The isotropic harmonic oscillator in an angular momemtum basis: An algebraic formulation — Source link

A. J. Bracken, H. I. Leemon

Published on: 01 Aug 1980 - Journal of Mathematical Physics (American Institute of Physics)

The isotropic harmonic oscillator in an angular momemtum basis: An algebraic formulation

A. J. Bracken and H. I. Leemon

Citation: Journal of Mathematical Physics 21, 2170 (1980); doi: 10.1063/1.524699
View online: http://dx.doi.org/10.1063/1.524699
View Table of Contents: http://scitation.aip.org/content/aip/journal/jmp/21/8?ver=pdfcov
Published by the AIP Publishing

The isotropic harmonic oscillator in an angular momentum basis: An algebraic formulation

A. J. Bracken and H. I. Leemon
Department of Mathematics, University of Queensland, St. Lucia, Queensland 4067 Australia

(Received 18 July 1979; accepted for publication 30 August 1979)

A completely algebraic and representation-independent solution is presented of the simultaneous eigenvalue problem for H, $\mathbf{L}^2$, and $\mathbf{L}_3$, where $\mathbf{H}$ is the Hamiltonian operator for the three-dimensional, isotropic harmonic oscillator, and $\mathbf{L}$ is its angular momentum vector. It is shown that H can be written in the form $\hbar\omega(2\nu^\dagger\nu + \boldsymbol{\lambda}^\dagger\cdot\boldsymbol{\lambda} + 3/2)$, where $\nu^\dagger$ and ν are raising and lowering (boson) operators for $\nu^\dagger\nu$, which has nonnegative integer eigenvalues k; and $\boldsymbol{\lambda}^\dagger$ and $\boldsymbol{\lambda}$ are raising and lowering operators for $\boldsymbol{\lambda}^\dagger\cdot\boldsymbol{\lambda}$, which has nonnegative integer eigenvalues l, the total angular momentum quantum number. Thus the eigenvalues of H appear in the familiar form $\hbar\omega(2k + l + 3/2)$, previously obtained only by working in the coordinate or momentum representation. The common eigenvectors are constructed by applying the operators $\nu^\dagger$ and $\boldsymbol{\lambda}^\dagger$ to a "vacuum" vector on which ν and $\boldsymbol{\lambda}$ vanish. The Lie algebra so(2,1) $\oplus$ so(3,2) is shown to be a spectrum-generating algebra for this problem. It is suggested that coherent angular momentum states can be defined for the oscillator, as the eigenvectors of the lowering operators ν and $\boldsymbol{\lambda}$. A brief discussion is given of the classical counterparts of ν, $\nu^\dagger$, $\boldsymbol{\lambda}$, and $\boldsymbol{\lambda}^\dagger$, in order to clarify their physical interpretation.

1. INTRODUCTION

The eigenvalue problem for the three-dimensional, isotropic harmonic oscillator Hamiltonian operator,

$$H = \frac{\mathbf{p}^2}{2M} + \tfrac{1}{2}M\omega^2\mathbf{x}^2\,, \tag{1}$$

is often solved algebraically (see for example Stehle,[1] Sec. 8). One introduces the boson creation and annihilation operators

$$\begin{aligned}\mathbf{a}^\dagger &= (2M\hbar\omega)^{-1/2}(-i\mathbf{p} + M\omega\mathbf{x})\,,\\ \mathbf{a} &= (2M\hbar\omega)^{-1/2}(i\mathbf{p} + M\omega\mathbf{x})\,,\end{aligned} \tag{2}$$

which are Hermitian conjugate to each other, and which satisfy the commutation relations

$$\begin{aligned}&[a_i, a_j] = 0 = [a_i^\dagger, a_j^\dagger]\,,\\ &[a_i, a_j^\dagger] = \delta_{ij}\,,\quad i,j = 1,2,3\,.\end{aligned} \tag{3}$$

Then one has

$$H = \hbar\omega(N + 3/2)\,, \tag{4}$$

where

$$N = \mathbf{a}^\dagger\cdot\mathbf{a} = N_1 + N_2 + N_3\,, \tag{5}$$

with, for example, $N_1 = a_1^\dagger a_1$. The usual boson calculus leads to the conclusion that the commuting operators N_1, N_2, and N_3, have simultaneous eigenvalues n_1, n_2, and n_3, running over all nonnegative integers independently, so that the eigenvalues of N appear in the form $n_1 + n_2 + n_3$. The corresponding normalized eigenvector may be denoted $|n_1, n_2, n_3\rangle$, and is nondegenerate. It may be obtained from a normalized "vacuum vector" $|0\rangle$ as

$$|n_1, n_2, n_3\rangle = (n_1! n_2! n_3!)^{-1/2}(a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}(a_3^\dagger)^{n_3}|0\rangle\,, \tag{6}$$

where

$$a_i|0\rangle = 0\,,\quad i = 1,2,3\,, \tag{7}$$

so that

$$N|0\rangle = 0 = N_i|0\rangle\,,\quad i = 1,2,3\,. \tag{8}$$

The eigenvalue problem for H (equivalently, for N) may be solved also in an "angular momentum basis." (See for example Davydov,[2] Sec. 37.) One works in either the coordinate or the momentum representation and looks for the common eigenfunctions of N, $\mathbf{L}^2$, and L_3, where

$$\begin{aligned}L_i &= \tfrac{1}{2}\hbar\epsilon_{ijk}l_{jk}\,,\\ l_{jk} &= (x_j p_k - x_k p_j)/\hbar\\ &= i(a_j a_k^\dagger - a_k a_j^\dagger)\,,\end{aligned} \tag{9}$$

so that

$$\begin{aligned}&[L_i, L_j] = i\hbar\epsilon_{ijk}L_k\,,\\ &i[l_{ij}, l_{km}] = \delta_{jk}l_{im} + \delta_{im}l_{jk} - \delta_{ik}l_{jm} - \delta_{jm}l_{ik}\,.\end{aligned} \tag{10}$$

