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# THE INVERTIBILITY OF THE RADON TRANSFORM ON ABSTRACT ROTATIONAL MANIFOLDS OF REAL TYPE

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**Abstract.**

Injectivity and support theorem are proved for the Radon transform on abstract rotational manifolds of real type. The transform is defined by integration over certain rotational submanifolds of codimension 1. Our technique is to use the theory of spherical harmonics. We also get unified closed inversion formulas for the spaces of constant curvature.

## 1. Introduction.

Nowadays the Radon transform is extensively studied in several settings [1, 5, 6, 7, 8, 9, 12]. The main question on every spaces are the invertibility of the transform and the support theorem [6].

We take the Radon transform on abstract rotational manifold $\mathcal{M}$ of real type [14] so that it integrates over the rotational submanifolds of codimension 1 by the natural measure induced by the original Riemannian metric. Such a rotational submanifold is obtained by rotating a geodesic around the orthogonal geodesic joining its closest point to the base point of $\mathcal{M}$. The manifold of these "hyperplanes" is denoted by $\mathcal{N}$. Precisely, the Radon transform of the function $f\colon \mathcal{M} \to \mathsf{R}$ is

$$Rf\colon \mathcal{N} \to \mathsf{R} \qquad Rf(\xi) = \int_{\xi} f(x)\,\mathrm{d}x,$$

where $\mathrm{d}x$ is the natural measure on $\xi \in \mathcal{N}$.

In this paper we will generalize and unify several results on the Radon transforms [5, 6, 8, 9] by proving the support theorem and the invertibility of this Radon transform. Then we prove inversion formulas on the spaces of constant curvature [8, 9] in this setting that makes the results of [8, 9] appear in unified form. From this point of view also the points of the proofs are more clear then they are in [8, 9].

Our method, to use the theory of spherical harmonics, is new on these spaces

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although the connection between the Radon transform and the spherical harmonics on the Euclidean space is well known since the middle of the century [10]. Roughly speaking, this connection is that the projection of the functions onto a one dimensional function space spanned by a spherical harmonic can be transposed with the Radon transform using a simple one dimensional integral transform. One can relatively easily handle these one dimensional integral transforms by using some facts about the Gegenbauer polynomials. We show out that the same argument works very well on the rotational manifolds and even more efficiently on the spaces of constant curvature.

## 2. Preliminaries.

We collect here the notations and facts we will use throughout this paper. First of all we recall the abstract rotational manifolds [14].

A complete Riemannian manifold $\mathscr{M}$ of dimension $n$ is called an abstract rotational manifold with base point $O \in \mathscr{M}$ if the induced linear action of the isotropy group of $O$ on $T_O\mathscr{M}$ is equivalent to $O(n)$.

The Riemannian metric on $\mathscr{M}$ is then completely described by a size function $g\colon \mathsf{R}_+ \to \mathsf{R}_+$ such that the geodesic sphere of radius $r$ in $\mathscr{M}$ is isometric to the Euclidean sphere of radius $g(r)$. This explains the notation $(\mathscr{M}, g)$. A complete abstract rotational manifold of real type is homogeneous if and only if either it is of constant sectional curvature $\kappa$ or, equivalently, the size function $g$ satisfies the ODE $\ddot{g} + \kappa g = 0$ for a suitable constant $\kappa$. Thus in these spaces the size functions are $\operatorname{sh} r$, $r$ and $\sin r$.

With the geodesic polar coordinatization (i.e. $(\omega, p) \in S^{n-1} \times [0, l_g) \to \operatorname{Exp}_O p\omega$) of the rotational manifold $\mathscr{M}$ and the Euclidean space $E$ of the same dimension one can define for every function $v\colon [0, l_g) \to [0, \infty)$ the map $(\omega, p) \to (\omega, v(p))$ from $\mathscr{M}$ into $E$. If the mapped geodesics are geodesics we call this function the "projector function" of $\mathscr{M}$ and usually we denote it by $\mu$. By Beltrami's theorem (L. P. Eisenhart: Riemannian Geometry) $\mathscr{M}$ must be of constant curvature if it has projector function because it makes a geodesic correspondence. On the other hand from the quadratic model of the spaces of constant curvature [7, p. 93] one can easily read off that in these cases the projector functions are $\operatorname{th} r$, $r$ and $\operatorname{tg} r$ as the curvatures are $-1$, 0 and 1.

We have the trigonometry on the rotational manifolds developed by Wu-Yi [14]. This trigonometry shows that a geodesic right triangle, where as the figure shows $H$ is the right angled vertex and $O$, $X$ are the not right angled vertexes, is determined by the angle $\alpha$ at the origin $O$ and by the distance $h$ of $H$ or ($x$ of $X$) from the origin. We have two equations for these data.

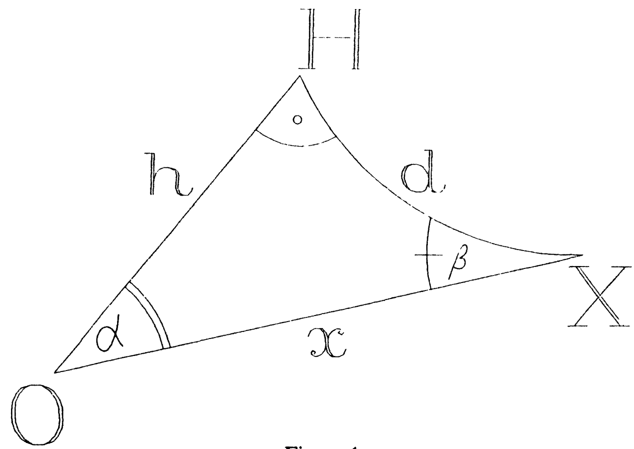


Figure 1.

$$\sin \beta = g(h)/g(x) \qquad \text{(sine law)},$$

$$d = \int_h^x \frac{g(r)}{\sqrt{g^2(r) - g^2(h)}} \, \mathrm{d}r \qquad \text{(cosine law)},$$

where $\beta$ denotes the angle at $X$ and $d$ denotes the distance between $H$ and $X$. In the following, we shall frequently use the angle $\beta$ and the distance $d$ as a function of $\alpha$ when the point $X$ or $H$ will be fixed.

