Теорія Ймовір. та Матем. Статист.
Вип. 70, 2004

Theor. Probability and Math. Statist.
No. 70, 2005, Pages 123–134
S 0094-9000(05)00636-8
Article electronically published on August 12, 2005

THE INVARIANCE PRINCIPLE FOR A CLASS OF DEPENDENT RANDOM FIELDS

UDC 519.21

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Abstract. Sufficient conditions for the tightness of a family of distributions of partial sum set-indexed processes constructed from symmetric random fields are obtained in this paper. We require that the moments of order s, $s > 2$, exist. The dependence structure of the field is described by the β_1-mixing coefficients decreasing with a power rate. Assuming that a field is stationary and applying a result of D. Chen (1991) on the convergence of finite-dimensional distributions of the processes we obtain the invariance principle.

1. Introduction

The asymptotic behavior of partial sum set-indexed processes is studied in a number of papers (see, for example, [2], [3], [6], [7], [11]). The results of the above papers are obtained for smoothed partial sums defined on a subclass $\mathfrak{A}$ of the Borel sets of the cube $[0,1]^d$. More precisely, let $X = \{X_j, j \in \mathbf{Z}^d\}$ be a stationary field defined on some probability space $(\Omega, \mathfrak{D}, \mathsf{P})$. Consider the processes

$$Z_n(A) = n^{-d/2} \sum_{j\in\mathbf{Z}^d} b_{nj}(A) X_j, \qquad A \in \mathfrak{A},\ n \in \mathbf{N},$$

where $j = (j_1, \dots, j_d)$, $C_j = (j_1 - 1, j_1] \times \cdots \times (j_d - 1, j_d]$ is a unit cube, $|\cdot|$ is Lebesgue measure in $\mathbf{R}^d$, $b_{nj}(A) = |(nA) \cap C_j|$, and $nA = \{nx \colon x \in A\}$.

Let $\overline{\mathfrak{A}}$ be the closure of $\mathfrak{A}$ with respect to the pseudometric $d_L(A,B) = |A \triangle B|$ defined for $A, B \in \overline{\mathfrak{A}}$. Denote by $C(\overline{\mathfrak{A}})$ the space of real continuous functions on $\overline{\mathfrak{A}}$ equipped with the sup-norm.

We consider symmetric fields X (see, for example, §4.2 in [12]) constructed from identically distributed random variables X_j. Recall that a field X is called symmetric if the finite-dimensional distributions of the fields X and

$$\varepsilon X = \{\varepsilon_j X_j, j \in \mathbf{Z}^d\}$$

coincide where $\varepsilon = \{\varepsilon_j, j \in \mathbf{Z}^d\}$ is the Rademacher field that does not depend on X.

We are interested in obtaining sufficient conditions for the invariance principle, that is, conditions for the convergence in distribution in the space $C(\overline{\mathfrak{A}})$ of the processes Z_n to the process $\sqrt{\eta} Z$ as $n \to \infty$, where Z is a standard Brownian motion,

$$\eta = \sum_{k\in\mathbf{Z}^d} \mathsf{E}(X_0 X_k \mid \mathfrak{I}),$$

2000 *Mathematics Subject Classification.* Primary 60F17, 60G60.
Supported in part by the RFFI grant 03-01-00724.

and $\mathfrak{I}$ is the σ-algebra of events that are invariant under the shifts of the field X (Z does not depend on η). The standard Brownian motion Z is defined as a mean zero Gaussian process with sample paths in $C(\overline{\mathfrak{A}})$ and such that $\mathsf{E}(Z(A)Z(B)) = |A \cap B|$ for $A, B \in \overline{\mathfrak{A}}$. The existence of such a process Z is proved in [9] under some entropy conditions posed on the class $\mathfrak{A}$. These conditions are given in terms of the so-called entropy with inclusion.

We need more notation to describe the dependence structure of the field X. Let $\beta(\sigma_1, \sigma_2)$ be the coefficient of absolute regularity of the σ-algebras $\sigma_1, \sigma_2 \subset \mathfrak{D}$ (see, for example, [1]). Put

$$\rho(G_1, G_2) = \inf\{\|x - y\| \colon x \in G_1, y \in G_2\}$$

where $G_1, G_2 \subset \mathbf{Z}^d$ and $\|x\| = \max_{1 \le i \le d} |x_i|$ for $x = (x_1, \dots, x_d) \in \mathbf{R}^d$. For $n \in \mathbf{N}$ and $k, m \in \mathbf{N} \cup \{\infty\}$ we introduce the mixing coefficients:

$$\beta_X(n, k, m) = \sup\{\beta(\sigma_X(G_1), \sigma_X(G_2)) \colon \sharp(G_1) \le k, \sharp(G_2) \le m, \rho(G_1, G_2) \ge n\} \tag{1}$$

where the sets G_1 and G_2 are separated by some hyperplane in $\mathbf{R}^d$, $\sigma_X(G)$ is the σ-algebra generated by the field X in the set $G \subset \mathbf{Z}^d$, and $\sharp(G)$ denotes the cardinality of G. For $x, y, z \ge 1$, we put $\beta_X(x, y, z) = \beta_X([x], [y], [z])$ where $[\cdot]$ is the integer part of a number.

The convergence of finite-dimensional distributions of the processes Z_n is proved in [7] under a condition on the dependence of $\sigma_X(\{j\})$ and $\sigma_X(G)$, such that

$$\rho(\{j\}, G) \ge n, \qquad n \in \mathbf{N}.$$

At the same time, the condition on the tightness of the family of distributions of the processes Z_n in $C(\overline{\mathfrak{A}})$ relies on the dependence of $\sigma_X(\{i, j\})$ and $\sigma_X(G)$ such that

$$\rho(\{i, j\}, G) \ge n, \qquad n \in \mathbf{N}.$$

The main aim of this paper is to obtain a sufficient condition for the tightness of the distributions of Z_n in terms of the coefficients $\beta_X(n, 1, m)$, so we avoid two-point subsets of $\mathbf{Z}^d$ in (1) (instead, the condition will involve those G_1 for which $\sharp(G_1) = 1$). There are, of course, some extra conditions on the moments of the field X and on the structure of the class $\mathfrak{A}$.

