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THE INVERSES OF BLOCK HANKEL AND BLOCK TOEPLITZ MATRICES*

GEORGE LABAHN†, DONG KOO CHOI‡, AND STAN CABAY§

Abstract. A set of new formulae for the inverse of a block Hankel (or block Toeplitz) matrix is given. The formulae are expressed in terms of certain matrix Padé forms, which approximate a matrix power series associated with the block Hankel matrix.

By using Frobenius-type identities between certain matrix Padé forms, the inversion formulae are shown to generalize the formulae of Gohberg–Heinig and, in the scalar case, the formulae of Gohberg–Semencul and Gohberg–Krupnik.

The new formulae have the significant advantage of requiring only that the block Hankel matrix itself be nonsingular. The other formulae require, in addition, that certain submatrices be nonsingular.

Since effective algorithms for computing the required matrix Padé forms are available, the formulae are practical. Indeed, some of the algorithms allow for the efficient calculation of the inverse not only of the given block Hankel matrix, but also of any nonsingular block principal minor.

Keywords. Hankel matrix, Toeplitz matrix, Padé fraction, power series, Padé form, Yule–Walker equation

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1. Introduction. Let

$$H_{m,n}=\begin{bmatrix} a_{m-n+1} & \cdots & a_m \\ \vdots & & \vdots \\ a_m & \cdots & a_{m+n-1}\end{bmatrix} \tag{1.1}$$

be a nonsingular block Hankel matrix with coefficients from the ring of $p\times p$ matrices over a field.[1] The special structure of Hankel matrices has resulted in a number of closed formulae for the inverse of $H_{m,n}$.

When $p=1$ (the scalar case) well-known formulae of Gohberg and Semencul [14] give $H_{m,n}^{-1}$ in terms of only the first and last columns of the inverse. Gohberg and Krupnik [15] give a formula for the inverse in terms of the last two columns of $H_{m,n}^{-1}$. Ben-Artzi and Shalom [3] give a series of inverse formulae, including one for determining the inverse once two adjacent columns, along with the last column, of the inverse are known.

When $p>1$, additional problems are encountered in obtaining a closed formula for the inverse of a block Hankel matrix. When the coefficients of $H_{m,n}$ come from a noncommutative algebra there are closed formulae due to Gohberg and Heinig [16]. These are given provided the first and last columns together with the first and last rows of the inverse are known.

All of the above formulae depend on the ability to perform certain bordering operations that lend themselves well to matrices with a Hankel structure. However, these bordering operations require the imposition of certain additional restrictions on

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† Department of Computing Science, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1.
‡ Department of Computing Science, University of Alberta, Edmonton, Alberta, Canada T6G 2H1.
§ Department of Computing Science, University of Alberta, Edmonton, Alberta, Canada T6G 2H1. The research of this author was partially supported by Natural Sciences and Engineering Council of Canada grant A8035.

[1] All results hold, with minor modifications, for block Toeplitz matrices.

$H_{m,n}$. For the Gohberg-Krupnik formula, the matrix $H_{m-1,n-1}$ must also be nonsingular; whereas, for the Gohberg-Semencul and Gohberg-Heinig formulae, the matrix $H_{m,n-1}$ must be nonsingular. Inverse formulae are then also given for the relevant submatrices.

In the case of the scalar Gohberg-Semencul formulae, there is a standard technique for overcoming the extra requirements. When $H_{m,n}$ is nonsingular but $H_{m,n-1}$ is singular, a larger nonsingular Hankel matrix, $H_{m,n+1}$, is created. An inverse formula is then obtained by using the formulae of Gohberg-Semencul for $H_{m,n}$ and $H_{m,n+1}$ (cf. Gohberg and Semencul [14], or Iohvidov [19]). For the nonscalar case, however, there is no known similar method for overcoming the added restriction in the Gohberg-Heinig formulae.

The primary contribution of this paper is a set of new closed formulae for $H_{m,n}^{-1}$. By avoiding bordering techniques, we require only that $H_{m,n}$ be nonsingular. When $p=1$, one of the formulae agrees with that obtained by Choi [12].

The representations for $H_{m,n}^{-1}$ depend on the concept of a matrix Padé form (Labahn and Cabay [22]) for the matrix polynomial

$$A(z)=\sum_{i=0}^{m+n} a_i z^i. \tag{1.2}$$

These matrix Padé forms are determined from solutions to equations of a Yule-Walker type. Central to our approach are commutativity relationships that are shown to exist between certain matrix Padé forms. These commutativity relationships allow us to overcome the traditional limitations imposed when using bordering techniques. Indeed, the conditions that we impose are both necessary and sufficient for the existence of an inverse.

When we add the condition that $H_{m,n-1}$ also be nonsingular, certain Frobenius-type identities for matrix Padé forms are used to show that our formulae yield the formulae of Gohberg and Heinig. On the other hand, when we add the condition that $H_{m-1,n-1}$ be nonsingular, a different set of Frobenius-type identities applied to our results yields inverse formulae, which in the scalar case corresponds to the Gohberg-Krupnik formulae. Finally, using somewhat different techniques, we show how our inverse formulae provide natural generalizations of the results of Ben-Artzi and Shalom to the nonscalar case.

A major advantage of a closed inverse formula is that it allows for efficient algorithms to calculate the inverses of Hankel matrices. This efficiency comes both in the cost complexity of calculating the inverse and also in the amount of storage required for the final result.

When our inverse formulae are used in conjunction with the MPADE algorithm of Labahn and Cabay [22], we obtain an algorithm for calculating $H_{m,n}^{-1}$. This algorithm has many advantages for our situation. It is successful without any preconditions placed on the original power series. As a by-product, we obtain inverses for all the principal minors of $H_{m,n}$ that are nonsingular. Also, it is iterative on n, allowing cost savings in implementation. The complexity of the MPADE algorithm is generically $O(p^3n^2)$, although there are pathological cases where it can be as high as $O(p^3n^3)$ (for example, when all the principal minors of $H_{m,n}$ are singular). This compares with other nonscalar methods (cf. Akaike [1], Watson [31], Rissanen [27], Bose and Basu [5]) that are also of complexity $O(p^3n^2)$, but that succeed only when all principal minors are nonsingular. In the scalar case, however, the cost complexity of MPADE is $O(n^2)$, regardless of the types of singularities found in $H_{m,n}$. This compares favorably with the method described by Rissanen [28] that is of complexity $O(n^2)$ and succeeds in the degenerate case. The $O(n^2)$ methods of Trench [30], Watson [31], Zohar [33], and Kailath, Kung, and Morf [20], on the other hand, fail whenever a principal minor of $H_{m,n}$ is singular.

When fast polynomial multiplication methods are available, in the scalar case, the required Padé forms can be calculated by the off-diagonal algorithm of Cabay and Choi [11] with a complexity of $O(n \log^2 n)$. The algorithm is also iterative on n and produces the inverses of some of the nonsingular principal minors as a bi-product. As a result of this, and some other factors, the performance is better than the $O(n \log^2 n)$ method of Brent, Gustavson, and Yun [6] and Sugiyama [29], both of which also succeed in the degenerate case. The $O(n \log^2 n)$ methods of Bitmead and Anderson [4], Ammar and Gragg [2], and de Hoog [18], on the other hand, succeed only in the nondegenerate case.

In the nonscalar case, fast algorithms can also be used to calculate the required Padé forms, but under some restrictions. If the block matrix is positive definite (or, more generally, if the associated power series is nearly-normal (cf. [21])), for example, and fast polynomial multiplication is allowed, then the inverse formulae can be calculated using the fast algorithm of Labahn [21] with complexity $O(p^3 \cdot n \log^2 n)$. This algorithm is also iterative and calculates the inverses of some of the nonsingular principal minors as a bi-product. The algorithm of Bitmead and Anderson, generalized to the nonscalar case using the formulae of Gohberg and Heinig, is also of complexity $O(p^3 \cdot n \log^2 n)$, but works only in the normal case.

For purposes of presentation, we adopt the following notation. We let D denote the noncommutative ring of $p \times p$ matrices over a field.[2] The domain of formal power series with coefficients over D and indeterminate z is denoted by $D[[z]]$. For any $A(z) \in D[[z]]$, $A(z)$ is formally represented by

$$A(z) = \sum_{i=0}^{\infty} a_i z^i, \tag{1.3}$$

where the coefficients $a_i \in D$ are always written in lower case. The domain of polynomials (finite power series) over D with indeterminant z is denoted by $D[z]$. Any polynomial $P_n(z) \in D[z]$ is represented formally by

$$P_n(z) = \sum_{i=0}^{n} p_i z^i, \tag{1.4}$$

where again the coefficients $p_i \in D$ are written in lower case. The degree of $P_n(z)$ (i.e., the largest i such that $p_i \neq 0$) is denoted by $\partial(P_n(z))$.

2. Matrix Padé forms. The inversion formulae derived in §§ 3 and 4 depend on the concept of a matrix Padé form for a matrix power series. This is a multidimensional generalization of scalar Padé forms (cf. Gragg [17]). Let

$$A(z) = \sum_{i=0}^{\infty} a_i z^i \in D[[z]] \tag{2.1}$$

be a formal power series with coefficients from the ring D of $p \times p$ matrices over some field. For nonnegative integers m and n, let

$$U_m(z) = \sum_{i=0}^{m} u_i z^i, \qquad V_n(z) = \sum_{i=0}^{n} v_i z^i \quad \in \quad D[z] \tag{2.2}$$

be $p \times p$ matrix polynomials.

[2] All the results of this paper can be presented in the more general setting where D is an arbitrary noncommutative algebra.

DEFINITION 2.1 (Labahn and Cabay [22]). The triple $(U_m(z), V_n(z), W(z))$ is defined to be a **Right Matrix Padé Form** (RMPFo) of type (m, n) for the power series $A(z)$ if

(I) $\partial(U_m(z)) \leqq m$, $\partial(V_n(z)) \leqq n$,

(II) $A(z) \cdot V_n(z) - U_m(z) = z^{m+n+1} W(z)$, where $W(z) \in D[[z]]$, and

(III) The columns of $V_n(z)$ are linearly independent over the field.[3]

The matrices $U_m(z)$, $V_n(z)$, and $W(z)$ are called the **right numerator, denominator**, and **residual** (all of type (m, n)), respectively. □

There is an equivalent definition for a left matrix Padé form (LMPFo). In condition (II), multiplication on the right by $V_n(z)$ is replaced by multiplication on the left. In addition, condition (III) is replaced by

(III) the rows of $V_n(z)$ are linearly independent over the field.

Condition (II) can be written as follows:

$$\begin{bmatrix} a_{-n} & \cdots & a_0 \\ \vdots & & \vdots \\ a_{m-n} & \cdots & a_m \end{bmatrix} \cdot \begin{bmatrix} v_n \\ \vdots \\ v_0 \end{bmatrix} = \begin{bmatrix} u_0 \\ \vdots \\ u_m \end{bmatrix} \tag{2.3}$$

and

$$\begin{bmatrix} a_{m-n+1} & \cdots & a_m & a_{m+1} \\ \vdots & & & \vdots \\ a_m & a_{m+1} & \cdots & a_{m+n} \end{bmatrix} \cdot \begin{bmatrix} v_n \\ \vdots \\ v_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{2.4}$$

Here $a_i = 0$ for $i < 0$. The matrix polynomial $V_n(z)$ can be determined by solving (2.4), and then $U_m(z)$ can be obtained from (2.3).

