The Inverse Tangent and Cotangent Functions, their Addition Formulas and their Values on their Branch Cuts

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Abstract

The principal inverse tangent and cotangent functions for complex arguments can be defined as formulas involving principal natural logarithms, but these are not odd on the imaginary axis, which they must be according to their definitions as inverse functions. These formulas are therefore modified in such a way that they become odd on the imaginary axis, by choosing the other branch on the lower branch cut, and the corresponding addition formulas for complex and real arguments are derived. With these addition formulas their values on their branch cuts are determined, confirming these modified formulas. Some new formulas for the (hyperbolic) inverse tangent and cotangent functions for complex arguments and some new addition formulas for these functions for real arguments are derived. Some new formulas for the inverse sine and cosine functions and their connections with the inverse tangent and cotangent functions for complex arguments are provided, and from these some new addition formulas for the inverse sine and cosine functions for real arguments are derived. Some duplication and bisection formulas for the inverse tangent, cotangent, sine and cosine functions are derived.

Keywords: inverse tangent and cotangent functions, addition formulas.
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Contents

1 Definitions and Basic Identities

A complex x can be represented by its absolute value $r = |x|$ and its principal angle with the positive real axis in the complex plane $\phi = \text{Arg}(x)$ where $-\pi < \text{Arg}(x) \leq \pi$:

$$x = re^{i\phi} \tag{1.1}$$

In this paper Arg(x) always means the principal angle here defined. The principal square root of a complex x is then defined by [1, 9]:

$$\sqrt{x} = \sqrt{r}\, e^{i\phi/2} \tag{1.2}$$

A definition which is used in this paper and in an earlier paper [4] is the function sg(x) for complex x [8].

Definition 1.1. *For complex x, let $\sqrt{x}$ be the principal square root of x, then:*

$$\text{sg}(x) = \begin{cases} \frac{\sqrt{x^2}}{x} = \frac{x}{\sqrt{x^2}} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases} \tag{1.3}$$

The result of this definition is:

$$\text{sg}(x) = \begin{cases} 1 & \text{if } \text{Re}(x) > 0 \\ -1 & \text{if } \text{Re}(x) < 0 \\ 1 & \text{if } \text{Re}(x) = 0 \text{ and } \text{Im}(x) \geq 0 \\ -1 & \text{if } \text{Re}(x) = 0 \text{ and } \text{Im}(x) < 0 \end{cases} \tag{1.4}$$

For complex x: $1/\text{sg}(x) = \text{sg}(x)$ and for complex $x \neq 0$: $\text{sg}(-x) = -\text{sg}(x)$.
For real x, $\text{sg}(ix) = \text{sg}(x)$.
For real x the function $\text{sg}(x)$ reduces to:

$$\text{sg}(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases} \tag{1.5}$$

From these identities follows that for complex x: $\text{sg}(x)\sqrt{x^2} = x$, $\text{sg}(x)x = \sqrt{x^2}$, and replacing x with ix: $\text{sg}(ix)\sqrt{-x^2} = ix$ and $\text{sg}(ix)ix = \sqrt{-x^2}$, and for real x, $\text{sg}(x)|x| = x$ and $\text{sg}(x)x = |x|$. For complex x, $\text{sg}(\sqrt{x}) = 1$, and for complex x, $(\sqrt{x})^2 = x$.
Let $f(x)$ and $g(x)$ be complex functions, and let:

$$g(x) = f^2(x) \tag{1.6}$$

Then:

$$\sqrt{g(x)} = \pm f(x) \tag{1.7}$$

Because for complex x: $\sqrt{x^2} = \text{sg}(x)x$, the sign is $\text{sg}(f(x))$:

$$\sqrt{g(x)} = \text{sg}(f(x))f(x) \tag{1.8}$$

For real x, $\text{sg}(x)x = |x|$, so for real $f(x)$ and $g(x)$: $\sqrt{g(x)} = |f(x)|$.
For real x, y [1, 9]:

$$\sqrt{x+iy} = \sqrt{\frac{\sqrt{x^2+y^2}+x}{2}} + i\,\text{sg}(y)\sqrt{\frac{\sqrt{x^2+y^2}-x}{2}} \tag{1.9}$$

For complex x:

$$\sqrt{-x} = i\,\text{sg}(ix)\sqrt{x} \tag{1.10}$$

This identity is proved below.
For complex x and real nonnegative α:

$$\sqrt{\alpha x} = \sqrt{\alpha}\sqrt{x} \tag{1.11}$$

For complex x and real positive α:

$$\text{sg}(\alpha x) = \text{sg}(x) \tag{1.12}$$

Using (1.10) and $\sqrt{x^2} = \text{sg}(x)x$ and $\sqrt{-x^2} = \text{sg}(ix)ix$:
For complex x:

$$\text{sg}(ix^2) = \text{sg}(x)\text{sg}(ix) \tag{1.13}$$

Replacing x with $(1-i)x$:
For complex x:

$$\text{sg}(x^2) = \text{sg}((1+i)x)\text{sg}((1-i)x) \tag{1.14}$$

The Iverson bracket notation [3, 7] is defined.

Definition 1.2. *Let S be a logical expression, then:*

$$[S] = \begin{cases} 1 & \text{if } S \text{ is true} \\ 0 & \text{if } S \text{ is false} \end{cases} \tag{1.15}$$

The function $\operatorname{sg}(x)$ for complex x can be written as:

$$\operatorname{sg}(x) = [\operatorname{Re}(x) > 0] - [\operatorname{Re}(x) < 0] + [\operatorname{Re}(x) = 0]([\operatorname{Im}(x) \geq 0] - [\operatorname{Im}(x) < 0]) \tag{1.16}$$

For complex $x \neq 0$ it is clear that:

$$\operatorname{sg}(\frac{1}{x}) = \operatorname{sg}(x) - 2[\operatorname{Re}(x) = 0]\operatorname{sg}(\operatorname{Im}(x)) \tag{1.17}$$

For the principal angle with the positive real axis in the complex plane $-\pi < \operatorname{Arg}(x) \leq \pi$, where $\operatorname{Arg}(0) = 0$ is defined, the following addition formula holds.
For complex x, y:

$$\operatorname{Arg}(x) + \operatorname{Arg}(y) = \operatorname{Arg}(xy) + \begin{cases} 2\pi & \text{if } \operatorname{Arg}(x) + \operatorname{Arg}(y) > \pi \\ -2\pi & \text{if } \operatorname{Arg}(x) + \operatorname{Arg}(y) \leq -\pi \\ 0 & \text{otherwise} \end{cases} \tag{1.18}$$

The following six identities are evident from the complex plane.
For real x, y:

$$\operatorname{Arg}(x + iy) = \arctan(\frac{y}{x}) + \pi[x < 0]\operatorname{sg}(y) \tag{1.19}$$

where when $x = 0$, for real y:

$$\arctan(\frac{y}{0}) = \frac{\pi}{2}([y > 0] - [y < 0]) \tag{1.20}$$

because $\arctan(\infty) = \pi/2$, $\arctan(-\infty) = -\pi/2$ and $\operatorname{Arg}(0) = 0$.
For complex $x \neq 0$:

$$\operatorname{Arg}(-x) = \operatorname{Arg}(x) + \pi \operatorname{sg}(ix) \tag{1.21}$$

For complex x:

$$\operatorname{Arg}(x^2) = 2\operatorname{Arg}(x) + \pi(\operatorname{sg}(x) - 1)\operatorname{sg}(\operatorname{Im}(x)) \tag{1.22}$$

$$[\operatorname{Arg}(x) \geq 0] = \frac{1}{2}(1 + \operatorname{sg}(\operatorname{Im}(x))) \tag{1.23}$$

$$[\operatorname{Arg}(x) > 0] = \frac{1}{2}(1 - \operatorname{sg}(ix)) \tag{1.24}$$

For complex x and real positive c:

$$\operatorname{Arg}(x) + \operatorname{Arg}(\frac{c}{x}) = 2\pi[\operatorname{Im}(x) = 0][\operatorname{Re}(x) < 0] \tag{1.25}$$

$$\operatorname{Arg}(x) + \operatorname{Arg}(-\frac{c}{x}) = \pi \operatorname{sg}(\operatorname{Im}(x)) \tag{1.26}$$

The principal natural logarithm function $\ln(x)$ for complex x is defined by [1]:

$$\ln(x) = \ln(|x|) + i\mathrm{Arg}(x) \tag{1.27}$$

In this paper $\ln(x)$ always means the principal natural logarithm function here defined. For complex x, y, application of (1.18) to this identity gives:

$$\ln(x) + \ln(y) = \ln(xy) + \begin{cases} 2\pi i & \text{if } \mathrm{Arg}(x) + \mathrm{Arg}(y) > \pi \\ -2\pi i & \text{if } \mathrm{Arg}(x) + \mathrm{Arg}(y) \leq -\pi \\ 0 & \text{otherwise} \end{cases} \tag{1.28}$$

For complex x, application of (1.21) gives:

$$\ln(-x) = \ln(x) + \pi i \,\mathrm{sg}(ix) \tag{1.29}$$

For complex x, y, because $-\pi/2 < \mathrm{Arg}(\sqrt{x}) \leq \pi/2$:

$$\ln(\sqrt{x}) + \ln(\sqrt{y}) = \ln(\sqrt{x}\sqrt{y}) \tag{1.30}$$

and when $x = y$ because $(\sqrt{x})^2 = x$:

$$2\ln(\sqrt{x}) = \ln(x) \tag{1.31}$$

Identity (1.10) can now be proved.

Theorem 1.1. *For complex x:*

$$\sqrt{-x} = i\,\mathrm{sg}(ix)\sqrt{x} \tag{1.32}$$

Proof. Using (1.29) and (1.31):

$$\frac{\sqrt{-x}}{\sqrt{x}} = e^{\ln(\sqrt{-x}) - \ln(\sqrt{x})} = e^{\frac{1}{2}(\ln(-x) - \ln(x))} = e^{\frac{1}{2}\pi i \,\mathrm{sg}(ix)} = i\,\mathrm{sg}(ix) \tag{1.33}$$

□

2 The Inverse Tangent and Cotangent Functions which Must Be Odd on the Imaginary Axis

The principal values of the inverse tangent and cotangent functions can be defined in the complex plane as the following formulas [1], where $\ln(x)$ is the principal natural logarithm function:

$$\arctan(x) = -\frac{i}{2}\ln\left(\frac{1+ix}{1-ix}\right) \tag{2.1}$$

$$\mathrm{arccot}(x) = \arctan\left(\frac{1}{x}\right) = -\frac{i}{2}\ln\left(\frac{ix-1}{ix+1}\right) \tag{2.2}$$

The power series expansion of the $\arctan(x)$ function is odd:

$$\arctan(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} \tag{2.3}$$

which indicates that the principal $\arctan(x)$ function should be odd. The functions $\tan(x)$ and $\cot(x)$ for complex x are defined as [1, 2]:

$$\tan(x) = -i\frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}} \tag{2.4}$$

$$\cot(x) = \frac{1}{\tan(x)} = \tan(\frac{\pi}{2} - x) \tag{2.5}$$

The principal inverse tangent function $\arctan(x)$ for complex x is defined by:

$$\arctan(\tan(x)) = x \tag{2.6}$$

Because both $\tan(x)$ and x are odd functions in the complex plane, from this equation follows that the principal $\arctan(x)$ function must also be odd in the complex plane, and similarly for the principal $\text{arccot}(x)$ function for $x \neq 0$. Replacing x by $-x$ in formulas (2.1) and (2.2) means replacing x by $1/x$ in the principal $\ln(x)$ function. From definition (1.27) and (1.25) it is clear that $\ln(1/x) = -\ln(x)$ except for x on the negative real axis, which is the branch cut of the principal $\ln(x)$ function [1], and where the angle is always π and never $-\pi$. Therefore these formulas are not odd there and must be made odd there explicitly. The following theorem determines for which arguments of the principal $\arctan(x)$ and $\text{arccot}(x)$ functions the arguments of the principal $\ln(x)$ function in (2.1) and (2.2) are on the negative real axis. The branch cuts of the principal $\arctan(x)$ and $\text{arccot}(x)$ functions are defined as in [1, 5] and not to include $\pm i$ which are singular points of these functions.

Theorem 2.1. *The argument x of the principal $\ln(x)$ function in (2.1) and (2.2) is on the negative real axis if and only if the argument x of the principal $\arctan(x)$ or $\text{arccot}(x)$ functions is on their branch cuts [1].*

Proof. For the $\arctan(x)$ function (2.1), let t be the argument of the principal $\ln(x)$ function in (2.1) and let t be real, then the following identity is solved:

$$\frac{1+ix}{1-ix} = t \tag{2.7}$$

which is easily checked to be:

$$x = i\frac{1-t}{1+t} \tag{2.8}$$

which means that x must be on the imaginary axis. Therefore x can be replaced with ix where x is real, and because t must be real and negative, the following identity is solved:

$$\frac{1-x}{1+x} < 0 \tag{2.9}$$

which is fulfilled if and only if $x < -1$ or $x > 1$. When $-1 \leq t < 0$, $x > 1$ is on the upper branch cut, and when $t < -1$, $x < -1$ is on the lower branch cut.
For the $\mathrm{arccot}(x)$ function (2.2), the following identity is solved:

$$\frac{ix - 1}{ix + 1} = t \tag{2.10}$$

which is easily checked to be:

$$x = i\frac{t + 1}{t - 1} \tag{2.11}$$

which means that x must be on the imaginary axis. Therefore x can be replaced with ix where x is real, and because t must be real and negative, the following identity is solved:

$$\frac{x + 1}{x - 1} < 0 \tag{2.12}$$

which is fulfilled if and only if $-1 < x < 1$. When $-1 < t < 0$, $-1 < x < 0$ is on the lower branch cut, and when $t \leq -1$, $0 \leq x < 1$ is on the upper branch cut. □

The principal $\arctan(x)$ and $\mathrm{arccot}(x)$ formulas (2.1) and (2.2) can be made odd on these branch cuts explicitly by defining the following functions that are π on the lower branch cuts of these functions and zero elsewhere, using the Iverson bracket notation definition 1.2.

$$\mathrm{oddtan}(x) = \pi[\mathrm{Re}(x) = 0][\mathrm{Im}(x) < -1] \tag{2.13}$$

$$\mathrm{oddcot}(x) = \pi[\mathrm{Re}(x) = 0][-1 < \mathrm{Im}(x) < 0] \tag{2.14}$$

The following are the formulas for the principal values of the $\arctan(x)$ and $\mathrm{arccot}(x)$ functions that are odd everywhere in the complex plane including on the imaginary axis, where $\ln(x)$ is the principal natural logarithm function.

Definition 2.1. *For complex x:*

$$\arctan(x) = -\frac{i}{2}\ln(\frac{1 + ix}{1 - ix}) - \pi[\mathrm{Re}(x) = 0][\mathrm{Im}(x) < -1] \tag{2.15}$$

$$\mathrm{arccot}(x) = -\frac{i}{2}\ln(\frac{ix - 1}{ix + 1}) - \pi[\mathrm{Re}(x) = 0][-1 < \mathrm{Im}(x) < 0] \tag{2.16}$$

This definition means that on the lower branch cut the other branch is chosen, and this definition will be confirmed by applying the addition formulas for determining the values of these functions on their branch cuts in section 5. In this paper from here this definition of the principal $\arctan(x)$ and $\mathrm{arccot}(x)$ functions is always used.

Theorem 2.2. *For real x, y:*

$$\begin{aligned}\arctan(x + iy) = &\frac{i}{4}\ln(\frac{x^2 + (y + 1)^2}{x^2 + (y - 1)^2}) + \frac{1}{2}\arctan(\frac{2x}{1 - x^2 - y^2}) \\ &+ \frac{\pi}{2}[x^2 + y^2 > 1]\mathrm{sg}(x) - \pi[x = 0][y < -1]\end{aligned} \tag{2.17}$$

$$\begin{aligned}\operatorname{arccot}(x+iy) =&\frac{i}{4}\ln(\frac{x^2+(y-1)^2}{x^2+(y+1)^2})+\frac{1}{2}\arctan(\frac{2x}{x^2+y^2-1})\\&+\frac{\pi}{2}[x^2+y^2<1]\operatorname{sg}(x)-\pi[x=0][-1<y<0]\end{aligned}\tag{2.18}$$

Proof. When evaluating (2.15) with (1.27):
For complex x:

$$\arctan(x)=-\frac{i}{4}\ln(\left|\frac{1+ix}{1-ix}\right|^2)+\frac{1}{2}\operatorname{Arg}(\frac{1+ix}{1-ix})-\pi[\operatorname{Re}(x)=0][\operatorname{Im}(x)<-1]\tag{2.19}$$

In this formula x is replaced by $x+iy$:
For real x, y:

$$\begin{aligned}\frac{|1-y+ix|^2}{|1+y-ix|^2}&=\frac{|(1-y+ix)(1+y+ix)|^2}{|(1+y-ix)(1+y+ix)|^2}=\frac{|(1-y)(1+y)-x^2+2ix|^2}{((1+y)^2+x^2)^2}\\&=\frac{((1-y)(1+y)-x^2)^2+4x^2}{((1+y)^2+x^2)^2}=\frac{(1-y)^2+x^2}{(1+y)^2+x^2}\end{aligned}\tag{2.20}$$

$$\operatorname{Arg}(\frac{1-y+ix}{1+y-ix})=\operatorname{Arg}(\frac{(1-y+ix)(1+y+ix)}{x^2+(1+y)^2})=\operatorname{Arg}(1-x^2-y^2+2ix)\tag{2.21}$$

With application of (1.19) the first identity is proved, and the proof of the second identity is similar. □

When $x=0$ and $y=\pm1$ these formulas yield $\arctan(i)=i\infty$, $\arctan(-i)=-i\infty$, $\operatorname{arccot}(i)=-i\infty$ and $\operatorname{arccot}(-i)=i\infty$.
For real x:

$$\arctan(ix)=\frac{i}{2}\ln(\left|\frac{1+x}{1-x}\right|)+\frac{\pi}{2}([x>1]-[x<-1])\tag{2.22}$$

$$\operatorname{arccot}(ix)=\frac{i}{2}\ln(\left|\frac{x-1}{x+1}\right|)+\frac{\pi}{2}([0\le x<1]-[-1<x<0])\tag{2.23}$$

The sum of the two functions (2.15) and (2.16) is now also odd in the complex plane (except for $x=0$ as mentioned above).