To solve the problem we generalize the method of the paper [3] to the case of weakly dependent fields. This generalization is due to the so-called reconstruction technique (see, for example, §1.2.2 in [8]) developed in the paper [5]. Our method also uses truncation of the original random variables, appropriate approximations of elements of the class $\mathfrak{A}$, and some maximal inequalities.

2. Entropy conditions

We introduce the following conditions posed on the entropy with inclusion for a family $\mathfrak{A}$. Let $g(\varepsilon)$, $\varepsilon \in [0, 1]$, be an increasing function such that $g(\varepsilon) \to \infty$ as $\varepsilon \to 0$ and

$$\int_0^1 (\varepsilon^{-1} H(\varepsilon))^{1/2} g(\varepsilon)\, d\varepsilon < \infty \tag{2}$$

where $H(\varepsilon) = \log N_I(\varepsilon, \mathfrak{A}, d_L)$ and $N_I(\varepsilon, \mathfrak{A}, d_L)$ denotes the minimal number $k \ge 1$ for which there are measurable sets $A_i^{(1)}$ and $A_i^{(2)}$ of $[0, 1]^d$, $1 \le i \le k$, such that for all $A \in \mathfrak{A}$ there exists i such that $A_i^{(1)} \subset A \subset A_i^{(2)}$ and $\bigl|A_i^{(2)} \setminus A_i^{(1)}\bigr| \le \varepsilon$.

It follows from metric entropy condition (2) that $\overline{\mathfrak{A}}$ is a compact set, and therefore $C(\overline{\mathfrak{A}})$ is a separable space. We define the exponent r of the metric entropy of $\mathfrak{A}$ by $r = \inf\{s > 0 \colon \log N_I(\varepsilon, \mathfrak{A}, d_L) = O(\varepsilon^{-s}) \text{ as } \varepsilon \to 0\}$. It is easy to see that metric entropy condition (2) holds if $r < 1$. Therefore all the classes of sets studied in [3] satisfy condition (2).

3. Tightness

Here is the main result of the paper.

Theorem 1. *Let X be a symmetric field of identically distributed random variables X_j. Assume that $\mathfrak{A}$ is a family of subsets of $[0,1]^d$ satisfying metric entropy condition* (2). *Moreover let*

(i) $\mathsf{E}\,|X_0|^s < \infty$ *for some* $s > 2$;

(ii) $\limsup_{\varepsilon\downarrow 0} f^{2\tau}(\varepsilon)\cdot\beta_X\left(g^{1/d}(\varepsilon)-1, 1, f^{2\tau}(\varepsilon)/g(\varepsilon)\right) < \infty$ *where* $\tau = s/(s-2)$ *and* $f(\varepsilon) = (\varepsilon^{-1}H(\varepsilon))^{1/2}g(\varepsilon)$.

Then the family of distributions of the processes $Z_n = \{Z_n(A)\colon A\in\mathfrak{A}\}$ *is tight in the space* $C(\overline{\mathfrak{A}})$.

Proof. Given $\delta > 0$ consider the family of sets

$$\mathbb{S}_\delta = \{A\setminus B\colon A, B\in\mathfrak{A} \text{ such that } |A\setminus B| \le \delta\}.$$

Since $N_I(\varepsilon, \mathbb{S}_\delta, d_L) \le N_I(\varepsilon/2, \mathfrak{A}, d_L)^2$, condition (2) holds for $\mathbb{S}_\delta$. Thus the process Z is continuous in the space $(\mathbb{S}_\delta, d_L)$. Let

$$\|f\|_D = \sup_{x\in D}|f(x)|$$

for all real functions f defined on the set D. The functional

$$\mathbf{w}(Z_n,\delta) = \sup\{|Z_n(B) - Z_n(C)|\colon B, C\in\mathfrak{A}, |B\triangle C| < \delta\}, \qquad \delta > 0,$$

is the modulus of continuity of the process Z_n in the space $C(\overline{\mathfrak{A}})$. It is clear that Z_n is a random element in the space $C(\overline{\mathfrak{A}})$. Since $\mathbf{w}(Z_n,\delta) \le 2\|Z_n\|_{\mathbb{S}_\delta}$, the family of distributions is tight if for all $M > 0$

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\mathsf{P}(\|Z_n\|_{\mathbb{S}_\delta} > M) = 0. \tag{3}$$

We split the proof of (3) into several steps.

Step 1. First we truncate the random variables X_j. According to (i) there is a sequence $\{c_n\}_{n\ge 1}$ such that $c_n \to 0$ and $\lim_{n\to\infty} n^d\,\mathsf{P}(|X_0|^s > c_n^2 n^d) = 0$. If

$$b_n^2 = n^{2d/s-d}c_n^{4/s},$$

then $\lim_{n\to\infty} n^d\,\mathsf{P}(X_0^2 > b_n^2 n^d) = 0$. Fix constants M and $\delta > 0$ and put

$$\gamma_n = \inf\left\{\gamma > 0\colon n^d\,\mathsf{P}(X_0^2 > \gamma^2 n^d) < M\delta b_n^{-1}\right\}\wedge b_n. \tag{4}$$

Let

$$Z_n' = n^{-d/2}\sum_j b_{nj}X_j\,\mathbb{1}_{\{|X_j| > b_n n^{d/2}\}}, \qquad Z_n'' = n^{-d/2}\sum_j b_{nj}X_j\,\mathbb{1}_{\{\gamma_n n^{d/2} < |X_j| \le b_n n^{d/2}\}}.$$

