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The inverse of a certain block matrix

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A simple formula for the inverse of a block matrix with non-zero blocks in the principal diagonal and the first sub-diagonal only is proved. The matrix had arisen in an investigation of a difference equation.

During an investigation of the general homogeneous linear difference equation

$$\sum_{s=0}^{r} a_s(n)u_{n-s} = 0 \ , \quad n \geq r \ ,$$

with $a_0(n) \neq 0$ for all $n \geq r$, it was found [2, equation (6)] that the solution involved the inverse of a non-singular block lower triangular matrix of the following type

$$A_{(N)} = \begin{bmatrix} A_1 & 0_r & 0_r & \cdots & 0_r & 0_r & 0_{r,s} \\ B_2 & A_2 & 0_r & \cdots & 0_r & 0_r & 0_{r,s} \\ 0_r & B_3 & A_3 & \cdots & 0_r & 0_r & 0_{r,s} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0_r & 0_r & 0_r & \cdots & B_{N-1} & A_{N-1} & 0_{r,s} \\ 0_{s,r} & 0_{s,r} & 0_{s,r} & \cdots & 0_{s,r} & B_N & A_N \end{bmatrix} .$$

Here N is the integral part of n/r, and $0_{p,q}$ denotes the null matrix of dimension p by q with $0_{r,r} \equiv 0_r$; the matrices A_k, B_k have the

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order r for $k = 1, \ldots, N-1$, while the dimensions of A_N and B_N are s by s and s by r respectively with $1 \le s \le r$.

In this note, we prove that the above matrix $A_{(N)}$ has the following inverse:

$$A_{(N)}^{-1} = [L_{ij}] , \quad i, j = 1, \ldots, N , \tag{1}$$

where

$$L_{ii} = A_i^{-1} , \quad i = 1, \ldots, N ,$$

$$L_{ij} = 0_r , \quad i < j \le N-1 , \quad L_{iN} = 0_{r,s} , \quad 1 \le i < N ,$$

and

$$L_{ij} = (-1)^{i+j}\left\{\prod_{k=i}^{j+1} \left(A_k^{-1}B_k\right)\right\}A_j^{-1} , \quad i = 2, \ldots, N , \quad j = 1, \ldots, i-1 .$$

The proof is by induction on N . For $N = 2$, formula (1) takes the form

$$A_{(2)}^{-1} = \begin{bmatrix} A_1^{-1} & 0_{r,s} \\ -A_2^{-1}B_2A_1^{-1} & A_2^{-1} \end{bmatrix} , \tag{2}$$

which is a special case of a well-known result [1, p. 109].

Suppose that (1) holds for a block matrix of order m ; that is, for $N = m$. Then, for a matrix of order $m + 1$, by (2) we have

$$A_{(m+1)}^{-1} = \begin{bmatrix} A_{(m)}^{-1} & 0_{mr,s} \\ -A_{m+1}^{-1}\left[0_{s,r}0_{s,r} \ldots B_{m+1}\right]A_{(m)}^{-1} & A_{m+1}^{-1} \end{bmatrix} .$$

Since

$$-A_{m+1}^{-1}\left[0_{s,r}0_{s,r} \ldots B_{m+1}\right]A_{(m)}^{-1} = -A_{m+1}^{-1}B_{m+1}\left[L_{m1}L_{m2} \ldots L_{mm}\right] ,$$

it is easy to see that, on account of the induction hypothesis, (1) holds for $N = m + 1$. The proof of (1) is thus complete.

References

[1] George F. Hadley, *Linear algebra* (Addison-Wesley, Reading, Massachusetts; London; 1961).

[2] V.N. Singh, "Solution of a general homogeneous linear difference equation", submitted.

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