Theorem 2.3. *For complex $x\neq\pm i$:*

$$\operatorname{arccot}(x)+\arctan(x)=\frac{\pi}{2}\operatorname{sg}(x)\tag{2.24}$$

where $\operatorname{sg}(x)$ *is defined by (1.4).*

Proof. Substituting formulas (2.15) and (2.16) and using (1.28) and $\ln(-1)=\pi i$:

$$\begin{aligned}&\operatorname{arccot}(x)+\arctan(x)\\&=-\frac{i}{2}[\ln(\frac{1+ix}{1-ix})+\ln(\frac{ix-1}{ix+1})]-\operatorname{oddtan}(x)-\operatorname{oddcot}(x)\\&=\frac{\pi}{2}-\operatorname{oddtan}(x)-\operatorname{oddcot}(x)+\begin{cases}\pi & \text{if }\operatorname{Arg}(\frac{1+ix}{1-ix})+\operatorname{Arg}(\frac{ix-1}{ix+1})>\pi\\-\pi & \text{if }\operatorname{Arg}(\frac{1+ix}{1-ix})+\operatorname{Arg}(\frac{ix-1}{ix+1})\le-\pi\\0 & \text{otherwise}\end{cases}\end{aligned}\tag{2.25}$$

In the last identity (1.26) with $c = 1$ can be applied, for which the imaginary part of $(1+ix)/(1-ix)$ can be determined by substituting $x = a + bi$:

$$\frac{1+i(a+bi)}{1-i(a+bi)} = \frac{(1-b+ia)(1+b+ia)}{(1+b-ia)(1+b+ia)} = \frac{1-a^2-b^2+2ia}{(1+b)^2+a^2} \tag{2.26}$$

From this follows that $\mathrm{Im}((1+ix)/(1-ix)) \geq 0$ if and only if $\mathrm{Re}(x) \geq 0$, and application of (1.26) with $c = 1$ yields:

$$\mathrm{Arg}(\frac{1+ix}{1-ix}) + \mathrm{Arg}(\frac{ix-1}{ix+1}) = \pi\,\mathrm{sg}(\mathrm{Re}(x)) \tag{2.27}$$

and consequently from (2.25) follows:

$$\mathrm{arccot}(x) + \arctan(x) = \frac{\pi}{2} - \mathrm{oddtan}(x) - \mathrm{oddcot}(x) - \pi[\mathrm{Re}(x) < 0] \tag{2.28}$$

which for $x \neq \pm i$ is exactly $\frac{\pi}{2}\mathrm{sg}(x)$ with $\mathrm{sg}(x)$ defined in (1.4). □

This result is different from [1] eq. 4.4.5 when $\mathrm{Re}(x) = 0$, because as mentioned the function definitions in this reference are (2.1) and (2.2) which are not odd on the imaginary axis.

Theorem 2.4. *For complex x:*

$$\arctan(\frac{1}{x}) = \mathrm{arccot}(x) + \pi[\mathrm{Re}(x) = 0]([-1 < \mathrm{Im}(x) < 0] - [0 < \mathrm{Im}(x) < 1]) \tag{2.29}$$

$$\mathrm{arccot}(\frac{1}{x}) = \arctan(x) + \pi[\mathrm{Re}(x) = 0]([\mathrm{Im}(x) < -1] - [\mathrm{Im}(x) > 1]) \tag{2.30}$$

Proof. Applying the definitions (2.15) and (2.16):

$$\begin{aligned} &\arctan(\frac{1}{x}) - \mathrm{arccot}(x) \\ =& -\frac{i}{2}[\ln(\frac{1+i\frac{1}{x}}{1-i\frac{1}{x}}) - \ln(\frac{ix-1}{ix+1})] + \mathrm{oddcot}(x) - \mathrm{oddtan}(\frac{1}{x}) \\ =& -\frac{i}{2}[\ln(\frac{ix-1}{ix+1}) - \ln(\frac{ix-1}{ix+1})] + \mathrm{oddcot}(x) - \mathrm{oddtan}(\frac{1}{x}) \\ =& \mathrm{oddcot}(x) - \mathrm{oddtan}(\frac{1}{x}) \end{aligned} \tag{2.31}$$

Because the lower branch cuts are on the imaginary axis, the following identity holds:

$$\mathrm{oddtan}(\frac{1}{x}) = \mathrm{oddcot}(-x) \tag{2.32}$$

and the first identity in the theorem is proved. The proof of the second identity is similar. □

Both sides of these identities are odd for $x \neq 0$. For real x, $\operatorname{arccot}(x) = \arctan(1/x)$. The following is a consequence of this theorem, using theorem 2.3.
For complex x:

$$\arctan(x) + \arctan(\frac{1}{x}) = \frac{\pi}{2}\operatorname{sg}(x) + \pi[\operatorname{Re}(x) = 0]([-1 < \operatorname{Im}(x) < 0] - [0 < \operatorname{Im}(x) < 1]) \tag{2.33}$$

$$\operatorname{arccot}(x) + \operatorname{arccot}(\frac{1}{x}) = \frac{\pi}{2}\operatorname{sg}(x) + \pi[\operatorname{Re}(x) = 0]([\operatorname{Im}(x) < -1] - [\operatorname{Im}(x) > 1]) \tag{2.34}$$

Theorem 2.5. *For complex x:*

$$\arctan(x) = -i\ln(\frac{1 + ix + \sqrt{1 + x^2}}{1 - ix + \sqrt{1 + x^2}}) \tag{2.35}$$

which is identical to (2.15), and therefore odd in the complex plane.

Proof. From definition (2.15) and (1.31) follows:

$$\arctan(x) = -i\ln(\sqrt{\frac{1 + ix}{1 - ix}}) - \pi[\operatorname{Re}(x) = 0][\operatorname{Im}(x) < -1] \tag{2.36}$$

Let:

$$F(x) = \frac{1 + ix + \sqrt{1 + x^2}}{1 - ix + \sqrt{1 + x^2}} \tag{2.37}$$

Then:

$$F^2(x) = \frac{2(1 + ix)(1 + \sqrt{1 + x^2})}{2(1 - ix)(1 + \sqrt{1 + x^2})} = \frac{1 + ix}{1 - ix} \tag{2.38}$$

Using (1.8) the result is:

$$\sqrt{\frac{1 + ix}{1 - ix}} = \operatorname{sg}(F(x))F(x) \tag{2.39}$$

Evaluation of $F(x)$ with (1.9) and:

$$\frac{a + ib}{c + id} = \frac{(a + ib)(c - id)}{(c + id)(c - id)} = \frac{ac + bd + i(bc - ad)}{c^2 + d^2} \tag{2.40}$$

yields for real x, y:

$$F(x + iy) = \frac{f(x, y) + ig(x, y)}{h(x, y)} \tag{2.41}$$

where:

$$z(x, y) = \sqrt{(1 + x^2 - y^2)^2 + (2xy)^2} = \sqrt{(x^2 + y^2 - 1)^2 + 4x^2} \tag{2.42}$$

$$f(x, y) = 1 - x^2 - y^2 + z(x, y) + 2\sqrt{(z(x, y) + 1 + x^2 - y^2)/2} \tag{2.43}$$

$$\begin{aligned} g(x, y) = &2x(1 + \sqrt{(z(x, y) + 1 + x^2 - y^2)/2}) \\ &+ 2y\operatorname{sg}(xy)\sqrt{(z(x, y) - 1 - x^2 + y^2)/2} \end{aligned} \tag{2.44}$$

$$
\begin{aligned}
h(x,y) =& x^2+(1+y)^2+z(x,y)+2(1+y)\sqrt{(z(x,y)+1+x^2-y^2)/2} \\
&-2x\,\mathrm{sg}(xy)\sqrt{(z(x,y)-1-x^2+y^2)/2}
\end{aligned}
\tag{2.45}
$$

where $h(x,y) \geq 0$. Because $z(x,y) \geq |x^2+y^2-1|$ and $x+|x| \geq 0$, it is concluded that $f(x,y) \geq 0$, so $\mathrm{sg}(F(x+iy)) = 1$ except when $f(x,y) = 0$ and $g(x,y) < 0$. Furthermore $f(x,y) = 0$ if and only if $x = 0$ and $y^2 \geq 1$. When $x = 0$, $g(x,y) < 0$ if and only if $y < 0$, so $\mathrm{sg}(F(x+iy)) = -1$ if and only if $x = 0$ and $y \leq -1$. Therefore the following results.
For complex x:

$$\sqrt{\frac{1+ix}{1-ix}} = (1-2[\mathrm{Re}(x)=0][\mathrm{Im}(x)\leq -1])F(x) \tag{2.46}$$

With (2.36) and (1.29) the theorem is proved. □

Theorem 2.6. *For complex x:*

$$\mathrm{arccot}(x) = \frac{\pi}{2}\mathrm{sg}(x) - i\ln(\frac{1-ix+\sqrt{1+x^2}}{1+ix+\sqrt{1+x^2}}) \tag{2.47}$$

which for $x \neq 0$ is odd in the complex plane.

Proof. From theorem 2.3 follows:
For complex x:

$$\mathrm{arccot}(x) = \frac{\pi}{2}\mathrm{sg}(x) - \arctan(x) \tag{2.48}$$

Substituting for $\arctan(x)$ the previous theorem, and because $\mathrm{Re}(F(x)) \geq 0$:

$$\ln(F(x)) = -\ln(1/F(x)) \tag{2.49}$$

gives the theorem. □

Theorem 2.7. *For the* $\arctan(x)$ *function in definition 2.1:*
For complex x:

$$\tan(\arctan(x)) = x \tag{2.50}$$

Proof. For the $\tan(x)$ function defined by (2.4), for integer n:

$$\tan(x \pm n\pi) = \tan(x) \tag{2.51}$$

and therefore using the $\arctan(x)$ function as defined in 2.1 and using for complex x: $e^{\ln(x)} = x$:

$$
\begin{aligned}
\tan(\arctan(x)) = \tan(-\frac{i}{2}\ln(\frac{1+ix}{1-ix})) &= -i\frac{e^{\frac{1}{2}\ln(\frac{1+ix}{1-ix})} - e^{-\frac{1}{2}\ln(\frac{1+ix}{1-ix})}}{e^{\frac{1}{2}\ln(\frac{1+ix}{1-ix})} + e^{-\frac{1}{2}\ln(\frac{1+ix}{1-ix})}} \\
= -i\frac{e^{\ln(\frac{1+ix}{1-ix})}-1}{e^{\ln(\frac{1+ix}{1-ix})}+1} &= -i\frac{\frac{1+ix}{1-ix}-1}{\frac{1+ix}{1-ix}+1} \\
= -i\frac{1+ix-1+ix}{1+ix+1-ix} &= -i\frac{2ix}{2} = x
\end{aligned}
\tag{2.52}
$$

□

Theorem 2.8. *For the* arccot(x) *function in definition 2.1:*
For complex x:

$$\tan(\text{arccot}(x)) = \frac{1}{x} \tag{2.53}$$

Proof. For the tan(x) function defined by (2.4), for integer n:

$$\tan(x \pm n\pi) = \tan(x) \tag{2.54}$$

and therefore using the arccot(x) function as defined in 2.1 and using for complex x: $e^{\ln(x)} = x$:

$$\begin{aligned}
\tan(\text{arccot}(x)) = \tan(-\frac{i}{2}\ln(\frac{ix-1}{ix+1})) &= -i\frac{e^{\frac{1}{2}\ln(\frac{ix-1}{ix+1})} - e^{-\frac{1}{2}\ln(\frac{ix-1}{ix+1})}}{e^{\frac{1}{2}\ln(\frac{ix-1}{ix+1})} + e^{-\frac{1}{2}\ln(\frac{ix-1}{ix+1})}} \\
&= -i\frac{e^{\ln(\frac{ix-1}{ix+1})} - 1}{e^{\ln(\frac{ix-1}{ix+1})} + 1} = -i\frac{\frac{ix-1}{ix+1} - 1}{\frac{ix-1}{ix+1} + 1} \\
&= -i\frac{ix-1-ix-1}{ix-1+ix+1} = -i\frac{-2}{2ix} = \frac{1}{x}
\end{aligned} \tag{2.55}$$

□

Theorem 2.9. *For the* arccot(x) *function in definition 2.1:*
For complex x:

$$\cot(\text{arccot}(x)) = x \tag{2.56}$$

Proof. For the $\cot(x) = 1/\tan(x)$ function defined by (2.4), for integer n:

$$\cot(x \pm n\pi) = \cot(x) \tag{2.57}$$

and therefore using the arccot(x) function as defined in 2.1 and using for complex x: $e^{\ln(x)} = x$:

$$\begin{aligned}
\cot(\text{arccot}(x)) = \cot(-\frac{i}{2}\ln(\frac{ix-1}{ix+1})) &= i\frac{e^{\frac{1}{2}\ln(\frac{ix-1}{ix+1})} + e^{-\frac{1}{2}\ln(\frac{ix-1}{ix+1})}}{e^{\frac{1}{2}\ln(\frac{ix-1}{ix+1})} - e^{-\frac{1}{2}\ln(\frac{ix-1}{ix+1})}} \\
&= i\frac{e^{\ln(\frac{ix-1}{ix+1})} + 1}{e^{\ln(\frac{ix-1}{ix+1})} - 1} = i\frac{\frac{ix-1}{ix+1} + 1}{\frac{ix-1}{ix+1} - 1} \\
&= i\frac{ix-1+ix+1}{ix-1-ix-1} = i\frac{2ix}{-2} = x
\end{aligned} \tag{2.58}$$

□

Theorem 2.10. *For the* arctan(x) *function in definition 2.1:*
For complex x:

$$\cot(\arctan(x)) = \frac{1}{x} \tag{2.59}$$

Proof. For the $\cot(x) = 1/\tan(x)$ function defined by (2.4), for integer n:

$$\cot(x \pm n\pi) = \cot(x) \tag{2.60}$$

and therefore using the $\arctan(x)$ function as defined in 2.1 and using for complex x: $e^{\ln(x)} = x$:

$$\begin{aligned}\cot(\arctan(x)) = \cot(-\frac{i}{2}\ln(\frac{1+ix}{1-ix})) &= i\frac{e^{\frac{1}{2}\ln(\frac{1+ix}{1-ix})} + e^{-\frac{1}{2}\ln(\frac{1+ix}{1-ix})}}{e^{\frac{1}{2}\ln(\frac{1+ix}{1-ix})} - e^{-\frac{1}{2}\ln(\frac{1+ix}{1-ix})}} \\ &= i\frac{e^{\ln(\frac{1+ix}{1-ix})}+1}{e^{\ln(\frac{1+ix}{1-ix})}-1} = i\frac{\frac{1+ix}{1-ix}+1}{\frac{1+ix}{1-ix}-1} \\ &= i\frac{1+ix+1-ix}{1+ix-1+ix} = i\frac{2}{2ix} = \frac{1}{x}\end{aligned} \tag{2.61}$$

□

Definition 2.2. *For real x:*

$$S(x) = \frac{1}{2}\text{Arg}(e^{2ix}) = \frac{\pi}{2} - ((\frac{\pi}{2} - x) \bmod \pi) \tag{2.62}$$

This function is periodic: for integer n: $S(x \pm n\pi) = S(x)$.

