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THE ISOMETRY GROUPS OF RIEMANNIAN MANIFOLDS ADMITTING STRICTLY CONVEX FUNCTIONS

BY TAKAO YAMAGUCHI

0. Introduction

A function f on a complete connected Riemannian manifold M is said to be *convex* if for any geodesic $\gamma : \mathbb{R} \to M$, any $t_1, t_2 \in \mathbb{R}$ and any $0 < \lambda < 1$, f satisfies the following inequality; $f \circ \gamma((1-\lambda)t_1 + \lambda t_2) \leqq (1-\lambda) f \circ \gamma(t_1) + \lambda f \circ \gamma(t_2)$. It is well known that a convex function is Lipschitz continuous on every compact subset. If the above inequality is strict for all γ, t_1, t_2 and λ, then f is said to be *strictly convex*. A function is said to be locally nonconstant if it is not constant on any open subset. If M admits a nontrivial convex function, then M is noncompact. Clearly strict convexity induces local nonconstancy. Recentry the topological structure of manifolds which admit locally nonconstant convex functions has been decided by Greene-Shiohama [4]. Since a convex function imposes a certain restriction to the Riemannian structure, it is natural to ask the influences of the existence of a convex function on the Riemannian structure. In this paper we will investigate the influences of the existence of strictly convex functions with compact levels on the isometry groups. According to [4], if a level set $f^{-1}(t)$ of a locally nonconstant convex function f on M is compact then all level sets are also compact. Such an f is said to be with compact levels. And corresponding to each $t \in f(M)$ the diameter $\delta(t)$ of $f^{-1}(t)$, the diameter function of f, $\delta : f(M) \to \mathbb{R}$, is well defined and is monotone nondecreasing. We will prove the following theorems.

THEOREM A. — *If* M *admits a strictly convex function with minimum, then each compact subgroup of the isometry group* I(M) *of* M *has a common fixed point.*

THEOREM B. — *If* M *admits a strictly convex function with compact levels and with no minimum, then all the isometric images of any level set intersect the level set. In particular,* I(M) *is compact.*

Cheeger-Gromoll [3] proved the following splitting theorem for complete manifolds of nonnegative sectional curvature by constructing an expanding filtration of M by compact totally convex sets which are sublevel sets of a convex function.

THEOREM [3]. — *A complete Riemannian manifold* M *of nonnegative sectional curvature splits uniquely as* $\overline{M} \times \mathbb{R}^k$, *where the isometry group of* $\overline{M}$ *is compact and* $I(M) = I(\overline{M}) \times I(\mathbb{R}^k)$.

Recently S. T. Yau [9] has obtained a similar result to Theorem A for strongly convex functions, which is stronger than strict convexity. A function $f: M \to \mathbb{R}$ is said to be *strongly convex* if for a given compact set K of M there exists a $\varepsilon > 0$ such that $\{f \circ \gamma(t) + f \circ \gamma(-t) - 2f \circ \gamma(0)\}/t^2 > \varepsilon$ for any geodesic γ with $\gamma(0) \in K$. Clearly $f(t) = t^4$ is not strongly convex but strictly convex. It will be clear from examples which we will construct later that Theorem A is a natural extention of a classical theorem due to E. Cartan which states that each compact subgroup of the isometry group of a simply connected complete Riemannian manifold of nonpositive sectional curvature has a common fixed point. We note that any manifold satisfying the hypothesis of Theorem A is diffeomorphic to $\mathbb{R}^n$ ($n = \dim M$), and in the situation of Theorem B M is homeomorphic to $N \times \mathbb{R}$, where N is a level set [4]. The key to the proof of Theorem B is to show that the metric projection onto any sublevel set is locally distance decreasing. This is done in paragraph 3.