THEOREM 2.2 (Existence of Matrix Padé Forms). *For any matrix power series $A(z)$ and for any pair of nonnegative integers (m, n), there exists an* RMPFo *and an* LMPFo *of type (m, n).*

Proof. The result follows from (2.3) and (2.4) by comparing the number of equations with the number of unknowns. For details see [22]. □

To distinguish between matrix Padé forms of different types, we introduce the following notation. For a given pair of positive integers (m, n), the triples $(U_m(z), V_n(z), W(z))$ and $(U_m^*(z), V_n^*(z), W^*(z))$ denote an RMPFo and an LMPFo, respectively, of type (m, n) for $A(z)$. For the same (m, n), an RMPFo and an LMPFo of type $(m-1, n-1)$ for $A(z)$ are represented, respectively, by $(P_{m-1}(z), Q_{n-1}(z), R(z))$ and $(P_{m-1}^*(z), Q_{n-1}^*(z), R^*(z))$. For these Padé forms, collectively, condition (II) becomes

$$A(z) V_n(z) - U_m(z) = z^{m+n+1} \cdot W(z), \tag{2.5}$$

$$V_n^*(z) A(z) - U_m^*(z) = z^{m+n+1} \cdot W^*(z), \tag{2.6}$$

$$A(z) Q_{n-1}(z) - P_{m-1}(z) = z^{m+n-1} \cdot R(z), \tag{2.7}$$

$$Q_{n-1}^*(z) A(z) - P_{m-1}^*(z) = z^{m+n-1} \cdot R^*(z). \tag{2.8}$$

In § 3, in the case that $H_{m,n}$ is nonsingular, the inverse is given in terms of these four matrix Padé forms.

THEOREM 2.3. *For a pair of positive integers (m, n), the following statements are equivalent*:

$$\det(H_{m,n}) \neq 0, \tag{2.9}$$

[3] When the leading term v_0 is nonsingular, then in [22] a RMPFo is called a Right Matrix Padé Fraction (RMPFr).

$$\det(r_0) \neq 0 \quad \text{and} \quad \det(v_0) \neq 0, \tag{2.10}$$

$$\det(r_0^*) \neq 0 \quad \text{and} \quad \det(v_0^*) \neq 0. \tag{2.11}$$

Proof. That (2.9) implies (2.10) and (2.9) implies (2.11) was proved in [22], and so we show only the converse here. To see that (2.10) implies (2.9), let $X=(x_1, \cdots, x_n)$ be a vector of length np and suppose that

$$X \cdot H_{m,n} = 0. \tag{2.12}$$

We shall show that $X=0$. We accomplish this by showing that (2.12) implies that $x_n = 0$ and

$$(0, x_1, \cdots, x_{n-1}) \cdot H_{m,n} = 0. \tag{2.13}$$

By repeated application of this property, it then follows that $x_{n-1} = \cdots = x_1 = 0$, and so $X=0$.

First observe that equating coefficients of z^i, for $m+1 \leqq i \leqq m+n$, in (2.5) yields

$$H_{m,n} \cdot \begin{bmatrix} v_n \\ \vdots \\ v_1 \end{bmatrix} = -\begin{bmatrix} a_{m+1} \\ \vdots \\ a_{m+n} \end{bmatrix} v_0 \tag{2.14}$$

where v_0 is invertible since we are assuming statement (2.10). Similarly, equating coefficients of z^i, for $m \leqq i \leqq m+n-1$, in (2.7) yields

$$H_{m,n} \cdot \begin{bmatrix} q_{n-1} \\ \vdots \\ q_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ r_0 \end{bmatrix} \tag{2.15}$$

where, by assumption, r_0 is invertible. From (2.12) and (2.15), it follows that

$$x_n \cdot r_0 = X \cdot \begin{bmatrix} 0 \\ \vdots \\ 0 \\ r_0 \end{bmatrix} = X \cdot H_{m,n} \begin{bmatrix} q_{n-1} \\ \vdots \\ q_0 \end{bmatrix} = 0. \tag{2.16}$$

Since r_0 is invertible, it then follows that $x_n = 0$.

Having shown that $x_n = 0$, (2.12) then yields

$$(x_1, \cdots, x_{n-1}) \cdot \begin{bmatrix} a_{m-n+2} & \cdots & a_m \\ \vdots & & \vdots \\ a_m & \cdots & a_{m+n-2} \end{bmatrix} = 0. \tag{2.17}$$

But, from (2.12) and (2.14), we have

$$(x_1, \cdots, x_{n-1}, 0) \cdot \begin{bmatrix} a_{m+1} \\ \vdots \\ a_{m+n} \end{bmatrix} \cdot v_0 = -X \cdot H_{m,n} \begin{bmatrix} v_n \\ \vdots \\ v_1 \end{bmatrix} = 0. \tag{2.18}$$

Since v_0 is invertible, (2.18) implies that

$$(x_1, \cdots, x_{n-1}) \cdot \begin{bmatrix} a_{m+1} \\ \vdots \\ a_{m+n-1} \end{bmatrix} = 0. \tag{2.19}$$

Equations (2.17) and (2.19) imply that

$$(2.20)\qquad (x_1, \cdots, x_{n-1}) \cdot \begin{bmatrix} a_{m-n+2} & \cdots & a_{m+1} \\ \vdots & & \vdots \\ a_m & \cdots & a_{m+n-1} \end{bmatrix} = 0,$$

which is equivalent to (2.13).

Thus, we have shown that (2.10) implies (2.9). A similar argument shows that (2.11) implies (2.9). □

Theorem 2.3 has important computational significance since the singularity of $H_{m,n}$ can be detected simply by recognizing a singular r_0 or a singular v_0. If both r_0 and v_0 are nonsingular, then we have Theorem 2.4.

THEOREM 2.4. *If* $\det(H_{m,n}) \neq 0$, *then the matrix Padé forms identified by* (2.5)-(2.8) *are unique, except for the specification of the nonsingular matrices* v_0, v_0^*, r_0, *and* r_0^*.

Proof. We refer the reader to Theorems 3.2 and 3.3 in [22] for a detailed proof of this result. □

As a consequence of Theorem 2.4, it can be assumed without loss of generality that

$$(2.21)\qquad v_0 = v_0^* = r_0 = r_0^* = I.$$

This nonrestrictive assumption simplifies the presentation of subsequent results.

The key relationship between matrix Padé forms that enables the presentation of the inverse of $H_{m,n}$ in §§ 3 and 4, is given by Lemma 2.5.

LEMMA 2.5. *Let* $\det(H_{m,n}) \neq 0$. *Then the matrix Padé forms identified by* (2.5)-(2.8) *and normalized according to* (2.21) *satisfy*

$$(2.22)\qquad \begin{bmatrix} Q_{n-1}^*(z) & -P_{m-1}^*(z) \\ -V_n^*(z) & U_m^*(z) \end{bmatrix} \cdot \begin{bmatrix} U_m(z) & P_{m-1}(z) \\ V_n(z) & Q_{n-1}(z) \end{bmatrix} = z^{m+n-1} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix},$$

$$(2.23)\qquad \begin{bmatrix} U_m(z) & P_{m-1}(z) \\ V_n(z) & Q_{n-1}(z) \end{bmatrix} \cdot \begin{bmatrix} Q_{n-1}^*(z) & -P_{m-1}^*(z) \\ -V_n^*(z) & U_m^*(z) \end{bmatrix} = z^{m+n-1} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix},$$

$$(2.24)\qquad \begin{bmatrix} R^*(z) & -Q_{n-1}^*(z) \\ -z^2 W^*(z) & V_n^*(z) \end{bmatrix} \cdot \begin{bmatrix} V_n(z) & Q_{n-1}(z) \\ z^2 W(z) & R(z) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix},$$

$$(2.25)\qquad \begin{bmatrix} V_n(z) & Q_{n-1}(z) \\ z^2 W(z) & R(z) \end{bmatrix} \cdot \begin{bmatrix} R^*(z) & -Q_{n-1}^*(z) \\ -z^2 W^*(z) & V_n^*(z) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

Proof. Multiplying (2.5) on the left by $Q_{n-1}^*(z)$ and (2.8) on the right by $V_n(z)$, and subtracting the first from the second, we obtain

$$(2.26)\qquad \begin{aligned} Q_{n-1}^*(z) \cdot U_m(z) - P_{m-1}^*(z) \cdot V_n(z) &= z^{m+n-1} \cdot (R^*(z) V_n(z) - z^2 Q_{n-1}^*(z) W(z)) \\ &= z^{m+n-1} r_0^* v_0 \\ &= z^{m+n-1} I. \end{aligned}$$

In (2.26), we have used the normalizing condition (2.21) and the fact that the left-hand side, and consequently the right-hand side, is a matrix polynomial of degree at most $m+n-1$.

Multiplying (2.5) on the left by $V_n^*(z)$ and (2.6) on the right by $V_n(z)$, and subtracting the second from the first, we obtain

$$(2.27)\qquad \begin{aligned} -V_{n-1}^*(z) \cdot U_m(z) + U_m^*(z) \cdot V_n(z) &= z^{m+n+1} \cdot (V_n^*(z) W(z) - W^*(z) V_n(z)) \\ &= 0. \end{aligned}$$

In (2.27), the last equality is true because the left-hand side, and consequently the right-hand side, is a matrix polynomial of degree at most $m+n$.

In a similar fashion, (2.7), (2.8), and (2.21) yield

$$\begin{aligned} Q_{n-1}^*(z)P_{m-1}(z) - P_{m-1}^*(z)Q_{n-1}(z) &= z^{m+n-1}\cdot(Q_{n-1}^*(z)R(z) - R^*(z)Q_{n-1}(z)) \\ &= 0; \end{aligned} \tag{2.28}$$

whereas, (2.6), (2.7), and (2.21) give

$$\begin{aligned} -V_n^*(z)P_{m-1}(z) + U_m^*(z)Q_{n-1}(z) &= z^{m+n-1}\cdot(V_n^*(z)R(z) - z^2W^*(z)Q_{n-1}) \\ &= z^{m+n-1}I. \end{aligned} \tag{2.29}$$

Equations (2.26)-(2.29) together comprise (2.22). Equation (2.23) follows directly from (2.22), since matrix inverses are two sided.

Equations (2.26) also gives

$$z^{m+n-1}\cdot(R^*(z)V_n(z) - z^2Q_{n-1}^*(z)W(z)) = z^{m+n-1}\cdot I, \tag{2.30}$$

from which we obtain

$$R^*(z)V_n(z) - z^2Q_{n-1}^*(z)W(z) = I. \tag{2.31}$$

Similarly, from (2.27), we obtain

$$V_n^*(z)W(z) - W^*(z)V_n(z) = 0. \tag{2.32}$$

From (2.28), we obtain

$$Q_{n-1}^*(z)R(z) - R^*(z)Q_{n-1}(z) = 0, \tag{2.33}$$

and (2.29) gives

$$V_n^*(z)R(z) - z^2W^*(z)Q_{n-1} = I. \tag{2.34}$$

Equations (2.31)-(2.34) comprise (2.24). As before, (2.25) follows from (2.24), since matrix inverses are two-sided. □

3. The off-diagonal inverse formula. The main result of this paper is Theorem 3.1.

THEOREM 3.1. *Let $H_{m,n}$ be the block Hankel matrix* (1.1). *If there are* RMPFO*s and* LMPFO*s of type $(m-1, n-1)$ and (m, n) for $A(z)$ satisfying the normalizing condition* (2.21), *then $H_{m,n}$ is nonsingular with inverse*

$$H_{m,n}^{-1} = \begin{bmatrix} v_{n-1} & \cdots & v_0 \\ \vdots & \cdot^{\cdot^{\cdot}} & \\ v_0 & & \end{bmatrix} \begin{bmatrix} q_{n-1}^* & \cdots & q_0^* \\ & \ddots & \vdots \\ & & q_{n-1}^* \end{bmatrix} - \begin{bmatrix} q_{n-2} & \cdots & q_0 & 0 \\ \vdots & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\ q_0 & \cdot^{\cdot^{\cdot}} & & \\ 0 & & & \end{bmatrix} \begin{bmatrix} v_n^* & \cdots & v_1^* \\ & \ddots & \vdots \\ & & v_n^* \end{bmatrix}, \tag{3.1}$$

or, equivalently,

$$H_{m,n}^{-1} = \begin{bmatrix} q_{n-1} & & \\ \vdots & \ddots & \\ q_0 & \cdots & q_{n-1} \end{bmatrix} \begin{bmatrix} v_{n-1}^* & \cdots & v_0^* \\ \vdots & \cdot^{\cdot^{\cdot}} & \\ v_0^* & & \end{bmatrix} - \begin{bmatrix} v_n & & \\ \vdots & \ddots & \\ v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} q_{n-2}^* & \cdots & q_0^* & 0 \\ \vdots & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\ q_0^* & \cdot^{\cdot^{\cdot}} & & \\ 0 & & & \end{bmatrix}. \tag{3.2}$$

