of the shell or plate respectively.

In order to carry out the integrations in (3.1)–(3.5), we must express all the quantities in the integrands as functions of x^0. These can be written as follows:

$$g_{\alpha\beta} = a_{\alpha\beta} + b_{\alpha\beta}x^0 + \tfrac{1}{2}c_{\alpha\beta}(x^0)^2, \quad (3.6)$$

$$g = a\{1 + 2Hx^0 + K(x^0)^2\}^2, \quad (3.7)$$

$$n_\alpha \frac{ds}{ds_0} = n_{[0]\alpha}(g/a)^{1/2} = n_{[0]\alpha}\{1 + 2Hx^0 + K(x^0)^2\}, \quad (3.8)$$

$$\lambda_0 = \lambda_{[0]0}, \qquad \lambda_\alpha = \lambda_{[0]\alpha} + \tfrac{1}{2}b^\beta{}_\alpha\lambda_{[0]\beta}x^0. \quad (3.9)$$

The first relation is well known; $a_{\alpha\beta}$, $(1/2)b_{\alpha\beta}$, $(1/2)c_{\alpha\beta}$ are the first, second and third fundamental tensors of the reference surface S_0 respectively. These tensors are not independent, but connected by six geometrical conditions of flat space ([5], p. 112). Three of these read

$$c_{\alpha\beta} = \tfrac{1}{2}a^{\pi\lambda}b_{\alpha\pi}b_{\beta\lambda}, \quad (3.10)$$

and the other three are the well known equations of Codazzi and Gauss,

$$b_{\alpha\beta|\gamma} - b_{\alpha\gamma|\beta} = 0, \quad (3.11)$$

$$4R_{\rho\alpha\beta\gamma} = b_{\rho\beta}b_{\alpha\gamma} - b_{\rho\gamma}b_{\alpha\beta}. \quad (3.12)$$

Here the single stroke indicates covariant differentiation with respect to x^α and the tensor $a_{\alpha\beta}$; $R_{\rho\alpha\beta\gamma}$ is the two dimensional curvature tensor formed from the tensor $a_{\alpha\beta}$ (sometimes called the Riemann Christoffel tensor of the surface S_0, see [6], p. 182, Eq. (50)).

The relation (3.7) can be obtained by direct calculation from the definition of g. Here K is the total curvature and H the mean curvature of the reference surface S_0. If R_1, R_2 are the principal radii of curvature, then

$$H = \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right), \qquad K = \frac{1}{R_1 R_2}; \tag{3.13a}$$

therefore, in tensor notation,

$$8K = a^{\pi\gamma}b_{\pi\gamma}a^{\lambda\delta}b_{\lambda\delta} - b^{\pi\lambda}b_{\pi\lambda}, \qquad 4H = a^{\pi\lambda}b_{\pi\lambda}. \tag{3.13b}$$

We also note the following relations, which are often used in later calculations,

$$\begin{aligned}
&\text{(a)} \qquad \epsilon^{\alpha\pi}\epsilon^{\beta\lambda}a_{\pi\lambda} = aa^{\alpha\beta}, \\
&\text{(b)} \qquad \epsilon^{\alpha\pi}\epsilon^{\beta\lambda}b_{\pi\lambda} = a(4Ha^{\alpha\beta} - b^{\alpha\beta}), \\
&\text{(c)} \qquad \epsilon^{\alpha\pi}\epsilon^{\beta\lambda}c_{\pi\lambda} = 2a\{(4H^2 - K)a^{\alpha\beta} - Hb^{\alpha\beta}\}.
\end{aligned} \tag{3.14}$$

The proof of the geometrical relation (3.8) is long, but not difficult. The relation (3.9) is obvious; since λ_i is a parallel vector field, we have

$$\lambda_{i\|j} = 0. \tag{3.15}$$

Here the double stroke indicates covariant differentiation with respect to the space coordinates x^i and the tensor g_{ij}. Putting $i=0$, $i=\alpha$, $j=0$ in (3.15), the relation (3.9) follows at once.

Besides all the relations (3.6)–(3.9), we also need the following geometrical results:

$$\frac{dS}{dS_0} = \left(\frac{g}{a}\right)^{1/2}, \quad \frac{d\sigma_{(+)}}{dS_0} = \frac{1}{|N^0_{(+)}|}\left(\frac{g_{(+)}}{a}\right)^{1/2}, \quad \frac{d\sigma_{(-)}}{dS_0} = \frac{1}{|N^0_{(-)}|}\left(\frac{g_{(-)}}{a}\right)^{1/2}, \tag{3.16}$$

the positive root being understood. Here $N_{(+)i}$, $N_{(-)i}$ are unit normal vectors, drawn out from the upper and lower boundary surfaces respectively, and $g_{(+)}$, $g_{(-)}$ are the values of g on the boundary surfaces.

Substituting (3.8), (3.9) in (3.1), we have

$$\begin{aligned}
n_{[0]\alpha}\lambda_{[0]\beta}&\left\{T^{\alpha\beta} - \int_h E^{\alpha\gamma}(a^\beta_\gamma + \tfrac{1}{2}b^\beta_\gamma x^0)\left(\frac{g}{a}\right)^{1/2}dx^0\right\} \\
&+ n_{[0]\alpha}\lambda_{[0]0}\left\{T^{\alpha 0} - \int_h E^{\alpha 0}\left(\frac{g}{a}\right)^{1/2}dx^0\right\} = 0.
\end{aligned} \tag{3.17}$$

But $n_{[0]\alpha}$ and $\lambda_{[0]i}$ are arbitrary, and consequently we have for the shearing stress tensor

$$T^{\alpha 0} = \int_h E^{\alpha 0}\left(\frac{g}{a}\right)^{1/2}dx^0, \tag{3.18}$$

and for the membrane stress tensor

$$T^{\alpha\beta} = \int_h E^{\alpha\gamma}(a_\gamma^\beta + \tfrac{1}{2}b_\gamma^\beta x^0)\left(\frac{g}{a}\right)^{1/2} dx^0. \tag{3.19}$$

Similarly, substituting (3.6)–(3.9) in (3.2), we obtain the bending moment tensor

$$L^{\alpha\beta} = \eta^{\lambda\beta}_{[0]}\, a_{\pi\lambda} \int_h E^{\alpha\gamma}(a_\gamma^\pi + \tfrac{1}{2}b_\gamma^\pi x^0)x^0\left(\frac{g}{a}\right)^{1/2} dx^0. \tag{3.20}$$

Furthermore, substituting (3.9), (3.16) in (3.3), (3.4), we have for the external force system

$$F^0 = \int_h X^0\left(\frac{g}{a}\right)^{1/2} dx_0 + \frac{Z^0_{(+)}}{\left| N^0_{(+)} \right|}\left(\frac{g_{(+)}}{a}\right)^{1/2} + \frac{Z^0_{(-)}}{\left| N^0_{(-)} \right|}\left(\frac{g_{(-)}}{a}\right)^{1/2}, \tag{3.21}$$

$$\begin{aligned} F^\alpha = \int_h X^\gamma(a_\gamma^\alpha + \tfrac{1}{2}b_\gamma^\alpha x^0)\left(\frac{g}{a}\right)^{1/2} dx^0 + \frac{Z^\gamma_{(+)}}{\left| N^0_{(+)} \right|}(a_\gamma^\alpha + \tfrac{1}{2}b_\gamma^\alpha h_{(+)})\left(\frac{g_{(+)}}{a}\right)^{1/2} \\ + \frac{Z^\gamma_{(-)}}{\left| N^0_{(-)} \right|}(a_\gamma^\alpha - \tfrac{1}{2}b_\gamma^\alpha h_{(-)})\left(\frac{g_{(-)}}{a}\right)^{1/2}, \end{aligned} \tag{3.22}$$

and for the external moment system

$$\begin{aligned} M^\alpha = \eta^{\lambda\alpha}_{[0]}\, a_{\pi\lambda}\Bigg\{ \int_h X^\gamma(a_\gamma^\pi + \tfrac{1}{2}b_\gamma^\pi x^0)\left(\frac{g}{a}\right)^{1/2} x^0\, dx^0 \\ + \frac{Z^\gamma_{(+)}}{\left| N^0_{(+)} \right|}(a_\gamma^\pi + \tfrac{1}{2}b_\gamma^\pi h_{(+)})h_{(+)}\left(\frac{g_{(+)}}{a}\right)^{1/2} \\ - \frac{Z^\gamma_{(-)}}{\left| N^0_{(-)} \right|}(a_\gamma^\pi - \tfrac{1}{2}b_\gamma^\pi h_{(-)})h_{(-)}\left(\frac{g_{(-)}}{a}\right)^{1/2}. \end{aligned} \tag{3.23}$$

It should be emphasized that the above expressions are exact, no approximation or assumption based on the thickness of the shell or plate being involved.

Now we assume that E^{ij} and X^i can be expanded in power series in x^0:

$$E^{ij} = \sum_0^\infty \frac{1}{m!} E^{ij}_{[m]}(x^0)^m, \qquad X^i = \sum_0^\infty \frac{1}{m!} X^i_{[m]}(x^0)^m. \tag{3.24}$$

We also introduce the abbreviations

$$B_\gamma^\beta = \tfrac{1}{2}(4Ha_\gamma^\beta + b_\gamma^\beta), \qquad C_\gamma^\beta = Ka_\gamma^\beta + Hb_\gamma^\beta, \qquad D_\gamma^\beta = \tfrac{1}{2}Kb_\gamma^\beta, \tag{3.25}$$

$$d^{(n)} = h^n_{(+)} - h^n_{(-)}, \qquad t^{(n)} = h^n_{(+)} + h^n_{(-)}, \qquad d^{(1)} = d, \quad t^{(1)} = t, \tag{3.26}$$

$$Q^{(n)i} = \frac{Z^i_{(+)}}{\left| N^0_{(+)} \right|} h^n_{(+)} - \frac{Z^i_{(-)}}{\left| N^0_{(-)} \right|} h^n_{(-)}, \qquad Q^{(0)i} = Q^i, \tag{3.27}$$

$$P^{(n)i} = \frac{Z^i_{(+)}}{\left| N^0_{(+)} \right|} h^n_{(+)} + \frac{Z^i_{(-)}}{\left| N^0_{(-)} \right|} h^n_{(-)}, \qquad P^{(0)i} = P^i. \tag{3.28}$$

Here H and K are mean and total curvature, as in (3.13), and t is the thickness of the shell or plate, measured normally to S_0. We substitute (3.7), (3.24) into (3.18)–(3.23), and carry out the integrations. This leads to series in t, $d^{(2)}$, $t^{(3)}$, $d^{(4)}$, $\cdots$; namely,

$$T^{\alpha 0} = E^{\alpha 0}_{[0]}t + (2HE^{\alpha 0}_{[0]} + E^{\alpha 0}_{[1]})\frac{d^{(2)}}{2!} + (2KE^{\alpha 0}_{[0]} + 4HE^{\alpha 0}_{(1)} + E^{\alpha 0}_{[2]})\frac{t^{(3)}}{3!} + R_{(1)}(E^{\alpha 0}), \tag{3.29}$$

$$T^{\alpha\beta} = E^{\alpha\beta}_{[0]}t + (B^{\beta}_{\gamma}E^{\alpha\gamma}_{[0]} + E^{\alpha\beta}_{[1]})\frac{d^{(2)}}{2!} + (2C^{\beta}_{\gamma}E^{\alpha\gamma}_{[0]} + 2B^{\beta}_{\gamma}E^{\alpha\gamma}_{[1]} + E^{\alpha\beta}_{[2]})\frac{t^{(3)}}{3!} + R_{(2)}(E^{\alpha\beta}), \tag{3.30}$$

$$L^{\alpha\pi} = \eta^{\lambda\pi}_{[0]}\, a_{\beta\lambda}\left\{E^{\alpha\beta}_{[0]}\frac{d^{(2)}}{2!} + 2(B^{\beta}_{\gamma}E^{\alpha\gamma}_{[0]} + E^{\alpha\beta}_{[1]})\frac{t^{(3)}}{3!} + R_{(3)}(E^{\alpha\beta})\right\}, \tag{3.31}$$

$$\begin{aligned} F^{0} = {} & X^{0}_{[0]}t + (2HX^{0}_{[0]} + X^{0}_{[1]})\frac{d^{(2)}}{2!} + (2KX^{0}_{[0]} + 4HX^{0}_{[1]} + X^{0}_{[2]})\frac{t^{(3)}}{3!} \\ & + P^{0} + 2HQ^{(1)0} + KP^{(2)0} + R_{(1)}(X^{0}), \end{aligned} \tag{3.32}$$

$$\begin{aligned} F^{\alpha} = {} & X^{\alpha}_{[0]}t + (B^{\alpha}_{\gamma}X^{\gamma}_{[0]} + X^{\alpha}_{[1]})\frac{d^{(2)}}{2!} + (2C^{\alpha}_{\gamma}X^{\gamma}_{[0]} + 2B^{\alpha}_{\gamma}X^{\gamma}_{[1]} + X^{\alpha}_{[2]})\frac{t^{(3)}}{3!} \\ & + P^{\alpha} + B^{\alpha}_{\gamma}Q^{(1)\gamma} + C^{\alpha}_{\gamma}P^{(2)\gamma} + D^{\alpha}_{\gamma}Q^{(3)\gamma} + R_{(2)}(X^{\alpha}), \end{aligned} \tag{3.33}$$

$$\begin{aligned} M^{\alpha} = \eta^{\lambda\alpha}_{[0]}a_{\beta\lambda}\Big\{ & X^{\beta}_{[0]}\frac{d^{(2)}}{2!} + 2(B^{\beta}_{\gamma}X^{\gamma}_{[0]} + X^{\beta}_{[1]})\frac{t^{(3)}}{3!} + R_{(3)}(X^{\beta}) \\ & + Q^{(1)\beta} + B^{\beta}_{\gamma}P^{(2)\gamma} + C^{\beta}_{\gamma}Q^{(3)\gamma} + D^{\beta}_{\gamma}P^{(4)\gamma}\Big\}. \end{aligned} \tag{3.34}$$

In these series the remainders are as follows:

$$\begin{aligned} R_{(1)}(E^{\alpha 0}) = {} & \sum \frac{1}{m+5}\left\{\frac{E^{\alpha 0}_{[m+4]}}{(m+4)!} + \frac{2HE^{\alpha 0}_{[m+3]}}{(m+3)!} + \frac{KE^{\alpha 0}_{[m+2]}}{(m+2)!}\right\}t^{(m+5)} \\ & + \sum \frac{1}{m+4}\left\{\frac{E^{\alpha 0}_{[m+3]}}{(m+3)!} + \frac{2HE^{\alpha 0}_{[m+2]}}{(m+2)!} + \frac{KE^{\alpha 0}_{[m+1]}}{(m+1)!}\right\}d^{(m+4)}, \end{aligned} \tag{3.35}$$

$$\begin{aligned} R_{(2)}(E^{\alpha\beta}) = {} & \sum \frac{1}{m+5}\left\{\frac{a^{\beta}_{\gamma}E^{\alpha\gamma}_{[m+4]}}{(m+4)!} + \frac{B^{\beta}_{\gamma}E^{\alpha\gamma}_{[m+3]}}{(m+3)!} + \frac{C^{\beta}_{\gamma}E^{\alpha\gamma}_{[m+2]}}{(m+2)!} + \frac{D^{\beta}_{\gamma}E^{\alpha\gamma}_{[m+1]}}{(m+1)!}\right\}t^{(m+5)} \\ & + \sum \frac{1}{m+4}\left\{\frac{a^{\beta}_{\gamma}E^{\alpha\gamma}_{[m+3]}}{(m+3)!} + \frac{B^{\beta}_{\gamma}E^{\alpha\gamma}_{[m+2]}}{(m+2)!} + \frac{C^{\beta}_{\gamma}E^{\alpha\gamma}_{[m+1]}}{(m+1)!} + \frac{D^{\beta}_{\gamma}E^{\alpha\gamma}_{(m)}}{m!}\right\}d^{(m+4)}, \end{aligned} \tag{3.36}$$

$$\begin{aligned} R_{(3)}(E^{\alpha\beta}) = {} & \sum \frac{1}{m+5}\left\{\frac{a^{\beta}_{\gamma}E^{\alpha\gamma}_{[m+3]}}{(m+3)!} + \frac{B^{\beta}_{\gamma}E^{\alpha\gamma}_{[m+2]}}{(m+2)!} + \frac{C^{\beta}_{\gamma}E^{\alpha\gamma}_{[m+1]}}{(m+1)!} + \frac{D^{\beta}_{\gamma}E^{\alpha\gamma}_{[m]}}{m!}\right\}t^{(m+5)} \\ & + \sum \frac{1}{m+4}\left\{\frac{a^{\beta}_{\gamma}E^{\alpha\gamma}_{[m+2]}}{(m+2)!} + \frac{B^{\beta}_{\gamma}E^{\alpha\gamma}_{[m+1]}}{(m+1)!} + \frac{C^{\beta}_{\gamma}E^{\alpha\gamma}_{[m]}}{m!}\right\}d^{(m+4)} \\ & + \sum \frac{1}{m+6}\,\frac{D^{\beta}_{\gamma}E^{\alpha\gamma}_{[m+1]}}{(m+1)!}d^{(m+6)}. \end{aligned} \tag{3.37}$$

These are accurate expressions, the summations being all for $m=0, 2, 4, \cdots$. The remainders $R_{(1)}(X^0)$, $R_{(2)}(X^\alpha)$, $R_{(3)}(X^\beta)$ are obtained simply by replacing $E^{\alpha 0}_{[m]}$ with $X^0_{[m]}$ in $R_{(1)}(E^{\alpha 0})$, $E^{\beta\gamma}_{[m]}$ with $X^\gamma_{[m]}$ in $R_{(2)}(E^{\beta\alpha})$, $E^{\alpha\gamma}_{[m]}$ with $X^\gamma_{[m]}$ in $R_{(3)}(E^{\alpha\beta})$ respectively.