The simultaneous eigenvalues are found to be

$$N: 2k + l\,,\quad \mathbf{L}^2: l(l+1)\hbar^2\,,\quad L_3: m\hbar\,, \tag{11}$$

where k and l run over the nonnegative integers independently, and for a given l, m runs over $l, l-1, \cdots, -l$. The corresponding normalized eigenfunction may be denoted ϕ_{klm} and is nondegenerate.

In this paper we show how the simultaneous eigenvalue problem for N, $\mathbf{L}^2$, and L_3 can be solved in a purely algebraic way, with the introduction of operators which raise and lower the values of k, l, and m, rather than n_1, n_2, and n_3. More precisely we find that N can be written in the form [contrast with Eq. (5)]

$$N = 2\nu^\dagger\nu + \boldsymbol{\lambda}^\dagger\cdot\boldsymbol{\lambda}\,, \tag{12}$$

where $\nu^\dagger$ and ν are raising and lowering operators for $\nu^\dagger\nu$, which has eigenvalues k; and $\boldsymbol{\lambda}^\dagger$ and $\boldsymbol{\lambda}$ are raising and lowering operators for $\boldsymbol{\lambda}^\dagger\cdot\boldsymbol{\lambda}$, which has eigenvalues l, the total angular momentum quantum number. The normalized common eigenvectors, denoted $|klm\rangle$, are obtained by applying suitable combinations of the raising operators to a normalized "vacuum vector" $|0\rangle$, which satisfies

$$N|0\rangle = \mathbf{L}^2|0\rangle = L_3|0\rangle = 0 . \tag{13}$$

[It is readily seen that this vector can be identified with the vector defined by Eqs. (7) or (8), hence the common notation.] In this approach, the fundamental dynamical variables in the problem are $\nu^\dagger$, ν, $\boldsymbol{\lambda}^\dagger$, and $\boldsymbol{\lambda}$ rather than $\mathbf{a}^\dagger$ and $\mathbf{a}$.

Of all the many investigations of the harmonic oscillator and related problems (for reviews and many references, see Kramer and Moshinsky[3] and McIntosh[4]), the closest in spirit to ours is that by Rose,[5] who examined the algebraic structure of operators χ_{kl}^m satisfying

$$|k\,l\,m\rangle = \chi_{kl}^m|0\rangle .$$

However, Rose did not identify the elementary operators ν, $\nu^\dagger$, $\boldsymbol{\lambda}$, and $\boldsymbol{\lambda}^\dagger$ in terms of which the Hamiltonian operator and all such χ_{kl}^m can be expressed [see our Eqs. (12) and (53)], and in terms of which the eigenvalue problem can be formulated and solved completely.

The algebraic solution of this problem is of some intrinsic interest, being independent of the choice of a particular representation space. Although one knows that any problem in quantum mechanics can be formulated in a variety of equivalent representations, and that the eigenvalues of any particular operator are determined by the structure of the relevant algebra of operators, rather than by the choice of representation space, few problems have been analyzed completely in a representation-independent way. (For examples, see the book of Green.[6] Of course, our constructions necessarily also define in the coordinate representation, for example, shift-operators associated with the differential operators N, $\mathbf{L}^2$, and L_3. There is a point of contact here with the so-called "factorization method."[7] We note however that the operator L which we introduce in the next section and which plays a central role in our analysis, is an integral operator, not a differential operator, in both the coordinate and the momentum representation.)

Having an algebraic formulation, we readily identify a hitherto unrecognized spectrum-generating algebra for this problem, namely the Lie algebra so (2,1) $\oplus$ so (3,2). However, our motivation for this work is primarily to set up an algebraic framework within which we can construct "coherent angular momentum states" for the oscillator. The investigation of such states will be the subject of a subsequent publication. They will be defined as common eigenvectors of the lowering operators ν and $\boldsymbol{\lambda}$, just as the usual coherent states can be defined as common eigenvectors of the lowering operators $\mathbf{a}$. They have many interesting properties in common with the usual coherent states, leading us to hope that they also will prove useful. Further motivation for the study of such states may be found in the work of Atkins and Dobson,[8] and of Delbourgo,[9] where the idea of superposing eigenvectors corresponding to all the possible values of the total angular momentum quantum number of a system, to form "coherent angular momentum states," has been proposed in a more general context.

In Sec. 2 we derive expressions for the operators $\nu^\dagger$, ν, $\boldsymbol{\lambda}^\dagger$ and $\boldsymbol{\lambda}$, and investigate some of their properties. Some proofs are relegated to Appendix A. The method used to determine $\boldsymbol{\lambda}^\dagger$ and $\boldsymbol{\lambda}$, in particular, depends heavily on techniques developed by Bracken and Green[10] for the analysis of vector operators. Indeed, the idea of constructing from the vector operators $\mathbf{a}^\dagger$ and $\mathbf{a}$ other vector operators which form "creation and annihilation operators for angular momentum" was partly developed some years ago by them.[11]

In Sec. 3, we present with the help of these operators the solution of the common eigenvalue problem for N, $\mathbf{L}^2$, and L_3, relegating some proofs to Appendix B. Then in Sec. 4, we discuss the time-dependence of these operators (in the Heisenberg picture) and the meaning of their classical counterparts.