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1. Preliminaries

Hereafter let M be a complete connected Riemannian manifold with $\dim M \geqq 2$ and let ρ be the distance function induced from the Riemannian metric. For an $r > 0$ and a point p of M let $B_r(p)$ denote the open metric ball of radius r around p. It is well known as the Whitehead Theorem (*see* [2]) that there exists a positive continuous function c on M, which is called a convexity radius function, such that for every point $p \in M$ (1) any open ball $B_r(p')$ contained in $B_{c(p)}(p)$ is a strongly convex set, (2) $\rho^2(p', .)$ is C^∞-strongly convex on $B_r(p')$. A set $A \subset M$ is called to be *strongly convex* if for any two points p and q of A there exists a unique minimizing geodesic from p to q and it is contained in A. A set $A \subset M$ is called to be *totally convex* if A contains all geodesic segments which join any two points of A, and a set $C \subset M$ is called to be *convex* if for any point p of the closure $\overline{C}$ of C there exists a positive number $\varepsilon(p)$, $0 < \varepsilon(p) \leqq c(p)$, such that $C \cap B(p)$ is strongly convex.

PROPOSITION (*cf.* [4], Prop. 1.2). — *If* C *is a closed convex set of* M *then there exists an open neighborhood* U *of* C *such that for any point* p *of* C *there exists a unique point* q *of* C *such that* $\rho(p, q) = \rho(p, C)$.

Then the map $\pi_c : U \to C$, which is called the metric projection of U onto C, can be defined by $\rho(p, \pi_c(p)) = \rho(p, C)$ and is continuous.

For a real valued function f on M and for arbitrary real numbers a and b, $a \leqq b$, we will denote $f([a, b])$ and $f((-\infty, a])$ by $M_a^b(f)$ and $M^a(f)$ respectively, or briefly M_a^b and M^a. If M_a^a (resp. M^a) is not empty, then it is called a level set of f (resp. a sublevel set of f). It is clear that every sublevel set of a convex function is totally convex.

Let C be a convex set of M and let $p \in C$. A tangent vector v to M at p is *normal* to C at p if for any smooth curve γ in C emanating from p we have $\langle \gamma'(0), v \rangle \leqq 0$. If $\pi_c : U \to C$ is a metric projection onto C and if $p \in U - C$ and if γ is a minimizing geodesic from $\pi_c(p)$ to p, then $\gamma'(0)$ is normal to C at $\pi_c(p)$. Conversely if v is a normal vector to C at p then

$\pi_c(\exp_p tv/\|v\|)=p$ for any sufficiently small $t>0$. We note that the set of all normal vectors to C at p is a closed subset of M_p.

2. Proof of Theorem A and examples

Proof of Theorem A. — Let f be a strictly convex function with minimum on M and let G be a compact subgroup of the isometry group of M. We note that $M^\alpha(f)$ is compact for any $\alpha \in f(M)$. Let μ denote the Haar measure on G normalized by $\int_G d\mu=1$. We define a function F on M by:

$$F(x)=\int_G f(gx)\,d\mu(g).$$

For every element g of G, $f \circ g$ is also strictly convex, and so is F. Now we will show that F has also minimum.

ASSERTION. — For any $a \in \mathbb{R}$ there is a $b \in \mathbb{R}$ such that $M^a(F) \subset M^b(f)$.

To prove the assertion, suppose that it is not true. Then there are some $a \in \mathbb{R}$ and a sequence $\{x_n\}$ in $M^a(F)$ so that $f(x_n) \to \infty$. It follows from the definition of F that for each n there is a $g_n \in G$ such that $f(g_n x_n) \leqq a$. Thus it turns out that $G \cdot M^a(f)$ is unbounded. This contradicts the compactness of G and $M^a(f)$.

The proof of Theorem A is complete since F has a unique minimum point by the strict convexity of F and since it is G-invariant.

Q.E.D.

Examples. — (*a*) Let H denote a simply connected Riemannian manifold of nonpositive sectional curvature. For a given point p of H $\rho^2(p, .)$ is C^∞-strongly convex with minimum.

(*b*) Palaboloid; $\{(x, y, z) \in \mathbb{R}^3;\ z=x^2+y^2\}$. $f(x, y, z)=z$ is strictly convex with minimum. The curvature is positive everywhere.