We shall parameterize in these spaces a hyperplane $\mathscr{W} \in \mathscr{N}$ by its distance $p$ from $O$ and the unit vector $\omega \in T_O\mathscr{M}$ so that the corresponding geodesic $\mathrm{Exp}_O\, t\omega$ is perpendicular to $\mathscr{W}$ at the point $\mathrm{Exp}_O\, p\omega$. This hyperplane is denoted by $\xi(\omega, p)$. There may be some problems with the uniqueness of this parameterization going away from the origin, therefore we have to modify a little bit the definition of the rotational manifold. We shall say that $I_g$ is the geodesic injectivity radius of the origin if it is the maximal number that the above parameterization of the geodesics is injective on $S^{n-1} \times (0, I_g)$. To avoid the non-uniqueness, we shall restrict the rotational manifolds to the set $\mathrm{Exp}_O\,[S^{n-1} \times (0, I_g)]$. As one can easily see $I_g$ is infinite on the hyperbolic space and $\pi/2$ on the unit sphere.

The normals of the hyperplanes make an obvious bijection between the set of hyperplanes passing through the point $x$ and the elements of the unit sphere in $T_x\mathscr{M}$. Therefore the surface measure of this unit sphere is projected onto the set of hyperplanes passing through the point $x$. Let $\mu_x$ be this projected measure and $F\colon \mathscr{N} \to \mathrm{R}$. Then the dual Radon transform of $F$ is

$$R_t F\colon \mathscr{M} \to \mathrm{R} \qquad R_t F(x) = \int_{x \in \xi} F(\xi)\, \mathrm{d}\mu_x(\xi).$$

To make easier our further investigations we now introduce the boomerang transform. A function $f$ on $\mathcal{M}$ define naturally a function $F$ on $\mathcal{N}$ by the equation $F(\xi) = f(x)$, where $x$ is the point of $\xi$ nearest to the origin. If this correspondence is denoted by $P$ then the boomerang transform $B$ is $R_t P$, i.e. $Bf = R_t Pf$.

Almost all of our calculations will be based on the theory of spherical harmonics. The most important facts we need about them are the following. A complete orthonormal system in the Hilbert space $L^2(S^{n-1})$ can be chosen consisting of spherical harmonics $Y_{l,m}$, where $Y_{l,m}$ is of degree $m$. If $Y_{l,m}$ is a member of such a system, $f \in C^\infty(S^{n-1} \times [0,\infty))$ and $p \in [0,\infty)$ let the corresponding coefficients of the series in this system for $f(\omega, p)$ be $f_{l,m}(p)$. Then the series

$$\sum_{l,m}^{\infty} f_{l,m}(p) Y_{l,m}(\omega)$$

converges uniformly absolutely on compact subset of $S^{n-1} \times [0,\infty)$ to $f(\omega, p)$. Our main tool in this theory is the Funk-Hecke theorem. If $\int_{-1}^{1} |f(t)|(1-t^2)^{\lambda-1/2}\,dt < \infty$ and $\lambda = (n-2)/2$, then

$$\int_{S^{n-1}} f(\langle \omega, \bar{\omega} \rangle) Y_{l,m}(\omega)\,d\omega = Y_{l,m}(\bar{\omega}) \frac{|S^{n-2}|}{C_m^\lambda(1)} \int_{-1}^{1} f(t) C_m^\lambda(t)(1-t^2)^{\lambda-1/2}\,dt,$$

where $|S^{n-2}|$ is the surface area of the unit sphere $S^{n-2}$, $\langle .,. \rangle$ is the usual scalar product and $C_m^\lambda$ is the Gegenbauer polynomial of degree $m$. For further details we refer to [11, 13].

## 3. Support theorem on rotational manifolds.

In this section we prove the support theorems for the Radon and the boomerang transform. We define for each real $m > 1/I_g$ the following function spaces:

$$L_m^2(\mathcal{M}) = \{f \in L_{\mathrm{loc}}^2(\mathcal{M}): d(O,X) \geqq 1/m \Rightarrow f(X) = 0\} \ \& \ L_*^2(\mathcal{M}) = \bigcup_{m>0} L_m^2(\mathcal{M})$$

$$L_m^2(\mathcal{N}) = \{F \in L_{\mathrm{loc}}^2(\mathcal{N}): \ p \geqq 1/m \Rightarrow F(\omega, p) = 0\}.$$

THEOREM 3.1. *The Radon transform* $R$: $L_m^2(\mathcal{M}) \to L_m^2(\mathcal{N})$ *is continuous and*
i) *if* $f \in L_*^2(\mathcal{M})$ *and* $Rf(\omega, p) = 0$ *when* $p \geqq 1/m$ *then* $f \in L_m^2(\mathcal{M})$
ii) $R$: $L_*^2(\mathcal{M}) \to L_*^2(\mathcal{N})$ *is one-to-one.*

PROOF. The continuity is clear and the support theorem i) clearly implies the injectivity ii) so we shall only deal with the statement i).