In the ordinary case where the body force is the force of gravity, we can regard X^i as a parallel vector field of constant magnitude. Then by (3.9), we have the following relations:

$$X^0_{[m]} = 0 \quad \text{for} \quad m \geqq 1, \tag{3.38}$$

$$X^\alpha_{[1]} = -\tfrac{1}{2}b^\alpha_\gamma X^\gamma_{[0]}, \qquad X^\alpha_{[m]} = 0 \quad \text{for} \quad m \geqq 2.$$

And consequently the tensor F^i, M^α in (3.32)–(3.34) can be simplified to the $(t^{(n)}, d^{(n)})$ polynomial of few terms. The most important equations of this section, for future use, are (3.29)–(3.34)

4. The tensors $T^{\alpha i}$, $L^{\alpha\beta}$ in terms of the six quantities $p_{\alpha\beta}$, $q_{\alpha\beta}$. We shall devote the present section to finding expressions for the tensors, (3.29)–(3.31), in terms of the six quantities $p_{\alpha\beta}$ and $q_{\alpha\beta}$. Here $p_{\alpha\beta}$ represents the extension of the reference surface S_0 and $q_{\alpha\beta}$ is closely connected to the change of curvature of the reference surface during the deformation; both were introduced in the paper [5] (p. 114, Eq. (44)), but will be formally defined below in (4.4).

We shall now proceed to determine $E^{i0}_{[m]}$, $E^{\alpha\beta}_{[m]}$ defined in (3.24) in terms of $p_{\alpha\beta}$ and $q_{\alpha\beta}$. This is accomplished by means of (i) the equations of microscopic equilibrium, (ii) the stress-strain relation, (iii) some geometrical relations, (iv) the conditions on the upper and lower boundary surfaces. The successive steps are as follows:

(I) By means of (i), we express $E^{i0}_{[m]}$ successively for $m=1, 2, \cdots$ in terms of $E^{i0}_{[0]}$ and $E^{\alpha\beta}_{[n]}$, where $n=0, 1, 2, \cdots, (m-1)$.

(II) With the aid of (ii), (iii) and the results of step (I), we determine $E^{\alpha\beta}_{[0]}$, $E^{i0}_{[1]}$, $E^{\alpha\beta}_{[1]}$, $E^{i0}_{[2]}$, $E^{\alpha\beta}_{[2]}$, $\cdots$ successively in terms of $E^{i0}_{[0]}$, $p_{\alpha\beta}$, $q_{\alpha\beta}$.

(III) Then using (iv) and the results in step (II), we determine $E^{i0}_{[0]}$, in terms of $p_{\alpha\beta}$, $q_{\alpha\beta}$. The surface force system (P^i, Q^i) is supposed to be given. Thus we have at once $E^{i0}_{[m]}$ and $E^{\alpha\beta}_{[m]}$ for all m in terms of $p_{\alpha\beta}$ and $q_{\alpha\beta}$.

(IV) Substituting $E^{\alpha\beta}_{[m]}$ from step (III) into (3.30), (3.31), we obtain the required expressions for $T^{\alpha\beta}$ and $L^{\alpha\beta}$. The expression for $T^{\alpha 0}$ can be found either by substituting $E^{\alpha 0}_{[m]}$ from step (III) into (3.29), or by using the equation of macroscopic equilibrium (2.5c).

The geometry of strain and the definitions of $p_{\alpha\beta}$ and $q_{\alpha\beta}$. Let us introduce comoving coordinates [7, 8]. The same coordinates are attached to each particle during deformation, and the coordinates form a normal system in the strained state. The fundamental tensor in the strained state is g_{ij} (satisfying (2.1), (2.2)), and in the unstrained state it is g'_{ij}. Let S'_0 be the surface in the unstrained state which is carried over into the reference surface S_0 in the strained state (after we reach section 6, we shall define S'_0 to be the middle surface in the unstrained state). The parametric lines of x^0 are not in general normal to the reference surface S'_0 in the unstrained state.

The strain tensor e_{ij} is defined as

$$e_{ij} = \tfrac{1}{2}(g_{ij} - g'_{ij}); \tag{4.1}$$

this is the definition usually adopted (cf. [7, 8, 9]). For small deformation, the principal part of the extension of an element in the direction λ^i is $e=e_{ij}\lambda^i\lambda^j$.

We shall throughout raise suffixes by means of g^{ij}; thus

$$g'^{mn} = g^{mi}g^{nj}g'_{ij}, \qquad g'^{i}_{m} = g^{ij}g'_{jm}, \tag{4.2a}$$

$$e^{mn} = g^{mi}g^{nj}e_{ij}, \qquad e^{i}_{m} = g^{ij}e_{jm}. \tag{4.2b}$$

Now we assume that e_{ij} is expansible in power series in x^0, so that

$$e_{ij} = \sum_{m=0}^{\infty} \frac{1}{m!} e_{[m]ij}(x^0)^m. \tag{4.3}$$

Let us define

$$p_{ij} = e_{[0]ij}, \qquad q_{ij} = e_{[1]ij}, \qquad r_{ij} = e_{[2]ij}; \tag{4.4}$$

obviously these tensors are symmetric. The six components $p_{\alpha\beta}$, $q_{\alpha\beta}$ are the basic dependent variables of our theory.

For small strain p_{ij} must be small, yet q_{ij}, r_{ij}, etc. may be finite in a thin shell or plate. These quantities are not independent but are connected by certain geometrical relations. Since space is flat, g_{ij} and g'_{ij} are not arbitrary functions of the coordinates. They must satisfy the equations

$$\widehat{R}_{ijkl} = 0, \tag{4.5a}$$

$$\widehat{R}'_{ijkl} = 0. \tag{4.5b}$$

Here $\widehat{R}_{ijkl}$ is the curvature tensor for g_{ij},

$$\widehat{R}_{ijkl} = \tfrac{1}{2}(g_{il,jk} + g_{jk,il} - g_{ik,jl} - g_{jl,ik}) + g^{mn}\{[il, m]_g[jk, n]_g - [ik, m]_g[jl, n]_g\}, \tag{4.6}$$

while $\widehat{R}_{ijkl}$ is the corresponding curvature tensor for g'_{ij},

$$\widehat{R}'_{ijkl} = \tfrac{1}{2}(g'_{il,jk} + g'_{jk,il} - g'_{ik,jl} - g'_{jl,ik}) + \bar{g}'^{mn}\{[il, m]_{g'}[jk, n]_{g'} - [ik, m]_{g'}[jl, n]_{g'}\}. \tag{4.7}$$

It should be noted that $\widehat{R}_{ijkl}$ is the curvature tensor in space, while $R_{\rho\alpha\beta\gamma}$ is the two dimensional curvature tensor of S_0 (cf. (3.11)). In (4.7), $\bar{g}'^{mn}$ by definition denotes the cofactor of g'_{mn} in g', divided by g'; namely,

$$\bar{g}'^{mn} = \frac{1}{2!g'} \epsilon^{mrs}\epsilon^{nkl}g'_{rk}g'_{sl}, \tag{4.8}$$

where ϵ^{mkl} is the usual permutation symbol, and g' is the determinant of g'_{ij},

$$g' = \frac{1}{3!} \epsilon^{rst}\epsilon^{mnp}g'_{rm}g'_{sn}g'_{tp}. \tag{4.9}$$

It follows that

$$\bar{g}'^{mm} = \frac{3!\eta^{mrt}\eta^{nkl}g'_{rk}g'_{tl}}{2!\eta^{ijs}\eta^{uvw}g'_{iu}g'_{jv}g'_{sw}}, \tag{4.10}$$

where

$$\eta^{rst} = \epsilon^{rst}g^{-1/2}; \tag{4.11}$$

this is a contravariant tensor and satisfies the relation

$$\eta^{rst}\eta_{rlk} = \delta_l^s\delta_k^t - \delta_k^s\delta_l^t. \tag{4.12}$$

The equations (4.5a, b) form a set of twelve independent geometrical conditions for g_{ij} and g'_{ij}. We shall now regard g_{ij} as given, and develop (4.5b) explicitly in terms of e_{ij} by using (4.5a), (4.1). The resulting expressions represent the six independent geometrical conditions for e_{ij}:

$$\begin{aligned}
&-(e_{il,jk}+e_{jk,il}-e_{ik,jl}-e_{jl,ik})\\
&+2e^{mn}\{[il,m]_g[jk,n]_g-[ik,m]_g[jl,n]_g\}\\
&-2g^{mn}\{[il,m]_g[jk,n]_e+[il,m]_e[jk,n]_g-[ik,m]_e[jl,n]_g-[ik,m]_g[jl,n]_e\}\\
&+\overset{(2)}{\Phi}_{ijkl}+\overset{(3)}{\Phi}_{ijkl}+\overset{(4)}{\Phi}_{ijkl}=0.
\end{aligned} \tag{4.13}$$

This is a polynomial in e_{ij}, with linear terms explicitly stated. The other terms proceed with increasing degree in e_{ij}, and have the following exact expressions:

$$\begin{aligned}
\Phi^{(2)}_{ijkl} = {} & 2e_t^t(e_{il,jk}+e_{jk,il}-e_{ik,jl}-e_{jl,ik})\\
&+4g^{mn}\{[il,m]_e[jk,n]_e-[ik,m]_e[jl,n]_e\}\\
&+4(e^{ns}e_s^m-e^{mn}e_s^s)\{[il,m]_g[jk,n]_g-[ik,m]_g[jl,n]_g\}\\
&+4(g^{mn}e_s^s-e^{mn})\{[il,m]_e[jk,n]_g+[il,m]_g[jk,n]_e\\
&\qquad-[ik,m]_e[jl,n]_g-[ik,m]_g[jl,n]_e\},
\end{aligned} \tag{4.13a}$$

$$\begin{aligned}
\Phi^{(3)}_{ijkl} = {} & -2(e_s^se_t^t-e^{st}e_{st})(e_{il,jk}+e_{jk,il}-e_{ik,jl}-e_{jl,ik})\\
&-\tfrac{2}{3}\eta^{stp}\eta^{uvw}e_{su}e_{tv}e_{pw}(g_{il,jk}+g_{jk,il}-g_{ik,jl}-g_{jl,ik})\\
&-8(g^{mn}e_s^s-e^{mn})\{[il,m]_e[jk,n]_e-[ik,m]_e[jl,n]_e\}\\
&-4\eta^{mrs}\eta^{ntp}e_{rt}e_{sp}\{[il,m]_e[jk,n]_g-[ik,m]_e[jl,n]_g\\
&\qquad+[il,m]_g[jk,n]_e-[ik,m]_g[jl,n]_e\},
\end{aligned} \tag{4.13b}$$

$$\Phi^{(4)}_{ijkl} = \tfrac{4}{3}\eta^{stp}\eta^{uvw}e_{su}e_{tv}e_{pw}(e_{il,jk}+e_{jk,il}-e_{ik,jl}-e_{jl,ik}), \tag{4.13c}$$

$$[ij,k]_g = \tfrac{1}{2}(g_{ik,j}+g_{jk,i}-g_{ij,k}), \tag{4.13d}$$

$$[ij,k]_e = \tfrac{1}{2}(e_{ik,j}+e_{jk,i}-e_{ij,k}). \tag{4.13e}$$

Three of (4.13) involve only e_{ij}, $e_{ij,0}$ and their derivatives with respect to x^α. When $x^0=0$, these three conditions become the equations of compatibility of the twelve quantities p_{ij} and q_{ij}; a detailed formulation of these equations will be given in section 5.

The other three of (4.13) involve not only e_{ij}, $e_{ij,0}$ and their derivatives with respect to x^α, but also $e_{\alpha\beta,00}$. In fact, by these equations, we express $e_{\alpha\beta,00}$ in terms of e_{ij}, $e_{ij,0}$ and their derivatives with respect to x^α. When $x^0=0$, these equations give us the expression of $r_{\alpha\beta}$ in terms of p_{ij}, q_{ij} and their derivatives with respect to x^α. Putting $i=\alpha$, $j=0$, $k=\beta$, $l=0$ and $x^0=0$ in (4.13), and solving for $r_{\alpha\beta}$, we obtain

$$
\begin{aligned}
r_{\alpha\beta} = e_{[2]\alpha\beta} = q_{\alpha 0|\beta} + q_{\beta 0|\alpha} - p_{00|\alpha\beta} + \tfrac{1}{2}(b_\alpha^\pi a_\beta^\lambda + b_\beta^\lambda a_\alpha^\pi)q_{\lambda\pi} + \tfrac{1}{2}b_{\alpha\beta}q_{00} \\
- \tfrac{1}{2}b_\beta^\lambda b_\alpha^\pi p_{\lambda\pi} + \tfrac{1}{2}(b_\beta^\pi a_\alpha^\lambda + b_\alpha^\pi a_\beta^\lambda - b_\beta^\lambda a_\alpha^\pi - b_\alpha^\lambda a_\beta^\pi)p_{\pi 0|\lambda} \\
- q_{00}q_{\alpha\beta} - q_\alpha^\pi q_{\beta\pi} + O_{(2)\alpha\beta}(\hat{p}^2, \hat{p}\hat{q}), \qquad (4.14)
\end{aligned}
$$

where $O_{(2)\alpha\beta}$ is a residual term, to be explained below.

To find $e_{[3]\alpha\beta}$, we have to differentiate $e_{\alpha\beta,00}$ with respect to x^0, and put $x^0=0$ in the resulting equations. By (4.14), this gives $e_{[3]\alpha\beta}$ in terms of p_{ij}, q_{ij}, r_{i0} and their derivatives with respect to x^α. To find the other $e_{[m]\alpha\beta}$, we merely repeat the process over and over again. Thus we can express $e_{[m]\alpha\beta}$ in terms of p_{ij}, q_{ij}, $e_{[n]i0}$ and their derivatives with respect to x^α, where $n<m$.

Actually, to obtain the principal parts of the final equations, we only require explicit calculation for $m=2$, as in (4.14).

All the above results are of a purely geometrical nature.

The symbol $O_{(2)}(\hat{p}^2, \hat{p}\hat{q})$ in (4.14) represents the terms which are not explicitly calculated. The quantities in parentheses indicate order of magnitude of these terms for small $\hat{p}$ and $\hat{q}$, which denote the magnitudes of the tensors p_{ij}, q_{ij} respectively. If $\hat{p}$, $\hat{q}$ approach zero simultaneously and independently, these terms converge to zero at least as fast as $\hat{p}\hat{q}$ or $\hat{p}^2$. Symbolically, we may write

$$O_{\alpha\beta}(\hat{p}^2, \hat{p}\hat{q}) = O_{\alpha\beta}(\hat{p}^2) + O_{\alpha\beta}(\hat{p}\hat{q}). \qquad (4.14a)$$

The label "(2)" is to distinguish this O-symbol from later symbols of the same type. The indices attached to $O_{(2)}$ are the tensorial indices of every terms involved. We assume throughout that differentiation with respect to the coordinates x^α does not change the order of magnitude; i.e., $p_{\alpha\beta}$, $p_{\alpha\beta,\gamma}$ are of the same order. On the other hand, we never make any assumption regarding the effect of differentiation with respect to x^0.

This O-symbol notation will be used extensively throughout the paper. The notations used inside the parentheses of O-symbols are given in the following table:

Tensors	$b_{\alpha\beta}$	$\mathbf{b}_{\alpha\beta}$	$e_{\alpha\beta}$	e_{ij}	E^{i0}	$E^{i0}_{0]}$	p_{ij}	$p_{\alpha\beta}$	$\mathbf{p}_{\alpha\beta}$	P^α	P^i	q_{ij}
Symbols for magnitudes	b	$\mathbf{b}$	e	$\hat{e}$	$\hat{E}$	E_0	$\hat{p}$	p	$\mathbf{p}$	P	$\hat{P}$	

Tensors	$\mathbf{q}_{\alpha\beta}$	$q_{\alpha\beta}$	Q^i	Q^α	$X^i_{[m]}$	$X^\alpha_{[m]}$
Symbols for magnitudes	$\mathbf{q}$	q	$\hat{Q}$	Q	X	X

The expressions of $E^{i0}_{[m]}$ in terms of $E^{i0}_{[0]}$ and $E^{\alpha\beta}_{[n]}$ $(n=0, 1, \cdots, (m-1))$.