(*c*) (*see* [8]). Let $0<a<b$ and $h : [0, \infty) \to [0, 1]$ be a C^∞-function such that (1) $h(v)=0$ for $v \leqq a$ and $h(v)=1$ for $v \geqq b$, (2) if we define g by $g(v)=v^2+h(v)$ for $v \geqq 0$, then $g'(v)>0$ for all $v>0$ and $g''(v_0)<0$ for some v_0, $a<v_0<b$. We consider a surface of revolution; $S=\{(v\cos u, v\sin u, g(v));\ 0 \leqq u \leqq 2\pi,\ v \geqq 0\}$ whose curvature is negative on a neighborhood of $\{(u, v_0);\ 0 \leqq u \leqq 2\pi\}$ and is positive on $\{(u, v);\ 0 \leqq u \leqq 2\pi,\ v \leqq a \text{ or } v \geqq b\}$. For each positive integer n we define a function f_n on S by $f_n(u, v)=g^n(v)$. Then f_n is strongly convex with minimum for any sufficiently large n.

3. The diameter functions for strictly convex functions

Let f be a locally nonconstant convex function with compact levels on M and let $m=\inf_M f$, then the diameter function $\delta : (m, \infty) \to \mathbb{R}$ is defined by $\delta(t)=\max\{\rho(x, y);\ x, y \in M_t^t\}$. δ is monotone nondecreasing [4]. In this section we will prove that if f is strictly convex with compact levels, then δ is strictly increasing. Hereafter we will fix a strictly convex function f with compact levels. Let a, $b \in (m, \infty)$, $a \leqq b$, be fixed and B be a sufficiently large compact neighborhood of M_a^b and let $r_0 = \min_B c$ where c is a convexity radius function on M. There exists a neighborhood U of the zero section of TM such that $\mathrm{Exp}\,|\,U$ is an embedding and $\mathrm{Exp}(U) \supset \overline{B_{r_0}(x)} \times \overline{B_{r_0}(x)}$ for any $x \in M_a^b$, where $\mathrm{Exp} : TM \to M \times M$ is the exponential mapping defined by $\mathrm{Exp}(v) = (\pi(v), \exp_{\pi(v)} v)$ and $\pi : TM \to M$ is the natural projection. For each $x \in B$ let:

$$L_x = \inf \{ L > 0;\ L^{-1} \leqq \| d(\mathrm{Exp}\,|\,U)^{-1} \,|\, \overline{B_{r_0}(x)} \times \overline{B_{r_0}(x)} \| \leqq L \}$$

and let $L = \sup \{ L_x;\ x \in B \}$. It is clear from compactness argument that $0 < L < \infty$. Let $\varkappa$ be the maximum of the absolute values of the sectional curvature on B. Let $\mu = \min \{ \delta(a)/8,\ r_0/8 \}$ and let $A = \{ (x, y) \in M_\beta^\beta \times M_\beta^\beta;\ \mu \leqq \rho(x, y) \leqq r_0/2,\ a \leqq \beta \leqq b \}$. For each $x \in M$ we denote the set of all unit normal vectors to $M^{f(x)}$ at x by $N_x^1(f)$. Now for each $(x, y) \in A$ and for each $v_1 \in N_x^1(f)$, $v_2 \in N_y^1(f)$ let γ_1 and γ_2 be the geodesics emanating from x and y whose velocity vectors are v_1 and v_2 respectively. Let $x' = \gamma_1(t_1)$ and $y' = \gamma_2(t_2)$ be arbitrary fixed points on γ_1 and γ_2 so that $t_1 > 0$, $\mu/4 \geqq t_1 \geqq t_2 \geqq 0$. We reparametrize the subarc of γ_1 and γ_2 by $\tau_1(s) = \gamma_1(s)$ and $\tau_2(s) = \gamma_2(t_2 s/t_1)$, $0 \leqq s \leqq t_1$. $\alpha : [0, 1] \times [0, t_1] \to M$ is the rectangle such that each $\alpha_s = \alpha(\ , s)$ is a unique minimizing geodesic from $\tau_1(s)$ to $\tau_2(s)$. Let $L(\alpha_s)$ be the length of α_s. The next lemma follows from a standard argument using the second variation formula and the Rauch comparison theorem. *See* [4] for details.