Proof. Using

$$U_m(z)Q_{n-1}^*(z) - P_{m-1}(z)V_n^*(z) = z^{m+n-1}I,$$

which is from (2.23), we can equate coefficients of z^i, $m \leqq i \leqq m+n-1$, to obtain

$$\begin{bmatrix} u_m & \cdots & u_{m-n+1} \\ & \ddots & \vdots \\ & & u_m \end{bmatrix} \cdot \begin{bmatrix} q_{n-1}^* & \cdots & q_0^* \\ & \ddots & \vdots \\ & & q_{n-1}^* \end{bmatrix} - \begin{bmatrix} p_{m-1} & \cdots & p_{m-n} \\ & \ddots & \vdots \\ & & p_{m-1} \end{bmatrix} \cdot \begin{bmatrix} v_n^* & \cdots & v_1^* \\ & \ddots & \vdots \\ & & v_n^* \end{bmatrix} = I. \tag{3.3}$$

Similarly,

$$V_n(z)Q^*_{n-1}(z) - Q_{n-1}(z)V^*_n(z) = 0,$$

also from (2.23), yields

$$\begin{bmatrix} & & v_n \\ & \iddots & \vdots \\ v_n & \cdots & v_1 \end{bmatrix}\begin{bmatrix} q^*_{n-1} & \cdots & q^*_0 \\ & \ddots & \vdots \\ & & q^*_{n-1} \end{bmatrix} - \begin{bmatrix} & & q_{n-1} \\ & \iddots & \vdots \\ q_{n-1} & \cdots & q_0 \end{bmatrix}\begin{bmatrix} v^*_n & \cdots & v^*_1 \\ & \ddots & \vdots \\ & & v^*_n \end{bmatrix} = 0. \tag{3.4}$$

Now, from (2.7), we obtain

$$H_{m,n}\cdot\begin{bmatrix} q_{n-2} & \cdots & q_0 & 0 \\ \vdots & \iddots & \iddots & \\ q_0 & \iddots & & \\ 0 & & & \end{bmatrix} = \begin{bmatrix} p_{m-1} & \cdots & p_{m-n} \\ & \ddots & \vdots \\ & & p_{m-1} \end{bmatrix} - H_{m-n,n}\cdot\begin{bmatrix} & & q_{n-1} \\ & \iddots & \vdots \\ q_{n-1} & \cdots & q_0 \end{bmatrix}. \tag{3.5}$$

Observe that, for $1 \leqq i, j \leqq n$, the (i, j) component in (3.5) is obtained by equating coefficients of $z^{m+i-j-1}$ in (2.7). Similarly, (2.5) yields

$$H_{m,n}\cdot\begin{bmatrix} v_{n-1} & \cdots & v_0 \\ \vdots & \iddots & \\ v_0 & & \end{bmatrix} = \begin{bmatrix} u_m & \cdots & u_{m-n+1} \\ & \ddots & \vdots \\ & & u_m \end{bmatrix} - H_{m-n,n}\cdot\begin{bmatrix} & & v_n \\ & \iddots & \vdots \\ v_n & \cdots & v_1 \end{bmatrix}. \tag{3.6}$$

Combining (3.5) and (3.6) and using (3.3) and (3.4), it then follows that

$$\begin{aligned}
&H_{m,n}\left\{\begin{bmatrix} v_{n-1} & \cdots & v_0 \\ \vdots & \iddots & \\ v_0 & & \end{bmatrix}\cdot\begin{bmatrix} q^*_{n-1} & \cdots & q^*_0 \\ & \ddots & \vdots \\ & & q^*_{n-1} \end{bmatrix} - \begin{bmatrix} q_{n-2} & \cdots & q_0 & 0 \\ \vdots & \iddots & \iddots & \\ q_0 & \iddots & & \\ 0 & & & \end{bmatrix}\begin{bmatrix} v^*_n & \cdots & v^*_1 \\ & \ddots & \vdots \\ & & v^*_n \end{bmatrix}\right\} \\
&= \left\{\begin{bmatrix} u_m & \cdots & u_{m-n+1} \\ & \ddots & \vdots \\ & & u_m \end{bmatrix} - H_{m-n,n}\cdot\begin{bmatrix} & & v_n \\ & \iddots & \vdots \\ v_n & \cdots & v_1 \end{bmatrix}\right\}\cdot\begin{bmatrix} q^*_{n-1} & \cdots & q^*_0 \\ & \ddots & \vdots \\ & & q^*_{n-1} \end{bmatrix} \\
&\quad - \left\{\begin{bmatrix} p_{m-1} & \cdots & p_{m-n} \\ & \ddots & \vdots \\ & & p_{m-1} \end{bmatrix} - H_{m-n,n}\begin{bmatrix} & & q_{n-1} \\ & \iddots & \vdots \\ q_{n-1} & \cdots & q_0 \end{bmatrix}\right\}\cdot\begin{bmatrix} v^*_n & \cdots & v^*_1 \\ & \ddots & \vdots \\ & & v^*_n \end{bmatrix} \\
&= \left\{\begin{bmatrix} u_m & \cdots & u_{m-n+1} \\ & \ddots & \vdots \\ & & u_m \end{bmatrix}\cdot\begin{bmatrix} q^*_{n-1} & \cdots & q^*_0 \\ & \ddots & \vdots \\ & & q^*_{n-1} \end{bmatrix} - \begin{bmatrix} p_{m-1} & \cdots & p_{m-n} \\ & \ddots & \vdots \\ & & p_{m-1} \end{bmatrix}\cdot\begin{bmatrix} v^*_n & \cdots & v^*_1 \\ & \ddots & \vdots \\ & & v^*_n \end{bmatrix}\right\} \\
&\quad - H_{m-n,n}\left\{\begin{bmatrix} & & v_n \\ & \iddots & \vdots \\ v_n & \cdots & v_1 \end{bmatrix}\cdot\begin{bmatrix} q^*_{n-1} & \cdots & q^*_0 \\ & \ddots & \vdots \\ & & q^*_{n-1} \end{bmatrix}\right. \\
&\quad \left. - \begin{bmatrix} & & q_{n-1} \\ & \iddots & \vdots \\ q_{n-1} & \cdots & q_0 \end{bmatrix}\cdot\begin{bmatrix} v^*_n & \cdots & v^*_1 \\ & \ddots & \vdots \\ & & v^*_n \end{bmatrix}\right\} \\
&= I.
\end{aligned} \tag{3.7}$$

Thus, $H_{m,n}$ is nonsingular with the inverse given by (3.1).

The second formula (3.2) for the inverse is proved using (2.6), (2.8), and the second column of (2.23). □

Remark 1. In the scalar case, (3.1) was first obtained by Choi [12].

Remark 2. The assumptions of Theorem 3.1 can be equivalently replaced by the requirement that we obtain solutions to

$$H_{m,n}\cdot[q_{n-1},\cdots,q_0]'=[0,\cdots,0,I]', \tag{3.8}$$

$$[q^*_{n-1},\cdots,q^*_0]\cdot H_{m,n}=[0,\cdots,0,I], \tag{3.9}$$

$$H_{m,n}\cdot[v_n,\cdots,v_1]'=-[a_{m+1},\cdots,a_{m+n-1},a_{m+n}]', \tag{3.10}$$

$$[v^*_n,\cdots,v^*_1]\cdot H_{m,n}=-[a_{m+1},\cdots,a_{m+n-1},a_{m+n}] \tag{3.11}$$

where a_{m+n} can be any $p\times p$ matrix. Equations (3.10) and (3.11) are block versions of the Yule-Walker equations.

4. The antidiagonal inverse formula. Theorem 3.1 provides inverse formulae for the block Hankel matrix $H_{m,n}$ in terms of RMPFo and LMPFo of type $(m-1,n-1)$ and (m,n) for the associated matrix polynomial $A(z)$. There are some algorithms (cf. [6], [24], [29]) that calculate Padé forms along an antidiagonal, rather than along an off-diagonal path of the Padé table. For this reason, it is useful to provide inverse formulae in terms of RMPFos and LMPFos of type $(m-1,n)$ and $(m,n-1)$ for $A(z)$.

Let $(E_m(z),F_{n-1}(z),G(z))$ and $(E^*_m(z),F^*_{n-1}(z),G^*(z))$ be an RMPFo and an LMPFo, respectively, of type $(m,n-1)$ for $A(z)$. Also, let $(B_{m-1}(z),C_n(z),D(z))$ and $(B^*_{m-1}(z),C^*_n(z),D^*(z))$ be an RMPFo and an LMPFo, respectively, of type $(m-1,n)$ for $A(z)$. Then, the following equations are satisfied:

$$A(z)F_{n-1}(z)-E_m(z)=z^{m+n}G(z), \tag{4.1}$$

$$F^*_{n-1}(z)A(z)-E^*_m(z)=z^{m+n}G^*(z), \tag{4.2}$$

$$A(z)C_n(z)-B_{m-1}(z)=z^{m+n}D(z), \tag{4.3}$$

$$C^*_n(z)A(z)-B^*_{m-1}(z)=z^{m+n}D^*(z). \tag{4.4}$$

COROLLARY 4.1. *Let $H_{m,n}$ be the block Hankel matrix* (1.1). *Then the following are equivalent*:

$$\det(H_{m,n})\neq 0, \tag{4.5}$$

$$\det(e_m)\neq 0 \quad \textit{and} \quad \det(c_n)\neq 0, \tag{4.6}$$

$$\det(e^*_m)\neq 0 \quad \textit{and} \quad \det(c^*_n)\neq 0. \tag{4.7}$$

If any (and therefore all) of (4.5), (4.6), or (4.7) hold, then the inverse is given by

$$H^{-1}_{m,n}=\begin{bmatrix} c_n & & \\ \vdots & \ddots & \\ c_1 & \cdots & c_n\end{bmatrix}\begin{bmatrix} f^*_{n-1} & \cdots & f^*_0 \\ \vdots & \iddots & \\ f^*_0 & & \end{bmatrix}-\begin{bmatrix} 0 & & & \\ f_{n-1} & \ddots & & \\ \vdots & \ddots & \ddots & \\ f_1 & \cdots & f_{n-1} & 0\end{bmatrix}\begin{bmatrix} c^*_{n-1} & \cdots & c^*_0 \\ \vdots & \iddots & \\ c^*_0 & & \end{bmatrix}, \tag{4.8}$$

or, equivalently,

$$H^{-1}_{m,n}=\begin{bmatrix} f_{n-1} & \cdots & f_0 \\ \vdots & \iddots & \\ f_0 & & \end{bmatrix}\begin{bmatrix} c^*_n & \cdots & c^*_1 \\ & \ddots & \vdots \\ & & c^*_n\end{bmatrix}-\begin{bmatrix} c_{n-1} & \cdots & c_0 \\ \vdots & \iddots & \\ c_0 & & \end{bmatrix}\begin{bmatrix} 0 & f^*_{n-1} & \cdots & f^*_1 \\ & \ddots & \ddots & \vdots \\ & & \ddots & f^*_{n-1} \\ & & & 0\end{bmatrix} \tag{4.9}$$

where we have normalized the Padé forms so that

$$e_m = e_m^* = c_n = c_n^* = I. \tag{4.10}$$

Proof. Let $a_i^\# = a_{2m-i}$, for $0 \leqq i \leqq m+n$, and define a truncated power series $A^\#(z) = \sum_{i=0}^{m+n} a_i^\# z^i$. Observe that, if

$$H_{m,n}^\# = \begin{bmatrix} a_{m-n+1}^\# & \cdots & a_m^\# \\ \vdots & & \vdots \\ a_m^\# & \cdots & a_{m+n-1}^\# \end{bmatrix}, \tag{4.11}$$

then

$$H_{m,n}^\# = J \cdot H_{m,n} \cdot J \tag{4.12}$$

where

$$J = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

Equating coefficients of z^i, for $m \leqq i \leqq m+n-1$, in (4.1), we obtain

$$H_{m,n} \begin{bmatrix} f_{n-1} \\ \vdots \\ f_0 \end{bmatrix} = \begin{bmatrix} e_m \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{4.13}$$

From (4.12) and (4.13), it then follows that

$$H_{m,n}^\# \begin{bmatrix} f_0 \\ \vdots \\ f_{n-1} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ e_m \end{bmatrix}. \tag{4.14}$$

Thus,

$$Q_{n-1}(z) = \sum_{i=0}^{n-1} f_{n-1+i} z^i \tag{4.15}$$

is a right denominator of type $(m-1, n-1)$ for $A^\#(z)$. Similarly, (4.2) yields

$$[f_0^*, \cdots, f_{n-1}^*] H_{m,n}^\# = [0, \cdots, 0, e_m^*], \tag{4.16}$$

and so

$$Q_{n-1}^*(z) = \sum_{i=0}^{n-1} f_{n-1+i}^* z^i \tag{4.17}$$

is a left denominator of type $(m-1, n-1)$ for $A^\#(z)$.

