The Annals of Statistics
1974, Vol. 2, No. 1, 49-62

THE INVARIANCE PRINCIPLE FOR ONE-SAMPLE RANK-ORDER STATISTICS[1]

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Analogous to the Donsker theorem on partial cumulative sums of independent random variables, for a broad class of one-sample rank order statistics, weak convergence to Brownian motion processes is studied here. A simple proof of the asymptotic normality of these statistics for random sample sizes is also presented. Some asymptotic results on renewal theory for one-sample rank order statistics are derived.

1. Introduction and the main theorem. Let $\{X_1, X_2, \cdots\}$ be a sequence of independent and identically distributed random variables (i.i.d. rv) having a continuous distribution function $F(x)$, $x \in R$, the real line $(-\infty, \infty)$. Let $c(u)$ be equal to 1 or 0 according as u is $\geqq$ or <0, and for every $n \geqq 1$, let

$$R_{ni} = \sum_{j=1}^{n} c(|X_i| - |X_j|)\,, \qquad 1 \leqq i \leqq n\,. \tag{1.1}$$

Consider then the usual one-sample rank order statistic

$$T_n = \sum_{i=1}^{n} c(X_i) J_n(R_{ni}/(n+1))\,, \qquad n \geqq 1\,, \tag{1.2}$$

where the *rank-scores* $J_n(i/(n+1))$ are defined in either of the following two ways:

$$\text{(a)}\quad J_n\left(\frac{i}{n+1}\right) = EJ(U_{ni}) \quad \text{or} \tag{1.3}$$

$$\text{(b)}\quad J_n\left(\frac{i}{n+1}\right) = J\left(\frac{i}{n+1}\right) = J(EU_{ni})\,, \qquad 1 \leqq i \leqq n\,,$$

$U_{n1} \leqq \cdots \leqq U_{nn}$ are the ordered random variables of a sample of size n from the rectangular (0, 1) distribution, and the *score-function* $J(u)$ is specified by

$$J(u) = J_{(1)}(u) - J_{(2)}(u) \quad \text{where} \quad J_{(i)}(u) \quad \text{is} \uparrow \text{in} \quad u; \qquad 0 < u < 1,\ i = 1, 2\,, \tag{1.4}$$

$J(u)$ is absolutely continuous inside (0, 1), and

$$\int_0^1 \{|J_{(1)}(u)| + |J_{(2)}(u)|\}\{u(1-u)\}^{-\frac{1}{2}}\, du < \infty\,. \tag{1.5}$$

The condition (1.5), due to Hoeffding (1973), is slightly more restrictive than

Received July 1971.

[1] Work sponsored by the Aerospace Research Laboratories, Air Force Systems Command, Contract F33615-71-C-1927. Reproduction in whole or in part permitted for any purpose of the U. S. Government.

AMS 1970 *subject classification*. 60B10, 62G99.

Key words and phrases. Brownian motion processes, invariance principles, martingales, one-sample rank order statistics, random sample sizes, renewal theory, weak convergence.

the square integrability condition of Hájek (1968), and it implies that

$$A_i^2 = \int_0^1 J_{(i)}^2(u)\,du < \infty\,, \qquad i = 1, 2\,. \tag{1.6}$$

Two well-known cases are (i) the Wilcoxon signed rank statistic for which $J(u) = u$: $0 \leqq u \leqq 1$, and (ii) the normal scores statistics, for which $J(u)$ is the inverse of the chi distribution with one degree of freedom.

Let us define $H(x) = F(x) - F(-x)$, $x \geqq 0$, and let

$$\mu_J(F) = \int_0^\infty J(H(x))\,dF(x)\,, \qquad \mu_J = \int_0^1 J(u)\,du\,, \qquad \text{and} \qquad A^2(J) = \int_0^1 J^2(u)\,du\,. \tag{1.7}$$

Note that $|\mu_J(F)| \leqq \int_0^1 |J(u)|\,du \leqq A(J) < \infty$. Asymptotic normality of $n^{-\frac{1}{2}}(T_n - n\mu_J(F))$ was studied by Govindarajulu (1960), Pyke and Shorack (1968), Puri and Sen (1969), Sen (1970), and Hušková (1970), among others. In the present paper, we are primarily interested in the classical invariance principle or weak convergence to Brownian motion processes for $\{T_n\}$.

For every $n \geqq 1$, let

$$Y_n(0) = 0\,, \qquad Y_n\left(\frac{k}{n}\right) = [T_k - k\mu_J(F)]/(\sigma n^{\frac{1}{2}})\,, \qquad k = 1, \cdots, n\,, \tag{1.8}$$

where σ^2 is defined by (3.10), and it is assumed that $0 < \sigma^2 < \infty$. Consider then a stochastic process $Y_n = \{Y_n(t): t \in I\}$, $I = \{t: 0 \leqq t \leqq 1\}$, where for $t \in [k/n, (k+1)/n]$, we let

$$Y_n(t) = Y_n(k/n) + (nt - k)[Y_n((k+1)/n) - Y_n(k/n)]\,, \qquad k = 0, \cdots, n-1\,. \tag{1.9}$$

Then Y_n belongs to the space $C[0, 1]$ of all continuous functions, on I, with which we associate the uniform topology defined by the metric

$$\rho(Y_n, Y_n^*) = \sup_{t \in I} \{|Y_n(t) - Y_n^*(t)| : Y_n, Y_n^* \in C\}\,. \tag{1.10}$$

Finally, let $W = \{W_t: t \in I\}$ be a standard Brownian motion, so that

$$EW_t = 0 \qquad \text{and} \qquad E(W_s W_t) = \min(s, t); \qquad s, t \in I\,. \tag{1.11}$$

Then, the main theorem of the paper is the following.

THEOREM 1. *If σ^2, defined by* (3.10) *is positive and finite, and* (1.3)–(1.5) *hold, then Y_n converges (as $n \to \infty$) in distribution in the uniform topology on* $C[0, 1]$ *to a standard Brownian motion W.*

We may remark that, in particular, the Wilcoxon signed rank statistic can be expressed as a von Mises' (1947) functional, and hence, the result follows from Miller and Sen (1972) who considered a similar theorem for Hoeffding's (1948) U-Statistics and von Mises' functionals. But, in general, this characterization is not possible for T_n, and hence, a different proof is needed. Our method of approach is based on a powerful polynomial approximation of $J(u)$ by Hájek (1968), a subsequent follow up by Hoeffding (1973), a martingale theorem on

$\{T_n\}$, and a recent functional central limit theorem for martingales by Brown (1971). The martingale theorem is considered in Section 2. The theorem for $J(u)$ having a bounded second derivative is proved in Section 3, while the general case is treated in the next section. The concluding section is devoted to a few applications having importance in the developing area of rank based sequential inference procedures.