We start the process outlined in the beginning of this section by writing down the microscopic equations of equilibrium under the body force X^i:

$$E^{ki}{}_{||k} + X^i = 0, \qquad (4.15)$$

the double stroke indicating covariant differentiation using g_{ij}. Putting in turn $i=0$ and $i=\alpha$, we get three equations which may be written as

$$E^{00}{}_{,0} = \tfrac{1}{2}g_{\pi\lambda,0}E^{\pi\lambda} - \tfrac{1}{2}g^{\pi\lambda}g_{\pi\lambda,0}E^{00} - E^{\pi 0}{}_{\underset{g}{|}\pi} - X^0, \qquad (4.16)$$

$$E^{\alpha 0}{}_{,0} = -\tfrac{1}{2}g^{\pi\lambda}g_{\pi\lambda,0}E^{\alpha 0} - g^{\alpha\pi}g_{\pi\lambda,0}E^{\lambda 0} - E^{\alpha\pi}{}_{\underset{g}{|}\pi} - X^\alpha, \qquad (4.17)$$

where g under the stroke indicates covariant differentiation using $g_{\alpha\beta}$,

$$E^{\pi 0}{}_{\underset{g}{|}\pi} = E^{\pi 0}{}_{,\pi} + \left\{ \begin{matrix} \pi \\ \lambda\pi \end{matrix} \right\}_g E^{\lambda 0}, \tag{4.18}$$

$$E^{\pi\alpha}{}_{\underset{g}{|}\pi} = E^{\pi\alpha}{}_{,\pi} + \left\{ \begin{matrix} \pi \\ \lambda\pi \end{matrix} \right\}_g E^{\lambda\alpha} + \left\{ \begin{matrix} \alpha \\ \lambda\pi \end{matrix} \right\}_g E^{\lambda\pi}. \tag{4.19}$$

This operation should be clearly distinguished from that indicated in (2.6). Putting $x^0=0$ in (4.16), (4.17), we get

$$E^{00}_{[1]} = -X^0_{[0]} - E^{\pi 0}_{[0]|\pi} + \tfrac{1}{2}b_{\pi\lambda}E^{\pi\lambda}_{[0]} - 2HE^{00}_{[0]}, \tag{4.19a}$$

$$E^{\alpha 0}_{[1]} = -X^\alpha_{[0]} - E^{\pi\alpha}_{[0]|\pi} - b^\alpha_\pi E^{\pi 0}_{[0]} - 2HE^{\alpha 0}_{[0]}. \tag{4.19b}$$

These are the expressions of $E^{i0}_{[1]}$ in terms of $E^{i0}_{[0]}$ and $E^{\alpha\beta}_{[0]}$, the body forces being supposed given. To find $E^{i0}_{[m]}$, we have to differentiate (4.16), (4.17) with respect to x^0 over and over again, and put $x^0=0$ in each of the resulting equations. We shall only write the expressions of $E^{i0}_{[2]}$ explicitly:

$$\begin{aligned} E^{00}_{[2]} = &-X^0_{[1]} + 2HX^0_{[0]} + X^\pi_{[0]|\pi} + E^{\pi\lambda}_{[0]|\pi\lambda} + \tfrac{1}{2}b_{\pi\lambda}E^{\pi\lambda}_{[1]} - a_{\pi\lambda}KE^{\pi\lambda}_{[0]} \\ &+ (8H^2 - 2K)E^{00}_{[0]} + b^\pi_\lambda E^{\lambda 0}_{[0]|\pi} + 4(HE^{\lambda 0}_{[0]})_{|\lambda}, \end{aligned} \tag{4.20a}$$

$$\begin{aligned} E^{0\alpha}_{[2]} = &-X^\alpha_{[1]} + (b^\alpha_\pi + 2Ha^\alpha_\pi)X^\pi_{[0]} - a^\alpha_\pi E^{\pi\lambda}_{[1]|\lambda} + (b^\alpha_\pi + 2Ha^\alpha_\pi)E^{\pi\lambda}_{[0]|\lambda} \\ &- \tfrac{1}{2}(4Ha^\alpha_\pi + b^\alpha_\pi)_{|\lambda}E^{\pi\lambda}_{[0]} + (10Hb^\alpha_\pi + 8H^2a^\alpha_\pi - 8Ka^\alpha_\pi)E^{\pi 0}_{[0]}. \end{aligned} \tag{4.20b}$$

This completes Step (I).

Stress-strain relations and expressions for $E^{i0}_{[m]}$, $E^{\alpha\beta}_{[m]}$ *in terms of* $p_{\alpha\beta}$, $q_{\alpha\beta}$, $E^{i0}_{[0]}$.

We shall accept, as the stress-strain relation for an isotropic body, the equation

$$E^{ij} = \frac{1}{(1+\sigma)(1-2\sigma)} \left\{ \sigma g^{ij}g^{kl} + (1-\sigma)g^{ik}g^{jl} \right\} e_{kl} + O^{ij}_{(3)}(\hat{e}^2), \tag{4.21}$$

where σ is Poisson's ratio, and E^{ij} is of course the reduced stress (see section 2); therefore Young's modulus does not appear. For small strain problems, $\hat{e}$ is small, and the terms in $O^{ij}(\hat{e}^2)$ are negligible in comparison with the terms linear in e^{ij} in (4.21); in that case, (4.21) becomes the usual linear stress-strain relation for an isotropic body (in rectangular Cartesians, see [1], p. 102, Eq. (18)). Any modification, such as replacement of g^{ij} by g'^{ij}, leads to no real change, because the difference is taken care of by the O-symbol.

From (4.21), we have

$$E^{00} = \frac{1}{(1+\sigma)(1-2\sigma)} \left\{ \sigma g^{\lambda\pi}e_{\lambda\pi} + (1-\sigma)e_{00} \right\} + O^{00}_{(3)}(\hat{e}^2), \tag{4.22}$$

$$E^{\alpha\beta} = \frac{1}{(1+\sigma)(1-2\sigma)} \left\{ \sigma g^{\alpha\beta}e_{00} + [\sigma g^{\alpha\beta}g^{\pi\lambda} + (1-2\sigma)g^{\alpha\pi}g^{\beta\lambda}]e_{\pi\lambda} \right\} + O^{\alpha\beta}_{(3)}(\hat{e}^2), \tag{4.23}$$

$$E^{\alpha 0} = \frac{1}{(1+\sigma)} g^{\alpha\pi}e_{\pi 0} + O^{\alpha 0}_{(3)}(\hat{e}^2). \tag{4.24}$$

It is evident from (4.21) that all the stress components are small, of at least as high an order as the largest strain components. The elimination of e_{0i} from the last three equations gives

$$E^{\alpha\beta} = \frac{\sigma}{1-\sigma} g^{\alpha\beta} E^{00} + \frac{1}{1-\sigma^2}\left\{\sigma g^{\alpha\beta}E^{\pi\lambda} + (1-\sigma) g^{\alpha\pi} g^{\beta\lambda}\right\} e_{\pi\lambda} + O^{\alpha\beta}_{(4)}(e^2, \widehat{E}^2, \widehat{E}e). \quad (4.25)$$

When $x^0=0$, the equation (4.25) becomes

$$E^{\alpha\beta}_{[0]} = \frac{\sigma}{1-\sigma} a^{\alpha\beta} E^{00}_{[0]} + A^{\alpha\beta\lambda\pi}_{(1)} p_{\pi\lambda} + O^{\alpha\beta}_{(4)}(p^2, \widehat{E}^2_0, \widehat{E}_0 p), \quad (4.26)$$

where the abbreviation $A^{\alpha\beta\pi\lambda}_{(1)}$ means

$$A^{\alpha\beta\pi\lambda}_{(1)} = \frac{1}{1-\sigma^2}\left\{\sigma a^{\alpha\beta} a^{\pi\lambda} + (1-\sigma) a^{\alpha\pi} a^{\lambda\beta}\right\}. \quad (4.27)$$

Equation (4.26) is the required expression for $E^{\alpha\beta}_{[0]}$.

The substitution of $E^{\alpha\beta}_{[0]}$ from (4.26) into (4.19a, b) gives

$$E^{00}_{[1]} = -X^0_{[0]} - E^{\pi 0}_{[0]|\pi} - \frac{2(1-2\sigma)}{(1-\sigma)} H E^{00}_{[0]} + \tfrac{1}{2} A^{\delta\gamma\pi\lambda}_{(1)} b_{\delta\gamma} p_{\pi\lambda}$$
$$+ O^{00}_{(5)}(bp^2, b\widehat{E}_0 p, b\widehat{E}^2_0), \quad (4.28)$$

$$E^{\alpha 0}_{[1]} = -X^{\alpha}_{[0]} - b^{\alpha}_{\pi} E^{\pi 0}_{[0]} - 2HE^{\alpha 0}_{[0]} - \frac{\sigma}{1-\sigma} a^{\alpha\pi} E^{00}_{[0]|\pi} - A^{\lambda\alpha\pi\delta}_{(1)} p_{\pi\delta|\lambda}$$
$$+ O^{\alpha 0}_{(6)}(p^2, p\widehat{E}_0, \widehat{E}^2_0). \quad (4.29)$$

Here $X^i_{[0]}$ are supposed to be given. These are the required expressions of $E^{i0}_{[1]}$.

Now, differentiating (4.25) with respect to x^0 and putting $x^0=0$ in the resulting equations, we obtain in consequence of (4.26), (4.28)

$$E^{\alpha\beta}_{[1]} = -\frac{\sigma}{1-\sigma} a^{\alpha\lambda} X^0_{[0]} - \frac{\sigma}{1-\sigma} a^{\alpha\beta} E^{\pi 0}_{[0]|\pi} - \frac{\sigma}{1-\sigma}\left\{b^{\alpha\beta} + \frac{2(1-2\sigma)}{(1-\sigma)} H a^{\alpha\beta}\right\} E^{00}_{[0]}$$
$$+ \left\{\frac{\sigma}{2(1-\sigma)} a^{\alpha\beta} A^{\lambda\delta\pi\gamma}_{(1)} - a^{\beta\delta} A^{\alpha\lambda\pi\gamma}_{(1)} - a^{\pi\beta} A^{\alpha\beta\delta\gamma}_{(1)}\right\} b_{\lambda\delta} p_{\pi\gamma}$$
$$+ A^{\alpha\beta\pi\lambda}_{(1)} q_{\pi\lambda} + O^{\alpha\beta}_{(7)}, \quad (4.30)$$

where

$$O^{\alpha\beta}_{(7)} = O^{\alpha\beta}_{(7)}(bp^2, bp\widehat{E}_0, b\widehat{E}^2_0, pq, q\widehat{E}_0, p\widehat{X}, \widehat{X}\widehat{E}_0). \quad (4.31)$$

Equation (4.30) is the required expression of $E^{\alpha\beta}_{[1]}$.

The substitution of $E^{\alpha\beta}_{[0]}$, $E^{\alpha\beta}_{[1]}$ from (4.26), (4.30) into (4.20a, b) gives the required expressions for $E^{i0}_{[2]}$:

$$E^{00}_{[2]} = \frac{2(1-2\sigma)}{(1-\sigma)} HX^{0}_{[0]} - X^{0}_{[1]} + X^{\pi}_{[0]|\pi}$$
$$+ \frac{2(1-2\sigma)}{(1-\sigma)^2}\left\{2(2-3\sigma)H^2 - (1-\sigma)K\right\}E^{00}_{[0]}$$
$$+ \frac{\sigma}{1-\sigma} a^{\pi\lambda}E^{00}_{[0]|\pi\lambda} + \frac{2(2-3\sigma)}{(1-\sigma)} HE^{\pi 0}_{[0]|\pi} + (b^{\lambda}_{\pi}E^{\pi 0}_{[0]})_{|\lambda} + A^{\lambda\delta\pi\gamma}_{(1)}p_{\pi\gamma|\lambda\delta}$$
$$- \frac{1}{1-\sigma^2}\left\{(4-3\sigma)Hb^{\pi\lambda} + \frac{4\sigma(2-3\sigma)}{1-\sigma}H^2a^{\pi\lambda} - (3-\sigma)Ka^{\pi\lambda}\right\}p_{\pi\lambda}$$
$$+ \frac{1}{2(1-\sigma^2)}\left\{4\sigma Ha^{\pi\lambda} + (1-\sigma)b^{\pi\lambda}\right\}q_{\pi\lambda} + O^{00}_{(8)}. \tag{4.32}$$

$$E^{\alpha 0}_{(2)} = (b^{\alpha}_{\pi} + 2Ha^{\alpha}_{\pi})X^{\pi}_{[0]} - X^{\alpha}_{[1]} + \frac{\sigma}{1-\sigma} a^{\alpha\pi}X^{0}_{[0]|\pi}$$
$$+ \frac{\sigma}{1-\sigma} a^{\alpha\lambda}E^{\pi 0}_{[0]|\pi\lambda} + \frac{2\sigma}{1-\sigma} b^{\alpha\pi}E^{00}_{[0]|\pi}$$
$$+ \frac{2\sigma(2-3\sigma)}{(1-\sigma)^2} a^{\alpha\pi}(HE^{00}_{[0]})_{|\pi} + (10Hb^{\alpha}_{\pi} + 8H^2a^{\alpha}_{\pi} - 8Ka^{\alpha}_{\pi})E^{\pi 0}_{[0]} - A^{\alpha\delta\pi\lambda}_{(1)}q_{\pi\lambda|\delta}$$
$$+ \frac{1}{1-\sigma^2}\left\{\tfrac{1}{2}\sigma b^{\pi\lambda}{}_{|\delta}\, a^{\alpha\delta} + \frac{1-\sigma}{2} b^{\alpha\lambda}{}_{|\delta}\, a^{\delta\pi}\right.$$
$$\left. - \frac{2\sigma^2}{1-\sigma} H_{|\delta}\, a^{\alpha\delta}a^{\pi\lambda} + 3(1-\sigma)H_{|\delta}\, a^{\alpha\pi}a^{\delta\lambda}\right\} p_{\pi\lambda}$$
$$+ \frac{1}{1-\sigma^2}\left\{\sigma(2b^{\alpha\delta}a^{\pi\lambda} + \tfrac{1}{2}b^{\pi\lambda}a^{\alpha\delta}) + \frac{2\sigma(1-2\sigma)}{(1-\sigma)} Ha^{\alpha\delta}a^{\pi\lambda}\right.$$
$$\left. + (1-\sigma)(2b^{\alpha\pi}a^{\delta\lambda} + a^{\alpha\pi}b^{\delta\lambda} + 2Ha^{\alpha\pi}a^{\delta\lambda})\right\} p_{\pi\lambda|\delta} + O^{\alpha 0}_{(9)}. \tag{4.33}$$

Here

$$O^{00}_{(8)} = O^{00}_{(8)}(p^2,\ p\widehat{E}_0,\ \widehat{E}^2_0,\ bpq,\ bq\widehat{E}_0,\ bp\widehat{X},\ b\widehat{X}\widehat{E}_0), \tag{4.34}$$

$$O^{\alpha 0}_{(9)} = O^{\alpha 0}_{(9)}(bp^2,\ bp\widehat{E}_0,\ b\widehat{E}^2_0,\ pq,\ q\widehat{E}_0,\ p\widehat{X},\ \widehat{X}\widehat{E}_0). \tag{4.35}$$

To find $E^{\alpha\beta}_{[2]}$, we have to determine p_{i0}, q_{i0} and then $r_{\alpha\beta}$ in terms of $p_{\alpha\beta}$, $q_{\alpha\beta}$, $E^{i0}_{[0]}$. Eliminating e_{00} from (4.22), (4.24) and solving the resulting expression for $e_{\alpha 0}$, we obtain

$$e_{\alpha 0} = (1+\sigma)g_{\alpha\beta}E^{0\beta} + O_{(10)\alpha}(e^2,\ e\widehat{E},\ \widehat{E}^2). \tag{4.36}$$

when $x^0 = 0$, this becomes

$$p_{\alpha 0} = (1+\sigma)a_{\alpha\beta}E^{\beta 0}_{[0]} + O_{(10)\alpha}(p^2,\ p\widehat{E}_0,\ \widehat{E}^2_0). \tag{4.37}$$

We now differentiate (4.36) with respect to x^0 and put $x^0 = 0$. In consequence of (4.26), (4.28), (4.29), this gives

$$q_{\alpha 0} = -\frac{1+\sigma}{1-\sigma}\left\{(1-\sigma)X_{[0]\alpha} + \sigma E^{00}_{[0]|\alpha} + 2(1-\sigma)Ha_{\alpha\beta}E^{\beta 0}_{[0]}\right\} - \frac{\sigma}{1-\sigma}a^{\pi\lambda}p_{\pi\lambda|\alpha}$$
$$- a^{\lambda\pi}p_{\alpha\pi|\lambda} + O_{(11)\alpha 0}(p^2, p\widehat{E}_0, \widehat{E}_0^2, pq, q\widehat{E}_0, p\widehat{X}, \widehat{X}E_0). \tag{4.38}$$

Similarly, from (4.22), the values of e_{00} and $e_{00,0}$ on $x^0=0$ are, by (4.28), (4.29)

$$p_{00} = \frac{(1+\sigma)(1-2\sigma)}{(1-\sigma)}E^{00}_{[0]} - \frac{\sigma}{1-\sigma}a^{\pi\lambda}p_{\pi\lambda} + O_{(12)00}(p^2, p\widehat{E}_0, \widehat{E}_0^2), \tag{4.39}$$

$$q_{00} = -\frac{(1+\sigma)(1-2\sigma)}{(1-\sigma)}\left\{X^0_{[0]} + E^{\pi 0}_{[0]|\pi} + \frac{2(1-2\sigma)}{1-\sigma}HE^{00}_{[0]}\right\} - \frac{\sigma}{1-\sigma}a^{\lambda\pi}q_{\pi\lambda}$$
$$+ \frac{1}{2(1-\sigma)}\left\{4\sigma(1-2\sigma)Ha^{\pi\lambda} + (1-\sigma)b^{\pi\lambda}\right\}p_{\pi\lambda} + O_{(13)00}, \tag{4.40}$$

where

$$O_{(13)00} = O_{(13)00}(bp^2, bp\widehat{E}_0, b\widehat{E}_0^2, pq, q\widehat{E}_0, p\widehat{X}, \widehat{X}\widehat{E}_0). \tag{4.41}$$

Substituting p_{i0}, q_{i0} from (4.37)–(4.40) into (4.41), we obtain

$$r_{\alpha\beta} = r^{(1)}_{\alpha\beta}(q) + r^{(2)}_{\alpha\beta}(p) + r^{(3)}_{\alpha\beta}(X) + r^{(4)}_{\alpha\beta}(\widehat{E}_0) + r^{(5)}_{\alpha\beta}(q^2) + O_{(14)\alpha\beta}, \tag{4.42}$$

where the abbreviations represent

$$r^{(1)}_{\alpha\beta}(q) = \frac{1}{2}\left(b^\pi_\alpha a^\lambda_\beta + b^\lambda_\beta a^\pi_\alpha - \frac{\sigma}{1-\sigma}a^{\lambda\pi}b_{\alpha\beta}\right)q_{\lambda\pi}, \tag{4.42a}$$

$$r^{(2)}_{\alpha\beta}(p) = -\left(a^{\pi\lambda}a^\gamma_\alpha a^\delta_\beta + a^{\pi\lambda}a^\gamma_\beta a^\delta_\alpha + \frac{\sigma}{1-\sigma}a^{\pi\gamma}a^\lambda_\alpha a^\delta_\beta\right)p_{\gamma\pi|\lambda\delta}$$
$$+ \frac{1}{4(1-\sigma)}\left\{\frac{4\sigma(1-2\sigma)}{(1-\sigma)}Hb_{\alpha\beta}a^{\pi\lambda} + b_{\alpha\beta}b^{\lambda\pi} - 2(1-\sigma)b^\lambda_\alpha b^\pi_\beta\right\}p_{\lambda\pi}, \tag{4.42b}$$

$$r^{(3)}_{\alpha\beta}(\widehat{X}) = -\frac{(1+\sigma)}{2(1-\sigma)}\left\{2(1-\sigma)(X_{[0]\alpha|\beta} + X_{[0]\beta|\alpha}) + (1-2\sigma)b_{\alpha\beta}X^0_{[0]}\right\}, \tag{4.42c}$$

$$r^{(4)}_{\alpha\beta}(\widehat{E}_0) = -\frac{1+\sigma}{1-\sigma}\left\{E^{00}_{[0]|\alpha\beta} + \frac{(1-2\sigma)^2}{(1-\sigma)}Hb_{\alpha\beta}E^{00}_{[0]}\right\}$$
$$- 2(1+\sigma)(a_{\alpha\pi}a^\lambda_\beta + a_{\beta\pi}a^\lambda_\alpha)(HE^{\pi 0}_{[0]})_{|\lambda} - \frac{1+\sigma}{2}\left\{b^\pi_\beta a_{\alpha\lambda} + b^\pi_\alpha a_{\beta\lambda}\right.$$
$$\left.- b_{\beta\lambda}a^\pi_\alpha - b_{\alpha\lambda}a^\pi_\beta + \frac{(1-2\sigma)}{1-\sigma}b_{\alpha\beta}a^\pi_\lambda\right\}E^{\lambda 0}_{[0]|\pi}, \tag{4.42d}$$

$$r^{(5)}_{\alpha\beta}(q^2) = \left\{\frac{\sigma}{1-\sigma}a^{\gamma\pi}a^\lambda_\alpha a^\delta_\beta - a^\lambda_\alpha a^{\pi\delta}a^\gamma_\beta\right\}q_{\lambda\delta}q_{\gamma\pi}, \tag{4.42e}$$

$$O_{(14)\alpha\beta} = O_{(14)\alpha\beta}(p^2, p\widehat{E}_0, \widehat{E}_0^2, pq, q\widehat{E}_0, p\widehat{X}, \widehat{X}\widehat{E}_0, q\widehat{X}). \tag{4.42f}$$

It will be noted that $r^{(1)}_{\alpha\beta}(q)$ is linear in the q's, $r^{(2)}_{\alpha\beta}(p)$ is linear in the p's, and so on.