Theorem 2.11. *For the* $\arctan(x)$ *function in definition 2.1:*
For complex x:

$$\arctan(\tan(x)) = \frac{\pi}{2} - ((\frac{\pi}{2} - \text{Re}(x)) \bmod \pi) - \pi[(\frac{\pi}{2} + \text{Re}(x)) \bmod \pi = 0][\text{Im}(x) < 0] + i\,\text{Im}(x) \tag{2.63}$$

$$\arctan(\cot(x)) = \frac{\pi}{2} - (\text{Re}(x) \bmod \pi) - [\text{Re}(x) \bmod \pi = 0][\text{Im}(x) > 0] - i\,\text{Im}(x) \tag{2.64}$$

Proof. For the first identity, the first part of definition 2.1 with (2.4):

$$\frac{1 + i\tan(x)}{1 - i\tan(x)} = \frac{1 + \dfrac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}}}{1 - \dfrac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}}} = \frac{e^{ix} + e^{-ix} + e^{ix} - e^{-ix}}{e^{ix} + e^{-ix} - e^{ix} + e^{-ix}} = \frac{2e^{ix}}{2e^{-ix}} = e^{2ix} \tag{2.65}$$

which results in:

$$\arctan(\tan(x)) = -\frac{i}{2}\ln(e^{2ix}) - \pi[\text{Re}(\tan(x)) = 0][\text{Im}(\tan(x)) < -1] \tag{2.66}$$

For the first part of this expression, for real a, b:

$$-\frac{i}{2}\ln(e^{2i(a+bi)}) = -\frac{i}{2}\ln(e^{2ia}e^{-2b}) = \frac{1}{2}\text{Arg}(e^{2ia}) + ib = S(a) + ib \tag{2.67}$$

For real a, b, the case $\text{Re}(\tan(a+ib)) = 0$ occurs when for integer n, $a = n\pi/2$:

$$\tan(n\frac{\pi}{2} + ib) = -i\frac{i^n e^{-b} - (-i)^n e^b}{i^n e^{-b} + (-i)^n e^b} = -i\frac{1 - (-1)^n e^{2b}}{1 + (-1)^n e^{2b}} \tag{2.68}$$

From this follows that $\mathrm{Im}(\tan(n\pi/2 + ib)) < -1$ if and only if n is odd and $b < 0$. As $S(-a) = -S(a)$ except for $a = \pm(2n+1)\pi/2$ when $S(a) = S(-a) = \pi/2$, this expression is odd in the complex plane, except when $\mathrm{Im}(x) = 0$ and $\mathrm{Re}(x) = \pm(2n+1)\pi/2$.
For the second identity, using (2.5), $\mathrm{Re}(x)$ is replaced by $\frac{\pi}{2} - \mathrm{Re}(x)$ and $\mathrm{Im}(x)$ by $-\mathrm{Im}(x)$. □

Theorem 2.12. *For the* arccot(x) *function in definition 2.1:*
For complex x:

$$\mathrm{arccot}(\cot(x)) = \frac{\pi}{2} - ((\frac{\pi}{2} - \mathrm{Re}(x)) \bmod \pi) - [(\frac{\pi}{2} + \mathrm{Re}(x)) \bmod \pi = 0][\mathrm{Im}(x) > 0] + i\,\mathrm{Im}(x) \tag{2.69}$$

$$\mathrm{arccot}(\tan(x)) = \frac{\pi}{2} - (\mathrm{Re}(x) \bmod \pi) - \pi[\mathrm{Re}(x) \bmod \pi = 0][\mathrm{Im}(x) < 0] - i\,\mathrm{Im}(x) \tag{2.70}$$

Proof. For the first identity, the first part of definition 2.1 with (2.4) and $\cot(x) = 1/\tan(x)$:

$$\frac{i\cot(x) - 1}{i\cot(x) + 1} = \frac{-\dfrac{e^{ix} + e^{-ix}}{e^{ix} - e^{-ix}} - 1}{-\dfrac{e^{ix} + e^{-ix}}{e^{ix} - e^{-ix}} + 1} = \frac{-e^{ix} - e^{-ix} - e^{ix} + e^{-ix}}{-e^{ix} - e^{-ix} + e^{ix} - e^{-ix}} = \frac{-2e^{ix}}{-2e^{-ix}} = e^{2ix} \tag{2.71}$$

which results in:

$$\mathrm{arccot}(\cot(x)) = -\frac{i}{2}\ln(e^{2ix}) - \pi[\mathrm{Re}(\cot(x)) = 0][-1 < \mathrm{Im}(\cot(x)) < 0] \tag{2.72}$$

As in the previous theorem, for the first part of this expression, for real a, b:

$$-\frac{i}{2}\ln(e^{2i(a+bi)}) = S(a) + ib \tag{2.73}$$

For real a, b, the case $\mathrm{Re}(\cot(a + ib)) = 0$ occurs when for integer n, $a = n\pi/2$:

$$\cot(n\frac{\pi}{2} + ib) = i\frac{i^n e^{-b} + (-i)^n e^b}{i^n e^{-b} - (-i)^n e^b} = i\frac{1 + (-1)^n e^{2b}}{1 - (-1)^n e^{2b}} \tag{2.74}$$

From this follows that $-1 < \mathrm{Im}(\cot(n\pi/2 + ib)) < 0$ if and only if n is odd and $b > 0$. As $S(-a) = -S(a)$ except for $a = \pm(2n+1)\pi/2$ when $S(a) = S(-a) = \pi/2$, this expression is odd in the complex plane, except when $\mathrm{Im}(x) = 0$ and $\mathrm{Re}(x) = \pm(2n+1)\pi/2$.
For the second identity, using (2.5) with x replaced by $\frac{\pi}{2} - x$, $\mathrm{Re}(x)$ is replaced by $\frac{\pi}{2} - \mathrm{Re}(x)$ and $\mathrm{Im}(x)$ by $-\mathrm{Im}(x)$. □

3 Addition Formulas for the Inverse Tangent and Cotangent Functions for Complex Arguments

For the addition formulas of the principal $\arctan(x)$ and $\mathrm{arccot}(x)$ functions for complex arguments, application of (1.28) to the arguments of (2.15) and (2.16) gives the following theorems.

Theorem 3.1. *For complex $x \neq \pm i$ and $y \neq \pm i$:*

$$\arctan(x) + \arctan(y) = \begin{cases} 0 & \text{if } y = -x \\ \frac{\pi}{2}\text{sg}(x) + \text{oddcot}(x) - \text{oddcot}(-x) & \text{if } y = 1/x \\ \arctan(\frac{x+y}{1-xy}) + \text{addtan}(x,y) + \text{oddtan}(x,y) & \text{otherwise} \end{cases} \tag{3.1}$$

where:

$$\text{addtan}(x,y) = \begin{cases} \pi & \text{if } \text{Arg}(\frac{1+ix}{1-ix}) + \text{Arg}(\frac{1+iy}{1-iy}) > \pi \\ -\pi & \text{if } \text{Arg}(\frac{1+ix}{1-ix}) + \text{Arg}(\frac{1+iy}{1-iy}) \leq -\pi \\ 0 & \text{otherwise} \end{cases} \tag{3.2}$$

$$\text{oddtan}(x,y) = \text{oddtan}(\frac{x+y}{1-xy}) - \text{oddtan}(x) - \text{oddtan}(y) \tag{3.3}$$

Proof. For $xy \neq 1$, application of (1.28) to (2.15):

$$\begin{aligned} &\arctan(x) + \arctan(y) \\ &= -\frac{i}{2}[\ln(\frac{1+ix}{1-ix}) + \ln(\frac{1+iy}{1-iy})] - \text{oddtan}(x) - \text{oddtan}(y) \\ &= -\frac{i}{2}\ln(\frac{(1+ix)(1+iy)}{(1-ix)(1-iy)}) + \text{addtan}(x,y) - \text{oddtan}(x) - \text{oddtan}(y) \\ &= -\frac{i}{2}\ln(\frac{1+iz}{1-iz}) + \text{addtan}(x,y) - \text{oddtan}(x) - \text{oddtan}(y) \end{aligned} \tag{3.4}$$

where z has to be solved. First the following equation is solved:

$$\frac{1+iz}{1-iz} = t \tag{3.5}$$

which is easily checked to be:

$$z = i\frac{1-t}{1+t} \tag{3.6}$$

The following t is now substituted:

$$t = \frac{(1+ix)(1+iy)}{(1-ix)(1-iy)} \tag{3.7}$$

and the solution is:

$$z = i\frac{1 - \frac{(1+ix)(1+iy)}{(1-ix)(1-iy)}}{1 + \frac{(1+ix)(1+iy)}{(1-ix)(1-iy)}} = i\frac{(1-ix)(1-iy) - (1+ix)(1+iy)}{(1-ix)(1-iy) + (1+ix)(1+iy)} = \frac{x+y}{1-xy} \tag{3.8}$$

Substituting (2.15):

$$-\frac{i}{2}\ln(\frac{1+iz}{1-iz}) = \arctan(z) + \text{oddtan}(z) \tag{3.9}$$

and the theorem is proved for $xy \neq 1$. For $xy = 1$, that is $y = 1/x$, (2.33) is used. □

When writing this theorem as $f(x, y) = g(x, y)$, because $f(-x, -y) = -f(x, y)$, the symmetry identity $g(-x, -y) = -g(x, y)$ must hold. When none of the inverse tangent arguments is on a branch cut, the arguments of the $\mathrm{Arg}(x)$ functions in $\mathrm{addtan}(x, y)$ are not on the negative real axis, and then by (1.25) $\mathrm{Arg}(1/x) = -\mathrm{Arg}(x)$, and this symmetry identity holds. When one of x or y is on a branch cut, then only the first or the third case in $\mathrm{addtan}(x)$ is possible, so $\mathrm{addtan}(x, y)$ changes from 0 to π or vice versa, and $\mathrm{oddtan}(x, y)$ changes from 0 to $-\pi$ or vice versa. In all of these cases the same identity holds. When both x and y are on a branch cut, then $\mathrm{addtan}(x, y) = \pi$ and $\mathrm{oddtan}(x, y)$ changes from 0 to -2π or vice versa or from $-\pi$ to $-\pi$, and the same identity holds. When only z is on a branch cut, then (3.7) holds with t real and negative, and using (1.26) then $\mathrm{addtan}(x, y)$ changes from 0 to $-\pi$ or vice versa, and $\mathrm{oddtan}(x, y)$ from 0 to π or vice versa, and the same identity holds. When z and x or y is on a branch cut, then $\mathrm{addtan}(x, y) = 0$ and $\mathrm{oddtan}(x, y)$ changes from 0 to 0 or from π to $-\pi$ or vice versa, and the same identity holds. The arguments x, y and z cannot all be on a branch cut, because the product of two real negative t values cannot be negative.

Theorem 3.2. *For complex $x \neq \pm i$ and $y \neq \pm i$:*

$$\mathrm{arccot}(x) + \mathrm{arccot}(y) = \begin{cases} \pi & \text{if } y = x = 0 \\ 0 & \text{if } y = -x \neq 0 \\ \frac{\pi}{2}\mathrm{sg}(x) + \mathrm{oddtan}(x) - \mathrm{oddtan}(-x) & \text{if } y = 1/x \\ \mathrm{arccot}(\dfrac{xy-1}{x+y}) + \mathrm{addcot}(x, y) + \mathrm{oddcot}(x, y) & \text{otherwise} \end{cases} \tag{3.10}$$

where:

$$\mathrm{addcot}(x, y) = \begin{cases} \pi & \text{if } \mathrm{Arg}(\frac{ix-1}{ix+1}) + \mathrm{Arg}(\frac{iy-1}{iy+1}) > \pi \\ -\pi & \text{if } \mathrm{Arg}(\frac{ix-1}{ix+1}) + \mathrm{Arg}(\frac{iy-1}{iy+1}) \leq -\pi \\ 0 & \text{otherwise} \end{cases} \tag{3.11}$$

$$\mathrm{oddcot}(x, y) = \mathrm{oddcot}(\frac{xy-1}{x+y}) - \mathrm{oddcot}(x) - \mathrm{oddcot}(y) \tag{3.12}$$

Proof. For $xy \neq 1$, application of (1.28) to (2.16):

$$\begin{aligned} &\mathrm{arccot}(x) + \mathrm{arccot}(y) \\ &= -\frac{i}{2}[\ln(\frac{ix-1}{ix+1}) + \ln(\frac{iy-1}{iy+1})] - \mathrm{oddcot}(x) - \mathrm{oddcot}(y) \\ &= -\frac{i}{2}\ln(\frac{(ix-1)(iy-1)}{(ix+1)(iy+1)}) + \mathrm{addcot}(x, y) - \mathrm{oddcot}(x) - \mathrm{oddcot}(y) \\ &= -\frac{i}{2}\ln(\frac{iz-1}{iz+1}) + \mathrm{addcot}(x, y) - \mathrm{oddcot}(x) - \mathrm{oddcot}(y) \end{aligned} \tag{3.13}$$

where z has to be solved. First the following equation is solved:

$$\frac{iz-1}{iz+1} = t \tag{3.14}$$

which is easily checked to be:

$$z = i\frac{t+1}{t-1} \tag{3.15}$$

The following t is now substituted:

$$t = \frac{(ix-1)(iy-1)}{(ix+1)(iy+1)} \tag{3.16}$$

and the solution is:

$$z = i\frac{\frac{(ix-1)(iy-1)}{(ix+1)(iy+1)}+1}{\frac{(ix-1)(iy-1)}{(ix+1)(iy+1)}-1} = i\frac{(ix-1)(iy-1)+(ix+1)(iy+1)}{(ix-1)(iy-1)-(ix+1)(iy+1)} = \frac{xy-1}{x+y} \tag{3.17}$$

Substituting (2.16):

$$-\frac{i}{2}\ln(\frac{iz-1}{iz+1}) = \text{arccot}(z) + \text{oddcot}(z) \tag{3.18}$$

and the theorem is proved for $xy \neq 1$. For $xy = 1$, that is $y = 1/x$, (2.34) is used. □

When $x \neq 0$ and $y \neq 0$, writing this theorem as $f(x,y) = g(x,y)$, because then $f(-x,-y) = -f(x,y)$, also $g(-x,-y) = -g(x,y)$, and the same reasoning as for the previous theorem can be given. When $y = 0$ this theorem and (2.32) and (2.34) yields:
For complex x:

$$\frac{1}{2}(1+\text{sg}(x)) + [\text{Re}(x)=0]([-1<\text{Im}(x)<0] - [\text{Im}(x)>1]) = [\text{Arg}(\frac{ix-1}{ix+1})>0] \tag{3.19}$$

and replacing x by $1/x$ and using (1.17) and (2.32):
For complex x:

$$\frac{1}{2}(1+\text{sg}(x)) + [\text{Re}(x)=0]([\text{Im}(x)<-1] - [0\leq\text{Im}(x)\leq 1]) = [\text{Arg}(\frac{1+ix}{1-ix})>0] \tag{3.20}$$

4 Addition Formulas for the Inverse Tangent and Cotangent Functions for Real Arguments

For the addition formulas of the principal $\arctan(x)$ and $\text{arccot}(x)$ functions for real arguments, theorems 3.1 and 3.2 are applied for real x and y.

Theorem 4.1. *For real x, y:*

$$\arctan(x) + \arctan(y) = \begin{cases} \frac{\pi}{2}\text{sg}(x) & \text{if } y = 1/x \\ \arctan(\frac{x+y}{1-xy}) + \pi[xy>1]\text{sg}(x) & \text{otherwise} \end{cases} \tag{4.1}$$

Proof. Applying theorem 3.1 for x and y real, $\text{oddtan}(x,y) = 0$, and for $\text{addtan}(x,y)$ in (3.2), because x and y are real:

$$\text{Arg}(\frac{1+ix}{1-ix}) + \text{Arg}(\frac{1+iy}{1-iy}) = 2[\text{Arg}(1+ix) + \text{Arg}(1+iy)] \tag{4.2}$$

Applying (1.18) because $-\pi < \mathrm{Arg}(1+ix) + \mathrm{Arg}(1+iy) < \pi$:

$$\mathrm{Arg}(1+ix) + \mathrm{Arg}(1+iy) = \mathrm{Arg}((1+ix)(1+iy)) = \mathrm{Arg}(1-xy+i(x+y)) \tag{4.3}$$

The addtan(x,y) in theorem 3.1 for real x, y thus becomes:

$$\mathrm{addtan}(x,y) = \begin{cases} \pi & \text{if } \mathrm{Arg}(1-xy+i(x+y)) > \pi/2 \\ -\pi & \text{if } \mathrm{Arg}(1-xy+i(x+y)) \leq -\pi/2 \\ 0 & \text{otherwise} \end{cases} \tag{4.4}$$

In the complex plane this is easily seen to be equivalent to the theorem, with the fact that in this case $xy \neq 1$. When $xy > 1$, x and y have the same sign, so $\mathrm{sg}(x+y)$ can be replaced by $\mathrm{sg}(x)$. □

Theorem 4.2. *For real x, y:*

$$\mathrm{arccot}(x) + \mathrm{arccot}(y) = \begin{cases} \pi & \text{if } y = x = 0 \\ 0 & \text{if } y = -x \neq 0 \\ \frac{\pi}{2}\mathrm{sg}(x) & \text{if } y = 1/x \\ \mathrm{arccot}(\frac{xy-1}{x+y}) + \pi[|xy| < \mathrm{sg}(x)\mathrm{sg}(y)]\mathrm{sg}(x+y) & \text{otherwise} \end{cases} \tag{4.5}$$

Proof. Applying theorem 3.2 for x and y real, $\mathrm{oddcot}(x,y) = 0$, and for $\mathrm{addcot}(x,y)$ in (3.11), because x and y are real and using $\mathrm{sg}(x)x = |x|$:

$$\frac{ix-1}{ix+1} = \frac{-i\,\mathrm{sg}(x)(ix-1)}{-i\,\mathrm{sg}(x)(ix+1)} = \frac{|x| + i\,\mathrm{sg}(x)}{|x| - i\,\mathrm{sg}(x)} \tag{4.6}$$

$$\mathrm{Arg}(\frac{|x| + i\,\mathrm{sg}(x)}{|x| - i\,\mathrm{sg}(x)}) + \mathrm{Arg}(\frac{|y| + i\,\mathrm{sg}(y)}{|y| - i\,\mathrm{sg}(y)}) = 2[\mathrm{Arg}(|x| + i\,\mathrm{sg}(x)) + \mathrm{Arg}(|y| + i\,\mathrm{sg}(y))] \tag{4.7}$$

Applying (1.18) because not $x = y = 0$ which is a special case in the theorem, $-\pi < \mathrm{Arg}(|x| + i\,\mathrm{sg}(x)) + \mathrm{Arg}(|y| + i\,\mathrm{sg}(y)) < \pi$:

$$\begin{aligned} &\mathrm{Arg}(|x| + i\,\mathrm{sg}(x)) + \mathrm{Arg}(|y| + i\,\mathrm{sg}(y)) \\ =&\mathrm{Arg}((|x| + i\,\mathrm{sg}(x))(|y| + i\,\mathrm{sg}(y))) \\ =&\mathrm{Arg}(|xy| - \mathrm{sg}(x)\mathrm{sg}(y) + i(\mathrm{sg}(x)|y| + \mathrm{sg}(y)|x|)) \end{aligned} \tag{4.8}$$

The addcot(x,y) in theorem 3.2 for real x, y thus becomes:

$$\mathrm{addcot}(x,y) = \begin{cases} \pi & \text{if } \mathrm{Arg}(|xy| - \mathrm{sg}(x)\mathrm{sg}(y) + i(\mathrm{sg}(x)|y| + \mathrm{sg}(y)|x|)) > \pi/2 \\ -\pi & \text{if } \mathrm{Arg}(|xy| - \mathrm{sg}(x)\mathrm{sg}(y) + i(\mathrm{sg}(x)|y| + \mathrm{sg}(y)|x|)) \leq -\pi/2 \\ 0 & \text{otherwise} \end{cases} \tag{4.9}$$