Let $\alpha_0$ be that angle where $x(\alpha_0, h)$ is just $l_g$ and let

$$S^{n-1}_{\bar\omega,t} = \{\omega \in S^{n-1}: t < \langle \bar\omega, \omega\rangle\}.$$

Let $(\bar\omega, h)$ be the geodesic coordinate of $H$ and let $X$ be a point on the hyperplane $\xi(\bar\omega, h)$ with coordinate $(\omega, x)$ (see Figure 1).

Let the dimension $n = 2$ first and let the elements of $S^1 \subset T_O\mathcal{M}$ be parameterized by their angle $\alpha$ to an arbitrary but fixed direction. As the Figure 1 shows then $X \in \xi(\bar\omega, h)$ is parameterized on the interval $[\bar\alpha - \alpha_0, \bar\alpha + \alpha_0]$, where $\bar\alpha$ is the angle of $\bar\omega$. In this meaning it is immediate that

$$Rf(\bar\alpha, h) = \int_{-\alpha_0}^{\alpha_0} f(\alpha + \bar\alpha, x(\alpha)) \frac{\mathrm{d}d}{\mathrm{d}\alpha}\,\mathrm{d}\alpha, \tag{1}$$

where $d(\alpha) = d(\alpha, h)$ comes from the cosines law.

If the dimension is more than 2 the new situation can be gotten from the two dimensional one by rotating around $OH$. The definition of the size function $g$ says that a geodesic sphere of radius $\rho$ is isometric with the Euclidean sphere of radius $g(\rho)$ therefore its surface measure is $g^{n-1}(\rho)\,\mathrm{d}\omega$. This means that the basis elements of the tangent space of the geodesic sphere are $g(\rho)$-times bigger than that of the unit sphere in the tangent space. Since the hyperplane $\xi(\bar\omega, h)$ is rotational manifold it follows from these that the surface element of $\xi(\bar\omega, h)$ at the point $X$ is just $g^{n-2}(x)\dfrac{\mathrm{d}d}{\mathrm{d}\alpha}\mathrm{d}\omega$, where $\cos\alpha = \langle \omega, \bar\omega\rangle$. Thus we have

$$Rf(\bar\omega, h) = \int_{S^{n-1}_{\bar\omega, \cos\alpha_0(h)}} f(\omega, x(\arccos\langle\omega,\bar\omega\rangle)) g^{n-2}(x(\arccos\langle\omega,\bar\omega\rangle)) \frac{\mathrm{d}d}{\mathrm{d}\alpha}(\arccos\langle\omega,\bar\omega\rangle)\,\mathrm{d}\omega. \tag{2}$$

Substitute now into (1) and (2) the Fourier and the spherical harmonic expansions of $f$ and $Rf$ according to the dimension being 2 or more. More precisely let

$$f(\alpha, q) = \sum_{m=-\infty}^{\infty} f_m(q)\exp(im\alpha) \quad \text{and } Rf(\alpha, q) = \sum_{m=-\infty}^{\infty} (Rf)_m(q)\exp(im\alpha)$$

for dimension 2 and let

$$f(\omega, q) = \sum_{l,m}^{\infty} f_{l,m}(q) Y_{l,m}(\omega) \quad \text{and} \quad Rf(\omega, q) = \sum_{l,m}^{\infty} (Rf)_{l,m}(q) Y_{l,m}(\omega)$$

for higher dimension.

If $n = 2$ we get immediately that

$$(Rf)_m(h) = \int_{-\alpha_0}^{\alpha_0} f_m(x(\alpha)) \frac{\mathrm{d}d}{\mathrm{d}\alpha} \exp(im\alpha)\,\mathrm{d}\alpha$$

and by the substitution $\alpha = \arccos t$ this results in

$$(Rf)_m(h) = 2 \int_{\cos\alpha_0}^{1} f_m(y(t)) \frac{\mathrm{d}d}{\mathrm{d}\alpha}(\arccos t) \frac{\cos(m \arccos t)}{\sqrt{1 - t^2}}\,\mathrm{d}t$$

where $y(t) = x(\arccos t)$. In higher dimension one has to use the Funk-Hecke theorem to get the equation

$$(Rf)_{lm}(h) = \frac{|S^{n-2}|}{C_m^\lambda(1)} \int_{\cos\alpha_0}^{1} f_{lm}(y(t)) g^{n-2}(y(t)) \frac{\mathrm{d}d}{\mathrm{d}\alpha}(\arccos t) C_m^\lambda(t)[1 - t^2]^{\frac{n-3}{2}}\,\mathrm{d}t.$$

Making use of $\frac{\mathrm{d}d}{\mathrm{d}\alpha} = \frac{\mathrm{d}d}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}\alpha}$ together with the cosine law the substitution $s = y(t)$ gives

$$(3) \qquad (Rf)_m(h) = 2 \int_h^{x(\alpha_0)} f_m(s) \frac{g(s)\sqrt{1 - \tilde{y}^2(s)}}{\sqrt{g^2(s) - g^2(h)}} \frac{\cos(m \arccos \tilde{y}(s))}{\sqrt{1 - \tilde{y}^2(s)}}\,\mathrm{d}s$$

for dimension 2 and

$$(4) \qquad (Rf)_{lm}(h) = \frac{2|S^{n-2}|}{C_m^\lambda(1)} \int_h^{x(\alpha_0)} f_{lm}(s)) g^{n-2}(s) \frac{g(s)\sqrt{1 - \tilde{y}^2(s)}}{\sqrt{g^2(s) - g^2(h)}}$$

$$C_m^\lambda(\tilde{y}(s))[1 - \tilde{y}^2(s)]^{\frac{n-3}{2}}\,\mathrm{d}s$$

for higher dimension, where the function $\tilde{y}$ is the inverse of $y$.