LEMMA 3.1. — *There exists a positive constant* $C_2 = C_2(r_0, L, \varkappa, \mu)$ *such that for any* $(x, y) \in A$ *and any* $v_1 \in N_x^1(f)$, $v_2 \in N_y^1(f)$, x', y' *as above and for any* $s \in [0, t_1]$, *we have* $|L''(\alpha_s)| \leqq C_2$.

Next we will estimate the first variation for α. By the first variation formula, we have:

$$L'(\alpha_s)|_{s=0} = (\langle t_2 v_2/t_1,\ \alpha_0'(1) \rangle - \langle v_1,\ \alpha_0'(0) \rangle).$$

From the definition of normal vectors, we have $\langle v_2, \alpha_0'(1) \rangle \geqq 0$, $\langle v_1, \alpha_0'(0) \rangle \leqq 0$. By the strict convexity of f, $f(\alpha_0(1/2)) < \beta$. Suppose that $\langle v_1, \alpha_0(0) \rangle = 0$ and let U_1 be a neighborhood of $\alpha_0(1/2)$ on which f takes values smaller than β. Take a point z of the intersection of the geodesic surface $\{ \exp_x(t_1 v_1 + t_2(\alpha_0'(0)));\ t_1, t_2 > 0 \}$ with U_1 and let γ be a unique minimizing geodesic segment from x to z. Then by the convexity of f, γ is contained in M^β. Since $\gamma'(0)$ makes an acute angle with v_1, this is a contradiction for v_1 to be a normal vector. It follows that $L'(\alpha_s)|_{s=0} > 0$. Now let:

$$C_1 = \inf \{ L'(\alpha_s)|_{s=0};\ (x, y) \in A,\ v_1 \in N_x^1(f),\ v_2 \in N_y^1(f),\ x', y' \text{ as above} \}.$$

It is easy to see that $C_1 > 0$. It follows from the preceding lemma that $L'(\alpha_s) = L'(0) + s L''(\theta s) \geqq C_1 - s C_2$ for some θ, $0 \leqq \theta \leqq 1$. Hence we have obtained:

LEMMA 3.2. — *For any* $(x, y) \in A$ *and any* $v_1 \in N_x^1(f)$, $v_2 \in N_y^1(f)$ *and for any* $x' = \gamma(t_1)$, $y' = \gamma(t_2)$ *such that* $C_1/C_2 \geqq t_1 \geqq t_2 \geqq 0$, $t_1 > 0$ *as before*, $L(\alpha_s)$ *is strictly increasing on* $[0, t_1]$.

For any $\beta \in [a, b]$ M^β is a totally convex set. If we set $U = \bigcup_{x \in M^\beta} B_{r_0/2}(x)$ then the metric projection π_{M^β} of U onto M^β, which we briefly denote by π_β, can be defined as in paragraph 1.

LEMMA 3.3. — *There exists a positive constant* ε_0 *such that for each* $\beta \in [a, b]$ *if* $x \in M^{\beta+\varepsilon_0} - M^\beta$ *and* $y \in M_\beta^{\beta+\varepsilon_0}$ *satisfy* $2\mu \leqq \rho(x, y) \leqq 3r_0/8$, *then we have* $\rho(x, y) > \rho(\pi_\beta(x), \pi_\beta(y))$.

Proof. — Let $\varepsilon_1 = \min\{\mu/4, C_1/C_2\}$ and let:

$$\varepsilon_0(\beta) = \inf\{f(\exp_x \varepsilon_1 v_x);\ x \in M_\beta^\beta,\ v_x \in N_x^1(f)\} - \beta.$$

The required constant will be obtained by $\varepsilon_0 = \inf\{\varepsilon_0(\beta);\ a \leqq \beta \leqq b\}$. We note that $\varepsilon_0 > 0$. Then for any x and y as in this lemma we have $\rho(\pi_\beta(x), x) \leqq \varepsilon_1$, $\rho(\pi_\beta(y), y) \leqq \varepsilon_1$ and $(\pi_\beta(x), \pi_\beta(y)) \in A$ by triangle inequalities. Therefore the preceding lemma completes the proof.

Q.E.D.