Because $|xy| = \mathrm{sg}(x)\mathrm{sg}(y)$ only occurs when $xy = 1$, in the complex plane this is easily seen to be equivalent to the theorem, with the fact that in this case $xy \neq 1$. Because $|xy| < \mathrm{sg}(x)\mathrm{sg}(y)$ can only occur when $\mathrm{sg}(x) = \mathrm{sg}(y)$, and because $\mathrm{sg}(x)|x| = x$, $\mathrm{sg}(x)|y| + \mathrm{sg}(y)|x|$ can be replaced by $x + y$. □

Theorem 4.3. *For real x, y:*

$$\operatorname{arccot}(x) + \operatorname{arccot}(y) = \begin{cases} \frac{\pi}{2}\operatorname{sg}(x) & \text{if } y = 1/x \\ \arctan(\dfrac{x+y}{xy-1}) + \pi[|xy| < \operatorname{sg}(x)\operatorname{sg}(y)]\operatorname{sg}(x+y) & \text{otherwise} \end{cases} \tag{4.10}$$

Proof. From (2.29) follows that for real x: $\operatorname{arccot}(x) = \arctan(1/x)$, so this theorem follows directly from the previous theorem, where the special cases $x = y = 0$ and $y = -x \neq 0$ give identical results. □

Theorem 4.4. *For real x, y:*

$$\arctan(x) + \arctan(y) = 2\arctan(\frac{x+y}{1 - xy + \sqrt{(1+x^2)(1+y^2)}}) \tag{4.11}$$

Proof. From definition (2.15), for real x:

$$\arctan(x) = -\frac{i}{2}\ln(\frac{1+ix}{1-ix}) \tag{4.12}$$

Using (1.31):

$$\arctan(x) = -i\ln(F(x)) \tag{4.13}$$

where:

$$F(x) = \sqrt{\frac{1+ix}{1-ix}} \tag{4.14}$$

For real x, y using (1.30):

$$\begin{aligned} \arctan(x) + \arctan(y) &= -i(\ln(F(x)) + \ln(F(y))) = -i\ln(F(x)F(y)) \\ &= -i\ln(\sqrt{\frac{1+ix}{1-ix}}\sqrt{\frac{1+iy}{1-iy}}) \end{aligned} \tag{4.15}$$

From definition 2.15, for real z:

$$2\arctan(z) = -i\ln(\frac{1+iz}{1-iz}) \tag{4.16}$$

From theorem 2.1, the solution of:

$$\frac{1+iz}{1-iz} = t \tag{4.17}$$

is:

$$z = i\frac{1-t}{1+t} \tag{4.18}$$

Substituting for t the result of (4.15), and using (1.11) and $(1-ix)(1+ix) = 1+x^2$:

$$z = i\frac{1 - \sqrt{\frac{1+ix}{1-ix}}\sqrt{\frac{1+iy}{1-iy}}}{1 + \sqrt{\frac{1+ix}{1-ix}}\sqrt{\frac{1+iy}{1-iy}}} = i\frac{\sqrt{(1+x^2)(1+y^2)} - \sqrt{(1+ix)^2}\sqrt{(1+iy)^2}}{\sqrt{(1+x^2)(1+y^2)} + \sqrt{(1+ix)^2}\sqrt{(1+iy)^2}} \tag{4.19}$$

Because for real x, $\mathrm{sg}(1+ix)=1$:

$$z = i\frac{\sqrt{(1+x^2)(1+y^2)}-(1+ix)(1+iy)}{\sqrt{(1+x^2)(1+y^2)}+(1+ix)(1+iy)} \tag{4.20}$$

Using (2.40):

$$\begin{aligned} z &= \frac{2(x+y)\sqrt{(1+x^2)(1+y^2)}}{(\sqrt{(1+x^2)(1+y^2)}+1-xy)^2+(x+y)^2} \\ &= \frac{(x+y)\sqrt{(1+x^2)(1+y^2)}}{(1+x^2)(1+y^2)+(1-xy)\sqrt{(1+x^2)(1+y^2)}} \\ &= \frac{x+y}{\sqrt{(1+x^2)(1+y^2)}+1-xy} \end{aligned} \tag{4.21}$$

□

In this theorem in the argument on the right side: $1-xy+\sqrt{(1+x^2)(1+y^2)} \geq 2$, with equal sign if and only if $x=y$, and squaring that argument yields:

$$[\frac{|x+y|}{1-xy+\sqrt{(1+x^2)(1+y^2)}} > 1] = [xy > 1] \tag{4.22}$$

Taking $x = 1/a$ and $y = 1/(4a^3+3a)$ and using $1+(4a^3+3a)^2 = (1+a^2)(1+4a^2)^2$, from this theorem the following known identity [6] results.
For real a:

$$\arctan(\frac{1}{a}) + \arctan(\frac{1}{4a^3+3a}) = 2\arctan(\frac{1}{2a}) \tag{4.23}$$

Theorem 4.5. *For real x, y:*

$$\mathrm{arccot}(x)+\mathrm{arccot}(y) = \begin{cases} \pi & \text{if } y=x=0 \\ 0 & \text{if } y=-x\neq 0 \\ 2\,\mathrm{arccot}(\frac{xy-1+\mathrm{sg}(x)\mathrm{sg}(y)\sqrt{(1+x^2)(1+y^2)}}{x+y}) & \text{otherwise} \end{cases} \tag{4.24}$$

Proof. This theorem directly follows from the previous theorem because for real x: $\mathrm{arccot}(x) = \arctan(1/x)$. □

Theorem 4.6. *For real x, y:*

$$\mathrm{arccot}(x)+\mathrm{arccot}(y) = \begin{cases} 0 & \text{if } y=-x\neq 0 \\ 2\,\mathrm{arccot}(\frac{x+y}{1-xy+\mathrm{sg}(x)\mathrm{sg}(y)\sqrt{(1+x^2)(1+y^2)}}) & \text{otherwise} \end{cases} \tag{4.25}$$

Proof. From definition (2.16) for real x:

$$\operatorname{arccot}(x) = -\frac{i}{2}\ln(\frac{ix-1}{ix+1}) \tag{4.26}$$

Using (1.31):

$$\operatorname{arccot}(x) = -i\ln(G(x)) \tag{4.27}$$

where:

$$G(x) = \sqrt{\frac{ix-1}{ix+1}} \tag{4.28}$$

For real x, y using (1.30):

$$\begin{aligned}\operatorname{arccot}(x) + \operatorname{arccot}(y) &= -i(\ln(G(x)) + \ln(G(y))) = -i\ln(G(x)G(y)) \\ &= -i\ln(\sqrt{\frac{ix-1}{ix+1}}\sqrt{\frac{iy-1}{iy+1}})\end{aligned} \tag{4.29}$$

From definition 2.16, for real z:

$$2\operatorname{arccot}(z) = -i\ln(\frac{iz-1}{iz+1}) \tag{4.30}$$

From theorem 2.1, the solution of:

$$\frac{iz-1}{iz+1} = t \tag{4.31}$$

is:

$$z = i\frac{t+1}{t-1} \tag{4.32}$$

Substituting for t the result of (4.29), and using (1.11) and $(1-ix)(1+ix) = 1+x^2$:

$$z = i\frac{\sqrt{\frac{ix-1}{ix+1}}\sqrt{\frac{iy-1}{iy+1}}+1}{\sqrt{\frac{ix-1}{ix+1}}\sqrt{\frac{iy-1}{iy+1}}-1} = i\frac{\sqrt{-(1-ix)^2}\sqrt{-(1-iy)^2}+\sqrt{(1+x^2)(1+y^2)}}{\sqrt{-(1-ix)^2}\sqrt{-(1-iy)^2}-\sqrt{(1+x^2)(1+y^2)}} \tag{4.33}$$

Using from section 1: $\sqrt{-x^2} = \operatorname{sg}(ix)ix$ and $i(1-ix) = i+x$ and for real x: $\operatorname{sg}(i+x) = \operatorname{sg}(x)$:

$$z = i\frac{\operatorname{sg}(x)\operatorname{sg}(y)(i+x)(i+y)+\sqrt{(1+x^2)(1+y^2)}}{\operatorname{sg}(x)\operatorname{sg}(y)(i+x)(i+y)-\sqrt{(1+x^2)(1+y^2)}} \tag{4.34}$$

and multiplying numerator and denominator with $\operatorname{sg}(x)\operatorname{sg}(y)$ and using $\operatorname{sg}^2(x) = 1$:

$$z = i\frac{(i+x)(i+y)+\operatorname{sg}(x)\operatorname{sg}(y)\sqrt{(1+x^2)(1+y^2)}}{(i+x)(i+y)-\operatorname{sg}(x)\operatorname{sg}(y)\sqrt{(1+x^2)(1+y^2)}} \tag{4.35}$$

Using (2.40):

$$
\begin{aligned}
z &= \frac{2(x+y)\mathrm{sg}(x)\mathrm{sg}(y)\sqrt{(1+x^2)(1+y^2)}}{(\mathrm{sg}(x)\mathrm{sg}(y)\sqrt{(1+x^2)(1+y^2)}+1-xy)^2+(x+y)^2} \\
&= \frac{(x+y)\mathrm{sg}(x)\mathrm{sg}(y)\sqrt{(1+x^2)(1+y^2)}}{(1+x^2)(1+y^2)+(1-xy)\mathrm{sg}(x)\mathrm{sg}(y)\sqrt{(1+x^2)(1+y^2)}} \\
&= \frac{x+y}{\mathrm{sg}(x)\mathrm{sg}(y)\sqrt{(1+x^2)(1+y^2)}+1-xy}
\end{aligned}
\tag{4.36}
$$

□

Theorem 4.7. *For real x, y:*

$$
\arctan(x)+\arctan(y) = \begin{cases} 0 & \text{if } y=-x \\ 2\ \arctan(\frac{xy-1+\sqrt{(1+x^2)(1+y^2)}}{x+y}) & \text{otherwise} \end{cases} \tag{4.37}
$$

Proof. This theorem directly follows from the previous theorem because for real x: $\mathrm{arccot}(x) = \arctan(1/x)$, and using (1.17). □

Let $s = \pm 1$, then by cross multiplication it is clear that:

$$
\frac{x+y}{1-xy+s\sqrt{(1+x^2)(1+y^2)}} = \frac{xy-1+s\sqrt{(1+x^2)(1+y^2)}}{x+y} \tag{4.38}
$$

From this follows that theorems 4.4 and 4.7 as well as theorems 4.5 and 4.6 are equivalent.

5 The Principal Values of the Inverse Tangent and Cotangent Functions on the Imaginary Axis

For determining the principal values of the $\arctan(x)$ and $\mathrm{arccot}(x)$ functions on the imaginary axis, and thus confirming the new definitions (2.15) and (2.16), the addition theorems for complex arguments are used to express these values as a sum of two principal $\arctan(x)$ or $\mathrm{arccot}(x)$ terms with arguments that are not on the imaginary axis. The following theorems state that this is possible.

Theorem 5.1. *Let t be on the imaginary axis and $t \neq \pm i$ and let x be real and positive, then from theorem 3.1 follows:*

$$
\arctan(t) = \arctan(x) + \arctan(\frac{t-x}{1+tx}) - \mathrm{oddtan}(t) \tag{5.1}
$$

Proof. In theorem 3.1 the solution of:

$$
\frac{x+y}{1-xy} = t \tag{5.2}
$$

is easily checked to be:

$$y = \frac{t-x}{1+tx} \tag{5.3}$$

When t is on the imaginary axis then it can be replaced by $t = iz$ where z is real:

$$y = \frac{iz-x}{1+izx} = \frac{(iz-x)(1-izx)}{1+z^2x^2} = \frac{x(z^2-1)+iz(1+x^2)}{1+z^2x^2} \tag{5.4}$$

Because x is real and positive, y is only on the imaginary axis when $z^2 = 1$, that is when $t = \pm i$ which are singular points and excluded in the theorem. Therefore y is never on the imaginary axis and $\text{oddtan}(y) = 0$. Because x is real, also $\text{oddtan}(x) = 0$, so in theorem 3.1 $\text{oddtan}(x, y) = \text{oddtan}(t)$. For evaluating $\text{addtan}(x, y)$ in that theorem, by substituting (5.3):

$$\frac{1+iy}{1-iy} = \frac{1+it}{1-it} \cdot \frac{1-ix}{1+ix} \tag{5.5}$$

When t is on the imaginary axis the first factor in the right side of this equation is real and is called c, and because $t \neq \pm i$, $c \neq 0$. Then in $\text{addtan}(x, y)$ in theorem 3.1, because x is real and positive, application of (1.25) and (1.26) yields:

$$\text{Arg}(\frac{1+ix}{1-ix}) + \text{Arg}(\frac{1+iy}{1-iy}) = \text{Arg}(\frac{1+ix}{1-ix}) + \text{Arg}(c\frac{1-ix}{1+ix}) = \begin{cases} 0 & \text{if } c > 0 \\ \pi & \text{if } c < 0 \end{cases} \tag{5.6}$$

This means that in theorem 3.1 $\text{addtan}(x, y) = 0$ and this theorem follows. □

Theorem 5.2. *Let t be on the imaginary axis and $t \neq \pm i$ and let x be real and positive, then from theorem 3.2 follows:*

$$\text{arccot}(t) = \text{arccot}(x) + \text{arccot}(\frac{1+tx}{x-t}) - \text{oddcot}(t) \tag{5.7}$$

Proof. In theorem 3.1 the solution of:

$$\frac{xy-1}{x+y} = t \tag{5.8}$$

is easily checked to be:

$$y = \frac{1+tx}{x-t} \tag{5.9}$$

When t is on the imaginary axis then it can be replaced by $t = iz$ where z is real:

$$y = \frac{1+izx}{x-iz} = \frac{(1+izx)(x+iz)}{x^2+z^2} = \frac{x(1-z^2)+iz(1+x^2)}{x^2+z^2} \tag{5.10}$$

Because x is real and positive, y is only on the imaginary axis when $z^2 = 1$, that is when $t = \pm i$ which are singular points and excluded in the theorem. Therefore y is never on the imaginary axis and $\text{oddcot}(y) = 0$. Because x is real, also $\text{oddcot}(x) = 0$, so in theorem 3.2 $\text{oddcot}(x, y) = \text{oddcot}(t)$. For evaluating $\text{addcot}(x, y)$ in that theorem, by substituting (5.9):

$$\frac{iy-1}{iy+1} = \frac{it-1}{it+1} \cdot \frac{ix+1}{ix-1} \tag{5.11}$$

When t is on the imaginary axis the first factor in the right side of this equation is real and is called c, and because $t \neq \pm i$, $c \neq 0$. Then in addcot(x, y) in theorem 3.2, because x is real and positive and using (4.6), application of (1.25) and (1.26) yields:

$$\begin{aligned}&\operatorname{Arg}(\frac{ix-1}{ix+1}) + \operatorname{Arg}(\frac{iy-1}{iy+1})\\ =&\operatorname{Arg}(\frac{|x|+i\operatorname{sg}(x)}{|x|-i\operatorname{sg}(x)}) + \operatorname{Arg}(c\frac{|x|-i\operatorname{sg}(x)}{|x|+i\operatorname{sg}(x)}) = \begin{cases}0 & \text{if } c>0\\ \pi & \text{if } c<0\end{cases}\end{aligned} \tag{5.12}$$

This means that in theorem 3.2 addcot$(x, y) = 0$ and this theorem follows. □

The two tables below are the computation of principal values of $\arctan(t)$ and $\operatorname{arccot}(t)$ on four points of the imaginary axis, where the principal $\arctan(y)$ and $\operatorname{arccot}(y)$ values can be computed with a computer algebra program. With these two tables it can be checked that for these values on the imaginary axis $\arctan(t)$ and $\operatorname{arccot}(t)$ are odd and that theorem 2.3 is valid. These values are also in agreement with definitions (2.15) and (2.16), which are therefore now confirmed with the addition formulas.