2. A martingale theorem. For every $n \geqq 1$, define the vectors

$$\mathbf{c}_n = (c(X_1), \cdots, c(X_n))' \quad \text{and} \quad \mathbf{R}_n = (R_{n1}, \cdots, R_{nn})' \tag{2.1}$$

of signs and ranks defined by (1.1). Note that for every F, the distribution of T_n is solely determined by the joint distribution of $(\mathbf{c}_n, \mathbf{R}_n)$. Let $\mathscr{F}_n$ be the σ-field generated by $(\mathbf{c}_n, \mathbf{R}_n)$, $n \geqq 1$, so that $\mathscr{F}_n$ is $\uparrow$ in n. Also, let $a_1 = E(T_1) = J_1(\frac{1}{2})P\{X_1 > 0\}$, and for $n \geqq 2$, let

$$a_n = \sum_{r=1}^{n} J_n\left(\frac{r}{n+1}\right) E\{F(X^*_{n-1,r}) - F(X^*_{n-1,r-1})\}\,, \tag{2.2}$$

where $X^*_{n-1,0} = 0$, $X^*_{n-1,n} = \infty$ and $X^*_{n-1,1} \leqq \cdots \leqq X^*_{n-1,n-1}$ are the ordered values of $|X_1|, \cdots, |X_{n-1}|$, $n \geqq 2$. Since $P\{X_n \in [X^*_{n-1,r-1}, X^*_{n-1,r}]\} = h_{n,r} = E\{F(X^*_{n-1,r}) - F(X^*_{n-1,r-1})\}$, we have

$$h_{n,r} = \binom{n-1}{r-1} \int_0^\infty [H(x)]^{r-1}[1 - H(x)]^{n-r}\, dF(x)\,, \qquad r = 1, \cdots, n\,, \tag{2.3}$$

So that

$$a_n = \sum_{r=1}^{n} J_n(r/(n+1))h_{n,r}\,, \qquad n \geqq 1\,. \tag{2.4}$$

For later use, we note that

$$h_{n,r} = n^{-1}E\{[dF(X^*_{n,r})/dH(X^*_{n,r})]\}; \qquad 0 \leqq h_{n,r} \leqq n^{-1}\,, \tag{2.5}$$

for all $r = 1, \cdots, n$. Finally, we define

$$T_n^* = T_n - a_n^*; \qquad a_n^* = \sum_{k=1}^{n} a_k\,. \tag{2.6}$$

Then, we have the following theorem which extends Theorem 4.5 of Sen and Ghosh (1971) to underlying df's $\{F\}$, not necessarily symmetric about zero.

THEOREM 2.1. *If* $\int_0^1 |J(u)|\, du < \infty$ *and the scores are defined by* (a) *in* (1.3), *then* $\{T_n^*, \mathscr{F}_n;\, n \geqq 1\}$ *is a martingale.*

PROOF. By (1.2) and (2.6), for every $n \geqq 2$,

$$\begin{aligned} E\{T_n^* \mid \mathscr{F}_{n-1}\} = {} & \sum_{i=1}^{n-1} c(X_i)E\{J_n(R_{ni}/(n+1)) \mid \mathscr{F}_{n-1}\} - a^*_{n-1} \\ & + E\{c(X_n)J_n(R_{nn}/(n+1)) \mid \mathscr{F}_{n-1}\} - a_n\,, \end{aligned} \tag{2.7}$$

as $\mathbf{c}_{n-1}$ is held fixed under $\mathscr{F}_{n-1}$. Also, given $\mathbf{R}_{n-1}$, R_{ni} can assume the two values R_{n-1i} and $(R_{n-1i} + 1)$ with respective conditional probabilities, $(1 - n^{-1}R_{n-1i})$ and $n^{-1}R_{n-1i}$, $1 \leqq i \leqq n-1$. Hence,

$$\begin{aligned} & E\{J_n((n+1)^{-1}R_{ni}) \mid \mathscr{F}_{n-1}\} \\ & = (n^{-1}R_{n-1i})J_n\left(\frac{R_{n-1i}+1}{n+1}\right) + (1 - n^{-1}R_{n-1i})J_n\left(\frac{R_{n-1i}}{n+1}\right) \\ & = J_{n-1}(n^{-1}R_{n-1i})\,, \qquad 1 \leqq i \leqq n-1\,, \end{aligned} \tag{2.8}$$

where the last equation follows from a well-known recursion relation among the expected order statistics (cf. [15] page 198). Thus, by (2.7) and (2.8),

$$(2.9) \qquad E\{T_n^* \mid \mathscr{F}_{n-1}\} = T_{n-1}^* + \{E[c(X_n)J_n((n+1)^{-1}R_{nn}) \mid \mathscr{F}_{n-1}] - a_n\} .$$

Now $c(X_n)J_n((n+1)^{-1}R_{nn})$ is either 0 (when $X_n < 0$), or equal to $J_n(r/(n+1))$, when $X_n \in [X_{n-1,r-1}^*, X_{n-1,r}^*]$, $r = 1, \cdots, n$. Thus, by (2.2) and (2.4),

$$(2.10) \qquad E\{c(X_n)J_n((n+1)^{-1}R_{nn}) \mid \mathscr{F}_{n-1}\} = \sum_{r=1}^n J_n\left(\frac{r}{n+1}\right) h_{n,r} = a_n .$$

Hence, from (2.9) and (2.10),

$$(2.11) \qquad E(T_n^* \mid \mathscr{F}_{n-1}) = T_{n-1}^* , \qquad n \geqq 2 .$$

Also, $E(T_1^*) = 0$. Hence the theorem follows,

Corollary 2.1. *$E(T_n) = a_n^*$ for all $n \geqq 1$.*

Proof. The result follows directly by noting that $E(T_n^*) = 0$, $\forall n \geqq 1$, and that a_n, $n \geqq 1$, are all non-stochastic constants.