To prove our assertion i) we now have to consider the kernel of the integral equations (3) (4). It is immediate from the L'Hospital law that

$$\lim_{\alpha\to 0} \frac{g(x(\alpha))\sin\alpha}{\sqrt{g^2(x(\alpha)) - g^2(h)}} = \lim_{\alpha\to 0} \frac{\sqrt{g^2(x) - g^2(h)}}{\dot{g}(h)\dot{x}} = \lim_{\alpha\to 0} \frac{g(h)\dot{x}}{\ddot{x}\sqrt{g^2(x) - g^2(h)}}.$$

Substituting $x(\alpha)$ for $s$ and multiplying the two last limit we obtain, that

$$\lim_{s\to h} \frac{g(s)\sqrt{1 - \tilde{y}^2(s)}}{\sqrt{g^2(s) - g^2(h)}} = \lim_{\alpha\to 0} \frac{g(x(\alpha))\sin\alpha}{\sqrt{\mathrm{g}^2(\mathrm{x}(\alpha)) - \mathrm{g}^2(\mathrm{h})}} = \lim_{\alpha\to 0}\left[\frac{g(h)}{\dot{g}(h)\ddot{x}}\right]^{1/2},$$

where the $x = x(\alpha)$ shorthand was used. This limit is not zero nor infinite because the function $x(\alpha)$ has real minimum at $\alpha = 0$ ($x(0) = h$). Therefore the kernels of our integral equations are of the form $K(s,h)[g^2(s) - g^2(h)]^{\frac{n-3}{2}} P(\tilde{y}(s))$, where $K(x,h) = Cg(x)\left[\frac{g^2(x)(1-\tilde{y}^2(x))}{g^2(x)-g^2(h)}\right]^{\frac{n-2}{2}}$, $K(h,h) = Cg(h)\lim_{\alpha\to 0}\left[\frac{g(h)}{\ddot{g}(h)\ddot{x}}\right]^{\frac{n-2}{2}} \neq 0$ and $P$ is a polynomial (Tschebyscheff or Gegenbauer) that satisfies $P(\tilde{y}(h)) = P(1) \neq 0$.

Since $f \in L^2_*(\mathcal{M})$ there must be a $k > 0$ that $f \in L^2_k(\mathcal{M})$ and so we only have to prove that $f_{lm}(s) = 0$ if $1/k > s > 1/m$ from the fact that $(Rf)_{lm}(h) = 0$ for $h > 1/m$. This is true because the kernels of our integral transformations satisfy the conditions of $B$ Theorem of Quinto's paper [12].

The theorem below is the corresponding result for the boomerang transform (it could also be formulated for the dual Radon transform). We only indicate its proof since it is so similar to the previous one.

THEOREM 3.2. *Let $\mathcal{M}$ be a rotational manifold, $f \in C^\infty(\mathcal{M})$ and $Bf(\omega,p) = 0$ for $0 \leqq p \leqq A$. Then $f(\omega,p) = 0$ for $0 \leqq p \leqq A$ too.*

PROOF. Let $(\bar{\omega}, x)$ be the geodesic coordinate of $X$ and $(\omega, h)$ be the geodesic coordinate of $H$ so that the hyperplane $\xi(\omega,h)$ passing through the point $X$ (See Figure 1).

First let the dimension $n = 2$ and $S^1$ be parameterized with the angle $\alpha$. Then as Figure 1 shows $H$ is parameterized on $[\bar{\alpha} - \pi/2, \bar{\alpha} + \pi/2]$, where $\bar{\alpha}$ denotes the angle of $\bar{\omega}$. We get

$$Bf(\bar{\alpha}, x) = \int_{-\pi/2}^{\pi/2} f(\alpha + \bar{\alpha}, h(\alpha))\frac{d\beta}{d\alpha}\,d\alpha, \tag{5}$$

where $\beta(\alpha)$ comes from the sine law.

For higher dimension the situation can be obtained by rotating Figure 1 around $OX$. Since the surface measure of the unit sphere in $T_X\mathcal{M}$ agree with that in $T_O\mathcal{M}$ we have

$$Bf(\bar{\omega}, x) = \int_{S^{n-1}_{\bar{\omega},0}} f(\omega, h(\arccos\langle\omega,\bar{\omega}\rangle))\frac{d\beta}{d\alpha}(\arccos\langle\omega,\bar{\omega}\rangle)\,d\omega. \tag{6}$$

On substituting the same expansions here as in the previous proof we obtain for dimension 2 that

$$(Bf)_m(x) = \int_{-\pi/2}^{\pi/2} f_m(h(\alpha)) \frac{d\beta}{d\alpha} \exp(im\alpha)\, d\alpha.$$