It is known[12] that the Lie algebra sp(6, R) is a relevant spectrum-generating algebra for the oscillator Hamiltonian when N_1, N_2, and N_3 are to be diagonalized. In Sec. 5, we show that the Lie algebra so(2, 1) $\oplus$ so (3, 2) [$\approx$sp(2, R) $\oplus$ sp(4, R)] is a more appropriate spectrum-generating algebra when N, $\mathbf{L}^2$, and L_3 are to be diagonalized.

2. THE APPROPRIATE DYNAMICAL VARIABLES

In order to introduce the operators $\nu^\dagger$, ν, $\boldsymbol{\lambda}^\dagger$, and $\boldsymbol{\lambda}$ with the desirable properties described above, it is necessary in the first place to define the operator $L + \frac{1}{2}$, as the positive, scalar, Hermitian square-root of the positive operator $\frac{1}{2}l_{ij}l_{ij} + \frac{1}{4}(= \hbar^{-2}\mathbf{L}^2 + \frac{1}{4})$, so that

$$\mathbf{L}^2 = L(L+1)\hbar^2 . \tag{14}$$

It follows from the nonnegativity of $\mathbf{L}^2$ that any of its eigenvalues can be written in the form $l(l+1)\hbar^2$, with l nonnegative. On the same eigenvector, the eigenvalue of L will then be l. Of course, it will turn out that l runs over all the nonnegative integers—but we deduce this, not assume it.

We define also the Hermitian operator

$$K = \tfrac{1}{2}(N - L) , \tag{15}$$

so that $N = 2K + L$. Like all scalar operators, N (and hence K) commutes with all l_{ij}, and therefore with L.

However, the vector operator $\mathbf{a}$ (and likewise $\mathbf{a}^\dagger$) can be resolved into the sum of a vector operator which shifts the eigenvalue of L up by one unit, and a vector operator which shifts it down by one unit. This may be seen with the help of the techniques developed by Bracken and Green[10] as follows: From Eqs. (3) and the definition (9) of l_{ij} we have

$$\epsilon_{ijk}a_i l_{jk} = 0 , \tag{16}$$

or, equivalently,

$$a_i l_{jk} + a_k l_{ij} + a_j l_{ki} = 0 . \tag{17}$$

Contracting on the right with $\frac{1}{2}l_{ij}$, and using the commutation relations (10) and the definition of L, we find

$$a_i l_{ij} l_{jk} + i a_i l_{ik} + a_k L(L+1) = 0 , \tag{18}$$

that is

$$\begin{aligned} 0 &= a_i[l_{ij} + i(L+1)\delta_{ij}][l_{jk} - iL\delta_{jk}] \\ &= a_i[l_{ij} - iL\delta_{ij}][l_{jk} + i(L+1)\delta_{jk}] . \end{aligned} \tag{19}$$

We define the operators $\mathbf{a}^{(\pm)}$ by

$$a_j^{(\pm)} = a_i[(L+\tfrac{1}{2})\delta_{ij} \pm \tfrac{1}{2}\delta_{ij} \mp i l_{ij}][2L+1]^{-1} \tag{20}$$

(noting that $[2L+1]$ has a well-defined inverse, since L is nonnegative). Then $\mathbf{a}^{(\pm)}$ is evidently a vector-operator so that

$$i[a_i^{(\pm)}, l_{jk}] = \delta_{ij} a_k^{(\pm)} - \delta_{ik} a_j^{(\pm)}, \tag{21}$$

and hence

$$[a_i^{(\pm)}, \tfrac{1}{2} l_{jk} l_{jk}] = 2i a_k^{(\pm)} l_{ki} - 2a_i^{(\pm)}. \tag{22}$$

But, according to Eqs. (19) and (20),

$$i a_k^{(\pm)} l_{ki} = a_i^{(\pm)} [\tfrac{1}{2} \mp (L+\tfrac{1}{2})]. \tag{23}$$

Combining Eqs. (22) and (23), and using again the definition of L, we have

$$[a_i^{(\pm)}, L(L+1)] = -a_i^{(\pm)}[1 \pm (2L+1)], \tag{24}$$

or, equivalently,

$$L(L+1)\mathbf{a}^{(\pm)} = \mathbf{a}^{(\pm)}(L \pm 1)(L \pm 1 + 1). \tag{25}$$

From the nonnegativty of L, it then follows that

$$L\mathbf{a}^{(\pm)} = \mathbf{a}^{(\pm)}(L \pm 1), \tag{26}$$

so that $\mathbf{a}^{(\pm)}$ is a vector shift-operator for L. We have from Eq. (20) that

$$\mathbf{a} = \mathbf{a}^{(+)} + \mathbf{a}^{(-)}, \tag{27}$$

which is the required resolution of $\mathbf{a}$.