Next, from (4.3), we obtain

$$H_{m,n} \begin{bmatrix} c_{n-1} \\ \vdots \\ c_0 \end{bmatrix} = \begin{bmatrix} a_{m-n} \\ \vdots \\ a_{m-1} \end{bmatrix} c_n, \tag{4.18}$$

and so (4.12) then gives

$$H_{m,n}^\# \begin{bmatrix} c_0 \\ \vdots \\ c_{n-1} \end{bmatrix} = \begin{bmatrix} a_{m+1}^\# \\ \vdots \\ a_{m+n}^\# \end{bmatrix} c_n. \tag{4.19}$$

Thus,

$$V_n(z) = \sum_{i=0}^{n} c_{n-i} z^i \tag{4.20}$$

is a right denominator of type (m, n) for $A^{\#}(z)$. Similarly, (4.4) can be used to obtain

$$[c_0^*, \cdots, c_{n-1}^*] H_{m,n}^{\#} = c_n^* [a_{m+1}^{\#}, \cdots, a_{m+n}^{\#}], \tag{4.21}$$

and so

$$V_n^*(z) = \sum_{i=0}^{n} c_{n-i}^* z^i \tag{4.22}$$

is a left denominator of type (m, n) for $A^{\#}(z)$. Since $\det(H_{m,n}^{\#}) \neq 0$ if and only if $\det(H_{m,n}) \neq 0$, the equivalence of (4.5)-(4.7) now follows from the equivalence of (2.9)-(2.11).

To prove (4.8), normalize according to (4.10) and substitute (4.15), (4.17), (4.20), and (4.22) into (3.1) to obtain

$$H_{m,n}^{\#-1} = \begin{bmatrix} c_1 & \cdots & c_n \\ \vdots & \cdot^{\cdot^{\cdot}} & \\ c_n & & \end{bmatrix} \begin{bmatrix} f_0^* & \cdots & f_{n-1}^* \\ & \ddots & \vdots \\ & & f_0^* \end{bmatrix} - \begin{bmatrix} f_1 & \cdots & f_{n-1} & 0 \\ \vdots & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\ f_{n-1} & \cdot^{\cdot^{\cdot}} & & \\ 0 & & & \end{bmatrix} \begin{bmatrix} c_0^* & \cdots & c_{n-1}^* \\ & \ddots & \vdots \\ & & c_0^* \end{bmatrix}. \tag{4.23}$$

By using (4.12), (4.8) follows immediately from (4.23). In a similar fashion, (4.9) can be obtained using (3.2). □

5. The Gohberg–Heinig inverse formulae. In this and the next section, we compare our inverse formulae (3.1) and (3.2) with other similar well-known formulae. In terms of matrix Padé forms of type $(m-1, n-1)$ and $(m, n-1)$, the inverse of $H_{m,n}$ is given by Corollary 5.1.

COROLLARY 5.1. *Let the matrix Padé forms identified by* (2.7), (2.8), (4.1), *and* (4.2) *be given. Then the following statements are equivalent*:

$$\det(H_{m,n-1}) \neq 0 \quad \textit{and} \quad \det(H_{m,n}) \neq 0, \tag{5.1}$$

$$\det(r_0) \neq 0 \quad \textit{and} \quad \det(f_0) \neq 0, \tag{5.2}$$

$$\det(r_0^*) \neq 0 \quad \textit{and} \quad \det(f_0^*) \neq 0. \tag{5.3}$$

In addition, if any (and therefore all) of conditions (5.1), (5.2), *or* (5.3) *are satisfied, then*

$$\begin{aligned} H_{m,n}^{-1} = & \begin{bmatrix} f_{n-1} & \cdots & f_0 \\ \vdots & \cdot^{\cdot^{\cdot}} & \\ f_0 & & \end{bmatrix} \begin{bmatrix} q_{n-1}^* & \cdots & q_0^* \\ & \ddots & \vdots \\ & & q_{n-1}^* \end{bmatrix} \\ & - \begin{bmatrix} q_{n-2} & \cdots & q_0 & 0 \\ \vdots & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\ q_0 & \cdot^{\cdot^{\cdot}} & & \\ 0 & & & \end{bmatrix} \begin{bmatrix} 0 & f_{n-1}^* & \cdots & f_1^* \\ & & \ddots & \vdots \\ & & \ddots & f_{n-1}^* \\ & & & 0 \end{bmatrix} \end{aligned} \tag{5.4}$$

where the Padé forms have been normalized by

$$r_0 = r_0^* = f_0 = f_0^* = I. \tag{5.5}$$

Proof. We first show that (5.1) implies (5.2). Since $\det(H_{m,n}) \neq 0$, Theorem 2.3 implies that $\det(r_0) \neq 0$. Since $\det(H_{m,n-1}) \neq 0$, Theorem 2.3 also implies that $\det(f_0) \neq 0$. Therefore (5.1) implies (5.2). In a similar fashion, (5.1) implies (5.3).

To show that (5.2) implies (5.1), let

$$U_m(z) = E_m(z) - z \cdot P_{m-1}(z) r_0^{-1} g_0, \tag{5.6}$$

$$V_n(z) = F_{n-1}(z) - z \cdot Q_{n-1}(z) r_0^{-1} g_0. \tag{5.7}$$

Then, $\partial(U_m(z)) \leqq m$ and $\partial(V_n(z)) \leqq n$. Also,

$$\begin{aligned} A(z)V_n(z) - U_m(z) &= \{A(z)F_{n-1}(z) - E_m(z)\} - z\{A(z)Q_{n-1}(z) - P_{m-1}(z)\} r_0^{-1} g_0 \\ &= z^{m+n}\{G(z) - R(z) r_0^{-1} g_0\} \\ &= z^{m+n+1} W(z) \end{aligned} \tag{5.8}$$

where

$$W(z) = z^{-1}\{G(z) - R(z) r_0^{-1} g_0\} \in D[[z]]. \tag{5.9}$$

Finally, the columns of $V_n(z)$ are linearly independent since from (5.7) $v_0 = f_0$, and by assumption f_0 is nonsingular. Thus, $(U_m(z), V_n(z), W(z))$ is an RMPFo of type (m, n) for $A(z)$, satisfying $\det(v_0) \neq 0$. From Theorem 2.3, it follows that $H_{m,n}$ is nonsingular since both $\det(r_0) \neq 0$ and $\det(v_0) \neq 0$. To see that $H_{m,n-1}$ is also nonsingular, observe that

$$\begin{bmatrix} a_{m-n+1} & \cdots & a_m \\ \vdots & & \vdots \\ a_m & \cdots & a_{m+n-1} \end{bmatrix} \begin{bmatrix} I & & & f_{n-1} \\ & \ddots & & \vdots \\ & & I & f_1 \\ 0 & \cdots & 0 & f_0 \end{bmatrix} = \begin{bmatrix} a_{m-n+1} & \cdots & a_{m-1} & e_m \\ a_{m-n+2} & \cdots & a_m & 0 \\ \vdots & & \vdots & \vdots \\ a_m & \cdots & a_{m+n-2} & 0 \end{bmatrix} \tag{5.10}$$

where the last column is determined by equating coefficients of z^i, for $m \leqq i \leqq m+n-1$ in (4.1). Thus, $\det(H_{m,n}) \neq 0$ and $\det(f_0) \neq 0$ implies that $\det(H_{m,n-1}) \neq 0$. Thus, (5.2) implies (5.1).

In a similar fashion, by defining

$$U_m^*(z) = E_m^*(z) - z g_0^* r_0^{*-1} P_{m-1}^*(z), \tag{5.11}$$

$$V_n^*(z) = F_{n-1}^*(z) - z g_0^* r_0^{*-1} Q_{n-1}^*(z), \tag{5.12}$$

it can be shown that (5.3) implies (5.1).

To obtain the inverse formula, substitution of (5.6), (5.7), (5.11), and (5.12), after normalization by (5.5), into equation (3.1) gives

$$\begin{aligned} H_{m,n}^{-1} = &\begin{bmatrix} f_{n-1} & \cdots & f_0 \\ \vdots & ⋰ & \\ f_0 & & \end{bmatrix} \begin{bmatrix} q_{n-1}^* & \cdots & q_0^* \\ & \ddots & \vdots \\ & & q_{n-1}^* \end{bmatrix} - \begin{bmatrix} q_{n-2} & \cdots & q_0 & 0 \\ \vdots & ⋰ & ⋰ & \\ q_0 & ⋰ & & \\ 0 & & & \end{bmatrix} \begin{bmatrix} 0 & f_{n-1}^* & \cdots & f_1^* \\ & & \ddots & \vdots \\ & & & f_{n-1}^* \\ & & & 0 \end{bmatrix} \\ &+ \begin{bmatrix} q_{n-2} & \cdots & q_0 & 0 \\ \vdots & ⋰ & ⋰ & \\ q_0 & ⋰ & & \\ 0 & & & \end{bmatrix} (g_0 - g_0^*) \begin{bmatrix} q_{n-1}^* & \cdots & q_0^* \\ & \ddots & \vdots \\ & & q_{n-1}^* \end{bmatrix}. \end{aligned} \tag{5.13}$$

But, (4.1) and (4.2) imply that

$$E_m^*(z)F_{n-1}(z) - F_{n-1}^*(z)E_m(z) = z^{m+n}\{F_{n-1}^*(z)G(z) - G^*(z)F_{n-1}(z)\}. \tag{5.14}$$

Consequently,

$$F^*_{n-1}(z)G(z)-G^*(z)F_{n-1}(z)=0 \tag{5.15}$$

and, in particular,

$$g_0=g_0^*. \tag{5.16}$$

Thus, (5.13) is exactly (5.4), since the last product cancels. □

Remark 1. Corollary 5.1 can be proved directly from (2.7), (2.8), (4.1), and (4.2). Indeed, using the same arguments as in Lemma 2.5, we can obtain

$$\begin{bmatrix} r_0^{*-1}Q^*_{n-1}(z) & -r_0^{*-1}P^*_{m-1}(z) \\ -f_0^{*-1}F^*_{n-1}(z) & f_0^{*-1}E^*_m(z) \end{bmatrix} \cdot \begin{bmatrix} E_m(z)f_0^{-1} & P_{m-1}(z)r_0^{-1} \\ F_{n-1}(z)f_0^{-1} & Q_{n-1}(z)r_0^{-1} \end{bmatrix} = z^{m+n-1}\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{5.17}$$

and the commutative relationship

$$\begin{bmatrix} E_m(z)f_0^{-1} & P_{m-1}(z)r_0^{-1} \\ F_{n-1}(z)f_0^{-1} & Q_{n-1}(z)r_0^{-1} \end{bmatrix} \cdot \begin{bmatrix} r_0^{*-1}Q^*_{n-1}(z) & -r_0^{*-1}P^*_{m-1}(z) \\ -f_0^{*-1}F^*_{n-1}(z) & f_0^{*-1}E^*_m(z) \end{bmatrix} = z^{m+n-1}\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{5.18}$$

Consequently, we can normalize our Padé forms according to (5.5) and the formulae will follow in a fashion similar to the proof of Theorem 3.1.

The actual proof, in addition to being simpler, serves to illustrate the existence of Frobenius-type relationships (generalized from the scalar case (cf. Gragg [17]) to the matrix case) between matrix Padé forms of types (m, n), $(m, n-1)$, and $(m-1, n-1)$. These relationships, which exist under the assumptions of Corollary 5.1, are given by (5.6), (5.7), (5.11), and (5.12) (see also [7]-[9]).