Taking the change of variable $\alpha = \arccos t$ this gives

$$(Bf)_m(x) = 2\int_0^1 f_m(y(t)) \frac{d\beta}{d\alpha}(\arccos t) \frac{\cos(m \arccos t)}{\sqrt{1 - t^2}}\, dt,$$

where $y(t) = h(\arccos t)$. For higher dimension the Funk-Hecke theorem implies

$$(Bf)_{lm}(x) = \frac{|S^{n-2}|}{C_m^\lambda(1)} \int_0^1 f_{lm}(y(t)) \frac{d\beta}{d\alpha}(\arccos t)\, C_m^\lambda(t)[1 - t^2]^{\frac{n-3}{2}}\, dt.$$

Using $\dfrac{d\beta}{d\alpha} = \dfrac{d\beta}{dh}\dfrac{dh}{d\alpha}$ with the sine law and taking $s = y(t)$ these lead to

$$(7) \qquad (Bf)_m(x) = 2\int_0^x f_m(s) \frac{g(s)\sqrt{1 - \tilde{y}^2(s)}}{\sqrt{g^2(x) - g^2(s)}} \frac{\cos(m \arccos \tilde{y}(s))}{\sqrt{1 - \tilde{y}^2(s)}}\, ds$$

for dimension 2 and to

$$(8) \quad (Bf)_{lm}(h) = \frac{2|S^{n-2}|}{C_m^\lambda(1)} \int_0^x f_{lm}(s) \frac{g(s)\sqrt{1 - \tilde{y}^2(s)}}{\sqrt{g^2(x) - g^2(s)}} C_m^\lambda(\tilde{y}(s))[1 - \tilde{y}^2(s)]^{\frac{n-3}{2}}\, ds$$

for higher dimension, where the function $\tilde{y}$ is the inverse of $y$.

One can conclude the proof here on the same way as we did in the previous one.

## 4. Spaces of constant curvature.

In this section we continue our considerations on the most important class of the rotational manifolds namely on the class of spaces of constant curvature. We make more precise the above results by using the projector function and then give inversion formulas. All the results below are proved in [8,9] using some specialties of the spaces. Now we present these in a unified and simplified form which shows more clearly the points of the proofs and theorems. We denote by $\mathsf{H}^n$ the $n$-dimensional hyperbolic space and by $\mathsf{P}^n$ the open half sphere. These and $\mathsf{R}^n$ are the spaces of constant curvature with size functions $\operatorname{sh} r$, $\sin r$ and $r$ and with projector functions $\operatorname{th} r$, $\operatorname{tg} r$ and $r$. In the following $\mathscr{M}^n$ will denote one of these three spaces, $g$ will be its size function, $\mu$ will be its projector and $\tilde{\mu}$ will be the inverse of the projector function $\mu$.

THEOREM 4.1. *If $f \in L^2(\mathcal{M}^n)$, then the Radon transform is*

$$Rf(\bar{\omega}, h) = \int_{S^{n-1}_{\bar{\omega}, \cos\alpha_0(h)}} f\left(\omega, \tilde{\mu}\left(\frac{\mu(h)}{\langle\omega, \bar{\omega}\rangle}\right)\right) \frac{(\langle\omega, \bar{\omega}\rangle^2/\mu^2(h) + \kappa)^{-n/2}}{g(h)} d\omega$$

and the boomerang transform is

$$Bf(\bar{\omega}, x) = \int_{S^{n-1}_{\bar{\omega}, 0}} f(\omega, \tilde{\mu}(\langle\omega, \bar{\omega}\rangle\mu(x))) \frac{(1 + \kappa\langle\omega, \bar{\omega}\rangle^2\mu^2(x))^{-1}}{\acute{g}(x)} d\omega.$$

PROOF. To obtain these formulas from (2) and (6)we only have to calculate the functions $x$ and $d$, respectively $h$ and $\beta$, if the point $H$, respectively $X$, is fixed (See Figure 1).

Since the projector function $\mu$ makes geodesic correspondence between the spaces $\mathcal{M}^n$ and $\mathrm{R}^n$, the functions $x$ and $h$ comes easily from the definition of $\mu$ to be $x(\alpha) = \tilde{\mu}\left(\frac{\mu(h)}{\cos\alpha}\right)$ and $h(\alpha) = \tilde{\mu}(\mu(x)\cos\alpha)$. Then one can immediately get the functions $d$ and $\beta$ from $x$ and $h$ by the corresponding sine law.

The following propositions come easily from (3, 4, 7, 8) by using the functions $x$ and $h$ determined above. For the sake of simplicity we denote the geodesic injectivity radius by $L$.

PROPOSITION 4.2. i) *If $f \in L^2(\mathcal{M}^2)$ then*

$$(9) \qquad (Rf)_m(h) = \frac{2}{\acute{g}(h)} \int_h^L f_m(q) \frac{\cos(m \arccos(\mu(h)/\mu(q)))}{\sqrt{1 - \mu^2(h)/\mu^2(q)}} dq.$$

ii) *If $f \in L^2(\mathcal{M}^n)$ then*

$$(10) \quad (Rf)_{l,m}(h) = \frac{|S^{n-2}|}{C_m^\lambda(1)\acute{g}(h)} \int_h^L f_{l,m}(q) C_m^\lambda\left(\frac{\mu(h)}{\mu(q)}\right) g^{n-2}(q) \left[1 - \frac{\mu^2(h)}{\mu^2(q)}\right]^{\frac{n-3}{2}} dq.$$

PROPOSITION 4.3. i) *If $f \in L^2(\mathcal{M}^2)$ then*

$$(11) \qquad (Bf)_m(x) = \frac{2}{g(x)} \int_0^x f_m(q) \frac{\cos(m \arccos(\mu(q)/\mu(x)))}{\sqrt{1 - \mu^2(q)/\mu^2(x)}} dq.$$

ii) *If* $f \in L^2(\mathcal{M}^n)$ *then*

$$(12) \qquad (Bf)_{l,m}(x) = \frac{|S^{n-2}|}{C_m^\lambda(1)g(x)} \int_0^x f_{l,m}(q) C_m^\lambda\left(\frac{\mu(q)}{\mu(x)}\right)\left[1 - \frac{\mu^2(q)}{\mu^2(x)}\right]^{\frac{n-3}{2}} dq.$$

For the following technical lemmas the function $\mu$ is only needed to be increasing. We shall use them only for the projector functions. The proofs are simple substitutions in the analogous formulas of [1] and [2].