In the same way we find

$$\mathbf{a}^\dagger = \mathbf{a}^{\dagger(+)} + \mathbf{a}^{\dagger(-)}, \tag{28}$$

with

$$\begin{gathered} a_j^{\dagger(\pm)} = a_i^\dagger [(L+\tfrac{1}{2})\delta_{ij} \pm \tfrac{1}{2}\delta_{ij} \mp i l_{ij}][2L+1]^{-1}, \\ i a_j^{\dagger(\pm)} l_{ji} = a_i^{\dagger(\pm)}[\tfrac{1}{2} \mp (L+\tfrac{1}{2})], \end{gathered} \tag{29}$$

and

$$L\mathbf{a}^{\dagger(\pm)} = \mathbf{a}^{\dagger(\pm)}(L \pm 1). \tag{30}$$

In Appendix A we prove that

$$[\mathbf{a}^{(\pm)}]^\dagger = \mathbf{a}^{\dagger(\mp)}. \tag{31}$$

Now

$$N\mathbf{a} = \mathbf{a}(N-1), \tag{32}$$

and since N commutes with l_{ij} and L, it follows from the definition (20) that

$$N\mathbf{a}^{(\pm)} = \mathbf{a}^{(\pm)}(N-1). \tag{33}$$

It then follows from Eq. (26) and the definition (15) of K, that

$$\begin{gathered} [K, \mathbf{a}^{(-)}] = 0, \\ K\mathbf{a}^{(+)} = \mathbf{a}^{(+)}(K-1). \end{gathered} \tag{34}$$

In a similar way [or by conjugation of Eqs. (34)] we deduce that

$$\begin{gathered} [K, \mathbf{a}^{\dagger(+)}] = 0, \\ K\mathbf{a}^{\dagger(-)} = \mathbf{a}^{\dagger(-)}(K+1). \end{gathered} \tag{35}$$

It is easily seen from Eqs. (3) that

$$N(\mathbf{a}\cdot\mathbf{a}) = (\mathbf{a}\cdot\mathbf{a})(N-2), \tag{36}$$

and that $(\mathbf{a}\cdot\mathbf{a})$, being a scalar, commutes with L. Hence, using Eq. (15) we have

$$K(\mathbf{a}\cdot\mathbf{a}) = (\mathbf{a}\cdot\mathbf{a})(K-1), \tag{37a}$$

and, by a similar argument

$$K(\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger) = (\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger)(K+1). \tag{37b}$$

It now follows that $\mathbf{a}^{(+)}$ and $\mathbf{a}^{\dagger(+)}(\mathbf{a}\cdot\mathbf{a})$ have the same shifting properties for N, K, and L, so it is not surprising to find that (see Appendix A for proofs)

$$\mathbf{a}^{(+)} = \mathbf{a}^{\dagger(+)}(\mathbf{a}\cdot\mathbf{a})(2K+2L+1)^{-1}, \tag{38a}$$

and similarly

$$\mathbf{a}^{\dagger(-)} = \mathbf{a}^{(-)}(\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger)(2K+2L+3)^{-1}. \tag{38b}$$

We therefore isolate as fundamental the operator $\mathbf{a}^{\dagger(+)}$ and its conjugate $\mathbf{a}^{(-)}$, which are raising and lowering operators for L, but which commute with K; and the operators $(\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger)$ and $(\mathbf{a}\cdot\mathbf{a})$, which are raising and lowering operators for K but which commute with L.
[The operators $\mathbf{a}^{(+)}$ and $\mathbf{a}^{\dagger(-)}$ are relegated to a secondary position, and they may be regarded as defined by Eqs. (38).] However, the operators $\boldsymbol{\lambda}$, ν, and their conjugates $\boldsymbol{\lambda}^\dagger$, $\nu^\dagger$, defined by

$$\begin{gathered} \boldsymbol{\lambda} = \mathbf{a}^{(-)} f(K,L) = f(K,L+1)\mathbf{a}^{(-)}, \\ \boldsymbol{\lambda}^\dagger = f(K,L)\mathbf{a}^{\dagger(+)} = \mathbf{a}^{\dagger(+)} f(K,L+1), \\ \nu = (\mathbf{a}\cdot\mathbf{a}) g(K,L) = g(K+1,L)(\mathbf{a}\cdot\mathbf{a}), \\ \nu^\dagger = g(K,L)(\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger) = (\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger) g(K+1,L), \end{gathered} \tag{39}$$

may equally well be regarded as fundamental, for any reasonable Hermitian operator functions f and g. They evidently have the same shifting properties for K and L as have $\mathbf{a}^{(-)}$, $\mathbf{a}^{\dagger(+)}$, $(\mathbf{a}\cdot\mathbf{a})$, and $(\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger)$, respectively, viz

$$\begin{gathered} L\boldsymbol{\lambda} = \boldsymbol{\lambda}(L-1), \quad L\boldsymbol{\lambda}^\dagger = \boldsymbol{\lambda}^\dagger(L+1), \\ [L,\nu] = 0 = [L,\nu^\dagger], \\ K\nu = \nu(K-1), \quad K\nu^\dagger = \nu^\dagger(K+1), \\ [K,\boldsymbol{\lambda}] = 0 = [K,\boldsymbol{\lambda}^\dagger]. \end{gathered} \tag{40}$$

Furthermore, for any f we have (see Appendix A)

$$\begin{gathered} [\lambda_i, \lambda_j] = 0 = [\lambda_i^\dagger, \lambda_j^\dagger], \\ \boldsymbol{\lambda}\cdot\boldsymbol{\lambda} = 0 = \boldsymbol{\lambda}^\dagger\cdot\boldsymbol{\lambda}^\dagger, \end{gathered} \tag{41}$$

$$\begin{gathered} i\lambda_k l_{ki} = \lambda_i(L+1), \\ -i\lambda_k^\dagger l_{ki} = \lambda_i^\dagger L, \end{gathered} \tag{42}$$

and also

$$\begin{gathered} i[\lambda_i, l_{jk}] = \delta_{ij}\lambda_k - \delta_{ik}\lambda_j, \\ i[\lambda_i^\dagger, l_{jk}] = \delta_{ij}\lambda_k^\dagger - \delta_{ik}\lambda_j^\dagger. \end{gathered} \tag{43}$$