Remark 2. From (5.17), it follows from equating coefficients of degree $m+n-1$ that

$$e_m^* q_{n-1} = f_0^* r_0 \tag{5.19}$$

and

$$q^*_{n-1} e_m = r_0^* f_0. \tag{5.20}$$

Thus, if the conditions of Corollary 5.1 are satisfied, then e_m^*, q_{n-1}, q^*_{n-1}, and e_m are all nonsingular. Normalizing (2.7), (2.8), (4.1), and (4.2) by setting

$$r_0 = r_0^* = e_m = e_m^* = I, \tag{5.21}$$

rather than by (5.5), we obtain

$$H_{m,n}\cdot[q_{n-1},\cdots,q_0]' = [0,\cdots,0,I]', \tag{5.22}$$

$$[q^*_{n-1},\cdots,q_0^*]\cdot H_{m,n} = [0,\cdots,0,I], \tag{5.23}$$

$$H_{m,n}\cdot[f_{n-1},\cdots,f_0]' = [I,0,\cdots,0]', \tag{5.24}$$

$$[f^*_{n-1},\cdots,f_0^*]\cdot H_{m,n} = [I,0,\cdots,0]. \tag{5.25}$$

These conditions, together with the requirement that $\det(q_{n-1}) \neq 0$ and $\det(q^*_{n-1}) \neq 0$, are exactly the conditions given by Gohberg and Heinig [16] in deriving the inverse formula (5.4). Because of the different normalization requirement, their formula includes the term q_{n-1}^{-1} between the first two matrices and q^{*-1}_{n-1} between the last two matrices. This is permissible because of (5.19)-(5.21). In the scalar case, this is the well-known formula of Gohberg and Semencul [14].

Remark 3. The assumptions of Corollary 5.1, which are equivalent to conditions (5.22)-(5.25) of Gohberg and Heinig, are far more restrictive than the assumptions of Theorem 3.1, which are equivalent to (3.8), (3.9) and the block Yule-Walker equations (3.10) and (3.11). The formula of Gohberg and Heinig has the additional requirement that q_{n-1} and q^*_{n-1} be nonsingular (which is equivalent to $H_{m,n-1}$ being nonsingular). Thus, for example, (3.1) can be used to obtain the inverse of

$$H_{2,3}=\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{bmatrix}, \tag{5.26}$$

whereas, (5.4) cannot be applied.

Remark 4. Since the assumptions of Corollary 5.1 require that not only $H_{m,n}$ but also $H_{m,n-1}$ be nonsingular, it should be possible to express the inverse of $H_{m,n-1}$ in closed form as well. Indeed, by deriving Frobenius-type identities similar to (5.6), (5.7), (5.11), and (5.12) (cf. Bultheel [7]-[9]), the matrix Padé form of type $(m-1, n-2)$ can be expressed in terms of matrix Padé forms of type $(m, n-1)$ and $(m-1, n-1)$. Then, substituting the Padé forms of type $(m, n-1)$ and $(m-1, n-2)$ into (3.1) (with n replaced by $n-1$) and simplifying, we obtain as another corollary to Theorem 3.1 the second inverse formula of Gohberg and Heinig, namely,

$$H^{-1}_{m,n-1}=\begin{bmatrix} f_{n-2} & \cdots & f_0 \\ \vdots & ⋰ & \\ f_0 & & \end{bmatrix}\begin{bmatrix} q^*_{n-1} & \cdots & q^*_1 \\ & \ddots & \vdots \\ & & q^*_{n-1} \end{bmatrix}-\begin{bmatrix} q_{n-2} & \cdots & q_0 \\ \vdots & ⋰ & \\ q_0 & & \end{bmatrix}\begin{bmatrix} f^*_{n-1} & \cdots & f^*_1 \\ & \ddots & \vdots \\ & & f^*_{n-1} \end{bmatrix}. \tag{5.27}$$

Here, we have again normalized according to (5.5). We also note that the Gohberg-Heinig formulae given here are both determined from (3.1). Additional formulae, based on (3.2) rather than (3.1), can also be derived.

Remark 5. Gohberg and Heinig prove their formulae with the coefficients over a noncommutative algebra. Our formulae and results also carry over with minor alterations. In particular, Theorem 2.3 and Corollary 5.1 would both require that (2.9) be equivalent to (2.10) and (2.11), simultaneously.

6. The inverse formulae of Gohberg-Krupnik. Let $(L_{m-2}(z), M_{n-2}(z), N(z))$ and $(L^*_{m-2}(z), M^*_{n-2}(z), N^*(z))$ be an RMPFo and an LMPFo, respectively, of type $(m-2, n-2)$ for $A(z)$. These matrix Padé forms then satisfy

$$A(z)M_{n-2}(z)-L_{m-2}(z)=z^{m+n-3}N(z), \tag{6.1}$$

$$M^*_{n-2}(z)A(z)-L^*_{m-2}(z)=z^{m+n-3}N^*(z). \tag{6.2}$$

The inverse of $H_{m,n}$ in terms of matrix Padé forms of types $(m-2, n-2)$ and $(m-1, n-1)$ is given by Corollary 6.1.

COROLLARY 6.1. *Let the matrix Padé forms identified by* (2.7), (2.8), (6.1), *and* (6.2) *be given. Then, the following statements are equivalent*:

$$\det(H_{m,n})\neq 0 \quad \textit{and} \quad \det(H_{m-1,n-1})\neq 0, \tag{6.3}$$

$$\det(n_0)\neq 0, \quad \det(q_0)\neq 0, \quad \textit{and} \quad \det(r_0)\neq 0, \tag{6.4}$$

$$\det(n^*_0)\neq 0, \quad \det(q^*_0)\neq 0 \quad \textit{and} \quad \det(r^*_0)\neq 0. \tag{6.5}$$

In addition, if any (and therefore all) of the conditions (6.3), (6.4), *or* (6.5) *are satisfied,* *then*

$$
\begin{aligned}
H_{m,n}^{-1} = &\begin{bmatrix} q_{n-2} & \cdots & q_0 & 0 \\ \vdots & & \cdot^{\cdot^{\cdot}} & \\ q_0 & \cdot^{\cdot^{\cdot}} & & \\ 0 & & & \end{bmatrix} q_0^{-1} \begin{bmatrix} m_{n-2}^* & \cdots & m_{-1}^* \\ & \ddots & \vdots \\ & & m_{n-2}^* \end{bmatrix} \\
&- \begin{bmatrix} m_{n-3} & \cdots & m_0 & 0 & 0 \\ \vdots & & \cdot^{\cdot^{\cdot}} & & \\ m_0 & \cdot^{\cdot^{\cdot}} & & & \\ 0 & & & & \\ 0 & & & & \end{bmatrix} q_0^{*-1} \begin{bmatrix} q_{n-1}^* & \cdots & q_0^* \\ & \ddots & \vdots \\ & & q_{n-1}^* \end{bmatrix} \\
&+ \begin{bmatrix} q_{n-1} \\ \vdots \\ q_0 \end{bmatrix} q_0^{-1} [q_{n-1}^*, \cdots, q_0^*].
\end{aligned} \tag{6.6}
$$

Here, the matrix Padé forms have been normalized so that[4]

$$n_0 = n_0^* = r_0 = r_0^* = I. \tag{6.7}$$

Proof. To prove that (6.3) is equivalent to (6.4), it follows directly from Theorem 2.3 that $\det(H_{m,n}) \neq 0$ implies that $\det(r_0) \neq 0$, while $\det(H_{m-1,n-1}) \neq 0$ implies that $\det(n_0) \neq 0$ and $\det(q_0) \neq 0$. Conversely, suppose that (6.4) holds. Again, from Theorem 2.3, we have that $\det(n_0) \neq 0$ and $\det(q_0) \neq 0$ implies $\det(H_{m-1,n-1}) \neq 0$. But, then

$$
\begin{bmatrix} a_{m-n+1} & \cdots & a_m \\ \vdots & & \vdots \\ a_m & \cdots & a_{m+n-1} \end{bmatrix} \begin{bmatrix} I & & & q_{n-1} \\ & \ddots & & \vdots \\ & & I & q_1 \\ 0 & \cdots & 0 & q_0 \end{bmatrix} = \begin{bmatrix} a_{m-n+1} & \cdots & a_{m-1} & 0 \\ \vdots & & \vdots & \vdots \\ a_{m-1} & \cdots & a_{m+n-2} & 0 \\ a_m & \cdots & a_{m+n-1} & r_0 \end{bmatrix}, \tag{6.8}
$$

together with the assumption that $\det(r_0) \neq 0$, implies that also $\det(H_{m,n}) \neq 0$.

A similar argument shows that (6.3) is equivalent to (6.5).

To prove (6.6), we first establish some identities. Observe that, under the normalization condition (6.7), $(L_{m-2}(z), M_{n-2}(z), N(z))$, $(L_{m-2}^*(z), M_{n-2}^*(z), N^*(z))$, $(P_{m-1}(z)q_0^{-1}, Q_{n-1}(z)q_0^{-1}, R(z)q_0^{-1})$, and $(q_0^{*-1}P_{m-1}^*(z), q_0^{*-1}Q_{n-1}^*(z), q_0^{*-1}R^*(z))$ satisfy the conditions of Lemma 2.5, with (m, n) replaced by $(m-1, n-1)$. Here, (2.25) becomes

$$
\begin{bmatrix} Q_{n-1}(z)q_0^{-1} & M_{n-2}(z) \\ z^2R(z)q_0^{-1} & N(z) \end{bmatrix} \begin{bmatrix} N^*(z) & -M_{n-2}^*(z) \\ -z^2q_0^{*-1}R^*(z) & q_0^{*-1}Q_{n-1}^*(z) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{6.9}
$$

and, in particular,

$$\{R(z)q_0^{-1}\}N^*(z) = N(z)\{q_0^{*-1}R^*(z)\}. \tag{6.10}$$

Note that the constant and linear terms in (6.10) yield

$$q_0 = q_0^* \tag{6.11}$$

and

$$q_0^{*-1}(n_1^* - r_1^*) = (n_1 - r_1)q_0^{-1}. \tag{6.12}$$

[4] Rather than normalizing with $r_0 = r_0^* = I$, it is equally proper to normalize with $q_0 = q_0^* = I$.

For later purposes, also observe the identity

$$\begin{bmatrix} q_{n-1} & \cdots & q_0 \\ \vdots & ⋰ & \\ q_0 & & \end{bmatrix} q_0^{-1} \begin{bmatrix} q_{n-1}^* & \cdots & q_0^* \\ & \ddots & \vdots \\ & & q_{n-1}^* \end{bmatrix} - \begin{bmatrix} q_{n-2} & \cdots & q_0 & 0 \\ \vdots & ⋰ & ⋰ & \\ q_0 & ⋰ & & \\ 0 & & & \end{bmatrix} q_0^{*-1} \begin{bmatrix} 0 & q_{n-1}^* \cdots q_1^* \\ & \ddots \quad \vdots \\ & q_{n-1}^* \\ & 0 \end{bmatrix}$$

$$(6.13) \qquad = \begin{bmatrix} q_{n-1} \\ \vdots \\ q_0 \end{bmatrix} q_0^{-1}[q_{n-1}^*, \cdots, q_0^*],$$

which follows using (6.11).

Next, we proceed as in Corollary 5.1 by constructing right and left matrix Padé forms of type (m, n) for $A(z)$. Set

$$(6.14) \qquad U_m(z) = \{P_{m-1}(z)[I + (n_1 - r_1)z] - L_{m-2}(z)z^2\}q_0^{-1}$$

and

$$(6.15) \qquad V_n(z) = \{Q_{n-1}(z)[I + (n_1 - r_1)z] - M_{n-2}(z)z^2\}q_0^{-1}.$$

Then, $U_m(z)$ and $V_n(z)$ provide an RMPFo of type (m, n) for $A(z)$. To see this, note that the degree requirements are clearly satisfied. In addition, the columns of $V_n(z)$ are linearly independent since, in (6.15), $v_0 = I$. Finally,

$$\begin{aligned} A(z)V_n(z) - U_m(z) &= \{[A(z)Q_{n-1}(z) - P_{m-1}(z)][I + (n_1 - r_1)z] \\ &\quad - z^2[A(z)M_{n-2}(z) - L_{m-2}(z)]\}q_0^{-1} \\ (6.16) \qquad &= \{z^{m+n-1}R(z)[I + (n_1 - r_1)z] - z^{m+n-1}N(z)\}q_0^{-1} \\ &= z^{m+n-1}\{(r_0 - n_0) + (r_1 + r_0(n_1 - r_1) - n_1)z + z^2\{\cdots\}\}q_0^{-1} \\ &= z^{m+n+1}\{\cdots\}q_0^{-1}, \end{aligned}$$

since $n_0 = r_0 = I$.