If the sequence $\{b_n\}$ is defined as above, then

$$n^d\,\mathsf{P}\left(|X_0| > \gamma_n n^{d/2}\right) \le M\delta b_n^{-1}$$

and $b_{nj}(\cdot) \le 1$. Thus by the Chebyshev inequality

$$\mathsf{P}\left(\|Z_n'\|_{\mathbb{S}_\delta} > M\right) \le \mathsf{P}\left(\bigcup_{0\le j\le n\mathbf{1}}\left\{|X_j| > b_n n^{d/2}\right\}\right) = o(1),$$

$$\mathsf{P}(\|Z_n''\|_{\mathbb{S}_\delta} > M) \le M^{-1}\,\mathsf{E}\,\|Z_n''\|_{\mathbb{S}_\delta} \le M^{-1}b_n\sum_{\mathbf{0}\le j\le n\mathbf{1}}\mathsf{P}\left(|X_j| > \gamma_n n^{d/2}\right) \le \delta.$$

Here $\mathbf{1} = (1, \dots, 1)$ and $\mathbf{0} \le j \le n\mathbf{1}$ is equivalent to the inequalities $0 \le j_i \le n$ for all $i = 1, \dots, d$. Therefore condition (3) holds in the general case if it holds for the truncated partial sum process

$$Z_n^T = Z_n - Z_n' - Z_n'' = n^{-d/2} \sum_j b_{nj} X_j \, \mathbb{1}_{\{|X_j| \le \gamma_n n^{d/2}\}}$$

instead of Z_n.

Step 2. We apply the so-called stratification procedure where the interval $(0, \gamma_n n^{d/2}]$ is partitioned into appropriate subintervals. Let $F(x) = \mathsf{P}(|X_0| > x)$, $x \in \mathbf{R}_+$. Put

$$Q_F(u) = \inf\{x \ge 0 \colon F(x) \le u\}, \qquad u \in (0, 1].$$

For $0 < \beta < 1$ let $\mu_k = \beta^k$ and $a_k = Q_F(\mu_k)$, $k = 0, \dots, k_n$, where

$$k_n = \max\left\{k \colon a_k < \gamma_n n^{d/2}\right\}.$$

Then $\mathsf{P}(|X_0| > a_k) \le \mu_k$ for $k = 0, \dots, k_n$. Consider the intervals $J_k = (a_k, a_{k+1}]$, $0 \le k \le k_n$, where $a_{k_n+1} = \gamma_n n^{d/2}$.

If $Y_j = |X_j|$, then

$$Z_n^T = \sum_{k \le k_n} \theta_k \nu_{nk}$$

where $\theta_k = a_{k+1} \mu_k^{1/2}$ and

$$\nu_{nk}(A) = \left(n^d \mu_k\right)^{-1/2} \sum_j b_{nj}(A) a_{k+1}^{-1} X_j \, \mathbb{1}_{\{Y_j \in J_k\}}.$$

It is clear that $a_{k+1}^{-1} Y_j \le 1$ if $Y_j \in J_k$. It is easy to see that

$$\sum_{k=0}^{k_n} \theta_k^2 = \sum_{k=0}^{k_n} a_{k+1}^2 \mu_k \le \sum_{k=0}^{\infty} Q_F^2(\beta^{k+1}) \beta^k \le 1/(\beta(1-\beta)) \, \mathsf{E}\, X_0^2 < \infty$$

for all $n \in \mathbf{N}$.

Below we use the functions f and H that are introduced above. The only properties of the functions H and f we use in the proof are that $H(\varepsilon)$ is an upper bound of $\log N_I(\varepsilon, \mathfrak{A}, d_L)$ and f is integrable. Thus without loss of generality we may assume that

$$H \text{ is continuous, decreases, and} \quad H(\varepsilon) \ge 1 + \log(\varepsilon^{-1}) \tag{5}$$

and f is a decreasing function. Hence its inverse function f^{inv} is well defined. Put $\delta_{nk_n} = f^{\mathrm{inv}}((n^d \mu_{k_n})^{1/2}/4)$. For $0 \le k < k_n$ we choose δ_{nk} such that

$$n^d \mu_k = 16 H(\delta_{nk}) g^2(\delta_{nk_n}) \delta_{nk}^{-1}. \tag{6}$$

This can be done, since H has the inverse function. Taking into account (4) we see that $n^d \mu_{k_n} \ge M\delta b_n^{-1} \to \infty$, whence $\delta_{nk_n} \to 0$ by (6). Note that the aim of the second truncation above, that is, the subtraction of the process Z_n'', is to satisfy the latter relation.

Step 3. Now we consider a finite net of subsets that approximate ν_{nk} on $A \in \mathbb{S}_\delta$ by its values on the subsets of the net that are close to A. The family $\mathfrak{A}$ is totally bounded, thus there are finite nets $\mathbb{D}_{nk}^{(l)}$, $l = 1, 2$, whose cardinalities are less than or equal to $\exp(2H(\delta_{nk}))$, respectively, and such that for all $A \in \mathbb{S}_\delta$ there is $D_{nk}^{(l)}(A) \in \mathbb{D}_{nk}^{(l)}$ for which $D_{nk}^{(1)}(A) \subset A \subset D_{nk}^{(2)}(A)$ and $\left|D_{nk}^{(2)}(A) \setminus D_{nk}^{(1)}(A)\right| \le 2\delta_{nk}$.

Since ν_{nk} is additive, we represent Z_n^T as follows:

$$Z_n^T(A) = \sum_{k \le k_n} \theta_k \nu_{nk}\left(D_{nk}^{(1)}(A)\right) + \sum_{k \le k_n} \theta_k \nu_{nk}\left(\left(A \setminus D_{nk}^{(1)}(A)\right)\right) = Z_n^{(1)}(A) + Z_n^{(2)}(A). \tag{7}$$

Step 4. Further we apply the reconstruction technique. Let

$$L_n = \{j/n\colon j \in \{1, 2, \dots, n\}^d\}$$

and $C_{n,j} = (\mathbf{j} - n^{-1}\mathbf{1}, \mathbf{j}]$. Let $[0,1]^d$ be represented as follows: $[0,1]^d = \bigcup_{l\in L_{p_n}} C_{p_n,l}$ where $p_n = [n/m_n]$, $m_n^d = N$, and $N = g(\delta_{nk_n})$. The intersections of an arbitrary cube $C_{p_n,l}$ with cubes of the family $\{C_{n,j}, j \in L_n\}$ form a family of smaller cubes. These small cubes are numbered in every cube of the family $\{C_{p_n,l}, l \in L_{p_n}\}$ and denoted by I_{nli}, $i = \{1, \dots, N\}$. Put

$$I_{ni} = \bigcup_{l\in L_{p_n}} I_{nli}.$$

Then every element I_{ni} is a union of cubes whose sides are of length $1/n$ and the distances between the cubes are at least $1/p_n - 1/n$.