We choose the functions f and g so that, in addition to Eqs. (40), (41), (42), and (43), the operators $\boldsymbol{\lambda}$, $\boldsymbol{\lambda}^\dagger$, ν, and $\nu^\dagger$ have other simple algebraic properties, which make them most useful for the solution of the problem at hand (and for the construction of coherent states–see the comments at the end of Sec. 5). Noting that $2K+2L+1(=2N+L+1)$ is positive definite, and so has well-defined negative powers, we take

$$f = [(2L+1)/(2K+2L+1)]^{1/2}, \quad g = (4K+4L+2)^{-1/2}, \tag{44}$$

and find (see Appendix A)

$$\begin{aligned} &[\nu, \nu^\dagger] = 1, \\ &[\lambda_i, \nu] = 0 = [\lambda_i^\dagger, \nu^\dagger], \\ &[\lambda_i, \nu^\dagger] = 0 = [\lambda_i^\dagger, \nu], \\ &(2\boldsymbol{\lambda}^\dagger\cdot\boldsymbol{\lambda}+1)[\lambda_i, \lambda_j^\dagger] = (2\boldsymbol{\lambda}^\dagger\cdot\boldsymbol{\lambda}+1)\delta_{ij} - 2\lambda_i^\dagger\lambda_j, \end{aligned} \tag{45}$$

and also

$$K = \nu^\dagger\nu, \quad L = \boldsymbol{\lambda}^\dagger\cdot\boldsymbol{\lambda}, \quad il_{ij} = \lambda_i^\dagger\lambda_j - \lambda_j^\dagger\lambda_i. \tag{46}$$

The definition of the operators $\boldsymbol{\lambda}$, $\boldsymbol{\lambda}^\dagger$, ν, and $\nu^\dagger$ in terms of $\mathbf{a}$ and $\mathbf{a}^\dagger$, as presented above, is rather complicated. However one may choose to regard them, rather than $\mathbf{a}$ and $\mathbf{a}^\dagger$, as the basic variables. Then Eqs. (15) and (46) become definitions of N, K, L, and l_{ij}, and it can be shown that all relations in the algebra, such as those in Eqs. (40), (42), and (43), follow from Eqs. (41) and (45). In particular, $\mathbf{a}$ and $\mathbf{a}^\dagger$, which from this point of view have the complicated *definitions*

$$\begin{aligned} \mathbf{a} &= \boldsymbol{\lambda}[(2K+2L+1)/(2L+1)]^{1/2} + \boldsymbol{\lambda}^\dagger\nu[2/(2L+3)]^{1/2}, \\ \mathbf{a}^\dagger &= \boldsymbol{\lambda}^\dagger[(2K+2L+3)/(2L+3)]^{1/2} + \boldsymbol{\lambda}\nu^\dagger[2/(2L+1)]^{1/2}, \end{aligned} \tag{47}$$

can be shown to satisfy the boson commutation relations (3).

The commutation relations satisfied by $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}^\dagger$ as given in Eqs. (41) and (45) make these operators more difficult to manipulate than the boson operators ν and $\nu^\dagger$. However, the last of Eqs. (45), although complicated in appearance, has an important property in common with boson commutation relations: It does permit an annihilation operator λ_i to be shuffled through a product of creation operators $\lambda_j^\dagger$ acting on a "vacuum" vector, with the accumulation of terms which are free of annihilation operators. Using these operators we are able to solve the common eigenvalue problem for $K, L,$an L_3 in a manner quite similar to that usually adopted for N_1, N_2, and N_3.

The algebraic relations satisfied by the operators $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}^\dagger$ as listed above, are the same as those satisfied by the "modified boson operators" introduced in a quite different context by Lohe and Hurst.[13] Accordingly the algebraic structure of the eigenvectors $|k\,l\,m\rangle$ defined in the next section, in so far as it involves the variables $\boldsymbol{\lambda}^\dagger$, is essentially the same as the structure of the vectors $|^{l}_{m}\rangle$ of Ref. 13.

However, there is an important difference between the two sets of operators (apart from the fact that no analogs of ν and $\nu^\dagger$ appear in the work of Lohe and Hurst). The operators $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}^\dagger$ have been defined in terms of boson operators $\mathbf{a}$ and $\mathbf{a}^\dagger$ and act in the same space as those operators. While this space can be taken to be that of the usual coordinate representation of quantum mechanics, $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}^\dagger$ have been defined in a representation-independent way, and are perhaps best thought of as acting in an abstract Hilbert space, not tied to any particular representation. In contrast, the operators of Lohe and Hurst are defined by modifying not only a set of boson operators $\mathbf{a}$ and $\mathbf{a}^\dagger$, but also the particular space in which they are taken to act. As a result, their modified boson operators are only defined in a space of harmonic functions of three variables. The reason that they satisfy the same algebraic relations as $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}^\dagger$ may be traced to the fact that equivalent representations of the Lie algebra so(3,2) underly the two structures. In our case this so(3,2) is a subalgebra of a spectrum-generating algebra for the oscillator (Sec. 5), whereas in the case of Lohe and Hurst, though not metioned by them, it arises as a well-known invariance algebra of Laplace's equation in three dimensions.