LEMMA 4.4. *If* $m \in \mathbf{Z}$ *then*

$$\int_t^q \frac{\cos(m \arccos(\mu(h)/\mu(q)))}{\sqrt{1 - \mu^2(h)/\mu^2(q)}} \times \frac{\mathrm{ch}(m\,\mathrm{arcch}(\mu(h)/\mu(t)))}{\sqrt{\mu^2(h)/\mu^2(t) - 1}} \frac{\dot{\mu}(h)}{\mu(h)} dh = \frac{\pi}{2}.$$

LEMMA 4.5. *If* $m \in \mathbf{Z}, n > 2$ *then*

$$M\left[\frac{1}{\mu(t)} - \frac{1}{\mu(q)}\right]^{n-2} = \int_t^q C_m^\lambda\left(\frac{\mu(h)}{\mu(t)}\right)\left[\frac{\mu^2(h)}{\mu^2(t)} - 1\right]^{\frac{n-3}{2}}$$

$$C_m^\lambda\left(\frac{\mu(h)}{\mu(q)}\right)\left[1 - \frac{\mu^2(h)}{\mu^2(q)}\right]^{\frac{n-3}{2}} \frac{\dot{\mu}(h)\,dh}{\mu^{n-1}(h)},$$

*where*

$$M = \pi 2^{3-n}\left[\frac{\Gamma(m+n-2)}{\Gamma(m+1)\Gamma(\lambda)}\right]^2 \frac{1}{\Gamma(n-1)}.$$

The two theorems below state our first inversion formulas in the sense of the spherical harmonic expansions. The proof for the boomerang transform is very similar to that for the Radon transform so we leave it to the reader.

THEOREM 4.6. i) *If* $f \in C_c^\infty(\mathcal{M}^2)$ *then*

$$(13) \qquad f_m(t) = \frac{-1}{\pi}\frac{d}{dt}\int_t^L (Rf)_m(h) \frac{\mathrm{ch}(m\,\mathrm{arcch}(\mu(h)/\mu(t)))}{g(h)\sqrt{\mu^2(h)/\mu^2(t) - 1}} dh.$$

ii) *If* $n > 2, f \in C_c^\infty(\mathcal{M}^n)$ *then*

$$(14)\, f_{l,m}(t) = (-1)^{n-1} \frac{\Gamma(m+1)\Gamma(\lambda)}{2\pi^{n/2}\Gamma(m+n-2)} \begin{cases} \frac{d}{dt}\delta_2\delta_4\ldots\delta_{n-2}F(t) & \text{if } n \text{ even} \\ \delta_1\delta_3\delta_5\ldots\delta_{n-2}F(t) & \text{if } n \text{ odd} \end{cases},$$

*where* $\quad \delta_k = \dfrac{\mathrm{d}^2}{\mathrm{d}t^2} + \kappa k^2 \ (k \in \mathrm{N})$ *and*

$$F(t) = g^{n-2}(t) \int_t^L (Rf)_{l,m}(h) C_m^\lambda\left(\frac{\mu(h)}{\mu(t)}\right)\left[\frac{\mu^2(\mathrm{h})}{\mu^2(\mathrm{t})} - 1\right]^{\frac{n-3}{2}} \frac{\mu(h)\dot{g}(h)}{\mu^{n-1}(h)}\mathrm{d}h.$$

PROOF. Writing the formulas of Proposition 4.2 into these formulas, then changing the order of the integrations the lemmas lead to the integral equation

$$F(t) = C\int_t^L f_{l,m}(q) g^{n-2}(q-t)\,\mathrm{d}q,$$

where $C$ is a suitable constant. One can easily prove from this the theorem by making use of the identity

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} g^k(q-t) = -\kappa k^2 g^k(q-t) + k(k-1) g^{k-2}(q-t).$$

THEOREM 4.7. i) *If* $f \in C^\infty(\mathcal{M}^2)$ *then*

$$f_m(t) = \frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (Bf)_m(x) \frac{\cos(m \arccos(\mu(x)/\mu(t)))}{\dot{g}(x)\sqrt{1-\mu^2(x)/\mu^2(t)}}\mathrm{d}x. \tag{15}$$

ii) *If* $n > 2$, $f \in C^\infty(\mathcal{M}^n)$ *then*

$$f_{l,m}(t) = \frac{\Gamma(m+1)\Gamma(\lambda)}{2\pi^{n/2}\Gamma(m+n-2)} g^{n-2}(t) \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\delta_2\delta_4\ldots\delta_{n-2}F(t) & \text{if } n \text{ even} \\ \delta_1\delta_3\delta_5\ldots\delta_{n-2}F(t) & \text{if } n \text{ odd} \end{cases}, \tag{16}$$

*where*

$$F(t) = \dot{g}^{n-2}(t)\int_0^t (Bf)_{l,m}(x) C_m^\lambda\left(\frac{\mu(t)}{\mu(x)}\right)\left[\frac{\mu^2(t)}{\mu^2(x)} - 1\right]^{\frac{n-3}{2}} \frac{1/\dot{g}(x)}{\mu^{n-2}(x)}\mathrm{d}x.$$

Our last theorem states the closed inversion formulas that are analogous for all the spaces of constant curvature. The proof is based on some properties of the Gegenbauer polynomials used for the above inversion formulas. In fact it turns out that the above formulas are the spherical expansions of the closed inversion formulas stated below.