PROPOSITION 3.4. — δ *is strictly increasing.*

Proof. — For a given $c \in (m, \infty)$ let ε_0 be the positive constant given in the preceding lemma for $a = b = c$. Fix an arbitrary s such that $0 < s \leqq \varepsilon_0$. Let x_0 and y_0 be two points of M_c^c such that $\rho(x_0, y_0) = \delta(c)$, and let $v_1 \in N_{x_0}^1(f)$, $v_2 \in N_{y_0}^1(f)$ and let x_1 and y_1 be two points of M_{c+s}^{c+s} at which two geodesics $\exp_{x_0} tv_1$, $\exp_{y_0} tv_2$, $t \geqq 0$, intersect M_{c+s}^{c+s} respectively. By $\sigma : [0, d] \to M$ we denote a minimizing unit speed geodesic from x_1 to y_1. We consider two cases.

Case 1. — $\sigma([0, d]) \cap M_c^c = \emptyset$.

We can choose a subdivision $0 = t_0 < t_1 < \ldots < t_k = d$ of $[0, d]$ such that $2\mu \leqq t_i - t_{i-1} \leqq 3r_0/8$ for all i, $1 \leqq i \leqq k$. Using Lemma 3.3 we have:

$$\rho(x_1, y_1) = \sum_1^k \rho(\sigma(t_{i-1}), \sigma(t_i)) > \sum_1^k \rho(\pi_c \sigma(t_{i-1}), \pi_c \sigma(t_i)) \geqq \rho(x_0, y_0).$$

Hence $\delta(c+s) > \delta(c)$.

Case 2. — $\sigma([0, d]) \cap M_c^c \neq \emptyset$.

Then there exist $s_1, s_2 \in (0, d)$, $s_1 \leqq s_2$, such that $\sigma([0, s_1))$ and $\sigma((s_2, d])$ are contained in $M^{c+s} - M^c$ and $\sigma([s_1, s_2])$ is contained in M^c. We can choose two subdivision, $0 = t_0 < t_1 < \ldots < t_{k_1} = s_1$ and $s_2 = u_0 < u_1 < \ldots < u_{k_2} = d$ of $[0, s_1]$ and $[s_2, d]$ which satisfy the following conditions:

$$2\mu \leqq t_i - t_{i-1} \leqq 3r_0/8 \quad \text{for} \quad i = 1, \ldots, k_1 - 1,\ s_1 - t_{k_1-1} < 2\mu,$$
$$2\mu \leqq u_i - u_{i-1} \leqq 3r_0/8 \quad \text{for} \quad i = 2, \ldots, k_2,\ u_1 - s_2 < 2\mu.$$

Since $\rho^2(\sigma(s_1), .)$ and $\rho^2(\sigma(s_2), .)$ are C^∞-strongly convex on $B_{r_0}(\sigma(s_1))$ and $B_{r_0}(\sigma(s_2))$ respectively, we have $\rho(\sigma(t_{k-1}), \sigma(s_1)) > \rho(\pi_c(\sigma(t_{k-1})), \sigma(s_1))$ and $\rho(\sigma(s_2), \sigma(u_1)) > \rho(\sigma(s_2), \pi_c\sigma(u_1))$. It follows from the same argument as in case 1 that $\rho(x_1, \sigma(s_1)) > \rho(\pi_c(x_1), \sigma(s_1))$ and $\rho(\sigma(s_2), y_1) > \rho(\sigma(s_2), \pi_c(y_1))$. It follows that $\rho(x_1, y_1) > \rho(\pi_c(x_1), \pi_c(y_1))$. Therefore $\delta(c+s) > \delta(c)$.

Q.E.D.

4. Proof of Theorem B

Let f be a strictly convex function on M with compact levels and with no minimum, and let $m = \inf_M f$. The proof of Theorem B is achieved by supposing that it is not true and then by deriving a contradiction. The contradiction, roughly speaking, comes as follows. By the fact that M is homeomorphic to $N \times \mathbb{R}$ where N is any level set (*see* [4], Theorem C), the isometric image of a level set must always separate M into two unbounded components. But by the diameter increasing property this is not possible if a low level set is moved to a higher level, where a larger diameter would be required.