The reasoning above implies that

$$\nu_{nk}(A) = \sum_{i=1}^{N} \nu_{nk}(A \cap I_{ni}) = \sum_{i=1}^{N} \sum_{l\in L_{p_n}} \nu_{nk}(A \cap I_{nli})$$

and

$$\nu_{nk}(A \cap I_{nli}) = (n^d\mu_k)^{-1/2} \sum_{j\in nS(n,l,i)} |n(A \cap I_{nli} \cap C_{n,j})| a_{k+1}^{-1} X_j \, \mathbb{1}_{\{Y_j\in J_k\}}$$

where

$$S(n,l,i) = \{j \in L_n\colon C_{n,j} \cap I_{nli} \neq \varnothing\}.$$

If n and i are fixed, then the set $S(n,l,i)$ contains only a single element and the distance between these sets is at least $1/p_n - 1/n$.

To every $j \in nS(n,l,i)$ there corresponds a triple (n,l,i). Then

$$Y_{nli} = Y_j$$

for $j \in nS(n,l,i)$ and $\nu_{nk}(A \cap I_{nli}) = (n^d\mu_k)^{-1/2} |n(A \cap I_{nli})| a_{k+1}^{-1} X_{nli} \, \mathbb{1}_{\{Y_{nli}\in J_k\}}$.

The following result plays the key role in the reconstruction technique.

Lemma 1 ([5])**.** *Let X and Y be two random variables assuming values in Polish spaces S_1 and S_2, respectively. Suppose that the probability space where X and Y are defined is essentially rich in the sense that there exists a random variable U that is uniform on the interval $[0,1]$ and independent of both X and Y. Then there exists a random variable Y^* having the same distribution as Y, independent of X, and such that*

$$\mathsf{P}(Y \neq Y^*) = \beta(\sigma(X), \sigma(Y)).$$

Moreover the random variable Y^* can be represented as $Y^* = f(X, Y, U)$ where

$$f\colon S_1 \times S_2 \times [0,1] \to S_2$$

is some measurable function.

Let ψ be a one-to-one mapping from $[1, \sharp(L_{p_n})] \cap \mathbf{N}$ to L_{p_n} such that $\psi(m) <_{\text{lex}} \psi(m')$ for all $1 \le m < m' \le \sharp(L_{p_n})$ where the symbol $<_{\text{lex}}$ stands for the lexicographic order. Then

$$\{X_{nli},\ l \in L_{p_n}\} = \{X_{n,\psi(m),i},\ m \in [1, \sharp(L_{p_n})] \cap \mathbf{N}\}.$$

Given numbers n and i we use an induction and construct the independent random variables

$$\{\widetilde{X}_{nli},\ l \in L_{p_n}\}$$

such that the distribution of $\widetilde{X}_{nli}$ coincides with that of X_{nli} and

$$\mathsf{P}(\widetilde{X}_{nli} \neq X_{nli}) \le \beta_X\left(m_n - 1, 1, p_n^d\right).$$

At the first step of the induction we put $\widetilde{X}_{n,\psi(1),i} = X_{n,\psi(1),i}$, while at the induction step r $(1 < r < \sharp(L_{p_n}))$ we apply Lemma 1 with

$$X = \left(\widetilde{X}_{n,\psi(m),i}\right)_{1\le m<r}, \qquad Y = X_{n,\psi(r),i} \qquad \widetilde{X}_{n,\psi(r),i} = Y^*.$$

As a result we get the inequalities

$$\mathsf{P}(\widetilde{X}_{nli} \ne X_{nli}) = \beta(\sigma\{X\},\sigma\{Y\}) \le \beta\big(\sigma\{X_{n,\psi(m),i}, 1\le m<r\},\sigma\{X_{n,\psi(r),i}\}\big).$$

Put $F_{ni} = \bigcup_{l\in L_{p_n}}\{\widetilde{X}_{nli} \ne X_{nli}\}$. Then

$$\mathsf{P}(F_{ni}) \le p_n^d \cdot \beta_X\left(m_n - 1, 1, p_n^d\right). \tag{8}$$

This completes the process of reconstruction of the field X.

Step 5. First we prove relation (3) for the process $Z_n^{(2)}(A)$. We represent $Z_n^{(2)}$ as follows:

$$Z_n^{(2)}(A) = \sum_{i=1}^N \sum_{k\le k_n} \theta_k \nu_{nk}\left(\left(A\setminus D_{nk}^{(1)}(A)\right)\cap I_{ni}\right).$$

If $U_{nli}^k(A) = n^{-d/2}|n(A\cap I_{nli})|\,\mathbb{1}_{\{Y_{nli}\in J_k\}}$, then

$$|\nu_{nk}(A)| \le \sum_{i=1}^N (\mu_k)^{-1/2}\sum_{l\in L_{p_n}} U_{nli}^k(A).$$