Similarly, it can be shown that

$$(6.17) \qquad U_m^*(z) = q_0^{*-1}\{[I + (n_1^* - r_1^*)z]P_{m-1}^*(z) - L_{m-2}^*(z)z^2\},$$

$$(6.18) \qquad V_n^*(z) = q_0^{*-1}\{[I + (n_1^* - r_1^*)z]Q_{n-1}^*(z) - M_{n-2}^*(z)z^2\}$$

provides an LMPFo of type (m, n) for $A(z)$.

Note that (6.15) and (6.18), respectively, yield

$$\begin{bmatrix} v_{n-1} & \cdots & v_0 \\ \vdots & ⋰ & \\ v_0 & & \end{bmatrix} = \begin{bmatrix} q_{n-1} & \cdots & q_0 \\ \vdots & ⋰ & \\ q_0 & & \end{bmatrix} q_0^{-1} + \begin{bmatrix} q_{n-2} & \cdots & q_0 & 0 \\ \vdots & ⋰ & ⋰ & \\ q_0 & ⋰ & & \\ 0 & & & \end{bmatrix} (n_1 - r_1)q_0^{-1}$$

$$(6.19) \qquad - \begin{bmatrix} m_{n-3} & \cdots & m_0 & 0 & 0 \\ \vdots & ⋰ & & & \\ m_0 & ⋰ & ⋰ & & \\ 0 & & & & \\ 0 & & & & \end{bmatrix} q_0^{-1}$$

and

$$
\begin{aligned}
\begin{bmatrix} v_n^* & \cdots & v_1^* \\ & \ddots & \vdots \\ & & v_n^* \end{bmatrix}
&= q_0^{*-1}\begin{bmatrix} 0 & q_{n-1}^* & \cdots & q_1^* \\ & & \iddots & \vdots \\ & \iddots & & q_{n-1}^* \\ & & & 0 \end{bmatrix}
+ q_0^{*-1}(n_1^* - r_1^*)\begin{bmatrix} q_{n-1}^* & \cdots & q_0^* \\ & \ddots & \vdots \\ & & q_{n-1}^* \end{bmatrix} \\
&\quad - q_0^{*-1}\begin{bmatrix} m_{n-2}^* & \cdots & m_{-1}^* \\ & \ddots & \vdots \\ & & m_{n-2}^* \end{bmatrix}
\end{aligned}
\tag{6.20}
$$

where $m_{-1}^* = 0$. Substituting (6.19) and (6.20) into (3.1), and rearranging terms, we obtain

$$
\begin{aligned}
H_{m,n}^{-1} &= \begin{bmatrix} q_{n-2} & \cdots & q_0 & 0 \\ \vdots & \iddots & & \\ q_0 & & & \\ 0 & & & \end{bmatrix} q_0^{-1} \begin{bmatrix} m_{n-2}^* & \cdots & m_{-1}^* \\ & \ddots & \vdots \\ & & m_{n-2}^* \end{bmatrix} \\
&\quad - \begin{bmatrix} m_{n-3} & \cdots & m_0 & 0 & 0 \\ \vdots & \iddots & & & \\ m_0 & & & & \\ 0 & & & & \\ 0 & & & & \end{bmatrix} q_0^{*-1} \begin{bmatrix} q_{n-1}^* & \cdots & q_0^* \\ & \ddots & \vdots \\ & & q_{n-1}^* \end{bmatrix} \\
&\quad + \begin{bmatrix} q_{n-1} & \cdots & q_0 \\ \vdots & \iddots & \\ q_0 & & \end{bmatrix} q_0^{-1} \begin{bmatrix} q_{n-1}^* & \cdots & q_0^* \\ & \ddots & \vdots \\ & & q_{n-1}^* \end{bmatrix} \\
&\quad - \begin{bmatrix} q_{n-2} & \cdots & q_0 & 0 \\ \vdots & \iddots & & \\ q_0 & & & \\ 0 & & & \end{bmatrix} q_0^{*-1} \begin{bmatrix} 0 & q_{n-1}^* & \cdots & q_1^* \\ & & \iddots & \vdots \\ & \iddots & & q_{n-1}^* \\ & & & 0 \end{bmatrix} \\
&\quad + \begin{bmatrix} q_{n-2} & \cdots & q_0 & 0 \\ \vdots & \iddots & & \\ q_0 & & & \\ 0 & & & \end{bmatrix} \{(n_1 - r_q)q_0^{-1} - q_0^{*-1}(n_1^* - r_1^*)\} \begin{bmatrix} q_{n-1}^* & \cdots & q_0^* \\ & \ddots & \vdots \\ & & q_{n-1}^* \end{bmatrix}
\end{aligned}
\tag{6.21}
$$

But, using (6.12) and (6.13), it is easy to see that (6.21) is exactly (6.6). □

Remark 1. The inverse formula (6.6) can also be determined by bordering techniques. Indeed, (6.8) can be further manipulated to obtain

$$
H_{m,n}^{-1} = \begin{bmatrix} H_{m-1,n-1}^{-1} & 0 \\ 0 \cdots & 0 \end{bmatrix} + \begin{bmatrix} q_{n-1} \\ \vdots \\ q_0 \end{bmatrix} q_0^{-1}[q_{n-1}^*, \cdots, q_0^*].
\tag{6.22}
$$

Equation (3.1) applied to $H_{m-1,n-1}^{-1}$, along with simplification using Lemma 2.5, converts (6.22) to (6.6).

The present proof takes its cue from the approach of §§ 4 and 5. In each case, the inverse formula is obtained from (3.1) using Frobenius-type identities for matrix

Padé forms. The Frobenius-type identities (6.14), (6.15), (6.17), and (6.18) used here can be found in [8] (see also [22]).

Remark 2. Note that, if the matrix Padé forms (2.7), (2.8), (6.1), and (6.2) satisfy the conditions of Corollary 6.1 and are normalized according to (6.7), then

$$H_{m,n}\cdot\left\{\begin{bmatrix} m_{n-2}\\ \vdots \\ m_0 \\ 0\end{bmatrix}-\begin{bmatrix} q_{n-1}\\ \vdots \\ q_0\end{bmatrix}\cdot n_1\right\}=\begin{bmatrix}0\\ \vdots\\ 0\\ I\\ n_1\end{bmatrix}-\begin{bmatrix}0\\ \vdots\\ \\ 0\\ I\end{bmatrix}n_1=\begin{bmatrix}0\\ \vdots\\ 0\\ I\\ 0\end{bmatrix}. \tag{6.23}$$

Thus, the second last column of $H_{m,n}^{-1}$ is a combination of the coefficients of $M_{n-2}(z)$ and $Q_{n-1}(z)$. Similarly, we can obtain the second last row of $H_{m,n}^{-1}$ as a combination of the coefficients of $M_{n-2}^*(z)$ and $Q_{n-1}^*(z)$.

Conversely, suppose $X=[x_{n-1},\cdots,x_0]'$ and $Q=[q_{n-1},\cdots,q_0]'$, respectively, represent the second last and last block columns of the inverse of $H_{m,n}$. Then, if $\det(q_0)\neq 0$, we have that

$$H_{m,n}\left\{\begin{bmatrix} x_{n-1}\\ \vdots \\ x_0\end{bmatrix}-\begin{bmatrix} q_{n-1}\\ \vdots \\ q_0\end{bmatrix}q_0^{-1}x_0\right\}=\begin{bmatrix}0\\ \vdots\\ 0\\ I\\ -q_0^{-1}x_0\end{bmatrix} \tag{6.24}$$

so that

$$H_{m-1,n-1}\left\{\begin{bmatrix} x_{n-1}\\ \vdots \\ x_1\end{bmatrix}-\begin{bmatrix} q_{n-1}\\ \vdots \\ q_1\end{bmatrix}q_0^{-1}x_0\right\}=\begin{bmatrix}0\\ \vdots\\ 0\\ I\end{bmatrix}. \tag{6.25}$$

This implies that

$$M_{n-2}(z)=z^{-1}\{X(z)-Q_{n-1}(z)q_0^{-1}x_0\} \tag{6.26}$$

is an RMPFo denominator of type $(m-2, n-2)$ for $A(z)$. Similarly, we can obtain an LMPFo denominator of type $(m-2, n-2)$ when we have the last and second last block rows of the inverse of $H_{m,n}$. Then substitution into (6.6) yields

$$\begin{aligned} H_{m,n}^{-1}=&\begin{bmatrix} q_{n-2} & \cdots & q_0 & 0\\ \vdots & & \iddots & \\ q_0 & \iddots & & \\ 0 & & & \end{bmatrix}q_0^{-1}\begin{bmatrix} x_{n-1}^* & \cdots & x_0^*\\ & \ddots & \vdots\\ & & x_{n-1}^*\end{bmatrix}\\ &-\begin{bmatrix} x_{n-2} & \cdots & x_0 & 0\\ \vdots & & \iddots & \\ x_0 & \iddots & & \\ 0 & & & \end{bmatrix}q_0^{*-1}\begin{bmatrix} q_{n-1}^* & \cdots & q_0^*\\ & \ddots & \vdots\\ & & q_{n-1}^*\end{bmatrix}\\ &+\begin{bmatrix} q_{n-2} & \cdots & q_0 & 0\\ \vdots & & \iddots & \\ q_0 & \iddots & & \\ 0 & & & \end{bmatrix}q_0^{-1}(x_0^*-x_0)q_0^{*-1}\begin{bmatrix} q_{n-1}^* & \cdots & q_0^*\\ & \ddots & \vdots\\ & & q_{n-1}^*\end{bmatrix} \end{aligned} \tag{6.27}$$

$$+\begin{bmatrix} q_{n-1} \\ \vdots \\ q_0 \end{bmatrix} q_0^{-1}[q_{n-1}^*, \cdots, q_0^*];$$

whereas, substitution into (3.1) gives

$$H_{m-1,n-1}^{-1} = \begin{bmatrix} q_{n-2} \cdots q_0 \\ \vdots \quad \cdot^{\cdot^{\cdot}} \\ q_0 \end{bmatrix} q_0^{-1} \begin{bmatrix} x_{n-1}^* \cdots x_1^* \\ \ddots \quad \vdots \\ x_{n-1}^* \end{bmatrix} - \begin{bmatrix} x_{n-2} \cdots x_0 \\ \vdots \quad \cdot^{\cdot^{\cdot}} \\ x_0 \end{bmatrix} q_0^{*-1} \begin{bmatrix} q_{n-1}^* \cdots q_1^* \\ \ddots \quad \vdots \\ q_{n-1}^* \end{bmatrix}. \tag{6.28}$$

Remark 3. In the scalar case, if $X = [x_{n-1}, \cdots, x_0]^t$ and $Q = [q_{n-1}, \cdots, q_0]^t$ represent the second last and last columns of the inverse of $H_{m,n}$ respectively, and $q_0 \neq 0$, then (6.27) and (6.28) reduce to

$$\begin{aligned} H_{m,n}^{-1} = q_0^{-1} \Bigg\{ & \begin{bmatrix} q_{n-2} & \cdots & q_0 & 0 \\ \vdots & & \cdot^{\cdot^{\cdot}} & \\ q_0 & \cdot^{\cdot^{\cdot}} & & \\ 0 & & & \end{bmatrix} \begin{bmatrix} x_{n-1} \cdots x_0 \\ \ddots \quad \vdots \\ x_{n-1} \end{bmatrix} \\ & - \begin{bmatrix} x_{n-2} & \cdots & x_0 & 0 \\ \vdots & & \cdot^{\cdot^{\cdot}} & \\ x_0 & \cdot^{\cdot^{\cdot}} & & \\ 0 & & & \end{bmatrix} \begin{bmatrix} q_{n-1} \cdots q_0 \\ \ddots \quad \vdots \\ q_{n-1} \end{bmatrix} + \begin{bmatrix} q_{n-1}^2 & \cdots & q_{n-1}q_0 \\ \vdots & & \vdots \\ q_{n-1}q_0 & \cdots & q_0^2 \end{bmatrix} \Bigg\} \end{aligned} \tag{6.29}$$

and

$$\begin{aligned} H_{m-1,n-1}^{-1} = q_0^{-1} \Bigg\{ & \begin{bmatrix} q_{n-2} & \cdots & q_0 \\ \vdots & \cdot^{\cdot^{\cdot}} & \\ q_0 & & \end{bmatrix} \begin{bmatrix} x_{n-1} \cdots x_1 \\ \ddots \quad \vdots \\ x_{n-1} \end{bmatrix} \\ & - \begin{bmatrix} x_{n-2} & \cdots & x_0 \\ \vdots & \cdot^{\cdot^{\cdot}} & \\ x_0 & & \end{bmatrix} \times \begin{bmatrix} q_{n-1} & \cdots & q_1 \\ & \ddots & \vdots \\ & & q_{n-1} \end{bmatrix} \Bigg\}. \end{aligned} \tag{6.30}$$

These are the original formulae of Gohberg and Krupnik [15].