Remark. If, in particular, $F(x) + F(-x) = 1$, $\forall x \geqq 0$, $dF(x) = \frac{1}{2} dH(x)$, then $h_{n,r} = (2n)^{-1}$, and hence, $a_n = \{\sum_{i=1}^n J_n(i/(n+1))\}/2n = \frac{1}{2}\int_0^1 J(u)\, du = \frac{1}{2}\mu_J$. Thus, $\{T_n - (n/2)\mu_J, \mathscr{F}_n; n \geqq 1\}$ forms a martingale; this was already observed by Sen and Ghosh (1971).

3. Proof of the theorem when J'' is bounded inside I. We define $\{T_n^*\}$ as in (2.6) with the scores defined by (a) in (1.3), and let $T_0^* = 0$. Let us then define two processes $\xi_n = \{\xi_n(t) : t \in I\}$ and $\xi_n^* = \{\xi_n^*(t) : t \in I\}$ by

$$(3.1) \qquad \xi_n(t) = \nu_n^{-1}\{T_k^* + (T_{k+1}^* - T_k^*)(t\nu_n^2 - \nu_k^2)/(\nu_{k+1}^2 - \nu_k^2)\} , \qquad (t \in I) ,$$

for $\nu_k^2 \leqq t\nu_n^2 \leqq \nu_{k+1}^2$, $k = 0, \cdots, n$, where

$$(3.2) \qquad \nu_k^2 = E\{T_k^*\}^2 , \qquad k \geqq 0 , \quad \text{so that} \quad \nu_0^2 = 0 ;$$

$$(3.3) \qquad \xi_n^*(t) = \{T_k^* + (nt - k)(T_{k+1}^* - T_k^*)\}/\{\sigma n^{\frac{1}{2}}\} \qquad \text{for} \quad t \in \left(\frac{k}{n}, \frac{k+1}{n}\right),$$

$k = 0, 1, \cdots, n-1$, where σ^2 is defined by (3.10). We shall approximate Y_n by ξ_n^* and subsequently by ξ_n, and the theorem will be proved for ξ_n. For this purpose, we let $V_k = T_k^* - T_{k-1}^*$, $k \geqq 1$, and define

$$(3.4) \qquad q_1^2 = EV_1^2 , \qquad q_k^2 = E\{V_k^2 \mid \mathscr{F}_{k-1}\} , \qquad k \geqq 2 ;$$

$$(3.5) \qquad Q_n = \sum_{k=1}^n q_k^2 , \qquad n \geqq 1 .$$

Since $\{T_k^*, \mathscr{F}_k; k \geqq 1\}$ has been shown in Theorem 3.1 to be a martingale, by Theorem 3 of Brown (1971), we obtain that

$$(3.6) \qquad \xi_n \to_{\mathscr{D}} W \qquad \text{as} \quad n \to \infty ,$$

provided that

$$(3.7) \qquad Q_n^2/\nu_n^2 \to_P 1 \qquad \text{as} \quad n \to \infty ,$$

and for every $\varepsilon > 0$,

$$\nu_n^{-2}\{\textstyle\sum_{i=1}^n E[V_i^2 I(|V_i| > \varepsilon\nu_n)]\} \to 0 \quad \text{as} \quad n \to \infty\,, \tag{3.8}$$

where $I(A)$ stands for the indicator function of the set A, and $\to_{\mathscr{D}}$ for convergence in distribution.

Now, by (2.6) and Corollary 2.1, for every $n \geqq 1$,

$$\nu_n^2 = E(T_n^*)^2 = \operatorname{Var}(T_n)\,. \tag{3.9}$$

Let us also define

$$\begin{aligned}\sigma^2 = {} & \textstyle\int_0^\infty J^2(H(x))\,dF(x) - (\int_0^\infty J(H(x))\,dF(x))^2 \\ & + 2\Big[\iint_{0<x<y<\infty} H(x)[1 - H(y)]J'(H(x))J'(H(y))\,dF(x)\,dF(y) \\ & - \iint_{0<x<y<\infty} H(x)J'(H(x))J(H(y))\,dF(x)\,dF(y) \\ & + \iint_{0<x<y<\infty} J(H(x))[1 - H(y)]J'(H(y))\,dF(x)\,dF(y)\Big]\,.\end{aligned} \tag{3.10}$$

Then by Lemma 2 of Hušková (1970), it follows that

$$[0 < \sigma < \infty] \Rightarrow [\operatorname{Var}(T_n)/n\sigma^2] \to 1 \quad \text{as} \quad n \to \infty\,, \tag{3.11}$$

and hence, from (3.9) through (3.11), we obtain that

$$[0 < \sigma < \infty] \Rightarrow \nu_n^2/(n\sigma^2) \to 1 \quad \text{as} \quad n \to \infty\,. \tag{3.12}$$

Thus, in the proof of (3.8), we may replace ν_n by $\sigma n^{\frac{1}{2}}$. Now, by (2.6)

$$|V_n| \leqq \sum_{i=1}^{n-1}\left|J_n\left(\frac{R_{ni}}{n+1}\right) - J_{n-1}\left(\frac{R_{n-1i}}{n}\right)\right| + \left|J_n\left(\frac{R_{nn}}{n+1}\right)\right| + |a_n|\,, \tag{3.13}$$

where by (2.4) and (2.5), $|a_n| < A(J)$ for all $n \geqq 1$. Also, by (1.4),

$$\left|J_n\left(\frac{R_{nn}}{n+1}\right)\right| \leqq \sum_{s=1}^2 \left\{\max_{1\leqq i\leqq n}\left|J_{(s)n}\left(\frac{i}{n+1}\right)\right|\right\}, \tag{3.14}$$

where $J_{(s)n}(i/(n+1)) = EJ_{(s)}(U_{ni})$, $1 \leqq i \leqq n$, $s = 1, 2$. Further, noting that for $1 \leqq i \leqq n-1$, $R_{n-1i} \leqq R_{ni} \leqq R_{n-1i} + 1$, and by (1.4), the $J_{(s)}$ are monotonic, we obtain by using (2.8) that

$$\begin{aligned}& \sum_{i=1}^{n-1}\left|J_n\left(\frac{R_{ni}}{n+1}\right) - J_{n-1}\left(\frac{R_{n-1i}}{n}\right)\right| \\ & \leqq \sum_{s=1}^2\left(\sum_{i=1}^{n-1}\left|J_{(s)n}\left(\frac{i+1}{n+1}\right) - J_{(s)n}\left(\frac{i}{n+1}\right)\right|\right) \\ & = \left|J_{(1)n}\left(\frac{n}{n+1}\right) - J_{(1)n}\left(\frac{1}{n+1}\right)\right| + \left|J_{(2)n}\left(\frac{n}{n+1}\right) - J_{(2)n}\left(\frac{1}{n+1}\right)\right|.\end{aligned} \tag{3.15}$$