Since $\mathsf{P}(Y_j\in J_k)\le \mu_k$, we obtain

$$(\mu_k)^{-1/2}\sum_{l\in L_{p_n}} \mathsf{E}\left(U_{nli}^k(A)\right) \le (n^d\mu_k)^{1/2}\sum_{l\in L_{p_n}}|A\cap I_{nli}|$$

where A is an arbitrary set belonging to $\mathfrak{B}([0,1]^d)$. This implies for $A\in\mathbb{S}_\delta$ the following estimate:

$$\begin{aligned}\left|\nu_{nk}\left(\left(A\setminus D_{nk}^{(1)}(A)\right)\cap I_{ni}\right)\right| &\le \sum_{i=1}^N (\mu_k)^{-1/2}\sum_{l\in L_{p_n}} U_{nli}^k\left(D_{nk}^{(2)}(A)\setminus D_{nk}^{(1)}(A)\right)\\ &\le \sum_{i=1}^N (\mu_k)^{-1/2}\sum_{l\in L_{p_n}} V_{nli}^k\left(D_{nk}^{(2)}(A)\setminus D_{nk}^{(1)}(A)\right) + (n^d\mu_k)^{1/2}\left|D_{nk}^{(2)}(A)\setminus D_{nk}^{(1)}(A)\right|\end{aligned}$$

where $V_{nli}^k = U_{nli}^k - \mathsf{E}\,U_{nli}^k$. Let $\widetilde{V}_{nli}^k = \widetilde{U}_{nli}^k - \mathsf{E}\,\widetilde{U}_{nli}^k$ where

$$\widetilde{U}_{nli}^k(A) = n^{-d/2}|n(A\cap I_{nli})|\,\mathbb{1}_{\{|\widetilde{X}_{nli}|\in J_k\}}$$

and $\lambda_{nk} = 16N(\delta_{nk}H(\delta_{nk}))^{1/2}$. Then we estimate $\mathsf{P}\big(\|Z_n^{(2)}\|_{\mathbb{S}_\delta} > M\big)$ by

$$\begin{aligned}&\mathsf{P}\left(\sum_{i=1}^N\sum_{k\le k_n}\theta_k\sup_{A\in\mathbb{S}_\delta}\left|\nu_{nk}\left(\left(A\setminus D_{nk}^{(1)}(A)\right)\cap I_{ni}\right)\right| > \sum_{k\le k_n}\theta_k\lambda_{nk}\right)\\ &\qquad\le \sum_{i=1}^N \mathsf{P}(F_{ni}) + \sum_{i=1}^N\sum_{k\le k_n} R_{ni}^k\end{aligned} \tag{9}$$

where

$$R_{ni}^k = \mathsf{P}\left(\sup_{A\in\mathbb{S}_\delta}\left|\sum_{l\in L_{p_n}}\widetilde{V}_{nli}^k\left(D_{nk}^{(2)}(A)\setminus D_{nk}^{(1)}(A)\right)\right| > p_{nkN}\right),$$

$$p_{nkN} = (\mu_k)^{1/2}\lambda_{nk}/N - n^{d/2}\mu_k 2\delta_{nk}/N.$$

Step 6. Now we are going to apply the Bernstein inequality (see [4]) to estimate R_{ni}^k. This can be done, since $\widetilde{V}_{nli}^k(E)$ for $E \in \mathbb{S}_\delta$ are independent random variables. We need the following estimates:

$$\left|\widetilde{V}_{nli}^k(E)\right| \le n^{-d/2} |n(E \cap I_{nli})| \le n^{-d/2}, \tag{10}$$

$$\operatorname{Var}\left(\sum_{l \in L_{p_n}} \widetilde{V}_{nli}^k(E)\right) \le \sum_{l \in L_{p_n}} \operatorname{Var}\left(\mathbb{1}_{\{Y_0 \in J_k\}}\right) |E \cap I_{nli}| \le |E \cap I_{ni}| \mu_k. \tag{11}$$

Putting $E = D_{nk}^{(2)}(A) \setminus D_{nk}^{(1)}(A)$ in (10) and (11) and applying the Bernstein inequality we obtain

$$\begin{aligned} R_{ni}^k &\le \left(\sharp\left(\mathbb{D}_{nk}^{(1)}\right)\right)^2 \max_{A \in \mathbb{S}_\delta} \mathsf{P}\left(\left|\sum_{l \in L_{p_n}} \widetilde{V}_{nli}^k\left(D_{nk}^{(2)}(A) \setminus D_{nk}^{(1)}(A)\right)\right| > p_{nkN}\right) \\ &\le \exp\left(4H(\delta_{nk}) - \frac{p_{nkN}^2}{2\left(\mu_k 2\delta_{nk} + 1/3 n^{-d/2} p_{nkN}\right)}\right). \end{aligned}$$

In view of (6) we have $n^{d/2}\mu_k\delta_{nk} = 4N(H(\delta_{nk})\delta_{nk})^{1/2}(\mu_k)^{1/2}$. Thus

$$R_{ni}^k \le \exp\left(4H(\delta_{nk}) - \frac{64\mu_k H(\delta_{nk})\delta_{nk}}{2(\mu_k 2\delta_{nk} + 2/3\delta_{nk}\mu_k)}\right) = \exp(-8H(\delta_{nk})).$$

Then it follows from (5) that

$$\sum_{i=1}^N R_{ni}^k \le N \exp(-8H(\delta_{nk})) \le N\delta_{nk} \le N(\delta_{nk} H(\delta_{nk}))^{1/2}. \tag{12}$$

Condition (2) implies that $\int_0^{\delta_{nk_n}} f(\varepsilon)\, d\varepsilon \to 0$ as $n \to \infty$. In turn, the latter relation implies that

$$\sum_{k \le k_n} N(\delta_{nk} H(\delta_{nk}))^{1/2} \to 0 \quad \text{as } n \to \infty. \tag{13}$$

Indeed, put

$$q_{nk} = \left(n^d \mu_k\right)^{1/2}/4 = N\left(H(\delta_{nk})/\delta_{nk}\right)^{1/2}$$

and note that $q_{nk} \le f(\delta_{nk})$ and $q_{n,k+1} = \beta^{1/2} q_{nk}$. Hence

$$\begin{aligned} \sum_{k \le k_n} N(\delta_{nk} H(\delta_{nk}))^{1/2} &\le q_{nk_n} f^{-1}(q_{nk_n}) + \left(1 - \beta^{1/2}\right)^{-1} \sum_{k < k_n} (q_{nk} - q_{n,k+1}) f^{-1}(q_{nk}) \\ &\le q_{nk_n} f^{-1}(q_{nk_n}) + \left(1 - \beta^{1/2}\right)^{-1} \int_{q_{nk_n}}^\infty f^{-1}(x)\, dx \\ &\le \left(1 - \beta^{1/2}\right)^{-1} \int_0^{\delta_{nk_n}} f(\varepsilon)\, d\varepsilon \to 0, \qquad n \to \infty, \end{aligned}$$

whence $\sum_{k \le k_n} \theta_k g(\delta_{nk_n})(\delta_{nk} H(\delta_{nk}))^{1/2} \to 0$ as $n \to \infty$. Therefore

$$\sum_{k \le k_n} \sum_{i=1}^N R_{ni}^k \to 0 \quad \text{and} \quad \sum_{k \le k_n} \theta_k \lambda_{nk} \to 0 \quad \text{as } n \to \infty. \tag{14}$$