Remark 4. Following the approach of § 4, we can also obtain conditions and inverse formulae for $H_{m-1,n-1}$ and $H_{m,n}$ when the first and second block column, along with the first and second block row, of the inverse of $H_{m,n}$ is given (cf. Iohvidov [19]). Here, conditions and inverse formulae for $H_{m-1,n-1}$ and $H_{m,n}$ are stated in terms of matrix Padé forms of type $(m, n-1)$ and $(m+1, n-2)$. Additional formulae, based on (3.2) rather than (3.1), can also be given.

7. The inverse formulae of Ben-Artzi and Shalom. As mentioned in § 3, the assumptions of Theorem 3.1 can be equivalently replaced by the requirement that we obtain solutions to

$$H_{m,n} \cdot [q_{n-1}, \cdots, q_0]^t = [0, \cdots, 0, I]^t, \tag{7.1}$$

$$[q_{n-1}^*, \cdots, q_0^*] \cdot H_{m,n} = [0, \cdots, 0, I], \tag{7.2}$$

$$H_{m,n} \cdot [v_n, \cdots, v_1]^t = -[a_{m+1}, \cdots, a_{m+n-1}, a_{m+n}]^t, \tag{7.3}$$

$$[v_n^*, \cdots, v_1^*] \cdot H_{m,n} = -[a_{m+1}, \cdots, a_{m+n-1}, a_{m+n}] \tag{7.4}$$

where a_{m+n} can be any $p \times p$ matrix. It is possible to alter the right-hand sides of (7.3) and (7.4) and still obtain inverse formulae for $H_{m,n}$. In particular, we may replace the right-hand sides by linear combinations of the rows and columns of $H_{m+1,n}$.

LEMMA 7.1. *Let $H_{m,n}$ be the block Hankel matrix* (1.1). *Suppose there are solutions to* (7.1) *and* (7.2) *along with solutions to*

$$H_{m,n}\cdot[x_{n-1},\cdots,x_0]^t=H_{m+1,n}\cdot[y_{n-1},\cdots,y_0]^t, \tag{7.5}$$

and

$$[x^*_{n-1},\cdots,x^*_0]\cdot H_{m,n}=[y^*_{n-1},\cdots,y^*_0]\cdot H_{m+1,n}, \tag{7.6}$$

with y_0 and y_0^* nonsingular. Then $H_{m,n}$ is nonsingular with inverse

$$\begin{aligned}&\begin{bmatrix} y_{n-1}-x_{n-2} & \cdots & y_1-x_0 & y_0\\ \vdots & & \cdot^{\cdot^{\cdot}} & \\ y_1-x_0 & \cdot^{\cdot^{\cdot}} & & \\ y_0 & & & \end{bmatrix} y_0^{-1}\begin{bmatrix} q^*_{n-1} & \cdots & q^*_0\\ & \ddots & \vdots\\ & & q^*_{n-1}\end{bmatrix}\\ &+\begin{bmatrix} q_{n-2} & \cdots & q_0 & 0\\ \vdots & \cdot^{\cdot^{\cdot}} & & \\ q_0 & \cdot^{\cdot^{\cdot}} & & \\ 0 & & & \end{bmatrix} y_0^{*-1}\begin{bmatrix} x^*_{n-1} & x^*_{n-2}-y^*_{n-1} & \cdots & x^*_0-y^*_1\\ & \ddots & \ddots & \vdots\\ & & & x^*_{n-2}-y^*_{n-1}\\ & & & x^*_{n-1}\end{bmatrix},\end{aligned} \tag{7.7}$$

or, equivalently, the inverse is given by

$$\begin{aligned}&\begin{bmatrix} q_{n-1} & & \\ \vdots & \ddots & \\ q_0 & \cdots & q_{n-1}\end{bmatrix} y_0^{*-1}\begin{bmatrix} y^*_{n-1}-x^*_{n-2} & \cdots & y^*_1-x^*_0 & y^*_0\\ \vdots & & \cdot^{\cdot^{\cdot}} & \\ y^*_1-x^*_0 & \cdot^{\cdot^{\cdot}} & & \\ y^*_0 & & & \end{bmatrix}\\ &+\begin{bmatrix} x_{n-1} & & & \\ x_{n-2}-y_{n-1} & & \ddots & \\ \vdots & \ddots & & \\ x_0-y_1 & \cdots & x_{n-2}-y_{n-1} & x_{n-1}\end{bmatrix} y_0^{-1}\begin{bmatrix} q^*_{n-2} & \cdots & q^*_0 & 0\\ \vdots & \cdot^{\cdot^{\cdot}} & & \\ q^*_0 & \cdot^{\cdot^{\cdot}} & & \\ 0 & & & \end{bmatrix}.\end{aligned} \tag{7.8}$$

Proof. Since

$$\begin{aligned}\begin{bmatrix} a_{m-n+1} & \cdots & a_m\\ \vdots & & \vdots\\ a_m & \cdots & a_{m+n-1}\end{bmatrix}\cdot\begin{bmatrix} x_{n-1}\\ \vdots\\ x_0\end{bmatrix}&=\begin{bmatrix} a_{m-n+2} & \cdots & a_{m+1}\\ \vdots & & \vdots\\ a_{m+1} & \cdots & a_{m+n}\end{bmatrix}\cdot\begin{bmatrix} y_{n-1}\\ \vdots\\ y_0\end{bmatrix}\\ &=\begin{bmatrix} a_{m-n+1} & \cdots & a_m\\ \vdots & & \vdots\\ a_m & & a_{m+n-1}\end{bmatrix}\cdot\begin{bmatrix} 0\\ y_{n-1}\\ \vdots\\ y_1\end{bmatrix}\\ &\quad+\begin{bmatrix} a_{m+1}\\ \vdots\\ a_{m+n}\end{bmatrix}\cdot y_0,\end{aligned} \tag{7.9}$$

we get that

$$\begin{bmatrix} v_n\\ \vdots\\ v_1\end{bmatrix}=\left\{\begin{bmatrix} 0\\ y_{n-1}\\ \vdots\\ y_1\end{bmatrix}-\begin{bmatrix} x_{n-1}\\ x_{n-2}\\ \vdots\\ x_0\end{bmatrix}\right\}\cdot y_0^{-1} \tag{7.10}$$

is a solution to (7.3). Similarly,

$$[v_n^*, \cdots, v_1^*] = y_0^{*-1} \cdot ([0, y_{n-1}^*, \cdots, y_1^*] - [x_{n-1}^*, x_{n-2}^*, \cdots, x_0^*]) \tag{7.11}$$

is a solution to (7.4). Substituting (7.10) and (7.11) into (3.1) gives (7.7), while substituting into (3.2) gives (7.8). □

Let $E^{(i)}$ denote the $n \times 1$ block matrix having the $p \times p$ identity matrix as its ith block row, and zeros elsewhere. Similarly, let $E^{*(i)}$ be the $1 \times n$ block matrix having the $p \times p$ identity matrix as its ith block column, with zeros elsewhere. Theorem 7.2 shows how to construct the inverse of a block Hankel matrix, knowing only the last block column and row, along with two successive block columns and rows of the inverse.

THEOREM 7.2. *Let $H_{m,n}$ be the block Hankel matrix* (1.1). *Suppose there are solutions to* (7.1) *and* (7.2), *along with solutions to*

$$H_{m,n} \cdot [x_{n-1}, \cdots, x_0]' = E^{(i)}, \tag{7.12}$$

$$H_{m,n} \cdot [y_{n-1}, \cdots, y_0]' = E^{(i+1)}, \tag{7.13}$$

$$[x_{n-1}^*, \cdots, x_0^*] \cdot H_{m,n} = E^{*(i)}, \tag{7.14}$$

$$[y_{n-1}^*, \cdots, y_0^*] \cdot H_{m,n} = E^{*(i+1)}. \tag{7.15}$$

If, in addition, y_0 and y_0^* are both nonsingular, then $H_{m,n}$ is nonsingular with inverse given by

$$\begin{aligned}
&\begin{bmatrix} y_{n-1}-x_{n-2} & \cdots & y_1-x_0 & y_0 \\ \vdots & ⋰ & ⋰ & \\ y_1-x_0 & ⋰ & & \\ y_0 & & & \end{bmatrix} y_0^{-1} \begin{bmatrix} q_{n-1}^* & \cdots & q_0^* \\ & \ddots & \vdots \\ & & q_{n-1}^* \end{bmatrix} \\
&+ \begin{bmatrix} q_{n-2} & \cdots & q_0 & 0 \\ \vdots & ⋰ & ⋰ & \\ q_0 & ⋰ & & \\ 0 & & & \end{bmatrix} y_0^{*-1} \begin{bmatrix} x_{n-1}^* & x_{n-2}^*-y_{n-1}^* & \cdots & x_0^*-y_1^* \\ & \ddots & \ddots & \vdots \\ & & & x_{n-2}^*-y_{n-1}^* \\ & & & x_{n-1}^* \end{bmatrix} \\
&+ \begin{bmatrix} q_{n-2} & \cdots & q_0 & 0 \\ \vdots & ⋰ & & \\ q_0 & & & \\ 0 & & & \end{bmatrix} y_0^{*-1}(y_{n-i}^*-y_{n-i}) \cdot y_0^{-1} \begin{bmatrix} q_{n-1}^* & \cdots & q_0^* \\ & \ddots & \vdots \\ & & q_{n-1}^* \end{bmatrix};
\end{aligned} \tag{7.16}$$

or, equivalently, the inverse is given by

$$\begin{aligned}
&\begin{bmatrix} q_{n-1} & & \\ \vdots & \ddots & \\ q_0 & \cdots & q_{n-1} \end{bmatrix} y_0^{*-1} \begin{bmatrix} y_{n-1}^*-x_{n-2}^* & \cdots & y_1^*-x_0^* & y_0^* \\ \vdots & ⋰ & ⋰ & \\ y_1^*-x_0^* & & & \\ y_0^* & & & \end{bmatrix} \\
&+ \begin{bmatrix} x_{n-1} & & & \\ x_{n-2}-y_{n-1} & & \ddots & \\ \vdots & \ddots & & \\ x_0-y_1 & \cdots & x_{n-2}-y_{n-1} & x_{n-1} \end{bmatrix} y_0^{-1} \begin{bmatrix} q_{n-2}^* & \cdots & q_0^* & 0 \\ \vdots & ⋰ & ⋰ & \\ q_0^* & & & \\ 0 & & & \end{bmatrix} \\
&+ \begin{bmatrix} q_{n-1} & & \\ \vdots & \ddots & \\ q_0 & \cdots & q_{n-1} \end{bmatrix} \cdot y_0^{*-1} \cdot (y_{n-i}-y_{n-i}^*) \cdot y_0^{-1} \cdot \begin{bmatrix} q_{n-2}^* & \cdots & q_0^* & 0 \\ \vdots & ⋰ & ⋰ & \\ q_0^* & & & \\ 0 & & & \end{bmatrix}.
\end{aligned} \tag{7.17}$$