3. SOLUTION OF THE EIGENVALUE PROBLEM

Since K and L cannot have negative eigenvalues, we see at once that there must be a vector on which the lowering operators $\boldsymbol{\lambda}$ and ν vanish. Thus we assert the existence of a normalized vector $|0\rangle$ such that

$$\nu|0\rangle = 0 = \lambda_i|0\rangle, \quad i = 1,2,3. \tag{48}$$

Since $K = \nu^\dagger\nu$, $L = \boldsymbol{\lambda}^\dagger\cdot\boldsymbol{\lambda}$, and $il_{ij} = \lambda_i^\dagger\lambda_j - \lambda_j^\dagger\lambda_i$, we have

$$\begin{aligned} &K\,|0\rangle = L\,|0\rangle = N\,|0\rangle = 0, \\ &l_{ij}|0\rangle = 0 = L_i|0\rangle, \quad i,j = 1,2,3. \end{aligned} \tag{49}$$

The other common eigenvectors of K, L, and L_3 can now be built up by applying the raising operators $\nu^\dagger$ and $\boldsymbol{\lambda}^\dagger$ to this "vacuum" vector. We define

$$\lambda_\pm = (\lambda_1 \pm i\lambda_2), \quad \lambda_\pm^\dagger = (\lambda_1^\dagger \pm i\lambda_2^\dagger), \tag{50}$$

so that

$$\begin{aligned} L_3\lambda_\pm &= \lambda_\pm(L_3 \pm \hbar), \\ L_3\lambda_\pm^\dagger &= \lambda_\pm^\dagger(L_3 \pm \hbar). \end{aligned} \tag{51}$$

Now let k, r, and s run over the nonnegative integers independently and let ϵ denote either $+$ or $-$. Then it is evident that on the vector

$$(\nu^\dagger)^k(\lambda_\epsilon^\dagger)^r(\lambda_3^\dagger)^s|0\rangle, \tag{52}$$

K, L, and L_3 have the eigenvalues k, $r+s$, and $\epsilon r\hbar$, respectively. Setting l equal to $r+s$, and m equal to ϵr, we write

$$|k\,l\,m\rangle = c_{klm}(\nu^\dagger)^k(\lambda_\epsilon^\dagger)^{|m|}(\lambda_3^\dagger)^{l-|m|}|0\rangle \tag{53}$$

as the normalized common eigenvector of these operators, corresponding to the eigenvalues k, l, and $m\hbar$. (We postpone for the moment discussion of the values of the normalization constants c_{klm}.) Here k and l run over the nonnegative integers independently, while for a given value of l, m runs over l, $l-1,\ldots,-l$, and ϵ is the sign of m. It is easily shown that for fixed k and l, the $2l+1$ vectors $|k\,l\,m\rangle$ form the basis for an irreducible representation of the Lie algebra so(3) spanned by the operators l_{ij} of Eqs. (46) (cf Ref. 13). [Alternatively one can consider for any fixed k, the vectors

$$|k;\alpha,\beta,\ldots,\tau\rangle = (\nu^\dagger)^k\lambda_\alpha^\dagger\lambda_\beta^\dagger\ldots\lambda_\tau^\dagger|0\rangle, \tag{54}$$

where the subscripts $\alpha, \beta, \ldots, \tau$ are l in number, and run over 1, 2, 3 independently. These vectors form a rank-l tensor basis for this representation of so(3). Note that in view of Eqs. (41), this tensor is automatically symmetric and traceless.]

For completeness, it is necessary to show that the vectors $|k\,l\,m\rangle$ are, up to multiplication by constants, the only common eigenvectors of K, L, L_3 which one can construct by applying to $|0\rangle$ any operator in the algebra generated by ν, $\nu^\dagger$, $\boldsymbol{\lambda}$, and $\boldsymbol{\lambda}^\dagger$. To do that, it suffices to show that the subspace of all finite linear combinations of the vectors $|k\,l\,m\rangle$ is invariant under the action of $\nu, \nu^\dagger, \lambda_3, \lambda_3^\dagger, \lambda_\pm$, and $\lambda_\pm^\dagger$; and this is true, for we see in Eqs. (59) below that when any of these operators is applied to any $|k\,l\,m\rangle$, a constant multiple of another such eigenvector is produced.

Turning to the calculation of the normalization constant c_{klm} in Eqs. (53), we see at once that

$$\langle <k\,l\,m|k\,l\,m\rangle = k!|c_{klm}|^2\langle 0|(\lambda_3)^{l-|m|} \times(\lambda_{-\epsilon})^{|m|}(\lambda_\epsilon^\dagger)^{|m|}(\lambda_3^\dagger)^{l-|m|}|0\rangle\,. \quad (55)$$