Step 7. Now we prove the relation

$$\sum_{i=1}^N \mathsf{P}(F_{ni}) \to 0 \quad \text{as } n \to \infty, \tag{15}$$

which, in view of (8), follows from $n^d\beta_X\left(m_n-1,1,p_n^d\right)\to 0$ as $n\to\infty$. The latter relation is equivalent to

$$n^d\beta_X\left(g^{1/d}(\delta_{nk_n})-1,1,n^d/g(\delta_{nk_n})\right)\to 0\quad\text{as } n\to\infty. \tag{16}$$

According to (6)

$$n^d=\left(\frac{n^d}{H(\delta_{nk_n})N^2}\right)^\tau\cdot\frac{H^\tau(\delta_{nk_n})N^{2\tau}}{n^{d(\tau-1)}}=\frac{16^\tau}{(\mu_{k_n})^\tau n^{d(\tau-1)}}\cdot f^{2\tau}(\delta_{nk_n})$$

where $\tau=s/(s-2)$ is defined in the statement of Theorem 1. The construction of μ_{k_n} and (4) imply that

$$\lim_{n\to\infty}(\mu_{k_n})^{-\tau}n^{-d(\tau-1)}\le\lim_{n\to\infty}n^d b_n^\tau/(M\delta)^\tau=0.$$

It is clear that the latter relation holds for b_n specified above. Then the relation

$$\lim_{n\to\infty}f^{2\tau}(\delta_{nk_n})\cdot\beta_X\left(g^{1/d}(\delta_{nk_n})-1,1,f^{2\tau}(\delta_{nk_n})/g(\delta_{nk_n})\right)<\infty$$

implies (16).

Taking (9), (14), and (15) into account, we see that $\mathsf{P}\big(\|Z_n^{(2)}\|_{\mathbb{S}_\delta}>M\big)\to 0$ as $n\to\infty$.

Step 8. Now it remains to consider the processes $Z_n^{(1)}$ defined by (7). When considering these processes we may face a problem that some of the approximating sets $D_{nk}^{(1)}(A)$ are too close together, and this may not allow us to obtain a suitable Gaussian approximation. To avoid this problem we apply the following idea. Let $\mathbb{S}_{nk}$ be the maximal subset of $\mathbb{S}_\delta$ such that $|C_1\triangle C_2|\ge 2\delta_{nk}$ for all $C_1\ne C_2$. Then for any $A\in\mathbb{S}_\delta$ there is an element $C_{nk}(A)$ of $\mathbb{S}_{nk}$ such that $|C_{nk}(A)\triangle A|<2\delta_{nk}$. Thus

$$\left|C_{nk}(A)\triangle D_{nk}^{(1)}(A)\right|<4\delta_{nk}. \tag{17}$$

Put

$$Z_n^{(3)}(A)=\sum_{k\le k_n}\theta_k\nu_{nk}(C_{nk}(A)),\qquad Z_n^{(4)}(A)=\sum_{k\le k_n}\theta_k\left\{\nu_{nk}\left(D_{nk}^{(1)}(A)\right)-\nu_{nk}(C_{nk}(A))\right\}.$$

Then $Z_n^{(1)}=Z_n^{(3)}(A)+Z_n^{(4)}(A)$.

Proceeding in the same way as in the case of the process $Z_n^{(2)}$ and substituting $D_{nk}^{(1)}(A)\triangle C_{nk}(A)$ instead of E in (10) and in (11) we obtain from (17) that

$$\mathsf{P}\left(\|Z_n^{(4)}\|_{\mathbb{S}_\delta}>M\right)\to 0\quad\text{as } n\to\infty.$$

Step 9. Recall that if a field $(X_j)_j$ is symmetric, then its distribution coincides with that of the field $(\varepsilon_jX_j)_j$ where $(\varepsilon_j)_j$ is a Rademacher field that does not depend on X. Without loss of generality one can thus consider the field $\widetilde{Z}_n^{(3)}(A)=\sum_j\varepsilon_jS_{nj}(A)$ instead of $Z_n^{(3)}(A)$ where $S_{nj}(A)=\sum_{k\le k_n}n^{-d/2}b_{nj}(C_{nk}(A))X_j\,\mathbb{1}_{\{Y_j\in J_k\}}$. We have

$$\mathsf{P}\left(\left\|Z_n^{(3)}\right\|_{\mathbb{S}_\delta}>M\right)=\mathsf{P}\left(\left\|\widetilde{Z}_n^{(3)}\right\|_{\mathbb{S}_\delta}>M\right)\le 1/M\,\mathsf{E}\left\|\widetilde{Z}_n^{(3)}\right\|_{\mathbb{S}_\delta}. \tag{18}$$

Step 10. Now we need

Lemma 2 ([3])**.** *Let $\{f_j,j\in T\}$ be a finite set of real functions defined on the set D and let $\{v_j,j\in T\}$ be a family of nonnegative random variables such that $\mathsf{E}(v_j)=1$. Further* *let $\{\varepsilon_j, j \in T\}$ be a family of random variables that do not depend on $\{v_j, j \in T\}$. Then*

$$\mathsf{E}\Big\|\sum_{j\in T}\varepsilon_j f_j\Big\|_D \le \mathsf{E}\Big\|\sum_{j\in T}\varepsilon_j v_j f_j\Big\|_D$$

provided the norms are measurable.