Proof. Equation (7.13) implies that

$$H_{m+1,n} \cdot Y = E^{(i)} + E^{(n)} \cdot c \tag{7.18}$$

where

$$c = a_{m+1} \cdot y_{n-1} + \cdots + a_{m+n} \cdot y_0. \tag{7.19}$$

Therefore

$$H_{m,n} \cdot (X + Q \cdot c) = H_{m+1,n} \cdot Y, \tag{7.20}$$

and similarly we can show that

$$(X^* + c^* \cdot Q^*) \cdot H_{m,n} = Y^* \cdot H_{m+1,n} \tag{7.21}$$

where

$$c^* = y^*_{n-1} \cdot a_{m+1} + \cdots + y^*_0 \cdot a_{m+n}. \tag{7.22}$$

Therefore, using Lemma 7.1, $H_{m,n}$ is nonsingular with inverse given according to (7.7) or (7.8) applied to equations (7.1), (7.2), (7.20), and (7.21). For example, substituting these expressions into (7.7) and expanding, we obtain the inverse of $H_{m,n}$ as

$$\begin{aligned}
&\begin{bmatrix} y_{n-1} - x_{n-2} & \cdots & y_1 - x_0 & y_0 \\ \vdots & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\ y_1 - x_0 & \cdot^{\cdot^{\cdot}} & & \\ y_0 & & & \end{bmatrix} y_0^{-1} \begin{bmatrix} q^*_{n-1} & \cdots & q^*_0 \\ & \ddots & \vdots \\ & & q^*_{n-1} \end{bmatrix} \\
&+ \begin{bmatrix} q_{n-2} & \cdots & q_0 & 0 \\ \vdots & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\ q_0 & \cdot^{\cdot^{\cdot}} & & \\ 0 & & & \end{bmatrix} y_0^{*-1} \begin{bmatrix} x^*_{n-1} & x^*_{n-2} - y^*_{n-1} & \cdots & x^*_0 - y^*_1 \\ & \ddots & \ddots & \vdots \\ & & & x^*_{n-2} - y^*_{n-1} \\ & & & x^*_{n-1} \end{bmatrix} \\
&+ \begin{bmatrix} q_{n-2} & \cdots & q_0 & 0 \\ \vdots & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\ q_0 & \cdot^{\cdot^{\cdot}} & & \\ 0 & & & \end{bmatrix} (y_0^{*-1} \cdot c^* - c \cdot y_0^{-1}) \begin{bmatrix} q^*_{n-1} & \cdots & q^*_0 \\ & \ddots & \vdots \\ & & q^*_{n-1} \end{bmatrix}.
\end{aligned} \tag{7.23}$$

To obtain formula (7.16), we note that

$$\begin{aligned}
y^*_{n-i} &= Y^* \cdot E^{(i)} \\
&= Y^* \cdot H_{m,n} \cdot X \\
&= Y^* \cdot \{H_{m+1,n} \cdot Y - H_{m,n} \cdot Q \cdot c\} \\
&= Y^* \cdot \{H_{m+1,n} \cdot Y - E^{(n)} \cdot c\} \\
&= Y^* \cdot H_{m+1,n} \cdot Y - y^*_0 \cdot c \\
&= Y^* H_{m,n} \cdot [0, y_{n-1}, \cdots, y_1]' + c^* y_0 - y^*_0 \cdot c \\
&= E^{*(i+1)} \cdot [0, y_{n-1}, \cdots, y_1]' + c^* y_0 - y^*_0 \cdot c \\
&= y_{n-i} + c^* y_0 - y^*_0 \cdot c.
\end{aligned} \tag{7.24}$$

Therefore

$$(y_0^{*-1} \cdot c^* - c \cdot y_0^{-1}) = y_0^{*-1} \cdot (y^*_{n-i} - y_{n-i}) \cdot y_0^{-1}. \tag{7.25}$$

Substituting (7.25) into (7.23) gives (7.16). Formula (7.17) is verified in a similar manner. □

Remark 1. In the scalar case, Theorem 7.2 gives the inverse of $H_{m,n}$ in terms of the last column along with an additional two successive columns of the inverse. In this case, (7.16) gives $H_{m,n}^{-1}$ as

$$y_0^{-1}\left\{\begin{bmatrix} y_{n-1}-x_{n-2} & \cdots & y_1-x_0 & y_0 \\ \vdots & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\ y_1-x_0 & & & \\ y_0 & & & \end{bmatrix}\begin{bmatrix} q_{n-1} & \cdots & q_0 \\ & \ddots & \vdots \\ & & q_{n-1} \end{bmatrix} + \begin{bmatrix} q_{n-2} & \cdots & q_0 & 0 \\ \vdots & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\ q_0 & & & \\ 0 & & & \end{bmatrix}\begin{bmatrix} x_{n-1} & x_{n-2}-y_{n-1} & \cdots & x_0-y_1 \\ & \ddots & \ddots & \vdots \\ & & & x_{n-2}-y_{n-1} \\ & & & x_{n-1} \end{bmatrix}\right\}. \tag{7.26}$$

Formula (7.26) is due to Ben-Artzi and Shalom [3] (in its Hankel formulation). Equation (7.17) reduces to an alternate formula in the scalar case.

Remark 2. Let S be the $n \times n$ shift matrix having 1's along the superdiagonal, and 0's elsewhere. Suppose in the scalar case there is a solution Q, to (7.1) along with a solution to

$$H_{m,n} \cdot Z = S \cdot H_{m,n} \cdot Y \tag{7.27}$$

where $y_0 \neq 0$. Then, there is also a solution to (7.5) since

$$H_{m,n} \cdot (Z + Q \cdot c) = H_{m+1,n} \cdot Y \tag{7.28}$$

where c is given by (7.19). Since $y_0 \neq 0$, Lemma 7.1 implies that $H_{m,n}$ is nonsingular, with inverse given by (7.7). After simplification, this inverse formula is

$$y_0^{-1}\left\{\begin{bmatrix} y_{n-1}-z_{n-2} & \cdots & y_1-z_0 & y_0 \\ \vdots & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\ y_1-z_0 & & & \\ y_0 & & & \end{bmatrix}\begin{bmatrix} q_{n-1} & \cdots & q_0 \\ & \ddots & \vdots \\ & & q_{n-1} \end{bmatrix} + \begin{bmatrix} q_{n-2} & \cdots & q_0 & 0 \\ \vdots & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\ q_0 & & & \\ 0 & & & \end{bmatrix}\begin{bmatrix} z_{n-1} & z_{n-2}-y_{n-1} & \cdots & z_0-y_1 \\ & \ddots & \ddots & \vdots \\ & & & z_{n-2}-y_{n-1} \\ & & & z_{n-1} \end{bmatrix}\right\}. \tag{7.29}$$

This is the main inverse formula of Ben-Artzi and Shalom [3] in the scalar case. They use this formula to give simple derivations of their own scalar formula (7.26), along with other inverse formulae including the formulae of both Gohberg-Krupnik and Gohberg-Semencul.

8. Conclusions. The Frobenius-type relationships given in this paper are but a small sample of similar recurrence relationships that exist between matrix Padé forms that have been developed by Bultheel [7]-[9]. All the relationships require the existence of inverses of certain coefficients in the Padé forms involved. These requirements are always satisfied for normal matrix power series (where $H_{m,n}$ is nonsingular for all m and n). For this restricted class of power series, many of the recursive relationships provide directly algorithms for the computation of Padé forms. Depending on the path (within the Padé table) determined by the recurrence, Bultheel observes that most previous algorithms [1], [5], [13], [23], [25]-[27], [32] that explicitly or implicitly compute the inverse of Hankel or Toeplitz matrices are equivalent to using an appropriate recurrence formula.

For a subset of these relationships, this paper shows that each recurrence yields a separate closed formula for the inverse of a block Hankel matrix. Algorithms based on recurrences that specify computations along an off-diagonal path (e.g., [1], [5], [27], [32]) yield closed formulae expressed by (3.1), (3.2), and (6.6). Those that specify computations along a staircase (e.g., [13], [23], [25]) yield formulae (5.4) and (5.27); whereas, those that specify computations along an antidiagonal path yield (4.8) and (4.9). Additional closed formulae can be derived corresponding to other recurrences given by Bultheel.

Formulae (5.4), (5.27), and (6.6) are equivalent to those given by Gohberg and Heinig and Gohberg and Krupnik, whereas (3.1), (3.2), (4.8), and (4.9) are new. A major advantage of the new formulae is that the underlying assumptions are far less restrictive than they are for (5.4), (5.27), and (6.6). Whereas, the new formulae require only that $H_{m,n}$ be nonsingular, the latter also require that an additional submatrix be nonsingular. In addition, necessary and sufficient conditions for the existence of $H_{m,n}^{-1}$ are directly available from the coefficients of Padé forms. This provides a significant computational advantage.

Relaxed conditions provide little computational gain, however, if the available algorithms can function only under the more severe restrictions of normality. Unfortunately, this is true for most algorithms that compute nonscalar Padé forms or decompose block Hankel matrices. One exception in this regard is the MPADE algorithm of Labahn and Cabay [22]. This algorithm is based on a recurrence relationship between Padé forms at successive nonsingular nodes along an off-diagonal path of the matrix Padé table (or, by reversing coefficients, along an antidiagonal path). When the power series is normal, or, less restrictively, when all principal minors of the associated Hankel matrix are nonsingular (e.g., when the block Hankel matrix is positive definite), all the nodes along the path are nonsingular and then their recurrence relationship reduces to (6.15), which is one of many given by Bultheel. The methods based on this relationship are special cases of the MPADE algorithm

For purposes of expressing the inverse of $H_{m,n}$ in terms of the new formulae (3.1), (3.2), (4.8), and (4.9), the MPADE algorithm is particularly suitable. Singularity is detected with no additional effort. When $H_{m,n}$ is nonsingular, the necessary Padé forms (i.e., the solutions of the associated block Yule-Walker equations) appearing in the formulae are simultaneously available on termination. The algorithm has no restrictions of normality. In addition, intermediate results enable the computation of the inverses of any nonsingular principal minors.

Using classical polynomial arithmetic, the cost of the MPADE algorithm is typically $O(p^3n^2)$, but can reach a complexity of $O(p^3n^3)$ in pathological cases (e.g., when all the principal minors are singular). When the power series is normal, this cost is the same as that of previously mentioned algorithms.

Using fast polynomial arithmetic in the normal case, Bitmead and Anderson [4] indicate that their scalar algorithm, based on a divide-and-conquer partitioning of the Hankel matrix, can be generalized to the nonscalar case with a cost complexity of $O(p^3n \log^2 n)$. Under somewhat relaxed normality conditions (i.e., near-normality), Labahn [21] also gives an algorithm, an adaptation of MPADE, with the same complexity.

In the scalar case, one call of an algorithm given by Cabay and Choi [11] can be used to construct the inverse formulae (3.1), (3.2), (4.8), or (4.9) with cost complexity $O(n \log^2 n)$ under no restrictions of normality. This is also true of other methods (cf. Sugiyama [29] for a survey) and, in particular, this is true of the method of Brent, Gustavson, and Yun [6]. They use two calls of a fast antidiagonal GCD algorithm, EMGCD, to determine the two Padé forms required by the Gohberg-Semencul formula (5.4). The algorithm succeeds immediately if both $H_{m,n}$ and $H_{m,n-1}$ are nonsingular. If $H_{m,n-1}$ is singular (but $H_{m,n}$ is not), then a nonsingular matrix $H_{m,n+1}$ is first constructed (it is not clear that this is always possible in the nonscalar case). Two additional calls of the antidiagonal algorithm are then made to yield the two Padé forms required by the second formula (5.27) of Gohberg and Semencul. By computing the inverse of $H_{m,n}$ using (4.8) or (4.9), their algorithm can now be altered so as to only require one call of their antidiagonal algorithm.

The use of (3.1) to express the inverse of $H_{m,n}$ avoids the immediate problem of potential numerical instabilities inherent when using instead the two formulae (5.4) and (5.27) according to the status of singularity of relevant minors (cf. Bunch [10]). However, this does not imply that the algorithm for determining the inverse of $H_{m,n}$ using (3.1) is stable, since this first requires the stable computation of $(P(z), Q(z))$ and $(U(z), V(z))$. The question of the stability of the algorithm MPADE for computing $(P(z), Q(z))$ and $(U(z), V(z))$ is an open question currently under investigation.

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