Table 1: Evaluation of $\arctan(t)$ with theorem 5.1

t	$2i$	$\frac{1}{2}i$	$-\frac{1}{2}i$	$-2i$
x	1	1	1	1
$y = \frac{t-x}{1+tx}$	$\frac{3}{5}+\frac{4}{5}i$	$-\frac{3}{5}+\frac{4}{5}i$	$-\frac{3}{5}-\frac{4}{5}i$	$\frac{3}{5}-\frac{4}{5}i$
$\frac{1+ix}{1-ix}$	i	i	i	i
$\frac{1+iy}{1-iy}$	$\frac{1}{3}i$	$-\frac{1}{3}i$	$-3i$	$3i$
$\arctan(x)$	$\frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{\pi}{4}$
$\arctan(y)$	$\frac{\pi}{4}+\frac{1}{2}i\ln(3)$	$-\frac{\pi}{4}+\frac{1}{2}i\ln(3)$	$-\frac{\pi}{4}-\frac{i}{2}\ln(3)$	$\frac{\pi}{4}-\frac{i}{2}\ln(3)$
$-\operatorname{oddtan}(t)$	0	0	0	$-\pi$
$\arctan(t)$	$\frac{\pi}{2}+\frac{1}{2}i\ln(3)$	$\frac{1}{2}i\ln(3)$	$-\frac{1}{2}i\ln(3)$	$-\frac{\pi}{2}-\frac{1}{2}i\ln(3)$

Table 2: Evaluation of arccot(t) with theorem 5.2

t	$2i$	$\frac{1}{2}i$	$-\frac{1}{2}i$	$-2i$
x	1	1	1	1
$y = \frac{1+tx}{x-t}$	$-\frac{3}{5}+\frac{4}{5}i$	$\frac{3}{5}+\frac{4}{5}i$	$\frac{3}{5}-\frac{4}{5}i$	$-\frac{3}{5}-\frac{4}{5}i$
$\frac{ix-1}{ix+1}$	i	i	i	i
$\frac{iy-1}{iy+1}$	$-3i$	$3i$	$\frac{1}{3}i$	$-\frac{1}{3}i$
$\operatorname{arccot}(x)$	$\frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{\pi}{4}$
$\operatorname{arccot}(y)$	$-\frac{\pi}{4}-\frac{1}{2}i\ln(3)$	$\frac{\pi}{4}-\frac{1}{2}i\ln(3)$	$\frac{\pi}{4}+\frac{i}{2}\ln(3)$	$-\frac{\pi}{4}+\frac{i}{2}\ln(3)$
$-\operatorname{oddcot}(t)$	0	0	$-\pi$	0
$\operatorname{arccot}(t)$	$-\frac{1}{2}i\ln(3)$	$\frac{\pi}{2}-\frac{1}{2}i\ln(3)$	$-\frac{\pi}{2}+\frac{1}{2}i\ln(3)$	$\frac{1}{2}i\ln(3)$

6 The Inverse Hyperbolic Tangent and Cotangent Functions and their Addition Formulas

The principal arctanh(x) and arccoth(x) functions for complex x are defined by:

$$\operatorname{arctanh}(x) = -i\arctan(ix) \tag{6.1}$$

$$\operatorname{arccoth}(x) = i\operatorname{arccot}(ix) \tag{6.2}$$

With these definitions from (2.15) and (2.16) follows:
For complex x:

$$\operatorname{arctanh}(x) = -\frac{1}{2}\ln(\frac{1-x}{1+x}) + \pi i[\operatorname{Im}(x)=0][\operatorname{Re}(x)<-1] \tag{6.3}$$

$$\operatorname{arccoth}(x) = \frac{1}{2}\ln(\frac{x+1}{x-1}) - \pi i[\operatorname{Im}(x)=0][-1<\operatorname{Re}(x)<0] \tag{6.4}$$

Theorem 6.1. *For complex x:*

$$\operatorname{arctanh}(x) = \ln(\frac{1+x+\sqrt{1-x^2}}{1-x+\sqrt{1-x^2}}) \tag{6.5}$$

$$\operatorname{arccoth}(x) = \frac{\pi}{2}i\operatorname{sg}(ix) + \ln(\frac{1+x+\sqrt{1-x^2}}{1-x+\sqrt{1-x^2}}) \tag{6.6}$$

Proof. This theorem directly follows from theorem 2.5 and definition (6.1) and theorem 2.6 and definition (6.2). □

Theorem 6.2. *For real x, y:*

$$\begin{aligned}\operatorname{arctanh}(x+iy) =& \frac{1}{4}\ln(\frac{y^2+(x+1)^2}{y^2+(x-1)^2}) + \frac{i}{2}\arctan(\frac{2y}{1-x^2-y^2}) \\ &- \frac{\pi}{2}i[x^2+y^2>1]\operatorname{sg}(-y) + \pi i[y=0][x<-1]\end{aligned} \tag{6.7}$$

$$\begin{aligned}\operatorname{arccoth}(x+iy) =&\frac{1}{4}\ln(\frac{y^2+(x+1)^2}{y^2+(x-1)^2})-\frac{i}{2}\arctan(\frac{2y}{x^2+y^2-1})\\&+\frac{\pi}{2}i[x^2+y^2<1]\operatorname{sg}(-y)-\pi i[y=0][-1<x<0]\end{aligned} \tag{6.8}$$

Proof. These two identities follow from theorem 2.2 with definitions (6.1) and (6.2) by replacing $x+iy$ with $i(x+iy)=-y+ix$, that is replacing x with $-y$ and y with x.

□

When $x=\pm 1$ and $y=0$ these formulas yield $\operatorname{arctanh}(1)=\infty$, $\operatorname{arctanh}(-1)=-\infty$, $\operatorname{arccoth}(1)=\infty$ and $\operatorname{arccoth}(-1)=-\infty$.
For real x:

$$\operatorname{arctanh}(x)=\frac{1}{2}\ln(\left|\frac{1+x}{1-x}\right|)-\frac{\pi}{2}i([x>1]-[x<-1]) \tag{6.9}$$

$$\operatorname{arccoth}(x)=\frac{1}{2}\ln(\left|\frac{x+1}{x-1}\right|)+\frac{\pi}{2}i([0\le x<1]-[-1<x<0]) \tag{6.10}$$

For complex $x\neq\pm 1$:

$$\operatorname{arccoth}(x)-\operatorname{arctanh}(x)=\frac{\pi}{2}i\operatorname{sg}(ix) \tag{6.11}$$

For complex x:

$$\operatorname{arctanh}(\frac{1}{x})=\operatorname{arccoth}(x)+\pi i[\operatorname{Im}(x)=0]([-1<\operatorname{Re}(x)<0]-[0<\operatorname{Re}(x)<1]) \tag{6.12}$$

$$\operatorname{arccoth}(\frac{1}{x})=\operatorname{arctanh}(x)-\pi i[\operatorname{Im}(x)=0]([\operatorname{Re}(x)<-1]-[\operatorname{Re}(x)>1]) \tag{6.13}$$

For complex $x\neq\pm 1$:

$$\begin{aligned}&\operatorname{arctanh}(x)-\operatorname{arctanh}(\frac{1}{x})\\=&-\frac{\pi}{2}i\operatorname{sg}(ix)-\pi i[\operatorname{Im}(x)=0]([-1<\operatorname{Re}(x)<0]-[0<\operatorname{Re}(x)<1])\end{aligned} \tag{6.14}$$

$$\operatorname{arccoth}(x)-\operatorname{arccoth}(\frac{1}{x})=\frac{\pi}{2}i\operatorname{sg}(ix)+\pi i[\operatorname{Im}(x)=0]([\operatorname{Re}(x)<-1]-[\operatorname{Re}(x)>1]) \tag{6.15}$$

The addition formulas for these functions for complex x and y can be taken from the addition formulas for complex x and y above by substituting ix for x and iy for y. The addition formulas for real x and y are given, using for real x: $\operatorname{sg}(ix)=\operatorname{sg}(x)$.

Theorem 6.3. *For real $x\neq\pm 1$ and $y\neq\pm 1$:*

$$\begin{aligned}&\operatorname{arctanh}(x)+\operatorname{arctanh}(y)\\&=\begin{cases}-\frac{\pi}{2}i\operatorname{sg}(x)-\pi i([-1<x<0]-[0<x<1]) & \text{if } y=-1/x\\ \operatorname{arctanh}(\frac{x+y}{1+xy})-\pi i(\operatorname{addtanh}(x,y)+\operatorname{oddtanh}(x,y)) & \text{otherwise}\end{cases}\end{aligned} \tag{6.16}$$

where:

$$\operatorname{addtanh}(x,y)=([x<-1]+[x>1])([y<-1]+[y>1]) \tag{6.17}$$

$$\operatorname{oddtanh}(x,y)=[\frac{x+y}{1+xy}<-1]-[x<-1]-[y<-1] \tag{6.18}$$

Proof. Application of theorem 3.1 by substituting definition (6.1) gives this theorem, where in $\text{addtan}(x, y)$ the sum of $\text{Arg}(x)$ functions becomes:

$$\text{Arg}(\frac{1-x}{1+x}) + \text{Arg}(\frac{1-y}{1+y}) = \begin{cases} 0 & \text{if } -1 < x < 1 \text{ and } -1 < y < 1 \\ \pi & \text{if one of } -1 < x < 1 \text{ or } -1 < y < 1 \\ 2\pi & \text{if } (x < -1 \text{ or } x > 1) \text{ and } (y < -1 \text{ or } y > 1) \end{cases} \tag{6.19}$$

Only in the last case is $\text{addtan}(x, y)$ π and otherwise zero, which is equivalent to $\text{addtanh}(x, y)$. □

Theorem 6.4. *For real $x \neq \pm 1$ and $y \neq \pm 1$:*

$$\begin{aligned} &\text{arccoth}(x) + \text{arccoth}(y) \\ &= \begin{cases} \pi i & \text{if } y = x = 0 \\ 0 & \text{if } y = -x \neq 0 \\ \frac{\pi}{2} i\, \text{sg}(x) + \pi i([x < -1] - [x > 1]) & \text{if } y = -1/x \\ \text{arccoth}(\frac{1+xy}{x+y}) + \pi i(\text{addcoth}(x, y) + \text{oddcoth}(x, y)) & \text{otherwise} \end{cases} \end{aligned} \tag{6.20}$$

where:

$$\text{addcoth}(x, y) = [-1 < x < 1][-1 < y < 1] \tag{6.21}$$

$$\text{oddcoth}(x, y) = [-1 < \frac{1+xy}{x+y} < 0] - [-1 < x < 0] - [-1 < y < 0] \tag{6.22}$$

Proof. Application of theorem 3.2 by substituting definition (6.2) gives this theorem, where in $\text{addtan}(x, y)$ the sum of $\text{Arg}(x)$ functions becomes:

$$\text{Arg}(\frac{x+1}{x-1}) + \text{Arg}(\frac{y+1}{y-1}) = \begin{cases} 0 & \text{if } (x < -1 \text{ or } x > 1) \text{ and } (y < -1 \text{ or } y > 1) \\ \pi & \text{if one of } -1 < x < 1 \text{ or } -1 < y < 1 \\ 2\pi & \text{if } -1 < x < 1 \text{ and } -1 < y < 1 \end{cases} \tag{6.23}$$

Only in the last case is $\text{addcot}(x, y)$ π and otherwise zero, which is equivalent to $\text{addcoth}(x, y)$. □

Theorem 6.5. *For real $x \neq \pm 1$ and $y \neq \pm 1$:*

$$\text{arccoth}(x) + \text{arccoth}(y) = \begin{cases} \frac{\pi}{2} i\, \text{sg}(x) + \pi i([x < -1] - [x > 1]) & \text{if } y = -1/x \\ \text{arctanh}(\frac{x+y}{1+xy}) + \pi i\, \text{addcoth}(x, y) & \text{otherwise} \end{cases} \tag{6.24}$$

where:

$$\begin{aligned} \text{addcoth}(x, y) = &[\frac{x+y}{1+xy} > 1] + [-1 < x < 1][-1 < y < 1] \\ &- [-1 < x < 0] - [-1 < y < 0] \end{aligned} \tag{6.25}$$

Proof. Using (6.12) and the identity:

$$[0 < \alpha < 1] = [\frac{1}{\alpha} > 1] \tag{6.26}$$

this theorem directly follows from the previous theorem, where the special cases $x = y = 0$ and $y = -x \neq 0$ give identical results. □

When $x = y$ these addition formulas reduce to the following duplication formulas, using $|2x/(1+x^2)| \leq 1$.
For real x:

$$2\operatorname{arctanh}(x) = \operatorname{arctanh}(\frac{2x}{1+x^2}) + \pi i([x < -1] - [x > 1]) \tag{6.27}$$

$$2\operatorname{arccoth}(x) = \operatorname{arccoth}(\frac{1+x^2}{2x}) + \pi i([0 \leq x < 1] - [-1 < x < 0]) \tag{6.28}$$

$$2\operatorname{arccoth}(x) = \operatorname{arctanh}(\frac{2x}{1+x^2}) + \pi i([0 \leq x < 1] - [-1 < x < 0]) \tag{6.29}$$

7 Conclusion

When the principal $\arctan(x)$ and the $\operatorname{arccot}(x)$ functions are defined by (2.15) and (2.16), by choosing the other branch on the lower branch cut, and the principal $\operatorname{arctanh}(x)$ and $\operatorname{arccoth}(x)$ functions by (6.1) and (6.2), then these functions are related by (2.24) and (6.11) with $\operatorname{sg}(x)$ defined by (1.4).
For complex x:

$$\operatorname{arccot}(x) = \frac{\pi}{2}\operatorname{sg}(x) - \arctan(x) \tag{7.1}$$

$$\operatorname{arccoth}(x) = \frac{\pi}{2} i \operatorname{sg}(ix) + \operatorname{arctanh}(x) \tag{7.2}$$

and not $\operatorname{arccot}(x) = \arctan(1/x)$ and $\operatorname{arccoth}(x) = \operatorname{arctanh}(1/x)$, which are then replaced by (2.29), (2.30), (6.12) and (6.13). This way the functions are odd everywhere in the complex plane (except at $x = 0$ for the $\operatorname{arccot}(x)$ and $\operatorname{arccoth}(x)$ functions) and consistent with the addition formulas as mentioned in section 5. These formulas also have the advantage that no inversion of the arguments is required. For implementation of these functions theorems 2.2 and 6.2 with (1.5) and (1.20) may be used.
The corresponding Mathematica® [10] program:

```
Sg[x_]:=If[Re[x]>0,1,If[Re[x]<0,-1,If[Im[x]>=0,1,-1]]]
```

8 The Inverse Sine and Cosine Functions

For complex x:

$$\sin(x) = \frac{1}{2i}(e^{ix} - e^{-ix}) \tag{8.1}$$

$$\cos(x) = \frac{1}{2}(e^{ix} + e^{-ix}) \tag{8.2}$$

$$\sin^2(x) + \cos^2(x) = 1 \tag{8.3}$$

$$\sin(\pi - x) = \sin(x) \tag{8.4}$$

$$\cos(\pi - x) = -\cos(x) \tag{8.5}$$

$$\sin(\frac{\pi}{2} + x) = \cos(x) \tag{8.6}$$

$$\cos(-\frac{\pi}{2} + x) = \sin(x) \tag{8.7}$$

Definition 8.1. *For complex x [1]:*

$$\arcsin(x) = -i\ln(ix + \sqrt{1 - x^2}) \tag{8.8}$$

$$\arccos(x) = -i\ln(x + i\sqrt{1 - x^2}) \tag{8.9}$$

Theorem 8.1. *For complex x:*

$$\mathrm{Re}(ix + \sqrt{1 - x^2}) \geq 0 \tag{8.10}$$

and therefore the $\arcsin(x)$ *function defined in (8.8) is odd in the complex plane:*
For complex x:

$$\arcsin(-x) = -\arcsin(x) \tag{8.11}$$

Proof. Because $\sqrt{-x^2} = ix\,\mathrm{sg}(ix)$:

$$ix + \sqrt{-x^2} = ix(\mathrm{sg}(ix) + 1) \tag{8.12}$$

Because for $\mathrm{Re}(ix) < 0$, $\mathrm{sg}(ix) = -1$, $\mathrm{Re}(ix + \sqrt{-x^2}) \geq 0$. Using (1.9) it is clear that $\mathrm{Re}(\sqrt{-x^2 + 1}) \geq \mathrm{Re}(\sqrt{-x^2})$ so the theorem is proved. From:

$$(ix + \sqrt{1 - x^2})(-ix + \sqrt{1 - x^2}) = 1 \tag{8.13}$$

follows:

$$-ix + \sqrt{1 - x^2} = \frac{1}{ix + \sqrt{1 - x^2}} \tag{8.14}$$

Because when $\mathrm{Re}(x) \geq 0$: $\ln(1/x) = -\ln(x)$, the $\arcsin(x)$ function is odd in the complex plane. □

Theorem 8.2. *For complex x:*

$$\arcsin(x) + \arccos(x) = \frac{\pi}{2} \tag{8.15}$$

Proof. For complex x:

$$(ix + \sqrt{1 - x^2})(x + i\sqrt{1 - x^2}) = i \tag{8.16}$$

therefore:

$$x + i\sqrt{1 - x^2} = \frac{i}{ix + \sqrt{1 - x^2}} \tag{8.17}$$

From (1.18) and (1.25) with $c = 1$ follows:

$$\mathrm{Arg}(x) + \mathrm{Arg}(\frac{i}{x}) = \frac{\pi}{2} - 2\pi[\mathrm{Arg}(x) < -\frac{\pi}{2}] \tag{8.18}$$

With (1.28), $-i\ln(i) = \pi/2$ and the previous theorem, this theorem follows. □

Corollary 8.1. *For complex x:*

$$\arccos(-x) = \pi - \arccos(x) \tag{8.19}$$

Proof. Combination of the previous two theorems:

$$\begin{aligned}\arccos(-x) &= \frac{\pi}{2} - \arcsin(-x) = \frac{\pi}{2} + \arcsin(x) \\ &= \frac{\pi}{2} + (\frac{\pi}{2} - \arccos(x)) = \pi - \arccos(x)\end{aligned} \tag{8.20}$$

□

Theorem 8.3. *For complex x:*

$$\sin(\arcsin(x)) = x \tag{8.21}$$

$$\cos(\arccos(x)) = x \tag{8.22}$$

Proof. Using $(\sqrt{x})^2 = x$ and $(ix + \sqrt{1-x^2})(-ix + \sqrt{1-x^2}) = 1$:

$$\begin{aligned}\sin(\arcsin(x)) &= \frac{1}{2i}(e^{\ln(ix+\sqrt{1-x^2})} - e^{-\ln(ix+\sqrt{1-x^2})}) \\ &= \frac{1}{2i}(ix + \sqrt{1-x^2} - \frac{1}{ix+\sqrt{1-x^2}}) \\ &= \frac{1}{2i}(ix + \sqrt{1-x^2} + ix - \sqrt{1-x^2}) = x\end{aligned} \tag{8.23}$$