Let $\{g_j, j \in \mathbf{Z}^d\}$ be independent standard Gaussian random variables that do not depend on $\{\varepsilon_j, j \in \mathbf{Z}^d\}$. Since the fields X, ε, and $g = \{g_j, j \in \mathbb{Z}^d\}$ are independent, we may assume that these fields are defined on different probability spaces with measures P_X, P_ε, and P_g, respectively. Denote by E_X, E_ε, and E_g the partial integration with respect to P_X, P_ε, and P_g, respectively. In particular, $\mathsf{E}(\cdot) = \mathsf{E}_X \mathsf{E}_\varepsilon \mathsf{E}_g(\cdot)$. Put $\mu = 1/\mathsf{E}|g_0|$. Since $\varepsilon_j|g_j| = g_j$ in law, we set $v_j = \mu|g_j|$ in Lemma 2 and obtain

$$\mathsf{E}_\varepsilon \left\|\widetilde{Z}_n^{(3)}\right\|_{\mathbb{S}_\delta} \le \mu\, \mathsf{E}_g \left\|Z_n^g\right\|_{\mathbb{S}_\delta} \tag{19}$$

where $Z_n^g(A) = \sum_j g_j S_{nj}(A)$. Note that all the norms are measurable, since the upper bounds are considered with respect to the finite set $\bigcup_{k\le k_n}\mathbb{S}_{nk}$.

The process Z_n^g is Gaussian with respect to the measure P_g. We compare the Gaussian process Z_n^g with another Gaussian process constructed from the Brownian motion Z. Namely let $G^{(k)}, k \ge 1$, be independent copies of the process Z. Let

$$G_n(A) = \sum_{k\le k_n} 2\theta_k G^{(k)}(C_{nk}(A)), \tag{20}$$

$$Q_{nk}(A) = \left(n^d\mu_k\right)^{-1}\sum_j b_{nj}(A)\,\mathbb{1}_{\{Y_j\in J_k\}}, \qquad \triangle_{nk} = \{E\triangle F\colon E \ne F \in \mathbb{S}_{nk}\}.$$

Then for $A, B \in \mathbb{S}_\delta$ we have $\mathsf{E}_g(Z_n^g(A) - Z_n^g(B))^2 = \sum_j (S_{nj}(A) - S_{nj}(B))^2$ and this does not exceed

$$\begin{aligned} n^{-d}&\sum_{k\le k_n}\sum_j\big(b_{nj}(C_{nk}(A)) - b_{nj}(C_{nk}(B))\big)^2 a_{k+1}^2\,\mathbb{1}_{\{Y_j\in J_k\}}\\ &\le \sum_{k\le k_n}\theta_k^2 Q_{nk}(C_{nk}(A)\triangle C_{nk}(B)).\end{aligned}$$

On the other hand

$$\mathsf{E}(G_n(A) - G_n(B))^2 = \sum_{k\le k_n} 4\theta_k^2|C_{nk}(A)\triangle C_{nk}(B)|.$$

Therefore

$$\mathsf{E}_g\left(Z_n^g(A) - Z_n^g(B)\right)^2 \le \mathsf{E}(G_n(A) - G_n(B))^2$$

on the event $D_n = \{Q_{nk}(A) \le 4|A| \text{ for all } A \in \triangle_{nk} \text{ and } k \le k_n\}$.

Step 11. Below we need the following result.

Lemma 3 ([10])**.** *Let $\{Y_i(t), t \in D\}$, $i = 1, 2$, be centered Gaussian processes indexed by a countable set D such that*

$$0 \in \{Y_1(t,\omega)\colon t \in D, \omega \in \Omega\} \quad \textit{almost surely.}$$

Assume that

$$\mathsf{E}(Y_1(t) - Y_1(s))^2 \le \mathsf{E}(Y_2(t) - Y_2(s))^2$$

for all $s, t \in D$. Then $\mathsf{E}\|Y_1\|_D \le 2\,\mathsf{E}\|Y_2\|_D$.

The condition $0 \in \{Y_1(t,\omega)\colon t \in D, \omega \in \Omega\}$ almost surely holds if D is a class of sets containing the empty set. Note that this is the case for our consideration (recall that $\varnothing \in \mathbb{S}_{nk}$).

By Lemma 3 we have

$$\mathsf{E}_g \, \|Z_n^g\|_{\mathbb{S}_\delta} \le 2\, \mathsf{E}\, \|G_n\|_{\mathbb{S}_\delta} \tag{21}$$

on the event D_n. It is convenient to introduce the following notation: $Z_{nk}(A) = \sum_{l=k}^{k_n} 2\theta_l G^{(l)}(A)$ and

$$W_{nk}(A) = Z_{nk}(C_{nk}(A)) - Z_{nk}(C_{n,k-1}(A)), \qquad v = 2\biggl(\sum_{l\ge 0} \theta_l^2\biggr)^{1/2}.$$

Changing the order of summation in (20) we get

$$\begin{aligned} G_n(A) &= \sum_{k\le k_n} 2\theta_k G^{(k)}(C_{n0}(A)) + \sum_{k\le k_n} \sum_{l=1}^{k} 2\theta_k \left\{ G^{(k)}(C_{nl}(A)) - G^{(k)}(C_{n,l-1}(A)) \right\} \\ &= Z_{n0}(C_{n0}(A)) + \sum_{k=1}^{k_n} W_{nk}(A). \end{aligned} \tag{22}$$

Lemma 4 ([9])**.** *There exists an universal constant K such that*

$$\mathsf{E}\, \|Z\|_{\mathbb{S}_\delta} \le K \int_0^\delta \left(\varepsilon^{-1} \log N_I(\varepsilon, \mathbb{S}_\delta, d_L)\right)^{1/2} \, d\varepsilon + K\delta.$$