Suppose that $M_c^c \cap \psi(M_c^c) = \emptyset$ for some $c \in f(M)$ and some $\psi \in I(M)$. It follows that $\psi(M_c^c) \cap M^c = \emptyset$ or $\psi(M_c^c) \subset M^c$. We consider two cases.

Proof of Theorem B in the case $\psi(M_c^c) \cap M^c = \emptyset$. — Let $a = \min\{f(x); x \in \psi(M_c^c)\}$ and $b = \max\{f(x); x \in \psi(M_c^c)\}$. Notice that $c < a$. Let ε_0 denote the constant obtained in Lemma 3.3 for these a and b. We choose subdivision $a = t_0 < t_1 < \ldots < t_k = b$ of $[a, b]$ such that $t_i - t_{i-1} \leqq \varepsilon_0$ for all i, $1 \leqq i \leqq k$. For each i, $1 \leqq i \leqq k-1$, let $\pi_{t_i} : M^{t_i+1} \to M^{t_i}$ be the metric projection and let $H = \pi_{t_0} \circ \ldots \circ \pi_{t_{k-1}} : M^b \to M^a$.

ASSERTION. — $d(H \circ \psi(M_c^c)) \leqq \delta(c)$, where $d(H \circ \psi(M_c^c))$ is by definition the diameter of $H \circ \psi(M_c^c)$.

Proof of Assertion. — We suppose that $d(H \circ \psi(M_c^c)) > \delta(c)$ and take two points x and y of $H \circ \psi(M_c^c)$ such that $\rho(x, y) = d(H \circ \psi(M_c^c))$. Let x' and y' be such points of $\psi(M_c^c)$ that $H(x') = x$ and $H(y') = y$. We may assume that $t_{i_0} \leqq f(x') < t_{i_0+1}$ and $t_{j_0} \leqq f(y') < t_{j_0+1}$ for $i_0 \geqq j_0$. Let $x_i = \pi_{t_i} \circ \ldots \circ \pi_{t_{i_0}}(x')$ for each $i \leqq i_0$ and let $y_j = \pi_{t_j} \circ \ldots \circ \pi_{t_{j_0}}(y')$ for each $j \leqq j_0$. In the proof of Proposition 3.4 if we replace $\mu = \min\{\delta(a)/8, r_0/8\}$ by $\min\{\delta(c)/8, r_0/8\}$ then we have $\rho(x, y) < \rho(x_1, y_1) < \ldots < \rho(x_{j_0}, y_{j_0}) < \rho(x_{j_0+1}, y')$. Let $\eta : [0, d] \to M$ be a unit speed minimizing geodesic from x' to y'. For each $i, j_0+1 \leqq i \leqq i_0$, let z_i be the point of intersection of η with $M_{t_i}^{t_i}$. In the same way as Proposition 3.4 we have $\rho(x', z_{i_0}) \geqq \rho(x_{i_0}, z_{i_0})$. It follows that:

$$\rho(x', z_{i_0-1}) \geqq \rho(x_{i_0}, z_{i_0}) + \rho(z_{i_0}, z_{i_0-1}) \geqq \rho(x_{i_0}, z_{i_0-1}).$$

Iterating this, we have:

$$\rho(x', z_{i_0-2}) \geqq \rho(x_{i_0-1}, z_{i_0-2}), \ldots, \rho(x', z_{j_0+1}) \geqq \rho(x_{j_0+2}, z_{j_0+1}) \geqq \rho(x_{j_0+1}, z_{j_0+1}).$$

It follows that:

$$\rho(x', y') = \rho(x', z_{j_0+1}) + \rho(z_{j_0+1}, y') \geqq \rho(x_{j_0+1}, z_{j_0+1}) + \rho(z_{j_0+1}, y') \geqq \rho(x_{j_0+1}, y').$$

Therefore we have:

$$\delta(c) \geqq \rho(x', y') \geqq \rho(x, y) = d(\mathrm{H} \circ \psi(\mathrm{M}_c^c))$$

which contradicts the first assumption.