The proof of the second identity is similar. □

Theorem 8.4. *For complex x:*

$$\sin(\arccos(x)) = \sqrt{1-x^2} \tag{8.24}$$

$$\cos(\arcsin(x)) = \sqrt{1-x^2} \tag{8.25}$$

Proof. Using $(\sqrt{x})^2 = x$ and $(x + i\sqrt{1-x^2})(x - i\sqrt{1-x^2}) = 1$:

$$\begin{aligned}\sin(\arccos(x)) &= \frac{1}{2i}(e^{\ln(x+i\sqrt{1-x^2})} - e^{-\ln(x+i\sqrt{1-x^2})}) \\ &= \frac{1}{2i}(x + i\sqrt{1-x^2} - \frac{1}{x+i\sqrt{1-x^2}}) \\ &= \frac{1}{2i}(x + i\sqrt{1-x^2} - x + i\sqrt{1-x^2}) = \sqrt{1-x^2}\end{aligned} \tag{8.26}$$

The proof of the second identity is similar. □

Theorem 8.5. *For complex x:*

$$\begin{aligned}\mathrm{sg}(\sin(x)) =& [0 < \mathrm{Re}(x) \bmod 2\pi < \pi] - [\pi < \mathrm{Re}(x) \bmod 2\pi < 2\pi] \\ &+ [\mathrm{Re}(x) \bmod 2\pi = 0]([\mathrm{Im}(x) \geq 0] - [\mathrm{Im}(x) < 0]) \\ &+ [(\mathrm{Re}(x) + \pi) \bmod 2\pi = 0]([\mathrm{Im}(x) \leq 0] - [\mathrm{Im}(x) > 0])\end{aligned} \tag{8.27}$$

$$\begin{aligned}\operatorname{sg}(\cos(x)) =&[0 < (\operatorname{Re}(x) + \frac{\pi}{2}) \bmod 2\pi < \pi] - [\pi < (\operatorname{Re}(x) + \frac{\pi}{2}) \bmod 2\pi < 2\pi] \\ &+ [(\operatorname{Re}(x) + \frac{\pi}{2}) \bmod 2\pi = 0]([\operatorname{Im}(x) \geq 0] - [\operatorname{Im}(x) < 0]) \\ &+ [(\operatorname{Re}(x) + \frac{3\pi}{2}) \bmod 2\pi = 0]([\operatorname{Im}(x) \leq 0] - [\operatorname{Im}(x) > 0])\end{aligned} \tag{8.28}$$

Proof. For the first identity, for real a, b:

$$\begin{aligned}\sin(a+ib) &= \frac{1}{2i}(e^{ia}e^{-b} - e^{-ia}e^{b}) \\ &= -\frac{i}{2}((\cos(a) + i\sin(a))e^{-b} - (\cos(a) - i\sin(a))e^{b}) \\ &= \frac{1}{2}\sin(a)(e^{-b} + e^{b}) + \frac{1}{2}i\cos(a)(e^{b} - e^{-b})\end{aligned} \tag{8.29}$$

Using (1.16):

$$\begin{aligned}\operatorname{sg}(\sin(a+ib)) &= [\sin(a) > 0] - [\sin(a) < 0] \\ &+ [\sin(a) = 0]([\cos(a)(e^{b} - e^{-b}) \geq 0] - [\cos(a)(e^{b} - e^{-b}) < 0])\end{aligned} \tag{8.30}$$

For the second identity, using (8.6), $\operatorname{Re}(x)$ is replaced by $\operatorname{Re}(x) + \frac{\pi}{2}$. □

Definition 8.2. *For real x:*

$$T(x) = \operatorname{Arg}(e^{ix}) = \pi - ((\pi - x) \bmod 2\pi) \tag{8.31}$$

Theorem 8.6. *For complex x:*

$$\begin{aligned}\arcsin(\sin(x)) =&[\operatorname{sg}(\cos(x)) = 1](\pi - ((\pi - \operatorname{Re}(x)) \bmod 2\pi) + i\operatorname{Im}(x)) \\ &+ [\operatorname{sg}(\cos(x)) = -1](\pi - (\operatorname{Re}(x) \bmod 2\pi) - i\operatorname{Im}(x))\end{aligned} \tag{8.32}$$

$$\begin{aligned}\arccos(\sin(x)) =&[\operatorname{sg}(\cos(x)) = 1](-\frac{\pi}{2} + ((\pi - \operatorname{Re}(x)) \bmod 2\pi) - i\operatorname{Im}(x)) \\ &+ [\operatorname{sg}(\cos(x)) = -1](-\frac{\pi}{2} + (\operatorname{Re}(x) \bmod 2\pi) + i\operatorname{Im}(x))\end{aligned} \tag{8.33}$$

Proof. For the first identity:

$$\begin{aligned}\arcsin(\sin(x)) &= -i\ln(i\sin(x) - \sqrt{1 - \sin^2(x)}) \\ &= -i\ln(i\sin(x) + \sqrt{\cos^2(x)}) \\ &= -i\ln(i\sin(x) + \operatorname{sg}(\cos(x))\cos(x)) \\ &= -i\ln(\frac{1}{2}(e^{ix} - e^{-ix}) + \operatorname{sg}(\cos(x))\frac{1}{2}(e^{ix} + e^{-ix})) \\ &= [\operatorname{sg}(\cos(x)) = 1](-i\ln(e^{ix})) + [\operatorname{sg}(\cos(x)) = -1](-i\ln(-e^{-ix}))\end{aligned} \tag{8.34}$$

Taking $x = a + bi$, for real a, b:

$$-i\ln(e^{i(a+bi)}) = -i\ln(e^{ia}e^{-b}) = \operatorname{Arg}(e^{ia}) + ib = T(a) + ib \tag{8.35}$$

$$
\begin{aligned}
-i\ln(-e^{-i(a+bi)}) &= -i\ln(-e^{-ia}e^{b}) = \operatorname{Arg}(-e^{-ia}) - ib = \operatorname{Arg}(e^{-ia+i\pi}) - ib \\
&= \operatorname{Arg}(e^{i(\pi-a)}) - ib = T(\pi - a) - ib
\end{aligned} \tag{8.36}
$$

The second identity follows from theorem 8.2. □

Theorem 8.7. *For complex x:*

$$
\begin{aligned}
\arcsin(\cos(x)) =& [\operatorname{sg}(\sin(x)) = 1](\pi - ((\frac{\pi}{2} + \operatorname{Re}(x)) \bmod 2\pi) - i\operatorname{Im}(x)) \\
&+ [\operatorname{sg}(\sin(x)) = -1](\pi - ((\frac{\pi}{2} - \operatorname{Re}(x)) \bmod 2\pi) + i\operatorname{Im}(x))
\end{aligned} \tag{8.37}
$$

$$
\begin{aligned}
\arccos(\cos(x)) =& [\operatorname{sg}(\sin(x)) = 1](-\frac{\pi}{2} + ((\frac{\pi}{2} + \operatorname{Re}(x)) \bmod 2\pi) + i\operatorname{Im}(x)) \\
&+ [\operatorname{sg}(\sin(x)) = -1](-\frac{\pi}{2} + ((\frac{\pi}{2} - \operatorname{Re}(x)) \bmod 2\pi) - i\operatorname{Im}(x))
\end{aligned} \tag{8.38}
$$

Proof.

$$
\begin{aligned}
\arcsin(\cos(x)) &= -i\ln(i\cos(x) - \sqrt{1-\cos^2(x)}) \\
&= -i\ln(i\cos(x) + \sqrt{\sin^2(x)}) \\
&= -i\ln(i\cos(x) + \operatorname{sg}(\sin(x))\sin(x)) \\
&= -i\ln(\frac{i}{2}(e^{ix}+e^{-ix}) - \operatorname{sg}(\sin(x))\frac{i}{2}(e^{ix}-e^{-ix})) \\
&= [\operatorname{sg}(\sin(x)) = 1](-i\ln(ie^{-ix})) + [\operatorname{sg}(\sin(x)) = -1](-i\ln(ie^{ix}))
\end{aligned} \tag{8.39}
$$

Taking $x = a + bi$, for real a, b:

$$
\begin{aligned}
-i\ln(ie^{-i(a+bi)}) &= -i\ln(ie^{-ia}e^{b}) = \operatorname{Arg}(ie^{-ia}) - ib = \operatorname{Arg}(e^{-ia+i\frac{\pi}{2}}) - ib \\
&= \operatorname{Arg}(e^{i(\frac{\pi}{2}-a)}) - ib = T(\frac{\pi}{2} - a) - ib
\end{aligned} \tag{8.40}
$$

$$
\begin{aligned}
-i\ln(ie^{i(a+bi)}) &= -i\ln(ie^{ia}e^{-b}) = \operatorname{Arg}(ie^{ia}) + ib = \operatorname{Arg}(e^{ia+i\frac{\pi}{2}}) + ib \\
&= \operatorname{Arg}(e^{i(\frac{\pi}{2}+a)}) + ib = T(\frac{\pi}{2} + a) + ib
\end{aligned} \tag{8.41}
$$

The second identity follows from theorem 8.2. □

9 Relations between the Inverse Sine and Cosine and the Inverse Tangent and Cotangent Functions

Theorem 9.1. *For complex x:*

$$
\sin(\arctan(x)) = \frac{x}{\sqrt{1+x^2}} \tag{9.1}
$$

Proof. Using theorem 2.5, the definition of $\sin(x)$ (8.1), $(\sqrt{x})^2 = x$, $e^{\ln(x)} = x$, $e^{-\ln(x)} = 1/x$ and:

$$
(1 + ix + \sqrt{1+x^2})(1 - ix + \sqrt{1+x^2}) = 2(1 + x^2 + \sqrt{1+x^2}) \tag{9.2}
$$

$$
\begin{aligned}
\sin(\arctan(x)) &= \frac{1}{2i}(e^{\ln(\frac{1+ix+\sqrt{1+x^2}}{1-ix+\sqrt{1+x^2}})} - e^{-\ln(\frac{1+ix+\sqrt{1+x^2}}{1-ix+\sqrt{1+x^2}})}) \\
&= \frac{1}{2i}(\frac{1+ix+\sqrt{1+x^2}}{1-ix+\sqrt{1+x^2}} - \frac{1-ix+\sqrt{1+x^2}}{1+ix+\sqrt{1+x^2}}) \\
&= \frac{1}{2i}\frac{(1+ix+\sqrt{1+x^2})^2 - (1-ix+\sqrt{1+x^2})^2}{2(1+x^2+\sqrt{1+x^2})} \\
&= \frac{1}{2i}\frac{4ix(1+\sqrt{1+x^2})}{2(1+x^2+\sqrt{1+x^2})} = \frac{x(1+\sqrt{1+x^2})}{\sqrt{1+x^2}(\sqrt{1+x^2}+1)} = \frac{x}{\sqrt{1+x^2}}
\end{aligned} \tag{9.3}
$$

□

Theorem 9.2. *For complex x:*

$$\cos(\arctan(x)) = \frac{1}{\sqrt{1+x^2}} \tag{9.4}$$

Proof. Using theorem 2.5, the definition of $\cos(x)$ (8.2), and the identities in the previous theorem:

$$
\begin{aligned}
\cos(\arctan(x)) &= \frac{1}{2}(e^{\ln(\frac{1+ix+\sqrt{1+x^2}}{1-ix+\sqrt{1+x^2}})} + e^{-\ln(\frac{1+ix+\sqrt{1+x^2}}{1-ix+\sqrt{1+x^2}})}) \\
&= \frac{1}{2}(\frac{1+ix+\sqrt{1+x^2}}{1-ix+\sqrt{1+x^2}} + \frac{1-ix+\sqrt{1+x^2}}{1+ix+\sqrt{1+x^2}}) \\
&= \frac{1}{2}\frac{(1+ix+\sqrt{1+x^2})^2 + (1-ix+\sqrt{1+x^2})^2}{2(1+x^2+\sqrt{1+x^2})} \\
&= \frac{1}{2}\frac{4(1+\sqrt{1+x^2})}{2(1+x^2+\sqrt{1+x^2})} = \frac{1+\sqrt{1+x^2}}{\sqrt{1+x^2}(\sqrt{1+x^2}+1)} = \frac{1}{\sqrt{1+x^2}}
\end{aligned} \tag{9.5}
$$

□

Theorem 9.3. *For complex x:*

$$\sin(\operatorname{arccot}(x)) = \frac{\operatorname{sg}(x)}{\sqrt{1+x^2}} \tag{9.6}$$

Proof. Using theorem 2.6, the definition of $\sin(x)$ (8.1), $(\sqrt{x})^2 = x$, $e^{\ln(x)} = x$, $e^{-\ln(x)} = 1/x$ and:

$$e^{\pm i\frac{\pi}{2}\operatorname{sg}(x)} = \pm i\operatorname{sg}(x) \tag{9.7}$$

$$(1+ix+\sqrt{1+x^2})(1-ix+\sqrt{1+x^2}) = 2(1+x^2+\sqrt{1+x^2}) \tag{9.8}$$

$$
\begin{aligned}
\sin(\operatorname{arccot}(x)) &= \frac{1}{2i}(e^{i\frac{\pi}{2}\operatorname{sg}(x)+\ln(\frac{1-ix+\sqrt{1+x^2}}{1+ix+\sqrt{1+x^2}})} - e^{-i\frac{\pi}{2}\operatorname{sg}(x)-\ln(\frac{1-ix+\sqrt{1+x^2}}{1+ix+\sqrt{1+x^2}})}) \\
&= \frac{1}{2i}(i\operatorname{sg}(x)\frac{1-ix+\sqrt{1+x^2}}{1+ix+\sqrt{1+x^2}} + i\operatorname{sg}(x)\frac{1+ix+\sqrt{1+x^2}}{1-ix+\sqrt{1+x^2}}) \\
&= \operatorname{sg}(x)\frac{1}{2}\frac{(1-ix+\sqrt{1+x^2})^2+(1+ix+\sqrt{1+x^2})^2}{2(1+x^2+\sqrt{1+x^2})} \\
&= \frac{1}{2}\frac{\operatorname{sg}(x)4(1+\sqrt{1+x^2})}{2(1+x^2+\sqrt{1+x^2})} = \frac{\operatorname{sg}(x)(1+\sqrt{1+x^2})}{\sqrt{1+x^2}(\sqrt{1+x^2}+1)} = \frac{\operatorname{sg}(x)}{\sqrt{1+x^2}}
\end{aligned}
\tag{9.9}
$$

□

Theorem 9.4. *For complex x:*

$$
\cos(\operatorname{arccot}(x)) = \frac{\operatorname{sg}(x)x}{\sqrt{1+x^2}} \tag{9.10}
$$

Proof. Using theorem 2.6, the definition of $\cos(x)$ (8.2), and the identities in the previous theorem:

$$
\begin{aligned}
\cos(\operatorname{arccot}(x)) &= \frac{1}{2}(e^{i\frac{\pi}{2}\operatorname{sg}(x)+\ln(\frac{1-ix+\sqrt{1+x^2}}{1+ix+\sqrt{1+x^2}})} + e^{-i\frac{\pi}{2}\operatorname{sg}(x)-\ln(\frac{1-ix+\sqrt{1+x^2}}{1+ix+\sqrt{1+x^2}})}) \\
&= \frac{1}{2}(i\operatorname{sg}(x)\frac{1-ix+\sqrt{1+x^2}}{1+ix+\sqrt{1+x^2}} - i\operatorname{sg}(x)\frac{1+ix+\sqrt{1+x^2}}{1-ix+\sqrt{1+x^2}}) \\
&= \operatorname{sg}(x)\frac{i}{2}\frac{(1-ix+\sqrt{1+x^2})^2-(1+ix+\sqrt{1+x^2})^2}{2(1+x^2+\sqrt{1+x^2})} \\
&= \frac{i}{2}\frac{-\operatorname{sg}(x)4ix(1+\sqrt{1+x^2})}{2(1+x^2+\sqrt{1+x^2})} = \frac{\operatorname{sg}(x)x(1+\sqrt{1+x^2})}{\sqrt{1+x^2}(\sqrt{1+x^2}+1)} = \frac{\operatorname{sg}(x)x}{\sqrt{1+x^2}}
\end{aligned}
\tag{9.11}
$$

□

Lemma 9.1. *For complex x:*

$$
\operatorname{sg}(\frac{1}{\sqrt{1+x^2}}) = 1 - 2[\operatorname{Re}(x)=0][|\operatorname{Im}(x)|>1] \tag{9.12}
$$

$$
\operatorname{sg}(\frac{1}{\sqrt{1-x^2}}) = 1 - 2[\operatorname{Im}(x)=0][|\operatorname{Re}(x)|>1] \tag{9.13}
$$

$$
\operatorname{sg}(\frac{x}{\sqrt{1+x^2}}) = \operatorname{sg}(x) \tag{9.14}
$$

$$
\operatorname{sg}(\frac{x}{\sqrt{1-x^2}}) = \operatorname{sg}(x)(1 - 2[\operatorname{Im}(x)=0][|\operatorname{Re}(x)|>1]) \tag{9.15}
$$

$$
[\operatorname{Re}(\frac{x}{\sqrt{1-x^2}})=0] = [\operatorname{Re}(x)=0] + [\operatorname{Im}(x)=0][|\operatorname{Re}(x)|>1] \tag{9.16}
$$

For complex $x \neq 0$:

$$
\operatorname{sg}(\frac{\sqrt{1-x^2}}{x}) = \operatorname{sg}(x)(1 - 2[\operatorname{Re}(x)=0]) \tag{9.17}
$$

Proof. From the complex plane it is clear that $\mathrm{sg}(1/\sqrt{x}) = -1$ if and only if $\mathrm{Im}(x) = 0$ and $\mathrm{Re}(x) < 0$, from which follows (9.12), and (9.13) is obtained by replacing x by ix. From the complex plane it is clear that for complex x:

$$\mathrm{sg}(\frac{\sqrt{x^2}}{\sqrt{x^2+1}}) = 1 \tag{9.18}$$

$$\mathrm{sg}(\frac{\sqrt{x^2}}{\sqrt{-x^2+1}}) = 1 - 2[\mathrm{Im}(x) = 0][|\mathrm{Re}(x)| > 1] \tag{9.19}$$