THEOREM 4.8. *Let* $n \geqq 2$ *and* $f \in C_c^\infty(\mathcal{M}^n)$. *If* $n$ *is odd, then*

$$f(\bar{\omega}, t) = (-1)^{\frac{n-1}{2}} \frac{2^{1-n}}{\pi^{n-1}} \delta_1 \delta_3 \ldots \delta_{n-2} \left[ R_t \left[ R_f(\omega, h) \frac{\dot{\mu}(h) g(h)}{\mu^{n-1}(h)} \right] (\bar{\omega}, t) g^{n-1}(t) \right].$$

*If n is even, the*

$$f(\bar{\omega}, t) = (-1)^{\frac{n}{2}} \frac{2^{1-n}}{\pi^n} \frac{\mathrm{d}}{\mathrm{d}t} \delta_2 \delta_4 \ldots \delta_{n-2} \left[ R_t \left[ \mathscr{H} \left\langle Rf(\omega, h) \frac{\dot{\mu}(h) g(h)}{\mu^{n-1}(h)} \right\rangle \right] (\bar{\omega}, t) g^{n-1}(t) \right],$$

*where* $\mathscr{H}\langle \cdot \rangle$ *is the distribution*

$$\mathscr{H}\langle f \rangle(\omega, h) = \frac{1}{g^2(h)} \int_{-L}^{L} f(\omega, r) \frac{1}{\mu(r) - \mu(h)} \mathrm{d}r.$$

PROOF. As a consequence of the definition of the boomerang transform the spherical harmonic expansion of the boomerang transform is in fact the same as that of the dual Radon transform. Thus one can certainly use the previous theorems to prove this one.

Let us start with the case of odd dimension, when Theorem 4.6 says

$$f_{l,m}(t) = C\mathscr{D}\left[ g^{n-2}(t) \int_t^L (Rf)_{l,m}(h) C_m^\lambda \left( \frac{\mu(h)}{\mu(t)} \right) \left[ \frac{\mu^2(h)}{\mu^2(t)} - 1 \right]^{\frac{n-3}{2}} \frac{\dot{\mu}(h) g(h)}{\mu^{n-1}(h)} \mathrm{d}h \right],$$

where $C = \dfrac{\Gamma(m+1)\Gamma(\lambda)}{2\pi^{n/2}\Gamma(m+n-2)}$ and $\mathscr{D} = \delta_1 \delta_3 \ldots \delta_{n-2}$. Breaking the integral $\int_t^L$ into two parts as $\int_t^L = \int_0^L - \int_0^t$ we obtain

$$(*) \qquad f_{l,m}(t) = I + C(-1)^{\frac{n-1}{2}} \mathscr{D}\left[ g^{n-2}(t) \int_0^t (Rf)_{l,m}(h) C_m^\lambda \left( \frac{\mu(h)}{\mu(t)} \right) \left[ 1 - \frac{\mu^2(h)}{\mu^2(t)} \right]^{\frac{n-3}{2}} \frac{\dot{\mu}(h) g(h)}{\mu^{n-1}(h)} \mathrm{d}h \right],$$

where

$$I = C\mathscr{D} \int_0^L (Rf)_{l,m}(h) C_m^\lambda \left( \frac{\mu(h)}{\mu(t)} \right) \left[ \frac{\mu^2(h)}{\mu^2(t)} - 1 \right]^{\frac{n-3}{2}} \frac{\dot{\mu}(h) g(h)}{\mu^{n-1}(h)} \mathrm{d}h.$$

Writing the formula (10) into this formula then changing the order of the integrals one gets that $I$ is proportional to

$$\int_0^L f_{l,m}(q) g^{2\lambda}(q) \int_0^q C_m^\lambda\left(\frac{\mu(h)}{\mu(q)}\right)\left[1 - \frac{\mu^2(h)}{\mu^2(q)}\right]^{\frac{n-3}{2}}$$

$$\mathscr{D}\left[C_m^\lambda\left(\frac{\mu(h)}{\mu(t)}\right)\left[\frac{\mu^2(h)}{\mu^2(t)} - 1\right]^{\frac{n-3}{2}} g^{2\lambda}(t)\right] \frac{\mu(h)\,\mathrm{d}h}{\mu^{n-1}(h)}\,\mathrm{d}q.$$