Using the last of the relations (45), it is straightforward to show by induction that

$$\begin{aligned}
&\lambda_i\lambda_\alpha^\dagger\lambda_\beta^\dagger\lambda_\gamma^\dagger\cdots\lambda_\sigma^\dagger\lambda_\tau^\dagger|0\rangle \\
&\quad=\{(\delta_{i\alpha}\lambda_\beta^\dagger\lambda_\gamma^\dagger\cdots\lambda_\sigma^\dagger\lambda_\tau^\dagger+\delta_{i\beta}\lambda_\alpha^\dagger\lambda_\gamma^\dagger\cdots\lambda_\sigma^\dagger\lambda_\tau^\dagger+\cdots \\
&\quad+\delta_{i\tau}\lambda_\alpha^\dagger\lambda_\beta^\dagger\lambda_\gamma^\dagger\cdots\lambda_\sigma^\dagger)-\frac{2}{(2l-1)}\lambda_i^\dagger(\delta_{\alpha\beta}\lambda_\gamma^\dagger\cdots\lambda_\sigma^\dagger\lambda_\tau^\dagger \\
&\quad+\delta_{\alpha\gamma}\lambda_\beta^\dagger\cdots\lambda_\sigma^\dagger\lambda_\tau^\dagger+\cdots+\delta_{\alpha\tau}\lambda_\beta^\dagger\lambda_\gamma^\dagger\cdots\lambda_\sigma^\dagger \\
&\quad+\delta_{\beta\gamma}\lambda_\alpha^\dagger\cdots\lambda_\sigma^\dagger\lambda_\tau^\dagger+\cdots+\delta_{\beta\tau}\lambda_\alpha^\dagger\lambda_\gamma^\dagger\cdots\lambda_\sigma^\dagger \\
&\quad+\cdots+\delta_{\sigma\tau}\lambda_\alpha^\dagger\lambda_\beta^\dagger\lambda_\gamma^\dagger\cdots)\}|0\rangle\,, \qquad (56)
\end{aligned}$$

where l is the number of creation operators $\lambda_\alpha^\dagger, \lambda_\beta^\dagger, \cdots, \lambda_\tau^\dagger$. With the help of this result, we are able to show (see Appendix B) that

$$\langle k\,l\,m|k\,l\,m\rangle = \frac{k!|c_{klm}|^2 2^l l!(l-m)!(l+m)!}{(2l)!}, \quad (57)$$

so that $|k\,l\,m\rangle$ as defined in Eq. (53) is normalized if we take (with a convenient choice of phases)

$$c_{klm} = (-\epsilon)^m\left(\frac{(2l)!}{k!l!(l-m)!(l+m)!2^l}\right)^{1/2}. \quad (58)$$

It is then found that (see Appendix B)

$$\begin{aligned}
\nu|k\,l\,m\rangle &= (k)^{1/2}|k-1\,l\,m\rangle\,, \\
\nu^\dagger|k\,l\,m\rangle &= (k+1)^{1/2}|k+1\,l\,m\rangle\,, \\
\lambda_3^\dagger|k\,l\,m\rangle &= \left(\frac{(l+1-m)(l+1+m)}{(2l+1)}\right)^{1/2}|k\,l+1\,m\rangle\,, \\
\lambda_\pm^\dagger|k\,l\,m\rangle &= \mp\left(\frac{(l\pm m+2)(l\pm m+1)}{(2l+1)}\right)^{1/2} \\
&\quad\times|k\,l+1\,m\pm1\rangle\,, \\
\lambda_3|k\,l\,m\rangle &= \left(\frac{(l-m)(l+m)}{(2l-1)}\right)^{1/2}|k\,l-1\,m\rangle\,, \\
\lambda_\pm|k\,l\,m\rangle &= \pm\left(\frac{(l\mp m)(l\mp m-1)}{(2l-1)}\right)^{1/2} \\
&\quad\times|k\,l-1\,m\pm1\rangle\,. \qquad (59)
\end{aligned}$$

We close this section by remarking that we have chosen phases in Eq. (58) in such a way that the vector $|k\,l\,m\rangle$ appears in the coordinate representation as

$$\phi_{klm} = (-1)^k\left(\frac{2a^3k!}{\Gamma(k+l+3/2)}\right)^{1/2}\xi^l e^{-(1/2)\xi^2} \times L_k^{(l+1/2)}(\xi^2)Y_{lm}(\theta,\phi)\,,$$

where $a=(M\omega/\hbar)^{1/2}$, $\xi = ar$ (r,θ and ϕ are the usual spherical polar coordinates), $L_k^{(l+1/2)}$ is the generalized Laguerre polynomial, defined as in Ref. 14, and the spherical harmonic Y_{lm} is defined as in Ref. 15.

4. TIME-DEPENDENCE AND INTERPRETATION

In the Heisenberg picture, the time-dependence of an observable A (or of any complex linear combination A of observables) is determined by

$$i\hbar\frac{dA}{dt} = [A,H]\,.$$

Now the operators ν, $\nu^\dagger$, $\boldsymbol{\lambda}$, and $\boldsymbol{\lambda}^\dagger$ are shift-operators for H, allowing us to deduce at once that

$$\frac{d\nu}{dt} = -2i\omega\nu\,,\quad \frac{d\nu^\dagger}{dt} = 2i\omega\nu^\dagger\,, \qquad \frac{d\boldsymbol{\lambda}}{dt} = -i\omega\boldsymbol{\lambda}\,,\quad \frac{d\boldsymbol{\lambda}^\dagger}{dt} = i\omega\boldsymbol{\lambda}^\dagger\,. \quad (60)$$

Thus

$$\nu = \nu_0 e^{-2i\omega t}\,,\quad \nu^\dagger = \nu_0^\dagger e^{2i\omega t}\,, \qquad \boldsymbol{\lambda} = \boldsymbol{\lambda}_0 e^{-i\omega t}\,,\quad \boldsymbol{\lambda}^\dagger = \boldsymbol{\lambda}_0^\dagger e^{i\omega t}\,, \quad (61)$$

where the (constant) operators ν_0, $\nu_0^\dagger$, $\boldsymbol{\lambda}_0$, and $\boldsymbol{\lambda}_0^\dagger$ satisfy the same algebraic relations as ν, $\nu^\dagger$, $\boldsymbol{\lambda}$, and $\boldsymbol{\lambda}^\dagger$.