Using $\sqrt{x^2} = \mathrm{sg}(x)x$ and $\mathrm{sg}(-x) = -\mathrm{sg}(x)$ (9.14) and (9.15) follow.
For (9.16):

$$\begin{aligned}[\mathrm{Re}(\frac{x}{\sqrt{1-x^2}}) = 0] &= [\mathrm{Im}(\frac{x^2}{1-x^2}) = 0][\mathrm{Re}(\frac{x^2}{1-x^2}) < 0] \\ &= [\mathrm{Re}(x) = 0] + [\mathrm{Im}(x) = 0][|\mathrm{Re}(x)| > 1]\end{aligned} \tag{9.20}$$

Using (1.17) and (9.16):

$$\begin{aligned}&\mathrm{sg}(\frac{\sqrt{1-x^2}}{x}) \\ &= \mathrm{sg}(\frac{x}{\sqrt{1-x^2}}) - 2([\mathrm{Re}(x) = 0] + [\mathrm{Im}(x) = 0][|\mathrm{Re}(x)| > 1])\mathrm{sg}(\mathrm{Im}(\frac{x}{\sqrt{1-x^2}})) \\ &= \mathrm{sg}(\frac{x}{\sqrt{1-x^2}}) - 2([\mathrm{Re}(x) = 0]\mathrm{sg}(\mathrm{Im}(x)) - [\mathrm{Im}(x) = 0][|\mathrm{Re}(x)| > 1]\mathrm{sg}(\mathrm{Re}(x)))\end{aligned} \tag{9.21}$$

which with (9.15) results in (9.17). □

Theorem 9.5. *For complex x:*

$$\arccos(\frac{1}{\sqrt{1+x^2}}) = \mathrm{sg}(x)\arctan(x) \tag{9.22}$$

Proof. Application of theorem 9.2:

$$\arccos(\frac{1}{\sqrt{1+x^2}}) = \arccos(\cos(\arctan(x))) \tag{9.23}$$

Application of theorem 8.7 with theorem 9.1 and lemma 9.1:

$$\mathrm{sg}(\sin(\arctan(x))) = \mathrm{sg}(\frac{x}{\sqrt{1+x^2}}) = \mathrm{sg}(x) \tag{9.24}$$

Using that $-\pi/2 \le \mathrm{Re}(\arctan(x)) \le \pi/2$ the theorem follows. □

Theorem 9.6. *For complex $x \neq 0$:*

$$\arctan(\frac{\sqrt{1-x^2}}{x}) = \arccos(x) - \pi[\mathrm{sg}(x)(1 - 2[\mathrm{Re}(x) = 0]) = -1] \tag{9.25}$$

Proof. In the previous theorem replacing x with $\sqrt{1-x^2}/x$ and using that $\mathrm{sg}(1/\sqrt{x}) = -1$ if and only if $\mathrm{Im}(x) = 0$ and $\mathrm{Re}(x) < 0$:

$$\frac{1}{\sqrt{1+\frac{1-x^2}{x^2}}} = \frac{1}{\sqrt{\frac{1}{x^2}}} = \sqrt{x^2}\,\mathrm{sg}(\frac{1}{\sqrt{x^2}}) = x\,\mathrm{sg}(x)(1-2[\mathrm{Re}(x)=0]) \tag{9.26}$$

Using lemma 9.1 and $\arccos(-x) = \pi - \arccos(x)$ gives the theorem. □

Theorem 9.7. *For complex x:*

$$\arcsin(\frac{1}{\sqrt{1+x^2}}) = \mathrm{sg}(x)\mathrm{arccot}(x) \tag{9.27}$$

Proof. Application of theorem 9.3:

$$\arcsin(\frac{1}{\sqrt{1+x^2}}) = \arcsin(\sin(\arctan(x))) \tag{9.28}$$

Application of theorem 8.6 with theorem 9.4 and lemma 9.1:

$$\mathrm{sg}(\cos(\mathrm{arccot}(x))) = \mathrm{sg}(\frac{\mathrm{sg}(x)x}{\sqrt{1+x^2}}) = \mathrm{sg}^2(x) = 1 \tag{9.29}$$

Using that $-\pi/2 \le \mathrm{Re}(\mathrm{arccot}(x)) \le \pi/2$ the theorem follows. □

Theorem 9.8. *For complex* $x \neq 0$*:*

$$\mathrm{arccot}(\frac{\sqrt{1-x^2}}{x}) = \arcsin(x) \tag{9.30}$$

Proof. In the previous theorem replacing x with $\sqrt{1-x^2}/x$ and using the same method as in theorem 9.6 and lemma 9.1 and $\arcsin(-x) = -\arcsin(x)$ gives the theorem. □

Theorem 9.9. *For complex x:*

$$\begin{aligned}\arcsin(\frac{x}{\sqrt{1+x^2}}) =& (1-2[\mathrm{Re}(x)=0][|\mathrm{Im}(x)|>1])\arctan(x) \\ &+ \pi[\mathrm{Re}(x)=0]([\mathrm{Im}(x)>1]-[\mathrm{Im}(x)<-1])\end{aligned} \tag{9.31}$$

Proof. Application of theorem 9.1:

$$\arcsin(\frac{x}{\sqrt{1+x^2}}) = \arcsin(\sin(\arctan(x))) \tag{9.32}$$

Application of theorem 8.6 with theorem 9.2 and lemma 9.1:

$$\mathrm{sg}(\cos(\arctan(x))) = \mathrm{sg}(\frac{1}{\sqrt{1+x^2}}) = 1-2[\mathrm{Re}(x)=0][|\mathrm{Im}(x)|>1] \tag{9.33}$$

Using that $-\pi/2 \le \mathrm{Re}(\arctan(x)) \le \pi/2$, with equal signs on the upper and lower branch cuts, the theorem follows. □

Theorem 9.10. *For complex x:*

$$\arctan(\frac{x}{\sqrt{1-x^2}}) = \arcsin(x) - \pi[\mathrm{Im}(x)=0]([\mathrm{Re}(x)>1] - [\mathrm{Re}(x)<-1]) \tag{9.34}$$

Proof. In the previous theorem replacing x with $x/\sqrt{1-x^2}$ and using lemma 9.1:

$$\frac{\frac{x}{\sqrt{1-x^2}}}{\sqrt{1+\frac{x^2}{1-x^2}}} = \frac{\frac{x}{\sqrt{1-x^2}}}{\sqrt{\frac{1}{1-x^2}}} = x\,\mathrm{sg}(\frac{1}{\sqrt{1-x^2}}) = x(1-2[\mathrm{Im}(x)=0][|\mathrm{Re}(x)|>1]) \tag{9.35}$$

From lemma 9.1:

$$[\mathrm{Re}(\frac{x}{\sqrt{1-x^2}})=0] = [\mathrm{Re}(x)=0] + [\mathrm{Im}(x)=0][|\mathrm{Re}(x)|>1] \tag{9.36}$$

In the second case only $|\mathrm{Im}(x/\sqrt{1-x^2})|>1$ where $\mathrm{Im}(x/\sqrt{1-x^2})>1$ if $\mathrm{Re}(x)<-1$. With $\arcsin(-x) = -\arcsin(x)$ follows:

$$\begin{aligned}\arcsin(x) &= (1-2[\mathrm{Im}(x)=0][|\mathrm{Re}(x)|>1])^2 \arctan(\frac{x}{\sqrt{1-x^2}}) \\ &+ \pi(1-2[\mathrm{Im}(x)=0][|\mathrm{Re}(x)|>1])[\mathrm{Im}(x)=0]([\mathrm{Re}(x)<-1]-[\mathrm{Re}(x)>1])\end{aligned} \tag{9.37}$$

which yields the theorem. □

Theorem 9.11. *For complex x:*

$$\arccos(\frac{x}{\sqrt{1+x^2}}) = (1-2[\mathrm{Re}(x)=0][|\mathrm{Im}(x)|>1])\mathrm{arccot}(x) + \pi[\mathrm{sg}(x)=-1] \tag{9.38}$$

Proof. Application of theorem 9.4:

$$\arccos(\frac{\mathrm{sg}(x)x}{\sqrt{1+x^2}}) = \arccos(\cos(\mathrm{arccot}(x))) \tag{9.39}$$

Application of theorem 8.7 with theorem 9.3 and $\mathrm{sg}(-x) = -\mathrm{sg}(x)$ and lemma 9.1:

$$\begin{aligned}\mathrm{sg}(\sin(\mathrm{arccot}(x))) &= \mathrm{sg}(\frac{\mathrm{sg}(x)}{\sqrt{1+x^2}}) = \mathrm{sg}(x)\mathrm{sg}(\frac{1}{\sqrt{1+x^2}}) \\ &= \mathrm{sg}(x)(1-2[\mathrm{Re}(x)=0][|\mathrm{Im}(x)|>1])\end{aligned} \tag{9.40}$$

Using that $-\pi/2 \le \mathrm{Re}(\mathrm{arccot}(x)) \le \pi/2$ and $\arccos(-x) = \pi - \arccos(x)$:

$$\begin{aligned}\arccos(\frac{\mathrm{sg}(x)x}{\sqrt{1+x^2}}) &= \pi[\mathrm{sg}(x)=-1] + \mathrm{sg}(x)\arccos(\frac{x}{\sqrt{1+x^2}}) \\ &= \mathrm{sg}(x)(1-2[\mathrm{Re}(x)=0][|\mathrm{Im}(x)|>1])\mathrm{arccot}(x)\end{aligned} \tag{9.41}$$

Multiplying this identity by $\mathrm{sg}(x)$ and using $\mathrm{sg}^2(x)=1$ and:

$$\mathrm{sg}(x)[\mathrm{sg}(x)=-1] = -[\mathrm{sg}(x)=-1] \tag{9.42}$$

gives the theorem. □

Theorem 9.12. *For complex x:*

$$\operatorname{arccot}(\frac{x}{\sqrt{1-x^2}}) = \arccos(x) - \pi[\operatorname{sg}(x) = -1] \tag{9.43}$$

Proof. In the previous theorem replacing x with $\sqrt{1-x^2}/x$ and using the same method as in theorem 9.10 and using lemma 9.1:

$$\begin{aligned}&\arccos(x(1-2[\operatorname{Im}(x)=0][|\operatorname{Re}(x)|>1]))\\&= (1-2[\operatorname{Im}(x)=0][|\operatorname{Re}(x)|>1])\operatorname{arccot}(\frac{x}{\sqrt{1-x^2}}) + \pi[\operatorname{sg}(\frac{x}{\sqrt{1-x^2}}) = -1]\end{aligned} \tag{9.44}$$

Multiplying by $1-2[\operatorname{Im}(x)=0][|\operatorname{Re}(x)|>1]$ and using lemma 9.1:

$$\begin{aligned}&\operatorname{arccot}(\frac{x}{\sqrt{1-x^2}}) = \arccos(x) + \pi(1-2[\operatorname{Im}(x)=0][|\operatorname{Re}(x)|>1])\\&\cdot([\operatorname{Im}(x)=0][|\operatorname{Re}(x)|>1] - [\operatorname{sg}(x)(1-2[\operatorname{Im}(x)=0][|\operatorname{Re}(x)|>1]) = -1])\end{aligned} \tag{9.45}$$

The second term on the right side is zero when $\operatorname{sg}(x) = 1$ and $-\pi$ when $\operatorname{sg}(x) = -1$, which is the theorem. □

10 Addition Formulas for the Inverse Sine and Cosine Functions for Real Arguments

Lemma 10.1. *For real $-1 \le x \le 1$ and $-1 \le y \le 1$ but not $x = y = -1$:*

$$\operatorname{sg}(x\sqrt{1-y^2} + y\sqrt{1-x^2}) = \operatorname{sg}(x+y) \tag{10.1}$$

For real $-1 \le x \le 1$ and $-1 \le y \le 1$:

$$\operatorname{sg}(\sqrt{(1-x^2)(1-y^2)} - xy) = 1 - 2[\operatorname{sg}(x)\operatorname{sg}(y) = 1][x^2+y^2 > 1] \tag{10.2}$$

$$[|xy| < \operatorname{sg}(x)\operatorname{sg}(y)\sqrt{(1-x^2)(1-y^2)}] = [\operatorname{sg}(x)\operatorname{sg}(y) = 1][x^2+y^2 < 1] \tag{10.3}$$

$$\operatorname{sg}(\operatorname{sg}(x)\operatorname{sg}(y) - xy + \sqrt{(1-x^2)(1-y^2)}) = \operatorname{sg}(x)\operatorname{sg}(y) + 2[y = -x \ne 0] \tag{10.4}$$

Proof. For the first identity, when $\operatorname{sg}(x) = \operatorname{sg}(y)$ the identity is obvious. When $\operatorname{sg}(x) = -\operatorname{sg}(y)$, suppose that x is nonnegative and y negative. Then as $-y$ is positive:

$$\begin{aligned}&[x\sqrt{1-y^2} + y\sqrt{1-x^2} \ge 0] = [x\sqrt{1-y^2} \ge -y\sqrt{1-x^2}]\\&= [x^2(1-y^2) \ge y^2(1-x^2)] = [x^2 \ge y^2] = [x \ge -y] = [x+y \ge 0]\end{aligned} \tag{10.5}$$

which confirms the first identity, and similarly when x is negative and y nonnegative.
For the second identity, when $\operatorname{sg}(x) = -\operatorname{sg}(y)$, the argument is always nonnegative, so then the result is one. When $\operatorname{sg}(x) = \operatorname{sg}(y)$:

$$\begin{aligned}&[\sqrt{(1-x^2)(1-y^2)} - xy < 0] = [\sqrt{(1-x^2)(1-y^2)} < xy]\\&= [(1-x^2)(1-y^2) < x^2y^2] = [x^2+y^2 > 1]\end{aligned} \tag{10.6}$$

which results in the second identity. A similar reasoning results in the third identity. For the fourth identity, because $-1 \leq xy \leq 1$ it is easily shown that:

$$-1 \leq \sqrt{(1-x^2)(1-y^2)} - xy \leq 1 \tag{10.7}$$

which is equal to 1 if and only if $y = -x$, which results in the identity. □

Theorem 10.1. *For real $-1 \leq x \leq 1$, $-1 \leq y \leq 1$:*

$$\begin{aligned} &\arcsin(x) + \arcsin(y) \\ &= (1 - 2[\mathrm{sg}(x)\mathrm{sg}(y) = 1][x^2 + y^2 > 1]) \arcsin(x\sqrt{1-y^2} + y\sqrt{1-x^2}) \\ &\quad + \pi[\mathrm{sg}(x)\mathrm{sg}(y) = 1][x^2 + y^2 > 1]\mathrm{sg}(x) \end{aligned} \tag{10.8}$$

$$\begin{aligned} &\arccos(x) + \arccos(y) \\ &= (1 - 2[\mathrm{sg}(x)\mathrm{sg}(y) = 1][x^2 + y^2 > 1]) \arccos(x\sqrt{1-y^2} + y\sqrt{1-x^2}) \\ &\quad + \pi(\frac{1}{2} + [\mathrm{sg}(x)\mathrm{sg}(y) = 1][x^2 + y^2 > 1](1 - \mathrm{sg}(x))) \end{aligned} \tag{10.9}$$

Proof. Application of theorem 4.1 and 9.10:

$$\begin{aligned} \arcsin(x) + \arcsin(y) = &\arctan(\frac{x\sqrt{1-y^2} + y\sqrt{1-x^2}}{\sqrt{(1-x^2)(1-y^2)} - xy}) \\ &+ \pi[xy > \sqrt{(1-x^2)(1-y^2)}]\mathrm{sg}(x) \end{aligned} \tag{10.10}$$

Application of theorem 9.9 and:

$$\frac{\alpha/\beta}{\sqrt{1 + (\alpha/\beta)^2}} = \frac{\alpha\,\mathrm{sg}(\beta)}{\sqrt{\alpha^2 + \beta^2}} \tag{10.11}$$

where in this case $\alpha^2 + \beta^2 = 1$, and from lemma 10.1:

$$[xy > \sqrt{(1-x^2)(1-y^2)}] = [\mathrm{sg}(x)\mathrm{sg}(y) = 1][x^2 + y^2 > 1] \tag{10.12}$$

$$\mathrm{sg}(\sqrt{(1-x^2)(1-y^2)} - xy) = 1 - 2[\mathrm{sg}(x)\mathrm{sg}(y) = 1][x^2 + y^2 > 1]) \tag{10.13}$$

yields the first identity. For the second identity, using theorem 8.2:

$$\begin{aligned} \arccos(x) + \arccos(y) &= \pi - (\arcsin(x) + \arcsin(y)) \\ &= \pi - ((1 - 2f(x,y)) \arcsin(g(x,y)) + \pi f(x,y)\mathrm{sg}(x)) \\ &= \pi - ((1 - 2f(x,y))(\frac{\pi}{2} - \arccos(g(x,y)))) - \pi f(x,y)\mathrm{sg}(x) \\ &= (1 - 2f(x,y)) \arccos(g(x,y)) + \pi(\frac{1}{2} + f(x,y)(1 - \mathrm{sg}(x))) \end{aligned} \tag{10.14}$$

□

Corollary 10.1. *For real* $-1 \le x \le 1$*:*

$$\arcsin(\sqrt{1-x^2}) = \arccos(|x|) \tag{10.15}$$

$$\arccos(\sqrt{1-x^2}) = \arcsin(|x|) \tag{10.16}$$

Proof. This corollary follows from the previous theorem taking $y = 1$:

$$\begin{aligned}\arccos(x) &= (1-2[x>0])\arccos(\sqrt{1-x^2}) + \pi(\frac{\pi}{2} + [x>0](1-\text{sg}(x)))\\ &= (-\text{sg}(x) + 2[x=0])\arccos(\sqrt{1-x^2}) + \frac{\pi}{2}\\ &= -\text{sg}(x)\arccos(\sqrt{1-x^2}) + \frac{\pi}{2}\end{aligned} \tag{10.17}$$

and using theorem 8.2:

$$\begin{aligned}\arccos(\sqrt{1-x^2}) &= -\text{sg}(x)(\arccos(x) - \frac{\pi}{2})\\ &= \text{sg}(x)\arcsin(x) = \arcsin(\text{sg}(x)x) = \arcsin(|x|)\end{aligned} \tag{10.18}$$

and again using theorem 8.2:

$$\begin{aligned}\arcsin(\sqrt{1-x^2}) &= \frac{\pi}{2} - \arccos(\sqrt{1-x^2}) = \frac{\pi}{2} - \arcsin(|x|)\\ &= \frac{\pi}{2} - (\frac{\pi}{2} - \arccos(|x|)) = \arccos(|x|)\end{aligned} \tag{10.19}$$