Now, (1.5) insures that $[J_{(s)}(u)(1-u)^{\frac{1}{2}}] \to 0$ as $u \to 1$, and hence,

$$J_{(s)}\left(\frac{n}{n+1}\right) = o(n^{\frac{1}{2}})\,, \qquad \text{for} \quad s = 1, 2\,. \tag{3.16}$$

Also, noting that J'' is bounded inside I, we have

$$\begin{aligned}\left|J_{(s)n}\left(\frac{n}{n+1}\right)\right| &= |EJ_{(s)}(U_{nn})| \\ &= |J_{(s)}(EU_{nn}) + E(U_{nn} - EU_{nn})J'_{(s)}(EU_{nn}) \\ &\quad + \tfrac{1}{2}E(U_{nn} - EU_{nn})^2K| \quad (\text{where } |K| < \infty) \\ &= \left|J_{(s)}\left(\frac{n}{n+1}\right)\right| + O(n^{-2}), \qquad s = 1, 2,\end{aligned} \tag{3.17}$$

as $E(U_{nn} - EU_{nn})^2 = n/(n+1)^2(n+2) = O(n^{-2})$. Thus, by (3.16) and (3.17),

$$\left|J_{(s)n}\left(\frac{n}{n+1}\right)\right| = o(n^{\frac{1}{2}}), \qquad s = 1, 2, \tag{3.18}$$

and a similar case holds for $|J_{(s)n}(1/(n+1))|$, $s = 1, 2$. Hence, from (3.13) through (3.18), we conclude that

$$|V_n| = o(n^{\frac{1}{2}}), \tag{3.19}$$

that is, for every $\varepsilon > 0$ and $0 < \sigma < \infty$, there exists an $n_0(\varepsilon, \sigma)$, such that

$$|V_n| < \varepsilon\sigma n^{\frac{1}{2}} \qquad \text{for all} \quad n \geqq n_0(\varepsilon, \sigma). \tag{3.20}$$

On the other hand, for every $k \geqq 1$,

$$\begin{aligned}\nu_n^{-2} \textstyle\sum_{i=1}^{k} E\{V_i^2 I(|V_i| > \varepsilon\nu_n)\} &\leqq \nu_n^{-2} \textstyle\sum_{i=1}^{k} E(V_i^2) \\ &= \nu_k^2/\nu_n^2 \to 0 \qquad \text{as} \quad n \to \infty.\end{aligned} \tag{3.21}$$

Hence, (3.8) readily follows from (3.12), (3.20) and (3.21).

We now proceed to the proof of (3.7). By (2.6), we have

$$\begin{aligned}q_n^2 &= \textstyle\sum_{i=1}^{n-1} c^2(X_i) E\left\{\left[J_n\left(\frac{R_{ni}}{n+1}\right) - J_{n-1}\left(\frac{R_{n-1i}}{n}\right)\right]^2 \middle| \mathscr{F}_{n-1}\right\} \\ &\quad + \textstyle\sum_{i\neq j=1}^{n-1} c(X_i)c(X_j) E\left\{\left[J_n\left(\frac{R_{ni}}{n+1}\right) - J_{n-1}\left(\frac{R_{n-1i}}{n}\right)\right]\right. \\ &\quad \left.\times \left[J_n\left(\frac{R_{nj}}{n+1}\right) - J_{n-1}\left(\frac{R_{n-1i}}{n}\right)\right] \middle| \mathscr{F}_{n-1}\right\} \\ &\quad + \left[E\left\{\left[c(X_n)J_n\left(\frac{R_{nn}}{n+1}\right)\right]^2 \middle| \mathscr{F}_{n-1}\right\} - a_n^2\right] \\ &\quad + 2 \textstyle\sum_{i=1}^{n-1} c(X_i) E\left\{\left[c(X_n)J_n\left(\frac{R_{nn}}{n+1}\right)\right]\right. \\ &\quad \left.\times \left[J_n\left(\frac{R_{ni}}{n+1}\right) - J_{n-1}\left(\frac{R_{n-1i}}{n}\right)\right] \middle| \mathscr{F}_{n-1}\right\}.\end{aligned} \tag{3.22}$$

Proceeding as in the proof of Theorem 2.1, the first term on the right-hand side (rhs) of (3.22) can be written as

$$\begin{aligned}
&\textstyle\sum_{i=1}^{n-1} c(X_i)\{R_{n-1i}(n - R_{n-1i})/n^2\}\left\{J_n\left(\frac{R_{n-1i}+1}{n+1}\right) - J_n\left(\frac{R_{n-1i}}{n+1}\right)\right\}^2 \\
(3.23)\qquad &\leqq \textstyle\sum_{i=1}^{n-1} \{i(n-i)/n^2\}\left\{J_n\left(\frac{i+1}{n+1}\right) - J_n\left(\frac{i}{n+1}\right)\right\}^2 \\
&\leqq \left\{\max_{1\leqq i\leqq n-1}[i^{\frac12}(n-i)^{\frac12}/n]\left|J_n\left(\frac{i+1}{n+1}\right) - J_n\left(\frac{i}{n+1}\right)\right|\right\} \\
&\quad\times\left\{\textstyle\sum_{i=1}^{n-1}\frac{i^{\frac12}(n-i)^{\frac12}}{n}\left|J_n\left(\frac{i+1}{n+1}\right) - J_n\left(\frac{i}{n+1}\right)\right|\right\}.
\end{aligned}$$

Let us now denote by

$$(3.24)\qquad K_1 = \sup_{0<t<1}|J'(u)| \qquad\text{and}\qquad K_2 = \sup_{0<t<1}|J''(u)|\,.$$

By our assumption, both K_1 and K_2 are finite, positive constants (depending only on J). Then for every i: $1 \leqq i \leqq n-1$

$$\begin{aligned}
(3.25)\qquad \left|J_n\left(\frac{i+1}{n+1}\right) - J_n\left(\frac{i}{n+1}\right)\right| &= \left|J\left(\frac{i+1}{n+1}\right) - J\left(\frac{i}{n+1}\right)\right| + O(n^{-1}) \\
&\quad(\leqq K_1(n+1)^{-1} + O(n^{-1}))\,.
\end{aligned}$$

Consequently, the first factor on the right-hand side of (3.23) is $O(n^{-1})$, while the second factor is bounded by $CK_1 \int_0^1 [x(1-x)]^{\frac12}\,dx < \infty$, where $C < \infty$. Thus, (3.23) converges to zero as $n \to \infty$, with probability one.