Substituting $x = \mu(h)/\mu(q)$ into the inner integral, $J$, one obtains that

$$J = \frac{1/2}{\mu^{2\lambda}(q)} \int_{-1}^{1} C_m^\lambda(x)[1 - x^2]^{\frac{n-3}{2}} \mathscr{D}\left[C_m^\lambda\left(\frac{\mu(q)}{\mu(t)}x\right)\left[\frac{\mu^2(q)}{\mu^2(t)}x^2 - 1\right]^{\frac{n-3}{2}} g^{2\lambda}(t) x^{1-n}\,\mathrm{d}x,$$

because the integrand is even function. This integral can be calculated by using two facts about the Gegenbauer polynomial. First, $C_m^\lambda(x)[1 - x^2]^{\frac{n-3}{2}}$ is polynomial of $(m + n - 3)$ degree. Second, the system $\{C_m^\lambda(x)\}$ is orthogonal on the interval $[-1, 1]$ with the weight function $[1 - x^2]^{\frac{n-3}{2}}$ [13]. Therefore it is enough to prove that the polynomial $P(x) = \mathscr{D}\left[C_m^\lambda\left(\frac{\mu(q)}{\mu(t)}x\right)\left[\frac{\mu^2(q)}{\mu^2(t)}x^2 - 1\right]^{\frac{n-3}{2}} g^{2\lambda}(t)\right]$ is homogeneous of degree $n - 1$ because then $P(x)x^{1-n}$ would also be a polynomial and so by the above facts $J = 0$ and $I = 0$ would be proved. Obviously the coefficient of $x^k$ is zero for $0 \leqq k \leqq n - 2$ if and only if

$$\delta_1\delta_3 \ldots \delta_{n-2}[g^{n-2}(t)/\mu^k(t)] = 0.$$

This can be easily prove by induction establishing that

$$\delta_k(\dot{g}^k(t)) = -k(k - 1)\dot{g}^{k-2}(t) \quad \text{and} \quad \delta_k(g^k(t)) = k(k - 1)g^{k-2}(t).$$

Since $I = 0$, the equation $(*)$ gives just the spherical expansion of the stated closed inversion formula.

Let us turn to the case of even dimension. In this case $\lambda$ is natural number and the differential operator $\mathscr{D}$ is $\frac{\mathrm{d}}{\mathrm{d}t}\delta_2\delta_4 \ldots \delta_{n-2}$. Then Theorem 4.6 says

$$(**)\quad f_{l,m}(t) = -C\mathscr{D}\left[g^{n-2}(t)\int_t^L (Rf)_{l,m}(h) C_m^\lambda\left(\frac{\mu(h)}{\mu(t)}\right)\left[\frac{\mu^2(h)}{\mu^2(t)} - 1\right]^{\frac{n-3}{2}} \frac{\mu(h)\dot{g}(h)}{\mu^{n-1}(h)}\,\mathrm{d}h\right].$$

To avoid the straightforward but very tedious calculations and explanations we shall simply refer to the numbers of the formulas of [3] and [4] in the following.

Just as in the case of even dimension it is easy to see now that

$$0 = C\mathscr{D}\left[g^{n-2}(t)\int_0^L (Rf)_{l,m}(h)E^{\lambda}_{m+2\lambda-1}\left(\frac{\mu(h)}{\mu(t)}\right)\frac{\dot\mu(h)g(h)}{\mu^{n-1}(h)}\,dh\right],$$

because $E^{\lambda}_{m+2\lambda-1}$ is polynomial of degree $(m+n-3)$ [4]. Take this integral as $\int_0^t + \int_t^L$ and write the equations (A.14) and (A.4) of [4] into these. Adding the result of this to the equation (**) one obtains that

$$f_{l,m}(t) = -C\mathscr{D}\left[g^{n-2}(t)\left[\int_t^L (Rf)_{l,m}(h)2D^{\lambda}_m\left(\frac{\mu(h)}{\mu(t)}\right)\left[\frac{\mu^2(h)}{\mu^2(t)}-1\right]^{\frac{n-3}{2}}\frac{\dot\mu(h)g(h)}{\mu^{n-1}(h)}\,dh\right.\right. +$$

$$\left.\left.+\int_0^t (Rf)_{l,m}(h)(-1)^{\lambda}D^{\lambda}_m\left(\frac{\mu(h)}{\mu(t)}\right)\left[1-\frac{\mu^2(h)}{\mu^2(t)}\right]^{\frac{n-3}{2}}\frac{\dot\mu(h)g(h)}{\mu^{n-1}(h)}\,dh\right]\right].$$

Now the equations (24) and (25) of [3] give

$$f_{l,m}(t) = -\frac{C}{\pi}(-1)^{\lambda}\mathscr{D}\left[g^{n-2}(t)\int_0^L (Rf)_{l,m}(h)I^{\lambda}_m\left(\frac{\mu(h)}{\mu(t)}\right)\frac{\dot\mu(h)g(h)}{\mu^{n-1}(h)}\,dh\right],$$

where by the definition of [3(22)]

$$I^{\lambda}_m(y) = \int_{-1}^{1} C^{\lambda}_m(x)(1-x^2)^{\frac{n-3}{2}}(y-x)^{-1}\,dx.$$

Substituting $x = \mu(q)/\mu(t)$, letting $\mu(-q) = -\mu(q)$ and then changing the order of the integration results in

$$f_{l,m}(t) = \frac{-C(-1)^{\lambda}}{\pi}\mathscr{D}\left[g^{2\lambda}(t)\int_{-t}^{t} C^{\lambda}_m\left(\frac{\mu(q)}{\mu(t)}\right)\left[1-\frac{\mu^2(q)}{\mu^2(t)}\right]^{\frac{n-3}{2}}\right.$$

$$\left.\frac{1}{g^2(q)}\int_0^L \frac{(Rf)_{l,m}(h)}{\mu(h)-\mu(q)}\frac{\dot\mu(h)g(h)}{\mu^{n-1}(h)}\,dh\,dq\right].$$

This formula is just the spherical expansion of the closed inversion formula that completes the proof.

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