We now differentiate (4.25) with respect to x^0 twice, and put $x^0=0$ in the resulting equation. In consequence of (4.28), (4.32), (4.42), this gives

$$E^{\alpha\beta}_{[2]} = E^{(1)\alpha\beta}_{[2]}(q) + E^{(2)\alpha\beta}_{[2]}(p) + E^{(3)\alpha\beta}_{[2]}(\widehat{X}) + E^{(4)\alpha\beta}_{[2]}(\widehat{E}_0) + E^{(5)\alpha\beta}_{[2]}(q^2) + O^{\alpha\beta}_{(15)}, \quad (4.43)$$

where

$$O^{\alpha\beta}_{(15)} = O^{\alpha\beta}_{(15)}(p^2, p\widehat{E}_0, \widehat{E}_0^2, pq, q\widehat{E}_0, p\widehat{X}, \widehat{X}\widehat{E}_0, q\widehat{X}); \quad (4.43a)$$

the other abbreviations are

$$E^{(1)\alpha\beta}_{[2]}(q) = -\frac{1}{2(1-\sigma^2)}\left\{\sigma a^{\alpha\beta}b^{\pi\lambda} + 5\sigma b^{\alpha\beta}a^{\pi\lambda} + 3(1-\sigma)(b^{\alpha\pi}a^{\beta\lambda} + b^{\beta\lambda}a^{\alpha\pi})\right\}q_{\lambda\pi}, \quad (4.43b)$$

$$\begin{aligned} E^{(2)\alpha\beta}_{[2]}(p) = &-\frac{1}{1-\sigma^2}\left\{\sigma(a^{\alpha\lambda}a^{\beta\delta}a^{\pi\gamma} + a^{\alpha\beta}a^{\delta\gamma}a^{\pi\lambda})\right. \\ &\left. + (1-\sigma)(a^{\alpha\gamma}a^{\beta\delta} + a^{\alpha\delta}a^{\beta\gamma})a^{\pi\lambda}\right\}p_{\pi\gamma|\lambda\delta} \\ &+ \frac{1}{1-\sigma^2}\left\{\sigma a^{\alpha\beta}b^{\pi\lambda}H - \frac{4\sigma^2}{1-\sigma}a^{\alpha\beta}a^{\pi\gamma}H^2 - \frac{\sigma(7-9\sigma)}{(1-\sigma)}a^{\alpha\beta}Ka^{\pi\gamma}\right. \\ &+ \frac{\sigma(7-12\sigma)}{(1-\sigma)}Ha^{\pi\gamma}b^{\alpha\beta} + \frac{(1-4\sigma)}{4}b^{\alpha\beta}b^{\pi\gamma} - \tfrac{1}{2}(1-\sigma)b^{\alpha\pi}b^{\beta\gamma} \\ &\left. + 6(1-\sigma)H(a^{\alpha\pi}b^{\beta\gamma} + b^{\alpha\pi}a^{\beta\gamma}) - (1-\sigma)12Ka^{\alpha\pi}a^{\beta\gamma}\right\}p_{\pi\gamma}, \end{aligned} \quad (4.43c)$$

$$\begin{aligned} E^{(3)\alpha\beta}_{[2]}(\widehat{X}) = &\frac{1}{1-\sigma}\left\{\tfrac{1}{2}(6\sigma-1)b^{\alpha\beta}X^0_{[0]} - \sigma a^{\alpha\beta}X^0_{[1]} - \sigma a^{\alpha\beta}X^\gamma_{[0]|\gamma}\right\} \\ &- a^{\alpha\pi}a^{\beta\gamma}(X_{[0]\pi|\gamma} + X_{[0]\gamma|\pi}), \end{aligned} \quad (4.43d)$$

$$\begin{aligned} E^{(4)\alpha\beta}_{[2]}(\widehat{E}_0) = &-\frac{1}{1-\sigma^2}\left\{(18\sigma^2 - 14\sigma + 1)Hb^{\alpha\beta}\right. \\ &\left. - 4\sigma(1-2\sigma)H^2a^{\alpha\beta} + 2\sigma(4-5\sigma)Ka^{\alpha\beta}\right\}E^{00}_{[0]} \\ &- \frac{1}{1-\sigma}\left\{a^{\alpha\pi}a^{\beta\gamma} + \sigma a^{\alpha\beta}a^{\pi\gamma}\right\}E^{00}_{[0]|\pi\gamma} + 2(a^{\beta\gamma}a^\alpha_\pi + a^{\alpha\gamma}a^\beta_\pi)(HE^{\pi 0}_{[0]})_{|\gamma} \\ &+ \frac{\sigma}{1-\sigma}a^{\alpha\beta}b^\pi_\gamma E^{\gamma 0}_{[0]|\pi} + \frac{1}{2(1-\sigma)}\left\{(6\sigma-1)b^{\alpha\beta} - 4\sigma Ha^{\alpha\beta}\right\}E^{\pi 0}_{[0]|\pi} \\ &+ \tfrac{1}{2}\left\{a^{\alpha\pi}b^\beta_\gamma + a^{\beta\pi}b_\gamma^{\ \alpha} - a^\beta_\gamma b^{\alpha\pi} - a^\alpha_\gamma b^{\beta\pi}\right\}E^{\gamma 0}_{[0]|\pi}, \end{aligned} \quad (4.43e)$$

$$E^{(5)\alpha\beta}_{[2]}(q^2) = A^{\alpha\beta\pi\lambda}_{(1)}\left(\frac{\sigma}{1-\sigma}a^{\xi\eta}a^\gamma_\pi a^\delta_\lambda - a^\xi_\pi a^\gamma a^{\eta\delta}\right)q_{\gamma\delta}q_{\xi\eta}. \quad (4.43f)$$

Equation (4.43) is the required expression of $E^{\alpha\beta}_{[2]}$.

By similar steps, we can express all the other $E^{i0}_{[m]}$ and $E^{\alpha\beta}_{[m]}$ in terms of $p_{\alpha\beta}$, $q_{\alpha\beta}$ and $E^{i0}_{[0]}$. However, for the purpose of the first approximation in the equations of equilibrium, the knowledge of the required expressions of $E^{ij}_{[0]}$, $E^{ij}_{[1]}$, $E^{ij}_{[2]}$ is sufficient. This completes Step (II).

Conditions of the boundary surfaces, and the expressions of $E^{i0}_{[0]}$ *in terms of* $p_{\alpha\beta}$ *and* $q_{\alpha\beta}$. We have now succeeded in expressing $E^{ij}_{[m]}$ in terms of $p_{\alpha\beta}$, $q_{\alpha\beta}$, $E^{i0}_{[0]}$. Our next task is to express $E^{i0}_{[0]}$ in terms of $p_{\alpha\beta}$, $q_{\alpha\beta}$ by applying the following six surface conditions:

$$N_{(+)k}E^{ki}_{(+)} = Z^i_{(+)} \quad \text{for} \quad x^0 = + h_{(+)}, \tag{4.44a}$$
$$N_{(-)k}E^{ki}_{(-)} = Z^i_{(-)} \quad \text{for} \quad x^0 = - h_{(-)}. \tag{4.44b}$$

The unit normal vectors $N_{(+)i}$, $N_{(-)i}$, drawn out from the shell or plate, are determined as functions of x^α by the equations

$$N_{(+)\alpha} = - N_{(+)0}h_{(+),\alpha}, \qquad N_{(-)\alpha} = N_{(-)0}h_{(-),\alpha}, \tag{4.45a}$$
$$N_{(+)0} = (1 - N_{(+)\alpha}N^\alpha_{(+)})^{1/2}, \qquad N_{(-)0} = (1 - N_{(-)\alpha}N^\alpha_{(-)})^{1/2}, \tag{4.45b}$$

the positive roots being understood. $Z_{(+)i}$, $Z_{(-)i}$ are tensor components of the given loads per unit area applied to the upper and lower surfaces.

Making use of (4.45a, b), we find that (4.44a, b) take the form

$$E^{i0}_{(+)} - E^{\pi i}_{(+)}h_{(+),\pi} = \frac{Z^i_{(+)}}{|N^0_{(+)}|}, \qquad E^{i0}_{(-)} + E^{\alpha i}_{(-)}h_{(-),\alpha} = -\frac{Z^i_{(-)}}{|N^0_{(-)}|}. \tag{4.46}$$

Substituting (3.24) into (4.46), and adding the resulting expressions, we obtain in consequence of (3.26), (3.28)

$$2E^{00}_{[0]} + E^{00}_{[1]}d + \frac{1}{2!}E^{00}_{[2]}t^{(2)} + \cdots - \left[E^{\pi 0}_{[0]}d_{,\pi} + \frac{1}{2!}E^{\pi 0}_{[1]}t^{(2)}_{,\pi} + \frac{1}{3!}E^{\pi 0}_{[2]}d^{(3)}_{,\pi} + \cdots\right] = Q^0, \tag{4.47a}$$

$$2E^{\alpha 0}_{[0]} + E^{\alpha 0}_{[1]}d + \frac{1}{2!}E^{\alpha 0}_{[2]}t^{(2)} + \cdots - \left[E^{\pi\alpha}_{[0]}d_{,\pi} + \frac{1}{2!}E^{\pi\alpha}_{[1]}t^{(2)}_{,\pi} + \frac{1}{3!}E^{\pi\alpha}_{[2]}d^{(3)}_{,\pi} + \cdots\right] = Q^\alpha. \tag{4.47b}$$

Substituting $E^{ij}_{[m]}$ from the results of Step (II) into (4.47a, b), we obtain a set of three equations in terms of nine quantities $p_{\alpha\beta}$, $q_{\alpha\beta}$, $E^{i0}_{[0]}$. Solving these equations for $E^{i0}_{[0]}$, we have

$$\begin{aligned}
E^{00}_{[0]} = {} & \tfrac{1}{2}Q^0 + \frac{(1-2\sigma)}{2(1-\sigma)}HQ^0d + \tfrac{1}{4}(Q^\pi d)_{|\pi} - \frac{\sigma}{8(1-\sigma)}a^{\lambda\pi}(Q^0_{|\lambda}t^{(2)})_{|\pi} \\
& - \frac{1-2\sigma}{4(1-\sigma)^2}\{(2-3\sigma)2H^2 - (1-\sigma)K\}Q^0t^{(2)} - \tfrac{1}{8}\{(b^\pi_\lambda + 2Ha^\pi_\lambda)Q^\lambda t^{(2)}\}_{|\pi} \\
& - \tfrac{1}{4}\left\{(HQ^\pi)_{|\pi} - \frac{\sigma}{1-\sigma}HQ^\pi_{|\pi}\right\}t^{(2)} + \tfrac{1}{2}X^0_{[0]}d \\
& + \tfrac{1}{4}\left\{X^0_{[1]} - \frac{2(1-2\sigma)}{1-\sigma}HX^0_{[0]}\right\}t^{(2)} - \tfrac{1}{4}(X^\pi_{[0]}t^{(2)})_{|\pi} \\
& - \frac{1}{2(1-\sigma^2)}\{2\sigma Ha^{\pi\lambda} + (1-\sigma)b^{\pi\lambda}\}p_{\pi\lambda}d - \tfrac{1}{4}A^{\lambda\pi\delta\gamma}_{(1)}(p_{\delta\gamma|\lambda}t^{(2)})_{|\pi} \\
& + \frac{1}{4(1-\sigma^2)}\left\{(4-3\sigma)Hb^{\pi\lambda} + \frac{4\sigma(2-3\sigma)}{(1-\sigma)}H^2a^{\pi\lambda} - (3-\sigma)Ka^{\pi\lambda}\right\}p_{\pi\lambda}t^{(2)} \\
& - \frac{1}{8(1-\sigma^2)}\{4H\sigma a^{\pi\lambda} + (1-\sigma)b^{\pi\lambda}\}q_{\pi\lambda}t^{(2)} + O^{00}_{(16)},
\end{aligned} \tag{4.48a}$$

$$E^{\alpha 0}_{[0]} = \tfrac{1}{2}Q^\alpha + \frac{\sigma}{4(1-\sigma)} a^{\alpha\pi}(Q^0 d)_{|\pi} + \tfrac{1}{4}(b^\alpha_\pi + 2Ha^\alpha_\pi)Q^\pi d$$
$$- \frac{\sigma}{8(1-\sigma)} a^{\alpha\pi}(Q^\lambda_{|\lambda} t^{(2)})_{|\pi} - \frac{\sigma}{8(1-\sigma)}\left\{\left(b^{\alpha\gamma} + \frac{2(1-2\sigma)}{(1-\sigma)} Ha^{\alpha\gamma}\right) Q^0 t^{(2)}\right\}_{|\gamma}$$
$$+ \tfrac{1}{2}X^\alpha_{[0]} d - \frac{\sigma}{4(1-\sigma)} a^{\alpha\lambda}(X^0_{[0]} t^{(2)})_{|\lambda}$$
$$+ \frac{\sigma}{16(1-\sigma)}\left\{(4Ha^{\alpha\lambda} + b^{\alpha\lambda})_{|\lambda} Q^0 - (b^{\alpha\lambda} + 2Ha^{\alpha\lambda})Q^0_{|\lambda}\right\} t^{(2)}$$
$$+ \tfrac{1}{4}(5Hb^\alpha_\lambda + 4H^2 a^\alpha_\lambda - 4Ka^\alpha_\lambda)Q^\lambda t^{(2)} + \tfrac{1}{4}\left\{X^\alpha_{[1]} - (b^\alpha_\pi + 2Ha^\alpha_\pi)X^\pi_{[0]}\right\} t^{(2)}$$
$$+ \tfrac{1}{4}A^{\alpha\beta\pi\lambda}_{(1)}(q_{\pi\lambda} t^{(2)})_{|\beta} + \tfrac{1}{2}A^{\alpha\pi\lambda\delta}_{(1)}(p_{\lambda\delta} d)_{|\pi} + \tfrac{1}{4}(4Ha^\alpha_\pi + b^\alpha_\pi)_{|\lambda} A^{\gamma\delta\pi\lambda}_{(1)} p_{\gamma\delta} t^{(2)}$$
$$- \frac{1}{4(1+\sigma)}\left\{\left[\frac{\sigma}{1-\sigma}(b^{\alpha\beta}a^{\pi\lambda} + \tfrac{1}{2}a^{\alpha\beta}b^{\pi\lambda}) - \frac{2\sigma^2}{(1-\sigma)^2} Ha^{\alpha\beta}a^{\pi\lambda}\right.\right.$$
$$\left.\left. + (b^{\alpha\pi}a^{\beta\lambda} + a^{\alpha\pi}b^{\beta\lambda})\right] p_{\pi\lambda} t^{(2)}\right\}_{|\beta}$$
$$- \tfrac{1}{2}(b^\alpha_\pi + 2Ha^\alpha_\pi)A^{\pi\gamma\lambda\delta}_{(1)} p_{\lambda\delta|\gamma} t^{(2)} + O^{\alpha 0}_{(17)}. \tag{4.48b}$$

Here $O^{00}_{(16)}$, $O^{\alpha 0}_{(17)}$ stand as usual for terms not explicitly calculated. It is possible to exhibit their orders of magnitude as in (4.43a), for example, but the expressions are very long and will therefore be omitted here; the full expressions may be found in the author's Ph.D. Thesis, "The intrinsic theory of elastic shells and plates" (University of Toronto Library). The important fact about these residual terms is that, in the case of small strain, they are small compared with the terms shown explicitly.

The substitution of $E^{i0}_{[0]}$ from (4.48a, b) into (4.26), (4.28), (4.29), (4.30), (4.32), (4.33), (4.43) etc., gives the expressions for $E^{ij}_{[m]}$ in terms of $p_{\alpha\beta}$ and $q_{\alpha\beta}$. This completes Step (III).