Let us now define for every $n \geqq 1$,

$$\begin{aligned}
(3.26)\qquad F_n{}^*(x) &= \frac{1}{n+1}\textstyle\sum_{i=1}^{n} c(x - X_i) \quad (-\infty < x < \infty)\,, \\
H_n{}^*(x) &= F_n{}^*(x) - F_n{}^*(-x-)\,, \quad x \geqq 0\,.
\end{aligned}$$

As $n/(n+1) \to 1$ with $n \to \infty$, by the Glivenko–Cantelli theorem, as $n \to \infty$

$$(3.27)\qquad \sup_{-\infty<x<\infty}|F_n{}^*(x) - F(x)| \to 0\,, \qquad \sup_{x\geqq 0}|H_n{}^*(x) - H(x)| \to 0\,,$$

almost surely (a.s.). For the second term on the rhs of (3.22), we note that for $R_{n-1i} < R_{n-1j}$, given $\mathscr{F}_{n-1}$, (R_{ni}, R_{nj}) can assume the values (R_{n-1i}, R_{n-1j}), $(R_{n-1i}, R_{n-1j}+1)$ and $(R_{n-1i}+1, R_{n-1j}+1)$ with respective conditional probabilities $(n - R_{n-1j})/n$, $(R_{n-1j} - R_{n-1i})/n$ and R_{n-1i}/n. A similar case follows for $R_{n-1i} > R_{n-1j}$. Hence, by some simple steps, the second term on the rhs of (3.22) can be expressed in the integral form as

$$\begin{aligned}
&(2n^2/(n+1)^2)\iint_{0<x<y<\infty} H_{n-1}^*(x)[1 - H_{n-1}^*(y)] \\
(3.28)\qquad &\times\left\{(n+1)\left[J_n\left(\frac{nH_{n-1}^*(x)+1}{n+1}\right) - J_n\left(\frac{nH_{n-1}^*(x)}{n+1}\right)\right]\right\} \\
&\times\left\{(n+1)\left[J_n\left(\frac{nH_{n-1}^*(y)+1}{n+1}\right)\right.\right. \\
&\qquad\left.\left. - J_n\left(\frac{nH_{n-1}^*(y)}{n+1}\right)\right]\right\} dF_{n-1}^*(x)\,dF_{n-1}^*(y)\,.
\end{aligned}$$

Now, by (3.24) and (3.25), the integrand in (3.28) is bounded (in absolute value), for all $0 < x < y < \infty$, by $\frac{1}{4}(K_1 + \frac{1}{2}K_2)^2$, and it converges a.s. (by (3.27)) to $H(x)[1 - H(y)]J'(H(x))J'(H(y))$ as $n \to \infty$. Consequently, we may write (3.28) as

$$(3.29) \qquad 2 \iint_{0<x<y<\infty} H(x)[1 - H(y)]J'(H(x))J'(H(y))\, dF^*_{n-1}(x)\, dF^*_{n-1}(y) + o(1)\,, \quad \text{a.s.},$$

as $n \to \infty$. Since, by (3.24), the integrand in (3.29) is bounded (in absolute value) by $\frac{1}{4}K_1^2$ for all $0 < x < y < \infty$, and (3.27) holds, (3.29) converges a.s. to

$$(3.30) \qquad 2 \iint_{0<x<y<\infty} H(x)[1 - H(y)]J'(H(x))J'(H(y))\, dF(x)\, dF(y)\,, \qquad \text{as } n \to \infty\,.$$

Proceeding as in the proof of Theorem 2.1, the third term on the rhs of (3.22) can be shown to be equal to

$$(3.31) \qquad \textstyle\sum_{r=1}^{n} J_n^2(r/(n+1))h_{n,r} - a_n^2\,.$$

Now, by the same method of proof as in Lemma 2 of Hušková (1970), it follows that under (3.24), as $n \to \infty$,

$$(3.32) \qquad |a_n - \mu_J(F)| \to 0\,,$$
$$\left|\textstyle\sum_{r=1}^{n} J_n^2\left(\frac{r}{n+1}\right) h_{n,r} - \int_0^\infty J^2(H(x))\, dF(x)\right| \to 0\,.$$

Consequently, (3.31) converges (as $n \to \infty$) to

$$(3.33) \qquad \textstyle\int_0^\infty J^2(H(x)\, dF(x) - [\int_0^\infty J(H(x))\, dF(x)]^2\,.$$

Finally, noting that under $\mathscr{F}_{n-1}$, $c(X_n) = 0$ with probability $F(0)$, and $c(X_n) = 1$ with (R_{nn}, R_{ni}) being either $(r, R_{n-1i} + 1)$, $1 \leqq r \leqq R_{n-1i}$ or (r, R_{n-1i}), $R_{n-1i} < r \leqq n$, with respective conditional probability $h_{n,r}$, $1 \leqq r \leqq n$, the last term on the rhs of (3.22) can be shown to be equal to

$$\begin{aligned}
&2 \textstyle\sum_{i=1}^{n-1} c(X_i) \left\{\sum_{j=1}^{R_{n-1i}} J_n\left(\frac{r}{n+1}\right)\right\} \left\{J_n\left(\frac{R_{n-1i}+1}{n+1}\right) - J_{n-1}\left(\frac{R_{n-i}}{n}\right)\right\} h_{n,r} \\
&\quad + 2 \textstyle\sum_{i=1}^{n-1} c(X_i) \left\{\sum_{j=R_{n-1i}+1}^{n} J_n\left(\frac{r}{n+1}\right)\right\} \left\{J_n\left(\frac{R_{n-1i}}{n+1}\right)\right. \\
(3.34) \qquad &\quad \left. - J_{n-1}\left(\frac{R_{n-1i}}{n}\right)\right\} h_{n,r} = 2 \textstyle\sum_{i=1}^{n-1} c(X_i) \left\{\left[\sum_{j=1}^{R_{n-1i}} J_n\left(\frac{r}{n+1}\right) h_{n,r}\right]\right. \\
&\quad \times \left(\frac{n - R_{n-1i}}{n}\right)\left.\left[J_n\left(\frac{R_{n-1i}+1}{n+1}\right) - J_n\left(\frac{R_{n-1i}}{n+1}\right)\right]\right\} \\
&\quad - 2 \textstyle\sum_{i=1}^{n-1} c(X_i) \left\{\left[\sum_{j-R_{n-1i}}^{n} J_n\left(\frac{r}{n+1}\right) h_{n,r}\right] \frac{R_{n-1i}}{n}\right. \\
&\quad \times \left.\left[J_n\left(\frac{R_{n-1i}+1}{n+1}\right) - J_n\left(\frac{R_{n-1i}}{n+1}\right)\right]\right\}
\end{aligned}$$