Q.E.D.

By Proposition 3.4 it is possible to take a point p_0 which belongs to $\mathrm{M}_a^a - \mathrm{H} \circ \psi(\mathrm{M}_c^c)$. Coosing :

$$p_1 \in \pi_{t_0}^{-1}(p_0) \cap \mathrm{M}_{t_1}^{t_1}, \qquad p_2 \in \pi_{t_1}^{-1}(p_1) \cap \mathrm{M}_{t_2}^{t_2}, \ \ldots, \ p_k \in \pi_{t_{k-1}}^{-1}(p_{k-1}) \cap \mathrm{M}_b^b$$

and joining p_0 to p_1, p_1 to p_2, ..., p_{k-1} to p_k in this order by minimizing geodesics we obtain a broken geodesic σ from p_0 to p_k which does not intersect $\psi(\mathrm{M}_c^c)$. It is easy to construct a continuous extention $\sigma_1 : \mathbb{R} \to \mathrm{M}$ of σ such that $\sigma_1(\mathbb{R}) \cap \psi(\mathrm{M}_c^c) = \emptyset$ and $f \circ \sigma_1(\mathbb{R}) = (m, \infty)$. Since M is topologically a product of a level set and $\mathbb{R}$, it turns out that $f \circ \psi^{-1} \circ \sigma_1(\mathbb{R}) = (m, \infty)$. This contradicts the fact that $\sigma_1(\mathbb{R}) \cap \psi(\mathrm{M}_c^c) = \emptyset$.

The rest of the proof of Theorem B is a direct consequence of the following:

COROLLARY C. — *Under the same hypothesis as in Theorem B, every isometry of* M *fixes each of the two ends of* M.

Proof. — If some $\psi \in \mathrm{I}(\mathrm{M})$ permutes the ends, then there is a compact set K of M such that ψ maps one component U_1 of $\mathrm{M} - \mathrm{K}$ into the other component U_2 and maps U_2 into U_1. It turns out that ψ maps a low level set to a much higher level. This is impossible.

Proof of Theorem B in the case $\psi(\mathrm{M}_c^c(f)) \subset \mathrm{M}^c(f)$. — We note that since $f \circ \psi^{-1}$ is strictly convex, it follows from Theorem A in [4] that every level set of $f \circ \psi^{-1}$ is connected. Let A be the closure of the component of $\mathrm{M} - \psi(\mathrm{M}_c^c(f))$ which does not contain $\mathrm{M}_c^c(f)$, then we get that $\mathrm{M}^c(f \circ \psi^{-1}) = \mathrm{A}$ or $\mathrm{M}^c(f \circ \psi^{-1}) = \overline{\mathrm{M} - \mathrm{A}}$. If $\mathrm{M}^c(f \circ \psi^{-1}) (= \psi(\mathrm{M}^c(f))) = \overline{\mathrm{M} - \mathrm{A}}$, it contradicts Corollary C. Hence $\mathrm{M}^c(f \circ \psi^{-1}) = \mathrm{A}$. We set $\alpha = \max\{f(x);\ x \in \psi(\mathrm{M}_c^c(f))\}$ and $d = \max\{f \circ \psi^{-1}(x);\ x \in \mathrm{M}_\alpha^\alpha(f)\}$. Notice that $\delta(\alpha) < \delta(c)$ and $\mathrm{M}_\alpha^\alpha(f) \subset \mathrm{M}_c^d(f \circ \psi^{-1})$. Now we can use the same argument as in the case $\psi(\mathrm{M}_c^c(f)) \cap \mathrm{M}^c(f) = \emptyset$ with $f \circ \psi^{-1}$ in place of f and define a projection from $\mathrm{M}^d(f \circ \psi^{-1})$ onto $\mathrm{M}^c(f \circ \psi^{-1})$ as before. Then projecting $\mathrm{M}_\alpha^\alpha(f)$ to $\mathrm{M}^c(f \circ \psi^{-1})$ derives a contradiction. This completes the proof of Theorem B.

Q.E.D.

In general, in the situation of Theorem B a level set is not invariant under the isometries. It is not difficult to exhibit the examples.

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