Expressions of $T^{\alpha 0}$, $T^{\alpha\beta}$, $L^{\alpha\beta}$ in terms of $p_{\alpha\beta}$, $q_{\alpha\beta}$. If we substitute the expressions of $E^{\alpha\beta}_{[m]}$ from the results of Step (III) into (3.30), (3.31), we immediately obtain the set of eight quantities $T^{\alpha\beta}$ and $L^{\alpha\beta}$ in terms of the six quantities $p_{\alpha\beta}$ and $q_{\alpha\beta}$; the quantities $X^i_{[m]}$, Q^i are supposed to be given. Therefore for the membrane stress tensor $T^{\alpha\beta}$ and the bending moment tensor $L^{\alpha\beta}$, we have

$$T^{\alpha\beta} = A^{\alpha\beta\pi\lambda}_{(1)} p_{\pi\lambda} t - B^{\alpha\beta\pi\lambda}_{(2)} q_{\pi\lambda} t t^{(2)} + B^{\alpha\beta\pi\lambda}_{(3)} q_{\pi\lambda} t^{(3)} + \tfrac{1}{2}A^{\alpha\beta\pi\lambda}_{(1)} q_{\pi\lambda} d^{(2)}$$
$$- A^{\alpha\beta\pi\gamma\lambda\delta}_{(2)} q_{\pi\gamma} q_{\lambda\delta} t^{(3)} + B^{\alpha\beta\pi\lambda}_{(4)} p_{\pi\lambda} d^{(2)} - \frac{\sigma}{1-\sigma} a^{\alpha\beta} A^{\lambda\pi\delta\gamma}_{(1)}(p_{\delta\gamma|\lambda} t^{(2)})_{|\pi} t$$
$$+ C^{\alpha\beta\pi\lambda}_{(1)} p_{\pi\lambda} t t^{(2)} - (a^{\beta\delta} A^{\alpha\lambda\gamma\pi}_{(1)} + a^{\alpha\lambda} A^{\alpha\beta\delta\gamma}_{(1)}) p_{\pi\gamma|\lambda\delta} t^{(3)} + C^{\alpha\beta\pi\lambda}_{(2)} p_{\pi\lambda} t^{(3)}$$
$$+ \frac{\sigma}{2(1-\sigma)} a^{\alpha\beta} Q^0 t + T^{\alpha\beta}_{(1)}\{X^0\} + T^{\alpha\beta}_{(2)}\{X\} + T^{\alpha\beta}_{(3)}\{Q\} + O^{\alpha\beta}_{(18)}, \tag{4.49}$$

$$L^{\alpha\beta} = \eta^{\gamma\beta}_{[0]} a_{\pi\gamma}\left\{\tfrac{1}{2}A^{\alpha\pi\lambda\delta}_{(1)} p_{\lambda\delta} d^{(2)} + B^{\alpha\pi\lambda\delta}_{(5)} p_{\lambda\delta} t^{(3)} + \tfrac{1}{3}A^{\alpha\pi\lambda\delta}_{(1)} q_{\lambda\delta} t^{(3)}\right\}$$
$$+ \frac{\sigma}{12(1-\sigma)} \eta^{\gamma\beta}_{[0]}\left\{a^\alpha_\gamma(3Q^0 d^{(2)} + \frac{4\sigma}{1-\sigma} HQ^0 t^{(3)} - 4X^0_{[0]} t^{(3)} - 2Q^\lambda_{|\lambda} t^{(3)})\right.$$
$$\left. - b^\alpha_\gamma Q^0 t^{(3)}\right\} + O^{\alpha\beta}_{(19)}, \tag{4.50}$$

where $O^{\alpha\beta}_{(18)}$, $O^{\alpha\beta}_{(19)}$ stand for the residual terms. The other abbreviations are

$$A^{\alpha\beta\pi\lambda}_{(1)} = \frac{1}{1-\sigma^2}\{\sigma a^{\alpha\beta}a^{\pi\lambda} + (1-\sigma)a^{\alpha\pi}a^{\beta\lambda}\}, \tag{4.51a}$$

$$A^{\alpha\beta\pi\gamma\lambda\delta}_{(2)} = \frac{1}{6(1-\sigma^2)}\left\{\sigma a^{\alpha\beta}a^{\pi\delta}a^{\gamma\lambda} + (1-\sigma)a^{\alpha\pi}a^{\beta\delta}a^{\gamma\lambda} - \frac{\sigma^2}{1-\sigma^2}a^{\alpha\beta}a^{\pi\gamma}a^{\delta\lambda}\right.$$
$$\left. - \sigma a^{\alpha\pi}a^{\beta\gamma}a^{\delta\lambda}\right\} = \tfrac{1}{6}\left\{A^{\alpha\beta\pi\delta}_{(1)}a^{\gamma\lambda} - \frac{\sigma}{1-\sigma}A^{\alpha\beta\pi\gamma}_{(1)}a^{\delta\lambda}\right\}, \tag{4.51b}$$

$$B^{\alpha\beta\pi\lambda}_{(2)} = \frac{\sigma}{8(1-\sigma)(1-\sigma^2)}\{4\sigma Ha^{\pi\lambda} + (1-\sigma)b^{\pi\lambda}\}a^{\alpha\beta}, \tag{4.51c}$$

$$B^{\alpha\beta\pi\lambda}_{(3)} = \frac{1}{12(1-\sigma^2)}\{\sigma(8Ha^{\alpha\beta}a^{\pi\lambda} - a^{\alpha\beta}b^{\pi\lambda} - 3b^{\alpha\beta}a^{\pi\lambda})$$
$$+ (1-\sigma)(8Ha^{\alpha\pi}a^{\beta\lambda} - 3b^{\alpha\pi}a^{\beta\lambda} - b^{\beta\lambda}a^{\alpha\pi})\}, \tag{4.51d}$$

$$B^{\alpha\beta\pi\lambda}_{(4)} = \frac{1}{4(1-\sigma^2)}\{\sigma(4Ha^{\alpha\beta}a^{\pi\lambda} - 3a^{\alpha\beta}b^{\pi\lambda} - a^{\pi\lambda}b^{\alpha\beta})$$
$$+ (1-\sigma)(4Ha^{\alpha\pi}a^{\beta\lambda} - 2b^{\alpha\pi}a^{\beta\lambda} - a^{\alpha\pi}b^{\beta\lambda})\}, \tag{4.51e}$$

$$B^{\alpha\pi\lambda\delta}_{(5)} = \frac{1}{6(1-\sigma^2)}\left\{\frac{4\sigma}{1-\sigma}Ha^{\alpha\pi}a^{\lambda\delta} + (1-\sigma)4Ha^{\alpha\lambda}a^{\pi\delta} - \sigma(a^{\lambda\delta}b^{\alpha\pi} + b^{\lambda\delta}a^{\alpha\pi})\right.$$
$$\left. - 2(1-\sigma)(b^{\alpha\lambda}a^{\pi\delta} + \tfrac{1}{2}a^{\alpha\lambda}b^{\pi\delta})\right\}, \tag{4.51f}$$

$$C^{\alpha\beta\pi\lambda}_{(1)} = \frac{\sigma}{4(1-\sigma^2)}\left\{(4-3\sigma)Hb^{\pi\lambda} + \frac{4\sigma(2-3\sigma)}{(1-\sigma)}H^2a^{\pi\lambda}\right.$$
$$\left. - (3-\sigma)Ka^{\pi\lambda}\right\}a^{\alpha\beta}, \tag{4.51g}$$

$$C^{\alpha\beta\pi\lambda}_{(2)} = \frac{1}{6(1-\sigma^2)}\left\{2(1-\sigma)H(a^{\alpha\pi}b^{\beta\lambda} + b^{\alpha\pi}a^{\beta\lambda}) - \frac{1-2\sigma}{4}b^{\alpha\beta}b^{\pi\lambda}\right.$$
$$- \tfrac{3}{2}(1-\sigma)b^{\alpha\pi}b^{\beta\lambda} + \frac{4\sigma^2}{1-\sigma}H^2a^{\alpha\beta}a^{\pi\lambda} - \sigma Ha^{\alpha\beta}b^{\pi\lambda} + \frac{\sigma(1-4\sigma)}{1-\sigma}Hb^{\alpha\beta}a^{\pi\lambda}$$
$$\left. - 8(1-\sigma)Ka^{\alpha\pi}a^{\beta\lambda} - \frac{\sigma(1-3\sigma)}{1-\sigma}Ka^{\alpha\beta}a^{\pi\lambda}\right\}, \tag{4.51h}$$

$$T^{\alpha\beta}_{(1)}\{X^0\} = \frac{\sigma}{4(1-\sigma)}a^{\alpha\beta}\left\{X^0_{[1]} - \frac{2(1-2\sigma)}{1-\sigma}HX^0_{[0]}\right\}tt^{(2)} - \frac{1}{2}\frac{\sigma}{1-\sigma}a^{\alpha\beta}X^0_{[0]}d^{(2)}$$
$$- \frac{1}{6(1-\sigma)}\{\tfrac{1}{2}(1-4\sigma)b^{\alpha\beta}X^0_{[0]} + 4\sigma Ha^{\alpha\beta}X^0_{[0]} + \sigma a^{\alpha\beta}X^0_{[1]}\}t^{(3)}, \tag{4.51i}$$

$$T^{\alpha\beta}_{(2)}\{X\} = -\frac{\sigma}{4(1-\sigma)}a^{\alpha\beta}(X^{\pi}_{[0]}t^{(2)})_{|\pi}t - \frac{\sigma}{6(1-\sigma)}a^{\alpha\beta}X^{\pi}_{[0]|\pi}t^{(3)}$$
$$- \tfrac{1}{6}a^{\alpha\pi}a^{\beta\lambda}(X_{[0]\pi|\lambda} + X_{[0]\lambda|\pi})t^{(3)}, \tag{4.51j}$$

$$T^{\alpha\beta}_{(3)}\{Q\} = -\frac{\sigma}{8(1-\sigma)} a^{\alpha\beta}\{(b^\pi_\lambda + 2Ha^\pi_\lambda)Q^\lambda t^{(2)}\}_{|\pi} t - \frac{\sigma}{4(1-\sigma)} a^{\alpha\beta} Q^\pi_{|\pi} d^{(2)}$$

$$- \tfrac{1}{6}\{a^{\beta\pi}(HQ^\alpha)_{|\pi} + a^{\alpha\pi}(HQ^\beta)_{|\pi}\} t^{(3)} + \frac{\sigma}{4(1-\sigma)} a^{\alpha\beta}(Q^\pi d)_{|\pi} t$$

$$- \frac{\sigma}{4(1-\sigma)} a^{\alpha\beta}(Q^\pi H)_{|\pi} t t^{(2)} + \frac{\sigma}{4(1-\sigma)^2} a^{\alpha\beta} H Q^\pi_{|\pi} t t^{(2)}$$

$$+ \frac{1}{24}\Big\{a^{\alpha\lambda}b^{\beta\pi} + a^{\beta\lambda}b^{\alpha\pi} - b^{\alpha\lambda}a^{\beta\pi} - b^{\beta\lambda}a^{\alpha\pi} + \frac{2\sigma}{1-\sigma} a^{\alpha\beta}b^{\pi\lambda}$$

$$+ \frac{4\sigma-1}{1-\sigma} a^{\lambda\pi}b^{\alpha\beta} - \frac{12\sigma}{1-\sigma} H a^{\alpha\beta}a^{\lambda\pi}\Big\} Q_{\pi|\lambda} t^{(3)}. \tag{4.51k}$$

Furthermore, by solving (2.5c), we have

$$T^{\alpha 0} = \eta^{\alpha\lambda}_{[0]} a_{\lambda\delta}(L^{\pi\delta}_{|\pi} + M^\delta), \tag{4.52}$$

in which $L^{\alpha\beta}$ is given by (4.50) and M^δ by (3.34).

Equations (4.49), (4.50) express $T^{\alpha\beta}$ and $L^{\alpha\beta}$ in terms of $p_{\alpha\beta}$ and $q_{\alpha\beta}$. When $L^{\alpha\beta}$ is known, $T^{\alpha 0}$ is calculated from (4.52). This completes the last step of the procedure outlined at the beginning of this section.

It should be noted that $p_{\alpha\beta}$ and $q_{\alpha\beta}$ correspond respectively to the extension and change of curvature of the reference surface S_0; $X^i_{[m]}$ is the normal derivatives of the mth order on S_0 of the body force, supposed to be given. If the form of the reference surface S_0 is given (in the strained state), the following quantities are known: $a_{\alpha\beta}$, $(1/2)b_{\alpha\beta}$, $(1/2)c_{\alpha\beta}$ are the first, second and third fundamental tensors, H the mean curvature and K the total curvature. Furthermore, if we know the positions of the boundary surfaces of the shell in the strained state and also the surface loads on the boundary surfaces, then the quantities $t^{(m)}$, $d^{(m)}$, $Q^{(m)i}$, $P^{(m)i}$ can be determined by using (3.26)–(3.28).

5. Equations of equilibrium and compatibility in terms of the six unknowns $p_{\alpha\beta}$ and $q_{\alpha\beta}$. Having now expressed the macroscopic stress tensors in terms of $p_{\alpha\beta}$ and $q_{\alpha\beta}$ we shall substitute these expressions into the macroscopic equations of equilibrium, (2.8a, b). In consequence of (3.32)–(3.34), this gives the following three equations in terms of the six unknowns $p_{\alpha\beta}$ and $a_{\alpha\beta}$:

$$- \tfrac{1}{2} b_{\rho\gamma} A^{\rho\gamma\pi\lambda}_{(1)} p_{\pi\lambda} t + \tfrac{1}{3} A^{\rho\gamma\pi\lambda}_{(1)} (q_{\pi\lambda} t^{(3)})_{|\rho\gamma} + \tfrac{1}{2} b_{\rho\gamma} B^{\rho\gamma\pi\lambda}_{(2)} q_{\pi\lambda} t t^{(2)}$$

$$- \tfrac{1}{2} b_{\rho\gamma} B^{\rho\gamma\pi\lambda}_{(3)} q_{\pi\lambda} t^{(3)} + \tfrac{1}{2} b_{\rho\gamma} A^{\rho\gamma\pi\beta\lambda\delta}_{(2)} q_{\pi\beta} q_{\lambda\delta} t^{(3)} - \tfrac{1}{4} A^{\rho\gamma\pi\lambda}_{(1)} b_{\rho\gamma} q_{\pi\lambda} d^{(2)}$$

$$- \tfrac{1}{2} b_{\rho\gamma} B^{\rho\gamma\pi\lambda}_{(4)} p_{\pi\lambda} d^{(2)} + \frac{\sigma}{2(1-\sigma)} a^{\rho\gamma} b_{\rho\gamma} A^{\lambda\pi\delta\beta}_{(1)} (p_{\delta\beta|\lambda} t^{(2)})_{|\pi} t$$

$$- \tfrac{1}{2} C^{\rho\gamma\pi\lambda}_{(1)} b_{\rho\gamma} p_{\pi\lambda} t t^{(2)} + \tfrac{1}{2} b_{\rho\gamma} (a^{\gamma\delta} A^{\rho\lambda\beta\pi}_{(1)} + a^{\pi\lambda} A^{\rho\gamma\delta\beta}_{(1)}) p_{\pi\beta|\lambda\delta} t^{(3)}$$

$$- \tfrac{1}{2} b_{\rho\gamma} C^{\rho\gamma\pi\lambda}_{(2)} p_{\pi\lambda} t^{(3)} + \tfrac{1}{2} A^{\rho\gamma\pi\lambda}_{(1)} (p_{\pi\lambda} d^{(2)})_{|\rho\gamma} + (B^{\rho\gamma\pi\lambda}_{(5)} p_{\pi\lambda} t^{(3)})_{|\rho\gamma} + P^0$$

$$- \frac{\sigma}{1-\sigma} H Q^0 t + 2HQ^{(1)0} + KP^{(2)0} + I^0_{(1)}\{Q^\pi\} + I^0_{(2)}\{P^\pi\} + I^0_{(3)}\{X^0\}$$

$$+ I^0_{(4)}\{X^\pi\} = O^0_{(20)}, \tag{5.1}$$

$$A^{\pi\alpha\delta\lambda}_{(1)}(p_{\lambda\delta}t)_{|\pi} + \{B^{\pi\alpha\lambda\delta}_{(3)}q_{\lambda\delta}t^{(3)} - B^{\pi\alpha\lambda\delta}_{(2)}q_{\lambda\delta}tt^{(2)}\}_{|\pi} - A^{\pi\alpha\gamma\rho\lambda\delta}_{(2)}(q_{\gamma\rho}q_{\lambda\delta}t^{(3)})_{|\pi}$$
$$+ \tfrac{1}{6}a^{\alpha\gamma}b_{\gamma\rho}A^{\pi\rho\lambda\delta}_{(1)}(q_{\lambda\delta}t^{(3)})_{|\pi} + (B^{\rho\alpha\pi\gamma}_{(\lambda)}p_{\pi\gamma}d^{(2)})_{|\rho} + (C^{\rho\alpha\pi\gamma}_{(1)}p_{\pi\gamma}tt^{(2)})_{|\rho}$$
$$- \frac{\sigma}{1-\sigma}a^{\rho\alpha}A^{\lambda\pi\delta\gamma}_{(1)}[(p_{\delta\gamma|\lambda}t^{(2)})_{|\pi}t]_{|\rho} - (a^{\alpha\delta}A^{\rho\gamma\lambda\pi}_{(1)} + a^{\pi\lambda}A^{\rho\alpha\delta\gamma}_{(1)})(p_{\pi\gamma|\lambda}t^{(3)})_{|\rho}$$
$$+ (C^{\rho\alpha\pi\gamma}_{(2)}p_{\pi\gamma}t^{(3)})_{|\rho} - \tfrac{1}{2}A^{\rho\alpha\pi\gamma}_{(1)}(q_{\pi\gamma}d^{(2)})_{|\rho} + P^{\alpha}$$
$$+ \frac{\sigma}{2(1-\sigma)}(Q^0t)_{|\pi}a^{\alpha\pi} + 3Hb^{\alpha}_{\gamma}P^{(2)\gamma} + K(Hb^{\alpha}_{\gamma} - Ka^{\alpha}_{\gamma})P^{(4)\gamma} + I^{\alpha}_{(1)}\{Q^{\pi}\}$$
$$+ I^{\alpha}_{(2)}\{X^0\} + I^{\alpha}_{(3)}\{X^{\pi}\} = O^{\alpha}_{(21)}. \qquad (5.2)$$