by using (2.8). Again writing the above in the integral form [on using (3.24)–

3.26)], it follows by using (3.27) that as $n \to \infty$, it converges a.s. to

$$(3.35)\qquad 2 \iint_{0<x<y<\infty} J(H(x))[1 - H(y)]J'(H(y))\, dF(x)\, dF(y) - 2 \iint_{0<x<y<\infty} H(x)J'(H(x))J(H(y))\, dF(x)\, dF(y)\,.$$

Thus, from (3.10), (3.22), (3.23), (3.30), (3.33) and (3.35), it follows that

$$(3.36)\qquad [0 < \sigma < \infty] \Rightarrow q_n^2/\sigma^2 \to 1 \quad \text{a.s.,} \qquad \text{as } n \to \infty\,.$$

Consequently, by (3.5), (3.12) and (3.7),

$$(3.37)\qquad Q_n^2/(n\sigma^2) \to_P 1 \qquad \text{as } n \to \infty\,,$$

which proves (3.7), and hence, (3.6) holds.

Now, by the tightness property of ξ_n (cf. Billingsley (1968), page 56), for every $\varepsilon > 0$ and $\eta > 0$ there exist a $\delta > 0$ and an $n_0 = n_0(\varepsilon, \eta)$, such that

$$(3.38)\qquad P\{\sup_{|t-s|<\delta} |\xi_n(t) - \xi_n(s)| > \varepsilon\} < \eta \qquad \text{for} \quad n > n_0\,.$$

Hence, by (3.1), (3.3), (3.12) and (3.38), we obtain that

$$(3.39)\qquad \rho(\xi_n, \xi_n^*) \to_P 0 \qquad \text{as} \quad n \to \infty\,,$$

so that by (3.6) and (3.39),

$$(3.40)\qquad \xi_n^* \to_{\mathscr{D}} W \qquad \text{as} \quad n \to \infty\,.$$

Let us now define another process in $C[0, 1]$ by

$$(3.41)\qquad \tilde{\xi}_n^* = \{\tilde{\xi}_n^*(t) : t \in I\}\,, \qquad \tilde{\xi}_n^*(t) = (\nu_n/n\sigma^2)\xi_n^*(t)\,, \qquad t \in I\,.$$

Then, by (3.12), (3.40) and a well-known theorem in Cramér (1946, page 254), we obtain that

$$(3.42)\qquad \tilde{\xi}_n^* \to_{\mathscr{D}} W \qquad \text{as} \quad n \to \infty\,.$$

Finally, by (1.9), (3.3), (3.41) and Corollary 2.1,

$$(3.43)\qquad \begin{aligned} \rho(Y_n, \tilde{\xi}_n^*) &= \sup_{t\in I} |Y_n(t) - \tilde{\xi}_n^*(t)| \\ &\leqq \{\max_{1\leqq k\leqq n} |a_k^* - k\mu_J(F)|/\sigma n^{\frac{1}{2}}\} \\ &= \{\max_{1\leqq k\leqq n} |ET_k - k\mu_J(F)|/\sigma n^{\frac{1}{2}}\} \\ &\leqq Mn^{-\frac{1}{2}}\,, \qquad \text{where} \quad M(<\infty) \quad \text{depends only on} \quad J, \end{aligned}$$

and the last inequality follows from Lemma 2 of Hušková (1970). Hence the proof is completed for scores defined by (a) in (1.3). If the scores are defined by (b) in (1.3), we note that

$$(3.44)\qquad \begin{aligned} & n^{-\frac{1}{2}}|\{\textstyle\sum_{i=1}^k c(X_i)J(R_{ki}/(k+1)) - \sum_{i=1}^k c(X_i)EJ(U_{kR_{ki}})\}| \\ &\quad \leqq n^{-\frac{1}{2}} \textstyle\sum_{i=1}^k |J(i/(k+1)) - EJ(U_{ki})| \\ &\quad = n^{-\frac{1}{2}} \textstyle\sum_{i=1}^k \Big| J\Big(\frac{i}{k+1}\Big) - J(EU_{ki}) - E(U_{ki} - EU_{ki})J'(EU_{ki}) \\ &\qquad\qquad - \tfrac{1}{2}E\{(U_{ki} - EU_{ki})^2 J''(\theta U_{ki} + (1-\theta)EU_{ki})\}\Big| \quad (0 < \theta < 1) \\ &\quad \leqq \tfrac{1}{2} n^{-\frac{1}{2}} K_2 k(k+1)^{-1} \qquad \text{for all} \quad k \leqq n\,, \qquad \text{by (3.24)}\,. \end{aligned}$$

Hence, the metric $\rho(Y_n, Y_n^*)$, defined by (1.10) for the two processes with the T_k defined by (a) and (b) in (1.3) is $O(n^{-\frac{1}{4}})$, and thereby tends to zero as $n \to \infty$. The proof of the theorem for bounded J'' is thus complete.

4. The proof for the general case. We now use the Hájek (1968) polynomial approximation of $J(u)$, as further studied by Hoeffding (1973). By Lemma 1 of Hoeffding (1973), under (1.4) and (1.5), for every $\alpha > 0$, there exists a decomposition

$$J(u) = \psi(u) + \phi^{(1)}(u) - \phi^{(2)}(u)\,, \qquad 0 < u < 1\,, \tag{4.1}$$

such that ψ is a polynomial, $\phi^{(1)}$ and $\phi^{(2)}$ are non-decreasing, and

$$\int_0^1 \{|\phi^{(1)}(u)| + |\phi^{(2)}(u)|\}\{u(1-u)\}^{-\frac{1}{2}}\, du < \alpha\,; \tag{4.2}$$

the last inequality, in turn, implies that

$$\int_0^1 [\{\phi^{(1)}(u)\}^2 + \{\phi^{(2)}(u)\}^2]\, du < \alpha\,. \tag{4.3}$$

In (1.2), on replacing J by ψ, $\phi^{(1)}$ and $\phi^{(2)}$, we define $T_n(\psi)$, $T_n^{(1)}$ and $T_n^{(2)}$, respectively, so that

$$T_n = T_n(\psi) + T_n^{(1)} - T_n^{(2)}\,, \qquad n \geqq 1\,; \tag{4.4}$$

the corresponding processes, defined by (1.8) and (1.9), are denoted by $Y_n(\psi)$, $Y_n^{(1)}$ and $Y_n^{(2)}$, so that

$$Y_n = Y_n(\psi) + Y_n^{(1)} - Y_n^{(2)}\,. \tag{4.5}$$

Note that in (4.5), all the processes have $(\sigma n^{\frac{1}{2}})$, in the denominator, defined in (1.8).