Then $\mathsf{E}\, \|Z_{n0}\|_{\mathbb{S}_\delta} \le v\, \mathsf{E}\, \|Z\|_{\mathbb{S}_\delta} \to 0$ as $\delta \to 0$. We see that $W_{nk}(A)$ is a Gaussian random variable, and (17) implies that $\mathsf{E}\, W_{nk}^2(A) \le 4v^2\delta_{nk}$. Thus

$$\begin{aligned} \mathsf{P}\left(\|W_{nk}\|_{\mathbb{S}_\delta} > t\left(4v^2\delta_{nk}\right)^{1/2} \right) &\le \sharp(\mathbb{S}_{nk})\sharp(\mathbb{S}_{n,k-1})(2/\sqrt{2\pi}) t^{-1} \exp\left(-t^2/2\right) \\ &\le \exp\left(-\left(t^2 - 8H(\delta_{n,k-1})\right)/2\right) \\ &\le \exp\left(-\left(t^2 - 8\beta^{-1} H(\delta_{nk})\right)/2\right) \le \exp\left(-t^2/4\right) \end{aligned}$$

for all $t \ge (16\beta^{-1}H(\delta_{nk}))^{1/2}$ if n is sufficiently large. Hence

$$\mathsf{E}\, \|W_{nk}\|_{\mathbb{S}_\delta} \le \left(4v^2\delta_{nk}\right)^{1/2} \left((16\beta^{-1}H(\delta_{nk}))^{1/2} + 4 \right).$$

Metric entropy condition (2) implies that $\sum_{1\le k\le k_n} \mathsf{E}\, \|W_{nk}\|_{\mathbb{S}_\delta} \to 0$ as $n\to\infty$. Combining (18)–(22) we see that

$$\mathsf{P}\left(\left\| Z_n^{(3)} \right\|_{\mathbb{S}_\delta} > M \right) \le \mathsf{P}\left(D_n^c\right) + o(1).$$

Step 12. To complete the proof of Theorem 1 it is sufficient to show that

$$\mathsf{P}(D_n^c) \to 0 \quad \text{as } n\to\infty. \tag{23}$$

As before

$$Q_{nk}(A) = n^{-d/2}\mu_k^{-1} \sum_{i=1}^{N} \sum_{l\in L_{p_n}} V_{nli}^k(A) + |A|.$$

Taking the above reasoning into account we obtain from the definition of the event D_n^c that

$$\begin{aligned} \mathsf{P}(D_n^c) &= \mathsf{P}(\text{there exist } k\le k_n \text{ and } A \in \triangle_{nk} \colon Q_{nk}(A) > 4|A|) \\ &\le \mathsf{P}\biggl(\bigcup_{i=1}^{N} F_{ni} \biggr) + \sum_{i=1}^{N} \sum_{k\le k_n} \hat{\mathrm{R}}_{ni}^k \end{aligned} \tag{24}$$

where

$$\hat{\mathrm{R}}_{ni}^{k} = \sharp(\triangle_{nk}) \sup_{A\in\triangle_{nk}} \mathsf{P}\left(n^{-d/2}(\mu_k)^{-1}\left| \sum_{l\in L_{p_n}} \widetilde{V}_{nli}^{k}(A)\right| > 3|A|/N \right).$$

As we have already shown before, conditions (i) and (ii) imply that the first term in (24) approaches 0 as $n\to\infty$. We estimate $\hat{\mathrm{R}}_{ni}^{k}$ in the second term by

$$\hat{\mathrm{R}}_{ni}^{k} \le \left(\sharp\left(\mathbb{D}_{nk}^{(1)}\right)\right)^2 \sup_{A\in\triangle_{nk}} \mathsf{P}\left(\left| \sum_{l\in L_{p_n}} \widetilde{V}_{nli}^{k}(A)\right| > 3|A|/Nn^{d/2}\mu_k \right).$$

Applying the Bernstein inequality once more and the estimate $|A|\ge 2\delta_{nk}$ for all $A\in\triangle_{nk}$ we get

$$\begin{aligned}\hat{\mathrm{R}}_{ni}^{k} &\le \sup_{A\in\triangle_{nk}} \exp\left(4H(\delta_{nk}) - \frac{9|A|^2/N^2n^d\mu_k^2}{2\left(\mu_k|A| + 1/3n^{-d/2}3|A|/Nn^{d/2}\mu_k\right)}\right)\\ &\le \exp\left(-68H(\delta_{nk})\right).\end{aligned}$$

The latter inequality holds, since (6) implies that $\mu_k n^d|A|/N \ge 32H(\delta_{nk})N$. Taking (12) and (13) into account, we get

$$\sum_{i=1}^{N}\sum_{k\le k_n}\hat{\mathrm{R}}_{ni}^{k} \to 0 \quad \text{as } n\to\infty. \tag{25}$$

Therefore (24) and (25) imply (23).

Theorem 1 is proved. □

4. Weak convergence

We use a result of the paper [7] to prove the convergence of finite-dimensional distributions of the processes Z_n.

We call a Borel set A regular if the Lebesgue measure of its boundary is zero.

Theorem 2. *Let $\mathfrak{A}$ be a family of regular sets of $[0,1]^d$ satisfying metric entropy condition* (2). *Let X be a symmetric strictly stationary field and let assumption* (i) *of Theorem 1 be satisfied. If*

(ii′) $\limsup_{\varepsilon\downarrow 0} f^{2\tau}(\varepsilon)\cdot\beta_X\left(g^{1/d}(\varepsilon),1,\infty\right) < \infty$,

then the invariance principle holds.

Proof. The tightness of the family of distributions of $\{Z_n\}_{n\in\mathbf{N}}$ follows from Theorem 1. To prove the convergence of finite-dimensional distributions of the fields $\{Z_n(B), B\in\mathfrak{A}\}$ to those of the field $\{Z(B), B\in\mathfrak{A}\}$ it is sufficient (see [7]) to check that

$$\sum_{n\in\mathbf{N}} n^{d-1}\alpha_X^{(s-2)/s}(n,1,\infty) < \infty. \tag{26}$$

The latter condition follows from (ii′). □

Acknowledgement. The author is grateful to Professor A. V. Bulinskiĭ for the setting of the problem and help in writing this paper.

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Received 27/FEB/2003

Translated by V. SEMENOV