The symbols $A^{\alpha\beta\pi\lambda}_{(1)}$, $A^{\alpha\beta\pi\gamma\lambda\delta}_{(2)}$, given by (4.51a, b), are functions of $a_{\alpha\beta}$; $B^{\alpha\beta\pi\lambda}_{(2)}$, $B^{\alpha\beta\pi\lambda}_{(3)}$, $B^{\alpha\beta\pi\lambda}_{(4)}$, $B^{\alpha\beta\pi\lambda}_{(5)}$, given by (4.51c, d, e, f), are linear functions of $b_{\alpha\beta}$; and $C^{\alpha\beta\pi\lambda}_{(1)}$, $C^{\alpha\beta\pi\lambda}_{(2)}$, given by (4.51g, h), are quadratic functions of $b_{\alpha\beta}$. The other abbreviations in (5.1), (5.2) are listed as follows:

$$I^0_{(1)}\{Q^{\pi}\} = \frac{\sigma}{4(1-\sigma)}H\{(b^{\pi}_{\lambda} + 2Ha^{\pi}_{\lambda})Q^{\lambda}t^{(2)}\}_{|\pi}t - \frac{\sigma}{2(1-\sigma)}HQ^{\pi}d_{|\pi}t$$
$$+ \frac{\sigma}{2(1-\sigma)}H\left\{(Q^{\pi}H)_{|\pi} - \frac{\sigma}{1-\sigma}HQ^{\pi}_{|\pi}\right\}tt^{(2)} - \frac{\sigma}{6(1-\sigma)}a^{\gamma\lambda}(Q^{\pi}_{|\pi}t^{(3)})_{|\gamma\lambda}$$
$$- \frac{1}{6}\left\{\frac{\sigma}{1-\sigma}Hb^{\lambda\pi} - 2H^2a^{\lambda\pi} - \frac{4\sigma-1}{1-\sigma}Ka^{\lambda\pi}\right\}Q_{\lambda|\pi}t^{(3)}$$
$$+ \tfrac{1}{6}b^{\pi}_{\lambda}(HQ^{\lambda})_{|\pi}t^{(3)} + \{Q^{(1)\pi} + (Ka^{\pi}_{\lambda} + Hb^{\pi}_{\lambda})Q^{(3)\lambda}\}_{|\pi}, \qquad (5.3a)$$

$$I^0_{(2)}\{P^{\pi}\} = \{\tfrac{1}{2}(4Ha^{\pi}_{\lambda} + b^{\pi}_{\lambda})P^{(2)\lambda} + \tfrac{1}{2}b^{\pi}_{\lambda}KP^{(4)\lambda}\}_{|\pi}, \qquad (5.3b)$$

$$I^0_{(3)}\{X^0\} = X^0_{[0]}t + (2HX^0_{[0]} + X^0_{[1]})\frac{d^{(2)}}{2} - \frac{\sigma}{3(1-\sigma)}a^{\lambda\pi}(X^0_{[0]}t^{(3)})_{|\lambda\pi}$$
$$+ \frac{1}{6(1-\sigma)}\{(4H^2 + 6\sigma K)X^0_{[0]} + 2(1-\sigma)HX^0_{[1]} + X^0_{[2]}\}t^{(3)}$$
$$- \frac{\sigma}{2(1-\sigma)}H\left\{X^0_{[1]} - \frac{2(1-2\sigma)}{1-\sigma}HX^0_{[0]}\right\}tt^{(2)}, \qquad (5.3c)$$

$$I^0_{(4)}\{X^{\pi}\} = \tfrac{1}{2}\{X^{\pi}_{[0]}d^{(2)}\}_{|\pi} + \tfrac{1}{3}\{\tfrac{1}{2}(4Ha^{\pi}_{\lambda} + b^{\pi}_{\lambda})X^{\lambda}_{[0]}t^{(3)} + X^{\pi}_{(1)}t^{(3)}\}_{|\pi}$$
$$+ \frac{\sigma}{2(1-\sigma)}H(X^{\pi}_{[0]}t^{(2)})_{|\pi}t + \frac{\sigma}{3(1-\sigma)}HX^{\pi}_{[0]|\pi}t^{(3)} + \tfrac{1}{6}b^{\pi\lambda}X_{[0]\pi|\lambda}t^{(3)}, \qquad (5.3d)$$

$$I^{\alpha}_{(1)}\{Q^{\pi}\} = \frac{\sigma}{8(1-\sigma)}a^{\alpha\beta}\Big\{[(b^{\pi}_{\lambda} + 2Ha^{\pi}_{\lambda})Q^{\lambda}t^{(2)}]_{|\pi}t + 2Q^{\pi}d_{|\pi}t$$
$$- 2\Big[(Q^{\pi}H)_{|\pi} - \frac{\sigma}{1-\sigma}HQ^{\pi}_{|\pi}\Big]tt^{(2)}\Big\}_{|\beta}$$
$$- \tfrac{1}{6}\{[a^{\beta\pi}(Q^{\alpha}H)_{|\pi} + a^{\alpha\pi}(Q^{\beta}H)_{|\pi}]t^{(3)}\}_{|\beta}$$

$$+ \frac{1}{24}\left\{\left[a^{\alpha\lambda}b^{\beta\pi} + a^{\beta\lambda}b^{\alpha\pi} - b^{\alpha\lambda}a^{\beta\pi} - b^{\beta\lambda}a^{\alpha\pi} + \frac{2\sigma}{1-\sigma}a^{\alpha\beta}b^{\lambda\pi}\right.\right.$$
$$\left.\left. + \frac{4\sigma - 1}{1-\sigma}b^{\alpha\beta}a^{\lambda\pi} - \frac{12\sigma}{1-\sigma}Ha^{\alpha\beta}a^{\lambda\pi}\right]Q_{\pi|\lambda}t^{(3)}\right\}_{|\beta}$$
$$+ (2Ha^{\alpha}_{\pi} + b^{\alpha}_{\pi})Q^{(1)\pi} + (Kb^{\alpha}_{\pi} + 2H^2b^{\alpha}_{\pi} - 2KHa^{\alpha}_{\pi})Q^{(3)\pi}, \tag{5.4a}$$

$$I^{\alpha}_{(2)}\{X^0\} = \frac{\sigma}{4(1-\sigma)}a^{\alpha\beta}\left\{\left[X^0_{[1]} - \frac{2(1-2\sigma)}{1-\sigma}HX^0_{[0]}\right]tt^{(2)}\right\}_{|\beta}$$
$$- \frac{\sigma}{6(1-\sigma)}\left\{\left[\tfrac{1}{2}(1-4\sigma)b^{\alpha\beta}X^0_{[0]} + 4\sigma Ha^{\alpha\beta}X^0_{[0]} + \sigma a^{\alpha\beta}X^0_{[1]}\right]t^{(3)}\right\}_{|\beta}$$
$$- \frac{\sigma}{6(1-\sigma)}b^{\alpha\beta}(X^0_{[0]}t^{(3)})_{|\beta}, \tag{5.4b}$$

$$I^{\alpha}_{(3)}\{X^{\pi}\} = X^{\alpha}_{[0]}t + \tfrac{1}{2}X^{\alpha}_{[1]}d^{(2)} - \tfrac{1}{6}(2Ha^{\alpha}_{\pi} + b^{\alpha}_{\pi})X^{\pi}_{[0]}d^{(2)} + \tfrac{1}{6}X^{\alpha}_{[2]}t^{(3)}$$
$$- \frac{\sigma}{4(1-\sigma)}a^{\alpha\beta}\left\{(X^{\pi}_{[0]}t^{(2)})_{|\pi}t\right\}_{|\beta} - \frac{\sigma}{6(1-\sigma)}a^{\alpha\beta}(X^{\lambda}_{[0]|\lambda}t^{(3)})_{|\beta}$$
$$+ Hb^{\alpha}_{\pi}X^{\pi}_{[0]}t^{(3)} - \tfrac{1}{6}a^{\alpha\pi}a^{\beta\lambda}\left\{(X_{[0]\pi|\lambda} + X_{[0]\lambda|\pi})t^{(3)}\right\}_{|\beta}$$
$$+ \tfrac{1}{3}(2Ha^{\alpha}_{\pi} + b^{\alpha}_{\pi})X^{\pi}_{[1]}t^{(3)}. \tag{5.4c}$$

Equations (5.1), (5.2) are the three equations of equilibrium in terms of the six unknowns $p_{\alpha\beta}$ and $q_{\alpha\beta}$. All the other quantities are supposed to be given.

The basic unknowns are the six quantities $p_{\alpha\beta}$ and $q_{\alpha\beta}$. To find them, we have to solve the three equations of equilibrium (5.1), (5.2), together with the three equations of compatibility, funished by the geometrical conditions.

We shall now convert (4.13) into a form in which $p_{\alpha\beta}$ and $q_{\alpha\beta}$ are the only unknowns. Let us put in turn $i=0, j=\alpha, k=\beta, l=\gamma$, and $i=\rho, j=\alpha, k=\beta, l=\gamma$ in (4.13), and then put $x^0=0$; we thus obtain three equations in p_{ij} and q_{ij}. We now substitute (4.37)–(4.40) for p_{00}, $p_{\alpha 0}$, q_{00}, $q_{\alpha 0}$, and (4.48a, b) for $E^{i0}_{[0]}$; this gives three equations in $p_{\alpha\beta}$ and $q_{\alpha\beta}$. Two of these, after being multiplied by $\eta^{\gamma\beta}_{[0]}$ and simplified by using (3.13), (3.14), can be written as follows:

$$\eta^{\gamma\beta}_{[0]}\left\{2q_{\alpha\gamma|\beta} - \frac{\sigma}{1-\sigma}a^{\rho\pi}b_{\alpha\beta}p_{\pi\rho|\gamma} + b^{\pi}_{\beta}(p_{\alpha\pi|\gamma} + p_{\gamma\pi|\alpha} - p_{\alpha\gamma|\pi})\right.$$
$$\left. + (1+\sigma)\left[Q_{\beta|\alpha\gamma} - a_{\alpha\beta}KQ_{\gamma} + \frac{1-2\sigma}{2(1-\sigma)}b_{\alpha\beta}Q^0_{|\gamma}\right]\right\} = O_{(22)\alpha}. \tag{5.5a}$$

The third equation, after being multiplied by $\eta^{\alpha\beta}_{[0]}\eta^{\rho\gamma}_{[0]}$, becomes

$$2\eta^{\alpha\beta}_{[0]}\eta^{\rho\gamma}_{[0]}p_{\rho\beta|\alpha\gamma} - \eta^{\alpha\beta}_{[0]}\eta^{\rho\gamma}_{[0]}q_{\rho\beta}q_{\alpha\gamma} + \frac{2(1-3\sigma)}{1-\sigma}a^{\pi\lambda}Kp_{\pi\lambda} + (b^{\pi\lambda} - 4Ha^{\pi\lambda})q_{\pi\lambda}$$
$$+ \frac{2(1+\sigma)(1-2\sigma)}{1-\sigma}Q^0K - (1+\sigma)(b^{\pi\lambda} - 4Ha^{\pi\lambda})Q_{\pi|\lambda} = O_{(23)}. \tag{5.5b}$$

Equations (5.5a, b) contain the three equations of compatibility in $p_{\alpha\beta}$ and $q_{\alpha\beta}$; these are to be associated with the three equations of equilibrium (5.1), (5.2) for the solution of plate and shell problems.

6. The equations of equilibrium and compatibility referred to the middle surface in the unstrained state. The quantities $a_{\alpha\beta}$, $b_{\alpha\beta}$, which occur in (5.1), (5.2), (5.5a, b), refer to the strained state. But it is usual in elasticity to regard the unstrained state as given, rather than the strained state, and from this point of view $a_{\alpha\beta}$, $b_{\alpha\beta}$ are unknown. We should use, instead of them, the fundamental tensors of that surface S_0' in the unstrained state, which passes over into the reference surface S_0 in the strained state. So far S_0 has been quite general. Now we shall follow the usual method by choosing S_0 so that S_0' is the middle surface of the shell or plate in the unstrained state. This means that S_0 is not accurately the middle surface in the strained state.

For the metric in the unstrained state, we have by (4.1)

$$g'_{ij} = g_{ij} - 2e_{ij}; \tag{6.1}$$

and so, if we define

$$a'_{ij} = (g'_{ij})_{x^0=0}, \qquad b'_{ij} = (g'_{ij,0})_{x^0=0}, \tag{6.2}$$

we have

$$\begin{aligned} a'_{\alpha\beta} &= a_{\alpha\beta} - 2p_{\alpha\beta}, & a'_{\alpha 0} &= -2p_{\alpha 0}, & a'_{00} &= 1 - 2p_{00}, \\ b'_{\alpha\beta} &= b_{\alpha\beta} - 2q_{\alpha\beta}, & b'_{\alpha 0} &= -2q_{\alpha 0}, & b'_{00} &= -2q_{00}, \end{aligned} \tag{6.3}$$

where p_{ij} and q_{ij} are defined as in (4.4). $a'_{\alpha\beta}$ is the fundamental tensor of S_0'; we regard it as given. We may substitute in the proceding theory

$$a_{\alpha\beta} = a'_{\alpha\beta} + 2p_{\alpha\beta}. \tag{6.4}$$

The tensor $b'_{\alpha\beta}$ does not represent the curvature of the middle surface S_0' in the natural state; since in general $g'_{\alpha 0} \neq 0$ on S_0', the parametric lines of x^0 cuts S_0' obliquely. Let us introduce the normal coordinates $\mathbf{x}^i$ based on S_0', choosing $\mathbf{x}^\alpha = x^\alpha$ on S_0', and $\mathbf{x}^0$ normal to S_0'. The metric $\mathbf{g}_{ij}$ corresponding to the coordinates $\mathbf{x}^i$ satisfies

$$\mathbf{g}_{00} = 1, \qquad \mathbf{g}_{\alpha 0} = 0. \tag{6.5}$$

Let us put

$$\mathbf{a}_{\alpha\beta} = (\mathbf{g}_{\alpha\beta})_{x^0=0}, \qquad \mathbf{b}_{\alpha\beta} = \left(\frac{\partial \mathbf{g}_{\alpha\beta}}{\partial \mathbf{x}^0}\right)_{\mathbf{x}^0=0}. \tag{6.6}$$

Now $\mathbf{g}_{ij}$ and g'_{ij} are metrics corresponding to the coordinate systems $\mathbf{x}^i$ and x^i respectively, for the description of the geometry of the unstrained state. Hence the tensor $\mathbf{g}_{ij}$ determines g'_{ij}, and vice versa.

It should be noted that the quantities $(1/2)\mathbf{b}_{\alpha\beta}$ are the coefficients of the second fundamental form of S_0'; they vanish if S_0' is a plane. The radius of curvature R in the direction of a unit vector $\boldsymbol{\mu}^\alpha_{[0]}$ (R counted positive when S_0' is convex in the sense of $\mathbf{x}^0$ increasing) is given by

$$\frac{1}{R} = \frac{1}{2}\, \mathbf{b}_{\alpha\beta}\, \boldsymbol{\mu}^\alpha_{[0]}\, \boldsymbol{\mu}^\beta_{[0]}. \tag{6.7}$$

We proceed to find in particular the quantities (6.6). In the first place, since $\mathbf{x}^\alpha = x^\alpha$ on S_0', we have

$$\mathbf{a}_{\alpha\beta} = a'_{\alpha\beta}, \qquad a_{\alpha\beta} = \mathbf{a}_{\alpha\beta} + 2p_{\alpha\beta} \tag{6.8}$$

We now follow a straight line normal to S_0', starting at the point $\mathbf{x}^\alpha$. Since it is a geodesic, we have (cf. 6, p. 301)

$$\frac{d^2x^i}{(d\mathbf{x}^0)^2} + \left\{ \begin{matrix} i \\ jk \end{matrix} \right\}_{g'} \frac{dx^j}{d\mathbf{x}^0}\frac{dx^k}{d\mathbf{x}^0} = 0, \tag{6.9}$$

and so we can develop x^i as power series in $\mathbf{x}^0$. For $\mathbf{x}^0 = 0$, we have

$$x^\alpha = \mathbf{x}^\alpha, \qquad x^0 = 0, \qquad \frac{dx^i}{d\mathbf{x}^0} = \frac{\tilde{g}'^{i0}}{(\tilde{g}'^{00})^{1/2}}, \tag{6.10}$$

where $\tilde{g}'^{ij}$ is the conjugate of g'_{ij}. The last follows from the fact that the line is normal to S_0', and so

$$g'_{\alpha i}\frac{dx^i}{d\mathbf{x}^0} = 0, \qquad g'_{ij}\frac{dx^j}{d\mathbf{x}^0}\frac{dx^i}{d\mathbf{x}^0} = 1. \tag{6.11}$$

On carrying out the development in power series, and using the transformation

$$\mathbf{g}_{ij} = g'_{mn}\frac{dx^m}{d\mathbf{x}^i}\frac{dx^n}{d\mathbf{x}^j}, \tag{6.12}$$

we obtain after a little calculation

$$b_{\alpha\beta} - 2q_{\alpha\beta} = b'_{\alpha\beta} = \mathbf{b}_{\alpha\beta}(\tilde{a}'^{00})^{1/2} + a'_{\alpha 0,\beta} + a'_{\beta 0,\alpha} + 2[\alpha\beta, \gamma]_{a'}\tilde{a}'^{\gamma 0}/\tilde{a}'^{00}. \tag{6.13}$$

where $\tilde{a}'^{ij} = \tilde{g}'^{ij}$ for $x^0 = 0$. The Christoffel symbol is calculated for $a'_{\alpha\beta}$. By (4.10) and (6.3), we have

$$\tilde{a}'^{00} = 1 + 2p_{00} + O^{00}_{(24)}(\hat{p}^2), \qquad \tilde{a}'^{0\alpha} = 2p^{0\alpha} + O^{0\alpha}_{(25)}(\hat{p}^2). \tag{6.14}$$

Thus (6.13) becomes

$$b_{\alpha\beta} - 2q'_{\alpha\beta} = b'_{\alpha\beta} = \mathbf{b}_{\alpha\beta} - \mathbf{b}_{\alpha\beta}p_{00} - 2p_{0\alpha|\beta}{}_{\mathbf{a}} - 2p_{0\beta|\alpha}{}_{\mathbf{a}} + O_{(26)\alpha\beta}(\hat{p}^2), \tag{6.15}$$

where $\mathbf{a}$ under the stroke indicates the covariant differentiation with respect to $\mathbf{x}^\alpha$ and $\mathbf{a}_{\alpha\beta}$.