Now from Hájek (1968) and Hušková (1970), along with (3.12), it follows that for every $\varepsilon > 0$, there exists an $\alpha > 0$, such that (4.2) holds, and

$$|1 - \{\mathrm{Var}\,(T_n(\psi))\}^{\frac{1}{2}}/(\sigma n^{\frac{1}{2}})| < \varepsilon \qquad \text{for} \quad n \geqq n_0(\varepsilon)\,. \tag{4.6}$$

Since ψ is a polynomial and (4.6) holds, by the results of Section 3,

$$Y_n(\psi) \to_{\mathscr{D}} W\,, \qquad \text{as } n \to \infty\,. \tag{4.7}$$

Consequently, it suffices to show that for every $\varepsilon > 0$ and $\eta > 0$, there exists a choice of $\alpha > 0$ in (4.2), such that with probability $> 1 - \eta$,

$$\sup_{t \in I}\{|Y_n^{(1)}(t)| + |Y_n^{(2)}(t)|\} < \varepsilon \qquad \text{for} \quad n \geqq n_0(\varepsilon, \eta)\,. \tag{4.8}$$

For each $i(=1, 2)$, we define $a_{n(i)}$ as in (2.4) (for $J = \phi^{(i)}$), and let

$$T_n^{(i)*} = T_n^{(i)} - a_{n(i)}^*\,; \qquad a_{n(i)}^* = \sum_{k=1}^n a_{k(i)}\,, \qquad n \geqq 1\,. \tag{4.9}$$

Then

$$\sup_{t \in I}|Y_n^{(i)}(t)| \leqq (\sigma n^{\frac{1}{2}})^{-1}\{\max_{1 \leqq j \leqq n}|T_j^{(i)*}| + \max_{1 \leqq j \leqq n}|a_{j(i)}^* - j\mu_J^{(i)}(F)|\}\,, \tag{4.10}$$

where

$$\mu_J^{(i)}(F) = \int_0^\infty \phi^{(i)}(H(x))\, dF(x)\,, \qquad i = 1, 2\,. \tag{4.11}$$

Let us first consider the case where the scores are defined by (a) in (1.3). Then, by Theorem 2.1, $\{T_k^{(i)*}, \mathscr{F}_k; k \geqq 1\}$ is a martingale, and hence, by the Kolmogorov inequality for martingales (viz., Feller (1965) page 235), for every $\eta > 0$,

$$(4.12) \qquad P\{\max_{1\leqq k\leqq n} |T_k^{*(i)}| > \eta\sigma n^{\frac{1}{2}}\} \leqq \operatorname{Var}(T_n^{(i)*})/(n\eta^2\sigma^2) = [\operatorname{Var}(T_n^{(i)})]/(n\eta^2\sigma^2), \quad \text{as } ET_n^{(i)*} = 0 .$$

Also, by Theorem 4 of Hušková (1970), for each $i(=1, 2)$

$$(4.13) \qquad \operatorname{Var}(T_n^{(i)}) \leqq 10 \sum_{k=1}^n \left[\phi_n^{(i)}\left(\frac{k}{n+1}\right)\right]^2 \leqq 10n \int_0^1 \{\phi^{(i)}(u)\}^2 \, du .$$

Consequently, by (4.3) and (4.13), (4.12) can be made arbitrarily small by proper choice of $\alpha(>0)$.

Now, under (1.5), we obtain on using our Corollary 2.1 and proceeding as in Proposition 2 of Hoeffding (1973) that

$$(4.14) \qquad |a_{k(i)}^* - k\mu_J^{(i)}(F)| = |ET_k^{(i)*} - k\mu_J^{(i)}(F)| \leqq Ck^{\frac{1}{2}}\alpha , \qquad k \geqq 1 , C < \infty ,$$

where C does not depend on α or the $\phi^{(i)}$. Consequently,

$$(4.15) \qquad \max_{1\leqq k\leqq n} |a_{k(i)}^* - k\mu_J^{(i)}(F)|/(\sigma n^{\frac{1}{2}}) \leqq (C/\sigma)\alpha , \qquad i = 1, 2 ,$$

and (4.15) can be made adequately small by proper choice of $\alpha(>0)$ in (4.2). Thus (4.8) holds, and the proof is completed for scores defined by (a) in (1.3).

We complete the proof of the theorem by considering the scores defined by (b) in (1.3). Since in (4.1), ϕ is the polynomial component on which the results of Section 3 apply, all we need to show is that on defining

$$(4.16) \qquad T_n^{(i)} = \sum_{i=1}^n c(X_i)\phi^{(i)}\left(\frac{R_{ni}}{n+1}\right), \qquad \tilde{T}_n^{(i)} = \sum_{i=1}^n c(X_i)E\phi^{(i)}(U_{nR_{ni}}) , \qquad n \geqq 1$$

$(i = 1, 2)$, that for every $\varepsilon > 0$ and $\eta > 0$, there exists an $\alpha(>0)$ in (4.2), such that

$$(4.17) \qquad P\{\max_{1\leqq k\leqq n} |T_k^{(i)} - \tilde{T}_k^{(i)}|/(\sigma n^{\frac{1}{2}}) > \varepsilon\} < \eta , \qquad i = 1, 2 ,$$

for all $n \geqq n_0(\varepsilon, \eta)$. Now, by our (4.16) and Proposition 1 of Hoeffding (1973),

$$(4.18) \qquad |T_k^{(i)} - \tilde{T}_k^{(i)}| \leqq \sum_{i=1}^k |\phi^{(i)}(i/(k+1)) - E\phi^{(i)}(U_{ki})| \leqq C_1 k^{\frac{1}{2}} \int_0^1 |\phi^{(i)}(u)|\{u(1-u)\}^{-\frac{1}{2}} \, du , \quad i = 1, 2 , C_1 < \infty ,$$

where C_1 does not depend on $\phi^{(i)}$. Hence, (4.17) readily follows from (4.18) and (4.2) by proper choice of $\alpha > 0$, and the proof is terminated.