□

Theorem 10.2. *For real* $-1 \le x \le 1$*,* $-1 \le y \le 1$*:*

$$\begin{aligned}\arcsin(x) + \arcsin(y) &= \text{sg}(x+y)\arcsin(xy - \sqrt{(1-x^2)(1-y^2)})\\ &+ \pi(1 - [x<0] - [y<0] + ([\text{sg}(x)\text{sg}(y) = -1] - \frac{1}{2})\text{sg}(x+y))\end{aligned} \tag{10.20}$$

$$\begin{aligned}\arccos(x) + \arccos(y) &= \text{sg}(x+y)\arccos(xy - \sqrt{(1-x^2)(1-y^2)})\\ &+ \pi([x<0] + [y<0] - [\text{sg}(x)\text{sg}(y) = -1]\text{sg}(x+y))\end{aligned} \tag{10.21}$$

Proof. Application of theorem 4.2 and 9.12:

$$\begin{aligned}&\arccos(x) + \arccos(y) = \text{arccot}(\frac{xy - \sqrt{(1-x^2)(1-y^2)}}{x\sqrt{1-y^2} + y\sqrt{1-x^2}})\\ &+ \pi([x<0] + [y<0] + [|xy| < \text{sg}(x)\text{sg}(y)\sqrt{(1-x^2)(1-y^2)}]\text{sg}(x+y))\end{aligned} \tag{10.22}$$

Application of theorem 9.11 and using the same method as in the previous theorem, and from lemma 10.1:

$$[|xy| < \text{sg}(x)\text{sg}(y)\sqrt{(1-x^2)(1-y^2)}] = [\text{sg}(x)\text{sg}(y) = 1][x^2 + y^2 < 1] \tag{10.23}$$

$$\text{sg}(x\sqrt{1-y^2} + y\sqrt{1-x^2}) = \text{sg}(x+y) \tag{10.24}$$

and when $y \neq -x$:

$$\begin{aligned}
&[\frac{xy - \sqrt{(1-x^2)(1-y^2)}}{x\sqrt{1-y^2} + y\sqrt{1-x^2}} < 0] \\
&= [x+y<0][xy > \sqrt{(1-x^2)(1-y^2)}] + [x+y>0][xy < \sqrt{(1-x^2)(1-y^2)}] \\
&= [x+y>0](1 - [xy = \sqrt{(1-x^2)(1-y^2)}]) + [xy > \sqrt{(1-x^2)(1-y^2)}]\mathrm{sg}(x+y) \\
&= [x+y>0](1 - [\mathrm{sg}(x)\mathrm{sg}(y)=1][x^2+y^2=1]) + [\mathrm{sg}(x)\mathrm{sg}(y)=1][x^2+y^2>1]\mathrm{sg}(x+y)
\end{aligned} \tag{10.25}$$

Using $[x^2+y^2<1] = 1 - [x^2+y^2=1] - [x^2+y^2>1]$ and using the special case $xy=1$ in theorem 4.2 and $1 - [\mathrm{sg}(x)\mathrm{sg}(y)=1] = [\mathrm{sg}(x)\mathrm{sg}(y)=-1]$ yields the theorem, which is also correct when $y=-x$. □

Theorem 10.3. *For real $-1 \leq x \leq 1$, $-1 \leq y \leq 1$, but not $x = y = \pm 1$:*

$$\arcsin(x) + \arcsin(y) = 2\arcsin(\frac{x\sqrt{1-y^2} + y\sqrt{1-x^2}}{\sqrt{2(1 - xy + \sqrt{(1-x^2)(1-y^2)})}}) \tag{10.26}$$

$$\arccos(x) + \arccos(y) = 2\arccos(\frac{x\sqrt{1-y^2} + y\sqrt{1-x^2}}{\sqrt{2(1 - xy + \sqrt{(1-x^2)(1-y^2)})}}) \tag{10.27}$$

When $x = y = \pm 1$, $2\arcsin(\pm 1) = \pm\pi$, $2\arccos(1) = 0$ and $2\arccos(-1) = 2\pi$.

Proof. Application of theorem 4.4 and 9.10:

$$\arcsin(x) + \arcsin(y) = 2\arctan(\frac{x\sqrt{1-y^2} + y\sqrt{1-x^2}}{1 - xy + \sqrt{(1-x^2)(1-y^2)}}) \tag{10.28}$$

Application of theorem 9.9 yields the first identity. For the second identity, using theorem 8.2:

$$\begin{aligned}
\arccos(x) + \arccos(y) &= \pi - (\arcsin(x) + \arcsin(y)) = \pi - 2\arcsin(f(x,y)) \\
&= \pi - 2(\frac{\pi}{2} - \arccos(f(x,y))) = 2\arccos(f(x,y))
\end{aligned} \tag{10.29}$$

□

Theorem 10.4. *For real $-1 \leq x \leq 1$, $-1 \leq y \leq 1$, but not $y = -x \neq 0$ or $x = y = \pm 1$:*

$$\begin{aligned}
&\arcsin(x) + \arcsin(y) \\
&= 2\,\mathrm{sg}(x)\mathrm{sg}(y)\arcsin(\frac{x\sqrt{1-y^2} + y\sqrt{1-x^2}}{\sqrt{2(1 - \mathrm{sg}(x)\mathrm{sg}(y)(xy - \sqrt{(1-x^2)(1-y^2)}))}}) \\
&\quad + \pi[\mathrm{sg}(x)\mathrm{sg}(y) = -1]\mathrm{sg}(x+y)
\end{aligned} \tag{10.30}$$

$$\begin{aligned}
&\arccos(x) + \arccos(y) \\
&= 2\,\mathrm{sg}(x)\mathrm{sg}(y)\arccos(\frac{x\sqrt{1-y^2}+y\sqrt{1-x^2}}{\sqrt{2(1-\mathrm{sg}(x)\mathrm{sg}(y)(xy-\sqrt{(1-x^2)(1-y^2)}))}}) \\
&\quad + \pi[\mathrm{sg}(x)\mathrm{sg}(y)=-1](1+2[x+y<0])
\end{aligned} \tag{10.31}$$

When $y=-x$, $\arcsin(x)+\arcsin(-x)=0$ *and* $\arccos(x)+\arccos(-x)=\pi$*.*

Proof. Application of theorem 4.6 and 9.12 and lemma 9.1, when not $y=-x\neq 0$:

$$\begin{aligned}
\arccos(x)+\arccos(y) =& 2\,\mathrm{arccot}(\frac{x\sqrt{1-y^2}+y\sqrt{1-x^2}}{\mathrm{sg}(x)\mathrm{sg}(y)-xy+\sqrt{(1-x^2)(1-y^2)}}) \\
&+\pi([\mathrm{sg}(x)=-1]+[\mathrm{sg}(y)=-1])
\end{aligned} \tag{10.32}$$

Application of theorem 9.11 and when $xy\neq 0$: $\mathrm{sg}(x/y)=\mathrm{sg}(x)\mathrm{sg}(y)$ and from lemma 10.1:

$$\mathrm{sg}(x\sqrt{1-y^2}+y\sqrt{1-x^2}) = \mathrm{sg}(x+y) \tag{10.33}$$

and when not $y=-x$:

$$\mathrm{sg}(\mathrm{sg}(x)\mathrm{sg}(y)-xy+\sqrt{(1-x^2)(1-y^2)}) = \mathrm{sg}(x)\mathrm{sg}(y) \tag{10.34}$$

and using $[\mathrm{sg}(x+y)=1]=1-[\mathrm{sg}(x+y)=-1]$:

$$\begin{aligned}
&\pi([\mathrm{sg}(x)=-1]+[\mathrm{sg}(y)=-1]+2([\mathrm{sg}(x)\mathrm{sg}(y)=-1]-[\mathrm{sg}(x)\mathrm{sg}(y)\mathrm{sg}(x+y)=-1])) \\
&= \pi[\mathrm{sg}(x)\mathrm{sg}(y)=-1](1+2[\mathrm{sg}(x+y)=-1])
\end{aligned} \tag{10.35}$$

For the first identity, using theorem 8.2: and using $\mathrm{sg}(x)\mathrm{sg}(y)=1-2[\mathrm{sg}(x)\mathrm{sg}(y)=-1]$:

$$\begin{aligned}
\arcsin(x)+\arcsin(y) &= \pi-(\arccos(x)+\arccos(y)) \\
&= \pi-(2\,\mathrm{sg}(x)\mathrm{sg}(y)\arccos(f(x,y))+\pi g(x,y)) \\
&= \pi-(2\,\mathrm{sg}(x)\mathrm{sg}(y)(\frac{\pi}{2}-\arcsin(f(x,y)))+\pi g(x,y)) \\
&= 2\,\mathrm{sg}(x)\mathrm{sg}(y)\arcsin(f(x,y))+\pi(1-\mathrm{sg}(x)\mathrm{sg}(y)-g(x,y))
\end{aligned} \tag{10.36}$$

With $1-2[\mathrm{sg}(x+y)=-1]=\mathrm{sg}(x+y)$ this gives the first identity. □

11 Duplication and Bisection Formulas for the Inverse Tangent, Cotangent, Sine and Cosine Functions

In theorems 4.1, 4.2 and 4.3, taking $y=x$:
For real x:

$$2\arctan(x) = \arctan(\frac{2x}{1-x^2})+\pi[x^2>1]\mathrm{sg}(x) \tag{11.1}$$

$$2\,\mathrm{arccot}(x) = \mathrm{arccot}(\frac{x^2-1}{2x})+\pi[x^2<1]\mathrm{sg}(x) \tag{11.2}$$

$$2\operatorname{arccot}(x) = \arctan(\frac{2x}{x^2-1}) + \pi[x^2 < 1]\operatorname{sg}(x) \tag{11.3}$$

These formulas also follow from theorem 2.2 when taking $y = 0$.
In theorems 4.4, 4.5, 4.6 and 4.7, taking $y = 0$ and for real x: $\operatorname{arccot}(1/x) = \arctan(x)$:
For real x:

$$\frac{1}{2}\arctan(x) = \arctan(\frac{x}{1+\sqrt{1+x^2}}) \tag{11.4}$$

$$\frac{1}{2}\arctan(x) = \arctan(x+\sqrt{1+x^2}) - \frac{\pi}{4} \tag{11.5}$$

$$\frac{1}{2}\operatorname{arccot}(x) = \operatorname{arccot}(x+\operatorname{sg}(x)\sqrt{1+x^2}) \tag{11.6}$$

$$\frac{1}{2}\operatorname{arccot}(x) = \operatorname{arccot}(\frac{x}{1+\operatorname{sg}(x)\sqrt{1+x^2}}) - \frac{\pi}{4} \tag{11.7}$$

From theorems 9.9, 9.10, 9.11 and 9.12:
For real $-1 \leq x \leq 1$:

$$\arcsin(x) = \arctan(\frac{x}{\sqrt{1-x^2}}) \tag{11.8}$$

$$\arccos(x) = \operatorname{arccot}(\frac{x}{\sqrt{1-x^2}}) + \pi[x < 0] \tag{11.9}$$

For real x:

$$\arctan(x) = \arcsin(\frac{x}{\sqrt{1+x^2}}) \tag{11.10}$$

$$\operatorname{arccot}(x) = \arccos(\frac{x}{\sqrt{1+x^2}}) - \pi[x < 0] \tag{11.11}$$

In theorem 10.1 taking $y = x$:
For real $-1 \leq x \leq 1$:

$$2\arcsin(x) = \operatorname{sg}(1-2x^2)\arcsin(2x\sqrt{1-x^2}) + \pi[2x^2 > 1]\operatorname{sg}(x) \tag{11.12}$$

$$2\arccos(x) = \operatorname{sg}(1-2x^2)\arccos(2x\sqrt{1-x^2}) + \pi(\frac{1}{2} + [2x^2 > 1](1-\operatorname{sg}(x))) \tag{11.13}$$

In theorem 10.2 taking $y = x$:
For real $-1 \leq x \leq 1$:

$$2\arcsin(x) = \operatorname{sg}(x)\arcsin(2x^2-1) + \frac{\pi}{2}\operatorname{sg}(x) \tag{11.14}$$

$$2\arccos(x) = \operatorname{sg}(x)\arccos(2x^2-1) + 2\pi[x < 0] \tag{11.15}$$

In theorem 10.3 taking $y = 1$:
For real $-1 \leq x \leq 1$:

$$\frac{1}{2}\arcsin(x) = \arcsin(\sqrt{\frac{1+x}{2}}) - \frac{\pi}{4} \tag{11.16}$$

$$\frac{1}{2}\arccos(x) = \arccos(\sqrt{\frac{1+x}{2}}) \tag{11.17}$$

In theorem 10.3 taking $y = 0$:
For real $-1 \le x \le 1$:

$$\begin{aligned}\frac{1}{2}\arcsin(x) &= \arcsin(\frac{x}{\sqrt{2(1+\sqrt{1-x^2})}}) \\ &= \mathrm{sg}(x)\arcsin(\sqrt{\frac{1-\sqrt{1-x^2}}{2}})\end{aligned} \tag{11.18}$$

$$\begin{aligned}\frac{1}{2}\arccos(x) &= \arccos(\frac{x}{\sqrt{2(1+\sqrt{1-x^2})}}) - \frac{\pi}{4} \\ &= \mathrm{sg}(x)\arccos(\sqrt{\frac{1-\sqrt{1-x^2}}{2}}) - \frac{\pi}{4} + \pi[x<0]\end{aligned} \tag{11.19}$$

In theorem 10.4 taking $y = 1$:
For real $-1 \le x \le 1$:

$$\begin{aligned}\frac{1}{2}\arcsin(x) &= \mathrm{sg}(x)\arcsin(\sqrt{\frac{1+|x|}{2}}) - \frac{\pi}{4} + \frac{\pi}{2}[x<0] \\ &= \arcsin(\frac{x+\mathrm{sg}(x)}{\sqrt{2(1+|x|)}}) - \frac{\pi}{4} + \frac{\pi}{2}[x<0]\end{aligned} \tag{11.20}$$

$$\begin{aligned}\frac{1}{2}\arccos(x) &= \mathrm{sg}(x)\arccos(\sqrt{\frac{1+|x|}{2}}) + \frac{\pi}{2}[x<0] \\ &= \arccos(\frac{x+\mathrm{sg}(x)}{\sqrt{2(1+|x|)}}) - \frac{\pi}{2}[x<0]\end{aligned} \tag{11.21}$$

In theorem 10.4 taking $y = 0$:
For real $-1 \le x \le 1$:

$$\begin{aligned}\frac{1}{2}\arcsin(x) &= \arcsin(\frac{|x|}{\sqrt{2(1+\mathrm{sg}(x)\sqrt{1-x^2})}}) - \frac{\pi}{2}[x<0] \\ &= \arcsin(\sqrt{\frac{1-\mathrm{sg}(x)\sqrt{1-x^2}}{2}}) - \frac{\pi}{2}[x<0]\end{aligned} \tag{11.22}$$

$$\begin{aligned}\frac{1}{2}\arccos(x) &= \arccos(\frac{|x|}{\sqrt{2(1+\mathrm{sg}(x)\sqrt{1-x^2})}}) - \frac{\pi}{4} + \frac{\pi}{2}[x<0] \\ &= \arccos(\sqrt{\frac{1-\mathrm{sg}(x)\sqrt{1-x^2}}{2}}) - \frac{\pi}{4} + \frac{\pi}{2}[x<0]\end{aligned} \tag{11.23}$$

Combination of identities (11.16) with (11.20) and (11.17) with (11.21) gives:
For real $-1 \le x \le 1$:

$$\arcsin(\sqrt{\frac{1+x}{2}}) + \arcsin(\sqrt{\frac{1-x}{2}}) = \frac{\pi}{2} \tag{11.24}$$

$$\arccos(\sqrt{\frac{1+x}{2}}) + \arccos(\sqrt{\frac{1-x}{2}}) = \frac{\pi}{2} \tag{11.25}$$

With theorem 8.2:
For real $-1 \leq x \leq 1$:

$$\arcsin(\sqrt{\frac{1-x}{2}}) = \arccos(\sqrt{\frac{1+x}{2}}) \tag{11.26}$$

$$\arcsin(\sqrt{\frac{1+x}{2}}) = \arccos(\sqrt{\frac{1-x}{2}}) \tag{11.27}$$

which also follows from corollary 10.1 by replacing x with $\sqrt{(1+x)/2}$.
Combination of identities (11.18) with (11.22) and (11.19) with (11.23) gives:
For real $-1 \leq x \leq 1$ but not $x = 0$:

$$\arcsin(\frac{x}{\sqrt{2(1+\sqrt{1-x^2})}}) + \arcsin(\frac{x}{\sqrt{2(1-\sqrt{1-x^2})}}) = \frac{\pi}{2}\text{sg}(x) \tag{11.28}$$

$$\arccos(\frac{x}{\sqrt{2(1+\sqrt{1-x^2})}}) + \arccos(\frac{x}{\sqrt{2(1-\sqrt{1-x^2})}}) = \frac{\pi}{2}\text{sg}(x) + 2\pi[x < 0] \tag{11.29}$$

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