Let us define $\mathbf{p}_{\alpha\beta}$, $\mathbf{q}_{\alpha\beta}$ so that

$$2\mathbf{q} = b_{\alpha\beta} - \mathbf{b}_{\alpha\beta}, \qquad 2\mathbf{p}_{\alpha\beta} = 2p_{\alpha\beta} = a_{\alpha\beta} - \mathbf{a}_{\alpha\beta}; \tag{6.16}$$

then the extension and change of curvature of the middle surface S_0' along the direction of a unit vector $\boldsymbol{\mu}^\alpha_{[0]}$ are given by

$$\mathbf{p}_{\alpha\beta}\boldsymbol{\mu}^\alpha_{[0]}\boldsymbol{\mu}^\beta_{[0]}, \qquad \mathbf{q}_{\alpha\beta}\boldsymbol{\mu}^\alpha_{[0]}\boldsymbol{\mu}^\beta_{[0]}. \tag{6.17}$$

From (6.15), (6.16), we have in consequence of (4.37), (4.39), (4.48a, b) and (6.8)

$$2q_{\alpha\beta} = 2\mathbf{q}_{\alpha\beta} - \frac{\sigma}{1-\sigma}\mathbf{a}^{\pi\lambda}\mathbf{b}_{\alpha\beta}\mathbf{p}_{\pi\lambda} + \frac{(1+\sigma)(1-2\sigma)}{2(1-\sigma)}\mathbf{b}_{\alpha\beta}Q^0 + (1+\sigma)Q_{\alpha|\beta \atop a} + (1+\sigma)Q_{\beta|\alpha \atop a} + O_{(27)\alpha\beta}, \tag{6.18}$$

$$b_{\alpha\beta} = 2q_{\alpha\beta} + \mathbf{b}_{\alpha\beta}, \qquad a_{\alpha\beta} = 2\mathbf{p}_{\alpha\beta} + \mathbf{a}_{\alpha\beta}, \qquad p_{\alpha\beta} = \mathbf{p}_{\alpha\beta}. \tag{6.19}$$

The symbol $O_{(27)\alpha\beta}$ represents the residual terms with the order of magnitude shown symbolically by

$$O_{(27)\alpha\beta} = O_{(27)\alpha\beta}(\mathbf{p}^2, \widehat{Q}\mathbf{p}, \widehat{Q}^2, \mathbf{p}d, \mathbf{b}\mathbf{p}t^{(2)}, \mathbf{q}t^{(2)}, Q^0d, \mathbf{b}Q^0t^{(2)}, \mathbf{b}Qd, Qt^{(2)}, \mathbf{q}Qd, \mathbf{b}X^0d, X^0t^{(2)}, Xd, Xt^{(2)}, \mathbf{q}X^0d). \tag{6.20}$$

Let us now denote the thickness of the shell or plate in the natural state by $2\boldsymbol{h}$. Then by (6.10), (4.10), (6.1), we have

$$\boldsymbol{h} = \int_0^{h(+)} \frac{dx^0}{(\tilde{g}'^{00})^{1/2}} = \int_0^{h(+)} [1 - \overset{00}{e} + \overset{00}{O}_{(28)}(\overset{2}{\hat{e}})]d\overset{0}{x}, \tag{6.21a}$$

$$\boldsymbol{h} = \int_{-h_{(-)}}^{0} \frac{dx^0}{(\hat{g}'^{00})^{1/2}} = \int_{-h_{(-)}}^{0} [1 - \overset{00}{e} + \overset{00}{O}_{(28)}(\overset{2}{\hat{e}})]d\overset{0}{x}. \tag{6.21b}$$

By (4.3), these become

$$\boldsymbol{h} = h_{(+)} - \overset{00}{p} h_{(+)} + O_{(29)}(\overset{2}{\hat{p}} h_{(+)}, \hat{q}\overset{2}{h}_{(+)}, \hat{q}\hat{p}\overset{2}{h}_{(+)}), \tag{6.22a}$$

$$\boldsymbol{h} = h_{(-)} - \overset{00}{p} h_{(-)} + O_{(30)}(\overset{2}{\hat{p}} h_{(-)}, \hat{q}\overset{2}{h}_{(-)}, \hat{q}\hat{p}\overset{2}{h}_{(-)}). \tag{6.22b}$$

Substituting (4.37)–(4.40) for p_{i0}, q_{i0}, (4.48a, b) for $E^{i0}_{[0]}$ and (6.18), (6.19) for $q_{\alpha\beta}$, $p_{\alpha\beta}$, $a_{\alpha\beta}$, $b_{\alpha\beta}$ into (6.22a, b), we have two equations for the determination of $h_{(+)}$ and $h_{(-)}$. We now solve these equations for $h_{(+)}$ and $h_{(-)}$. In deciding what terms to retain explicitly, we note that, for a thin or plate undergoing small strain, the quantities $\boldsymbol{h}$, $h_{(+)}$, $h_{(-)}$, $\mathbf{p}_{\alpha\beta}$, Q^i, $X^i_{[m]}$ are small. We obtain

$$h_{(+)} = \boldsymbol{h} - \frac{\sigma}{1-\sigma}\mathbf{a}^{\pi\lambda}\mathbf{p}_{\pi\lambda}\boldsymbol{h} + \frac{(1+\sigma)(1-2\sigma)}{2(1-\sigma)}Q^0\boldsymbol{h} + O_{(31)}, \tag{6.23a}$$

$$h_{(-)} = \boldsymbol{h} - \frac{\sigma}{1-\sigma}\mathbf{a}^{\pi\lambda}\mathbf{p}_{\pi\lambda}\boldsymbol{h} + \frac{(1+\sigma)(1-2\sigma)}{2(1-\sigma)}Q^0\boldsymbol{h} + O_{(32)}. \tag{6.23b}$$

The O-symbols represent the residual terms with orders of magnitude shown symbolically by

$$\mathbf{p}^2\boldsymbol{h}, \widehat{Q}^2\boldsymbol{h}, \mathbf{p}\widehat{Q}\boldsymbol{h}, \mathbf{p}\boldsymbol{h}^2, \widehat{Q}\boldsymbol{h}^2, \widehat{X}\boldsymbol{h}^2, \mathbf{q}\boldsymbol{h}^2. \tag{6.24}$$

Hence from (6.23a, b), we have immediately, in the notation of (3.26),

$$t^{(n)} = 2\left(1 - \frac{n\sigma}{1-\sigma}\mathbf{a}^{\pi\lambda}\mathbf{p}_{\pi\lambda} + \frac{n(1+\sigma)(1-2\sigma)}{2(1-\sigma)}Q^0\right) + O_{(33)}, \tag{6.25a}$$

$$d^{(n)} = O_{(34)}, \tag{6.25b}$$

where

$$O_{(33)} = O_{(33)}(\mathbf{p}^2\boldsymbol{h}^n, \widehat{Q}^2\boldsymbol{h}^n, \mathbf{p}\widehat{Q}\boldsymbol{h}^n, \widehat{Q}\boldsymbol{h}^{n+1}, \widehat{X}\boldsymbol{h}^{n+1}, \mathbf{p}\boldsymbol{h}^{n+1}, \mathbf{q}\boldsymbol{h}^{n+1}), \tag{6.26a}$$

$$O_{(34)} = O_{(34)}(\widehat{X}\boldsymbol{h}^{n+1}, \widehat{Q}\boldsymbol{h}^{n+1}, \mathbf{q}\boldsymbol{h}^{n+1}, \mathbf{p}\boldsymbol{h}^{n+1}). \tag{6.26b}$$

Similarly, substituting (6.23a, b) into (3.27), (3.28), we get

$$P^{(n)i} = \overset{i}{P}\overset{n}{h} + \overset{i}{O}_{(35)}, \tag{6.27a}$$

$$Q^{(n)i} = \overset{i}{Q}\overset{n}{h} + \overset{i}{O}_{(36)}, \tag{6.27b}$$

where

$$O^i_{(35)} = O^i_{(35)}(\widehat{P}ph^n, \widehat{PQ}h^n, \widehat{P}\widehat{X}h^{n+1}, \widehat{P}qh^{n+1}), \tag{6.28a}$$

$$O^i_{(36)} = O^i_{(36)}(\widehat{Q}ph^n, \widehat{Q}^2h^n, \widehat{Q}\widehat{X}h^{n+1}, \widehat{Q}qh^{n+1}), \tag{6.28b}$$

where P^i and Q^i defined by (3.28), represent the sum and difference of the components of the surface loads with respect to the surface S_0 in the strained state.

With (6.18), (6.19), (6.25), (6.27) established, the expression for $T^{\alpha\beta}$, $L^{\alpha\beta}$, $T^{\alpha 0}$ in (4.49), (4.50), (4.52) will now be reduced to forms involving $\mathbf{p}_{\alpha\beta}$, $\mathbf{q}_{\alpha\beta}$ instead of $p_{\alpha\beta}$, $q_{\alpha\beta}$. The results are as follows:

(i) *The membrane stress tensor* $T^{\alpha\beta}$,

$$T^{\alpha\beta} = 2A^{\alpha\beta\pi\lambda}_{(1)}\mathbf{p}_{\pi\lambda}h + A^{\alpha\beta\pi\gamma\lambda\delta}_{(\lambda)}b_{\lambda\delta}\mathbf{q}_{\pi\gamma}h^3 - A^{\alpha\beta\pi\gamma\lambda\delta}_{(3)}\mathbf{q}_{\pi\gamma}\mathbf{q}_{\lambda\delta}h^3 + \frac{\sigma}{1-\sigma}a^{\alpha\beta}Q^0h + O^{\alpha\beta}_{(37)}. \tag{6.29}$$

(ii) *The bending moment tensor* $L^{\alpha\beta}$,

$$\begin{aligned} L^{\alpha\beta} = {} & \mathfrak{n}^{\gamma\beta}_{(0)}a_{\pi\gamma}\left\{\tfrac{2}{3}A^{\alpha\pi\lambda\delta}_{(1)}\mathbf{q}_{\lambda\delta} + 2A^{\alpha\pi\lambda\delta\rho\omega}_{(5)}b_{\lambda\delta}\mathbf{p}_{\rho\omega}\right\}h^3 \\ & + \frac{\sigma}{6(1-\sigma)}\mathfrak{n}^{\gamma\beta}_{[0]}\left\{a^\alpha_\gamma\left(\frac{4\sigma}{1-\sigma}HQ^0 - 4X^0_{[0]} - 2Q^\lambda_{\ |\lambda \atop a}\right) - b^\alpha_\gamma Q^0\right\}h^3 + O^{\alpha\beta}_{(38)}. \end{aligned} \tag{6.30}$$

(iii) *The shearing stress tensor* $T^{\alpha 0}$,

$$\begin{aligned} T^{\alpha 0} = {} & 2\left\{A^{\pi\alpha\lambda\delta\rho\gamma}_{(5)}b_{\lambda\delta}\mathbf{p}_{\rho\gamma}h^3 + \tfrac{1}{3}A^{\pi\alpha\lambda\delta}_{(1)}\mathbf{q}_{\lambda\delta}h^3\right\}_{|\pi \atop a} + Q^\alpha h \\ & + \tfrac{1}{2}(4HP^\alpha + a^{\alpha\pi}b_{\pi\gamma}P^\gamma)h^2 + (a^{\pi\gamma}P^\alpha + a^{\alpha\pi}P^\gamma)\mathbf{q}_{\pi\gamma}h^2 \\ & + \frac{\sigma}{6(1-\sigma)}\left\{\left[a^{\alpha\pi}\left(\frac{4\sigma}{1-\sigma}HQ^0 - 4X^0_{[0]} + \frac{2\sigma}{1-\sigma}a^{\lambda\delta}\mathbf{q}_{\lambda\delta}Q^0\right)\right.\right. \\ & \left.\left. - b^{\alpha\pi}Q^0 - 2\mathbf{q}^{\alpha\pi}Q^0\right]h^3\right\}_{|\pi \atop a} + \tfrac{1}{3}\left\{(4Ha^\alpha_\pi + b^\alpha_\pi)X^\pi_{[0]} + X^\alpha_{[1]}\right\}h^3 \\ & + \tfrac{2}{3}(a^{\pi\gamma}\mathbf{q}_{\pi\gamma}a^\alpha_\lambda + a^{\alpha\pi}\mathbf{q}_{\pi\lambda})X^\lambda_{[0]}h^3 + O^{\alpha 0}_{(39)}. \end{aligned} \tag{6.31}$$

The residual terms in (6.29)–(6.31) are

$$\begin{aligned} O^{\alpha\beta}_{(38)} = {} & O^{\alpha\beta}_{(38)}(bp^2h^3, \widehat{Q}^2h^3, p\widehat{Q}h^3, \widehat{X}\widehat{Q}h^3, qph^3, \widehat{X}ph^3, \widehat{Q}qh^3, bph^5, qh^5, \\ & bQ^0h^5, Qh^5, \widehat{X}h^5), \end{aligned} \tag{6.32a}$$

$$O^{\alpha\beta}_{(37)} = O^{\alpha\beta}_{(37)}(p^2h, \widehat{Q}^2h, \widehat{Q}ph, ph^3, Q^0h^3, bQh^3, qQh^3, \widehat{X}h^3, bqph^3, q^2ph^3, qh^5), \tag{6.32b}$$

$$\begin{aligned} O^{\alpha 0}_{(39)} = {} & O^{\alpha 0}_{(39)}(Qph, Q\widehat{Q}h, Q\widehat{X}h^2, Qqh^2, bPph^2, qPph^2, bP\widehat{Q}h^2, qP\widehat{Q}h^2, pqh^3, \\ & bPqh^3, Pq^2h^3, bPh^4, bph^5, qh^5, b^3Q^0h^5, b^2Qh^5, \widehat{X}h^5). \end{aligned} \tag{6.32c}$$

The abbreviations are as follows:

$$A_{(1)}^{\alpha\beta\pi\lambda} = \frac{1}{1-\sigma^2}\left\{\sigma a^{\alpha\beta}a^{\pi\lambda} + (1-\sigma)a^{\alpha\pi}a^{\beta\lambda}\right\}, \tag{6.33a}$$

$$A_{(3)}^{\alpha\beta\pi\gamma\lambda\delta} = \frac{2(2\sigma-1)}{3(1-\sigma)}a^{\delta\lambda}A_{(1)}^{\alpha\beta\pi\gamma} + \tfrac{5}{3}a^{\delta\pi}A_{(1)}^{\alpha\beta\lambda\gamma}, \tag{6.33b}$$

$$A_{(4)}^{\alpha\beta\pi\gamma\lambda\delta} = \tfrac{1}{3}a^{\delta\lambda}A_{(1)}^{\alpha\beta\pi\gamma} - \frac{\sigma}{2(1-\sigma)}a^{\pi\gamma}A_{(1)}^{\alpha\beta\delta\lambda} - \tfrac{1}{6}a^{\delta\gamma}A_{(1)}^{\alpha\beta\pi\lambda} - \tfrac{1}{2}a^{\pi\lambda}A_{(1)}^{\alpha\beta\delta\gamma}, \tag{6.33c}$$

$$A_{(5)}^{\alpha\pi\lambda\delta\rho\gamma} = \tfrac{1}{6}\left\{\frac{\sigma}{1-\sigma}a^{\alpha\pi}A_{(1)}^{\lambda\delta\rho\gamma} - 2a^{\rho\delta}A_{(1)}^{\alpha\pi\lambda\gamma} - a^{\pi\delta}A_{(1)}^{\alpha\lambda\rho\gamma} + a^{\lambda\delta}A_{(1)}^{\alpha\pi\rho\gamma}\right\}, \tag{6.33d}$$

$$\mathbf{n}_{[0]}^{\alpha\beta} = (a)^{-1/2}\epsilon^{\alpha\beta}, \qquad a = \det.(a_{\alpha\beta}), \qquad \epsilon^{11} = \epsilon^{22} = 0, \tag{6.33e}$$

$$\epsilon^{12} = -\epsilon^{21} = 1,$$

$$H = \tfrac{1}{4}a^{\pi\lambda}b_{\pi\lambda}, \tag{6.33f}$$

where H is the mean curvature of the middle surface S_0' in the unstrained state. All these quantities are determined by the geometry of the middle surface (in the unstrained state) of the shell or plate.