REMARK. If F is symmetric about 0, $F(x) + F(-x) = 1$, $\forall x \geqq 0$, so that $a_{k(i)} = \mu_J^{(i)}(F) = \frac{1}{2}\int_0^1 \phi^{(i)}(u) \, du$, for $i = 1, 2$, $k \geqq 1$. Thus, we do not require (4.14) and (4.15), and hence, for scores defined by (a) in (1.3), the square integrability condition (1.6) and (1.4) suffice our purpose. However, in general, for arbitrary F or for scores defined by (b) in (1.3), (1.5) may not be replaced by (1.6). For certain stronger results for F symmetric about 0, we may refer to Sen and Ghosh (1973).

5. Weak convergence for random sample sizes. For every $t > 0$, consider now a positive integer-valued random variable N_t, and for $N_t = n(\geqq 1)$ define T_n and Y_n as in (1.2), (1.8) and (1.9). If N_t satisfies the condition

$$\lim_{t\to\infty} (N_t/t) = \theta\,, \qquad \text{in probability,} \tag{5.1}$$

where θ is a positive random variable defined on the same probability space, then analogous to Theorem 1, we have under (1.3)–(1.5)

$$Y_{N_t} \to_{\mathscr{D}} W\,, \qquad \text{as } t \to \infty\,. \tag{5.2}$$

The proof is quite similar to Theorem 17.2 of Billingsley (1968, page 146) and follows as a corollary to our main theorem in Section 1. For brevity, the details are omitted. In particular, we have from (5.2) that

$$\mathscr{L}(N_t^{-\frac{1}{2}}(T_{N_t} - N_t\mu_J(F))/\sigma) \to \mathscr{N}(0, 1) \qquad \text{as } t \to \infty\,; \tag{5.3}$$

for a different proof of (5.3) under slightly different regularity conditions, we may refer to Pyke and Shorack (1968).

We conclude this section with the following problems. As in Section 2, define $T_n^* = T_n - a_n^*$, $n \geqq 1$. For every $\tau > 0$, define a positive number K_τ, such that

$$\lim_{\tau\to\infty} \tau^{-\frac{1}{2}} K_\tau = K^*: \qquad 0 < K^* < \infty\,. \tag{5.4}$$

Let then

$$T_\tau^+ = \inf\{n: T_n^* \geqq K_\tau\}\,, \qquad \tau > 0\,, \tag{5.5}$$

i.e., T_τ^+ is the first time (n), T_n^* exceeds or reaches K_τ. We want to find an expression for

$$P\{T_\tau^+ \leqq t_\tau\} \qquad \text{for} \quad t_\tau > 0\,. \tag{5.6}$$

Note that on denoting by $[s]$ the greatest integer contained in s,

$$\begin{aligned} P\{T_\tau^+ \leqq t_\tau\} &= P\{T_n^* \geqq K_\tau \ \text{ for at least one } n: \ 1 \leqq n \leqq [t_\tau]\} \\ &= P\{\max_{1\leqq n\leqq [t_\tau]} T_n^*/\sigma[t_\tau]^{\frac{1}{2}} \geqq K_\tau/\sigma[t_\tau]^{\frac{1}{2}}\}\,, \end{aligned} \tag{5.7}$$

where σ^2 is defined by (3.10). Thus, if $t_\tau = c^2\tau + o(\tau)$, $0 < c < \infty$, we obtain from (5.4), (5.7) and Theorem 1 that

$$\lim_{\tau\to\infty} P\{T_\tau^+ \leqq t_\tau\} = P\{\sup_{u\in I} W_u \geqq (K^*/c\sigma)\} \tag{5.8}$$

when $W = \{W_u: u \in I\}$ is standard Brownian motion on I. Hence, by a well-known result on W, we have from (5.8) that

$$\lim_{\tau\to\infty} P\{T_\tau^+ \leqq t_\tau\} = 2\left\{\frac{1}{(2\pi)^{\frac{1}{2}}} \int_\beta^\infty e^{-\frac{1}{2}u^2}\, du\right\}; \qquad \beta = K^*/c\sigma\,. \tag{5.9}$$

Similarly, on defining $T_\tau^* = \inf\{n: |T_n^*| \geqq K_\tau\}$, $\tau > 0$, we have

$$\text{(5.10)}\qquad \lim_{\tau\to\infty} P\{T_\tau^* \leqq t_\tau\} = \sum_{k=-\infty}^{\infty} (-1)^k \int_{(2k-1)\beta}^{(2k+1)\beta} \frac{1}{(2\pi)^{\frac{1}{2}}} e^{-\frac{1}{2}u^2}\, du\,, \qquad \beta = K^*/c\sigma\,.$$

Theorem 1 also provides a sequential analogue of the fixed sample size rank order test for symmetry. Suppose we want to test the null hypothesis (H_0) that F is symmetric about 0 against the alternative (H_1) that F is symmetric about some positive θ. Instead of basing our test on a fixed sample size (n), we may consider the following sequential procedure which may lead to a termination at an early stage. Continue sampling so long as $T_m - (m/2)\mu_J$, $m \geqq 1$, lies below $C_{n,\alpha}$, $C_{n,\alpha} > 0$. If N is the smallest positive integer ($\leqq n$) for which $T_N - (N/2)\mu_J$ exceeds $C_{n,\alpha}$, we reject H_0 and accept H_1. If $N > n$, the terminal decision is based on T_n where we reject or accept H_0 according as $T_n - (n/2)\mu_J$ is $\geqq$ or $< C_{n,\alpha}$. By virtue of Theorem 1, under H_0, $2n^{-\frac{1}{2}}[\max_{1\leqq k\leqq n}\{(T_k - (k/2)\mu_j)\}]/A(J)$ converges in law to $M = \sup_{0\leqq t\leqq 1}\{W(t)\}$, where the distribution of M is well known [viz., Billingsley (1968) page 79]. Thus, if M_α be the upper 100α % point of the distribution of M, we may approximate $2n^{-\frac{1}{2}}C_{n,\alpha}$ by M_α for large n. The case of two-sided alternatives follows on parallel lines.

For some alternative sequential tests based on one-sample rank order statistics, we may refer to Sen and Ghosh (1973, 1974).

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