With (6.18), (6.19), (6.25), (6.27) established, (5.1), (5.2) can be reduced by direct substitution to the form involving $\mathbf{p}_{\alpha\beta}$ and $\mathbf{q}_{\alpha\beta}$ instead of $p_{\alpha\beta}$ and $q_{\alpha\beta}$. Thus we have the following *three equations of equilibrium in terms of* $\mathbf{p}_{\alpha\beta}$ *and* $\mathbf{q}_{\alpha\beta}$:

$$\begin{aligned}
&- A_{(1)}^{\rho\gamma\pi\lambda}b_{\rho\gamma}p_{\pi\lambda}h - 2A_{(1)}^{\rho\gamma\pi\lambda}q_{\rho\gamma}p_{\pi\lambda}h + \tfrac{2}{3}A_{(1)}^{\rho\gamma\pi\lambda}(q_{\pi\lambda}h^3)_{\underset{a}{|\rho\gamma}} \\
&\quad - \tfrac{1}{2}A_{(\lambda)}^{\rho\gamma\pi\omega\lambda\delta}b_{\rho\gamma}b_{\lambda\delta}q_{\pi\omega}h^3 + A_{(6)}^{\rho\gamma\pi\omega\lambda\delta}q_{\pi\omega}q_{\lambda\delta}b_{\rho\gamma}h^3 + A_{(3)}^{\rho\gamma\pi\omega\lambda\delta}q_{\pi\omega}q_{\lambda\delta}q_{\rho\gamma}h^3 \\
&\quad + P^0 + 2X_{[0]}^0 h + (Q^\pi h)_{\underset{a}{|\pi}} + \frac{2(1-2\sigma)}{1-\sigma}HQ^0h \\
&\quad + \frac{1-2\sigma}{1-\sigma}q_{\pi\lambda}a^{\pi\lambda}Q^0h = O_{(40)}^0,
\end{aligned} \tag{6.34}$$

$$\begin{aligned}
&2A_{(1)}^{\rho\alpha\pi\lambda}(p_{\pi\lambda}h)_{\underset{a}{|\rho}} + A_{(\lambda)}^{\rho\alpha\pi\gamma\lambda\delta}(b_{\lambda\delta}q_{\pi\gamma}h^3)_{\underset{a}{|\rho}} + \tfrac{1}{3}A_{(1)}^{\gamma\beta\lambda\delta}a^{\alpha\pi}b_{\pi\gamma}(q_{\lambda\delta}h^3)_{\underset{a}{|\beta}} \\
&\quad - A_{(3)}^{\rho\alpha\pi\gamma\delta\lambda}(q_{\pi\gamma}q_{\lambda\delta}h^3)_{\underset{a}{|\rho}} + \tfrac{2}{3}a^{\alpha\pi}q_{\pi\gamma}A_{(1)}^{\gamma\beta\lambda\delta}(q_{\lambda\delta}h^3)_{\underset{a}{|\beta}} + \frac{\sigma}{1-\sigma}a^{\alpha\beta}(Q^0h)_{\underset{a}{|\beta}} \\
&\quad + P^\alpha + 2X_{[0]}^\alpha h + (2Ha_\pi^\alpha + b_\pi^\alpha)Q^\pi h \\
&\quad + (a^{\pi\lambda}q_{\pi\lambda}a_\delta^\alpha + 2a^{\alpha\pi}q_{\pi\delta})Q^\delta h = O_{(41)}^\alpha,
\end{aligned} \tag{6.35}$$

where

$$\begin{aligned}
O_{(40)}^0 = O_{(40)}^0(&Qph,\ Q\widehat{Q}h,\ Q\widehat{X}h^2,\ Qqh^2,\ bPh^2,\ qPh^2,\ b^2P^0h^2,\ q^2P^0h^2,\ qbP^0h^2, \\
&bph^3,\ bQ^0h^3,\ qQ^0h^3,\ Qh^3,\ \widehat{X}h^3,\ qph^3,\ qh^5),
\end{aligned} \tag{6.36a}$$

$$O_{(41)}^\alpha = O_{(41)}^\alpha(p^2h,\ \widehat{Q}^2h,\ \widehat{Q}ph,\ b^2Ph^2,\ bqPh^2,\ q^2Ph^2,\ ph^3,\ \widehat{Q}h^3,\ \widehat{X}h^3,\ qph^3,\ qh^5). \tag{6.36b}$$

The abbreviations $A_{(1)}^{\rho\gamma\pi\lambda}$, $A_{(3)}^{\rho\gamma\pi\omega\lambda\delta}$, $A_{(4)}^{\rho\gamma\pi\omega\lambda\delta}$ are given by (6.33a, b, c), while $A_{(6)}^{\rho\gamma\pi\omega\lambda\delta}$ is given by

$$A_{(6)}^{\rho\gamma\pi\omega\lambda\delta} = \frac{1}{6(1-\sigma^2)}\left\{\frac{(9\sigma-4)\sigma}{1-\sigma} a^{\rho\gamma}a^{\pi\omega}a^{\lambda\delta} - (2-7\sigma)a^{\delta\pi}a^{\lambda\omega}a^{\rho\gamma} + 9(1-\sigma)a^{\omega\delta}a^{\pi\rho}a^{\lambda\gamma} + (11\sigma-2)a^{\pi\omega}a^{\lambda\rho}a^{\delta\gamma}\right\}. \tag{6.37}$$

All these tensors are determined by the geometry of the unstrained state.

If we were to substitute (6.18), (6.19), (6.25), (6.27) into (5.5a, b), we would immediately obtain three equations of compatibility for $p_{\alpha\beta}$, $q_{\alpha\beta}$, with certain terms not explicitly calculated. However it is wiser to adopt an entirely different method, for the required equations of compatibility can be obtained in an exact form by a purely geometrical method.

We first introduce the equations of Codazzi and Gauss for the reference surface S_0 in the strained state,

$$b_{\alpha\beta|\gamma} - b_{\alpha\gamma|\beta} = 0, \tag{6.38a}$$

$$R_{\rho\alpha\beta\gamma} = \tfrac{1}{4}(b_{\rho\beta}b_{\alpha\gamma} - b_{\rho\gamma}b_{\beta\alpha}), \tag{6.38b}$$

and the corresponding equations for the middle surface S_0' in the unstrained state,

$$\mathbf{b}_{\alpha\beta\underset{\mathbf{a}}{|}\gamma} - \mathbf{b}_{\alpha\gamma\underset{\mathbf{a}}{|}\beta} = 0, \tag{6.39a}$$

$$\mathbf{R}_{\rho\alpha\beta\gamma} = \tfrac{1}{4}(\mathbf{b}_{\rho\beta}\mathbf{b}_{\alpha\gamma} - \mathbf{b}_{\rho\gamma}\mathbf{b}_{\beta\alpha}). \tag{6.39b}$$

Here we recall that

$$b_{\alpha\beta|\gamma} = b_{\alpha\beta,\gamma} - a^{\pi\rho}[\alpha\gamma,\rho]_a b_{\beta\pi} - a^{\pi\rho}[\beta\gamma,\rho]_a b_{\alpha\pi}, \tag{6.40a}$$

$$\mathbf{b}_{\alpha\beta\underset{\mathbf{a}}{|}\gamma} = \mathbf{b}_{\alpha\beta,\gamma} - \mathbf{a}^{\pi\rho}[\alpha\gamma,\rho]_{\mathbf{a}}\mathbf{b}_{\beta\pi} - \mathbf{a}^{\pi\rho}[\beta\gamma,\rho]_{\mathbf{a}}\mathbf{b}_{\alpha\pi}, \tag{6.40b}$$

$$R_{\rho\alpha\beta\gamma} = \tfrac{1}{2}(a_{\rho\gamma,\alpha\beta} + a_{\alpha\beta,\rho\gamma} - a_{\rho\beta,\alpha\gamma} - a_{\alpha\gamma,\rho\beta}) + a^{\pi\lambda}\{[\rho\gamma,\pi]_a[\alpha\beta,\lambda]_a - [\rho\beta,\pi]_a[\alpha\gamma,\lambda]_a\}, \tag{6.40c}$$

$$\mathbf{R}_{\rho\alpha\beta\gamma} = \tfrac{1}{2}(\mathbf{a}_{\rho\gamma,\alpha\beta} + \mathbf{a}_{\alpha\beta,\rho\gamma} - \mathbf{a}_{\rho\beta,\alpha\gamma} - \mathbf{a}_{\alpha\gamma,\rho\beta}) + \mathbf{a}^{\pi\lambda}\{[\rho\gamma,\pi]_{\mathbf{a}}[\alpha\beta,\lambda]_{\mathbf{a}} - [\rho\beta,\pi]_{\mathbf{a}}[\alpha\gamma,\lambda]_{\mathbf{a}}\}. \tag{6.40d}$$

Furthermore by definition, we have

$$a^{\pi\omega} = \tfrac{1}{2}\epsilon^{\pi\lambda}\epsilon^{\omega\delta}a_{\lambda\delta}, \qquad a = \tfrac{1}{2}\epsilon^{\pi\lambda}\epsilon^{\omega\delta}a_{\lambda\delta}a_{\pi\omega}, \tag{6.41a}$$

$$\mathbf{a}^{\pi\omega} = \tfrac{1}{2}\epsilon^{\pi\lambda}\epsilon^{\omega\delta}\mathbf{a}_{\lambda\delta}, \qquad \mathbf{a} = \tfrac{1}{2}\epsilon^{\pi\lambda}\epsilon^{\omega\delta}\mathbf{a}_{\lambda\delta}\mathbf{a}_{\pi\omega}, \tag{6.41b}$$

$$a_{\gamma}^{\pi} = \delta_{\gamma}^{\pi} = \tfrac{1}{2}\epsilon^{\pi\lambda}\epsilon^{\rho\delta}a_{\lambda\delta}a_{\rho\gamma}, \qquad \mathbf{a}_{\gamma}^{\pi} = \delta_{\gamma}^{\pi} = \tfrac{1}{2}\epsilon^{\pi\lambda}\epsilon^{\rho\delta}\mathbf{a}_{\lambda\delta}\mathbf{a}_{\rho\gamma}, \tag{6.41c}$$

where δ_{γ}^{π} is the Kronecker delta. Substitution of $a_{\lambda\delta}$ from (6.19) into (6.41a, c) gives, with (6.41b)

$$a^{\pi\omega} = \frac{\mathbf{a}}{a}(2\eta_{[0]}^{\pi\lambda}\eta_{[0]}^{\omega\delta}p_{\lambda\delta} + \mathbf{a}^{\pi\omega}), \tag{6.42a}$$

$$\frac{a}{\mathbf{a}} = 1 + 2\eta_{[0]}^{\pi\lambda}\eta_{[0]}^{\gamma\delta}p_{\lambda\delta}p_{\pi\gamma} + 2p_{\pi\lambda}\mathbf{a}^{\pi\delta}, \tag{6.42b}$$

$$\delta_\gamma^\pi = \frac{\boldsymbol{a}}{a}\left\{\delta_\gamma^\pi + 2p_{\lambda\gamma}a^{\lambda\pi} + 2\eta_{[0]}^{\pi\lambda}\eta_{[0]}^{\rho\delta}p_{\lambda\delta}a_{\rho\gamma} + 4\eta_{[0]}^{\pi\lambda}\eta_{[0]}^{\rho\delta}p_{\lambda\delta}p_{\rho\gamma}\right\}, \tag{6.42c}$$

where $\mathfrak{n}_{[0]}^{\pi\lambda}$ is given as in (6.33f).

We now multiply (6.38a), (6.38b) respectively by $(a/\boldsymbol{a})\mathfrak{n}_{[0]}^{\beta\gamma}$, $(a/\boldsymbol{a})\mathfrak{n}_{[0]}^{\beta\gamma}\mathfrak{n}_{[0]}^{\rho\alpha}$, and substitute $a_{\alpha\beta}$, $b_{\alpha\beta}$ from (6.19) into the resulting equations. This gives, in consequence of (6.42a, b, c) and (6.39a, b), the following three equations in $\boldsymbol{p}_{\alpha\beta}$ and $\boldsymbol{q}_{\alpha\beta}$:

$$\begin{aligned}&2\mathfrak{n}_{[0]}^{\beta\gamma}\,\boldsymbol{q}_{\alpha\beta\,|\,\underset{a}{\gamma}}(1 + 2\eta_{[0]}^{\pi\lambda}\eta_{[0]}^{\omega\delta}\boldsymbol{p}_{\pi\omega}\boldsymbol{p}_{\lambda\delta} + 2a^{\pi\omega}\boldsymbol{p}_{\pi\omega})\\ &\qquad - \mathfrak{n}_{[0]}^{\beta\gamma}(2\boldsymbol{q}_{\beta\pi} + \boldsymbol{b}_{\beta\pi})(a^{\pi\omega} + 2\mathfrak{n}_{[0]}^{\pi\lambda}\mathfrak{n}_{[0]}^{\omega\delta}\boldsymbol{p}_{\lambda\delta})(\boldsymbol{p}_{\alpha\omega\,|\,\underset{a}{\gamma}} + \boldsymbol{p}_{\gamma\omega\,|\,\underset{a}{\alpha}} - \boldsymbol{p}_{\alpha\gamma\,|\,\underset{a}{\omega}}) = 0,\end{aligned} \tag{6.43}$$

$$\begin{aligned}&(1 + 2\mathfrak{n}_{[0]}^{\pi\lambda}\mathfrak{n}_{[0]}^{\gamma\delta}\boldsymbol{p}_{\lambda\delta}\boldsymbol{p}_{\pi\gamma} + 2a^{\pi\gamma}\boldsymbol{p}_{\pi\gamma})\left\{2\mathfrak{n}_{[0]}^{\rho\alpha}\mathfrak{n}_{[0]}^{\beta\omega}\boldsymbol{p}_{\rho\omega\,|\,\underset{a}{\alpha\beta}}\right.\\ &\qquad\left. + \mathfrak{n}_{[0]}^{\rho\alpha}\mathfrak{n}_{[0]}^{\beta\omega}\boldsymbol{q}_{\rho\omega}\boldsymbol{q}_{\alpha\beta} + 2a^{\alpha\beta}\boldsymbol{p}_{\alpha\beta}K - (4Ha^{\alpha\beta} - b^{\alpha\beta})\boldsymbol{q}_{\alpha\beta}\right\}\\ &+ \mathfrak{n}_{[0]}^{\rho\alpha}\mathfrak{n}_{[0]}^{\beta\gamma}(a^{\pi\omega} + 2\mathfrak{n}_{[0]}^{\pi\lambda}\mathfrak{n}_{[0]}^{\omega\delta}\boldsymbol{p}_{\lambda\delta})(\boldsymbol{p}_{\pi\rho\,|\,\underset{a}{\gamma}} + \boldsymbol{p}_{\gamma\pi\,|\,\underset{a}{\rho}} - \boldsymbol{p}_{\gamma\rho\,|\,\underset{a}{\pi}})(\boldsymbol{p}_{\alpha\omega\,|\,\underset{a}{\beta}} + \boldsymbol{p}_{\beta\omega\,|\,\underset{a}{\alpha}} - \boldsymbol{p}_{\alpha\beta\,|\,\underset{a}{\omega}}) = 0.\end{aligned} \tag{6.44}$$

Equations (6.43), (6.44) are *the three equations of compatibility for $\boldsymbol{p}_{\alpha\beta}$ and $\boldsymbol{q}_{\alpha\beta}$*; these are to be associated with *the three equations of equilibrium*, (6.34), (6.35) for the solutions of plate or shell problems. We note that $\boldsymbol{K}$ is the total curvature of the middle surface S_0' in the unstrained state, and satisfies

$$\boldsymbol{K} = \tfrac{1}{8}(a^{\pi\gamma}\boldsymbol{b}_{\pi\gamma}a^{\lambda\delta}\boldsymbol{b}_{\lambda\delta} - \boldsymbol{b}^{\pi\lambda}\boldsymbol{b}_{\pi\lambda}). \tag{6.45}$$

Conclusion. Equations (6.34), (6.35), (6.43), (6.44) are the final forms of six differential equations in the six unknowns $\boldsymbol{p}_{\alpha\beta}$ and $\boldsymbol{q}_{\alpha\beta}$. Here $\boldsymbol{p}_{\alpha\beta}$ and $\boldsymbol{q}_{\alpha\beta}$, as indicated in (6.17), represent the extension and change of curvature of S_0', the middle surface in the unstrained state. With $\boldsymbol{p}_{\alpha\beta}$ and $\boldsymbol{q}_{\alpha\beta}$ known, the macroscopic tensors $T^{\alpha\beta}$, $T^{\alpha 0}$, $L^{\alpha\beta}$ can be calculated from (6.29)–(6.31).

We recall that the tensors $\boldsymbol{a}_{\alpha\beta}$ and $(1/2)\boldsymbol{b}_{\alpha\beta}$ are respectively the first and second fundamental tensors of the middle surface S_0' in the unstrained state; $\boldsymbol{H}$ and $\boldsymbol{K}$ are the mean and total curvature of S_0' as in (6.33e), (6.45); σ is Poisson's ratio; $2h$ is the thickness of the shell or plate in the unstrained state; P^i and Q^i represent the sum and difference of the surface forces on the upper and lower surfaces of the shell or plate in the strained state as in (5.28); $X_{[m]}^i$ are the normal derivatives of the body force on the reference surface S_0 in the strained state as in (3.24). All these quantities may be regard as given. The covariant differentiations are calculated for $\boldsymbol{a}_{\alpha\beta}$ and $\mathbf{x}^\alpha$.

We also note that the six equations (6.34), (6.35), (6.43), (6.44) are exact, in the sense that no terms have been omitted, but of course the residual terms, represented by O-symbols, have not been calculated explicitly. However, it will be shown in Parts II and III that in all cases of small thickness and small strain, the residual terms are small compared with those shown explicitly, and it is legitimate to neglect them in a first approximation.

(To be continued)

References

[1] A. E. H. Love, *A treatise on the mathematical theory of elasticity*, Cambridge, 1927.

[2] S. Timoshenko, *Theory of plates and shells*, McGraw-Hill, 1940.

[3] W. Flügge, *Statik und Dynamik der Schalen*, Springer, 1934. (A complete list of literature up to 1933 can be found in the end of this book.)

[4] E. Reissner, *A new derivation of the equations for the deformation of elastic shells*, Amer. J. Math. **63**, 177–184 (1941).

[5] J. L. Synge and W. Z. Chien, *The intrinsic theory of elastic shells and plates*, Applied Mechanics, Theodore von Kármán Anniversary Volume (1941), pp. 103–120.

[6] A. J. MacConnell, *Applications of the absolute differential calculus*, Blackie, 1931.

[7] L. Brillouin, *Les tenseurs en mécanique et en élasticité*, Paris, 1928.

[8] F. D. Murnaghan, *Finite deformations of an elastic solid*, Amer. J. Math., **59**, 235–260 (1937).