We gain some insight into the physical interpretation of these variables by considering their classical counterparts. Denoting the classical coordinate and momentum vectors by $\hat{\mathbf{x}}$ and $\hat{\mathbf{p}}$ respectively and the classical Hamiltonian by $\hat{H}$, we define

$$\hat{\mathbf{a}} = (2M\omega)^{-1/2}(i\hat{\mathbf{p}} + M\omega\hat{\mathbf{x}})\,, \quad (62)$$

and its *complex* conjugate $\hat{\mathbf{a}}^*$. [Note the extra factor of $(\hbar)^{1/2}$ in comparison with Eq. (2).] Then

$$\hat{H} = \omega\hat{\mathbf{a}}^*\cdot\hat{\mathbf{a}}\,. \quad (63)$$

Introducing the classical angular momentum vector

$$\hat{\mathbf{L}} = \hat{\mathbf{x}}\times\hat{\mathbf{p}}\,, \quad (64)$$

with length $\hat{L}$, we define

$$\hat{K} = \frac{1}{2\omega}(\hat{H}-\omega\hat{L})\,. \quad (65)$$

In the definitions of ν, $\nu^\dagger$, $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}^\dagger$ above, we let $\hbar\to0$, with $H\to\hat{H}$ and

$$\begin{aligned}
&(\hbar)^{1/2}\mathbf{a}\to\hat{\mathbf{a}}\,,\quad (\hbar)^{1/2}\mathbf{a}^\dagger\to\hat{\mathbf{a}}^*\,, \\
&\hbar l_{ij}\to\epsilon_{ijk}\hat{L}_k\,, \\
&\hbar L\to\hat{L}\,,\quad \hbar K\to\hat{K}\,,
\end{aligned} \quad (66)$$

in order to obtain the classical variables corresponding to ν and $\boldsymbol{\lambda}$,

$$\hat{\nu} = \tfrac{1}{2}(\hat{\mathbf{a}}\cdot\hat{\mathbf{a}})(\hat{K}+\hat{L})^{-1/2}$$

$$= \frac{(\hat{K}+\hat{L})^{-1/2}}{4M\omega}[(M^2\omega^2\hat{\mathbf{x}}^2-\hat{\mathbf{p}}^2)+i(2M\omega\hat{\mathbf{x}}\cdot\hat{\mathbf{p}})]\,, \tag{67}$$

$$\begin{aligned}\hat{\boldsymbol{\lambda}} &= \tfrac{1}{2}[\hat{L}(\hat{K}+\hat{L})]^{-1/2}(\hat{L}\hat{\mathbf{a}}+i\hat{\mathbf{L}}\times\hat{\mathbf{a}}) \\ &= [8M\omega\hat{L}(\hat{K}+\hat{L})]^{-1/2}[(M\omega\hat{L}\hat{\mathbf{x}}-\hat{\mathbf{L}}\times\hat{\mathbf{p}}) \\ &\quad + i(\hat{\mathbf{L}}\hat{\mathbf{p}}+M\omega\hat{\mathbf{L}}\times\hat{\mathbf{x}})]\,,\end{aligned}$$

and their complex conjugates $\hat{v}^*$ and $\hat{\boldsymbol{\lambda}}^*$, which correspond to $v^\dagger$ and $\boldsymbol{\lambda}^\dagger$. Apart from the overall factors involving the constants of the motion $\hat{K}$ and $\hat{L}$, these expressions are reasonably simple. It is straightforward to verify in particular that

$$\begin{aligned}&\hat{v}^*\hat{v}=\hat{K}\,,\quad \hat{\boldsymbol{\lambda}}^*\cdot\hat{\boldsymbol{\lambda}}=\hat{L}\,,\\ &\hat{\boldsymbol{\lambda}}\cdot\hat{\boldsymbol{\lambda}}=0=\hat{\boldsymbol{\lambda}}^*\cdot\hat{\boldsymbol{\lambda}}^*\,,\\ &\hat{H}=\omega(2\hat{v}^*\hat{v}+\hat{\boldsymbol{\lambda}}^*\cdot\hat{\boldsymbol{\lambda}})\,,\\ &\hat{\boldsymbol{\lambda}}\times\hat{\mathbf{L}}=i\hat{L}\hat{\boldsymbol{\lambda}}\,,\quad \hat{\boldsymbol{\lambda}}^*\times\hat{\mathbf{L}}=-i\hat{L}\hat{\boldsymbol{\lambda}}^*\,,\\ &\hat{\boldsymbol{\lambda}}^*\times\hat{\boldsymbol{\lambda}}=i\hat{\mathbf{L}}\,,\end{aligned} \tag{68}$$

and also that the time-dependence of the classical variables is the same as that of their counterparts, as in Eqs. (61).

From the relations (68), it can be seen that if $\hat{\boldsymbol{\alpha}}$ and $\hat{\boldsymbol{\beta}}$ denote the real and imaginary parts of $(\sqrt{2})\hat{\boldsymbol{\lambda}}$, then $\hat{\boldsymbol{\alpha}}$ and $\hat{\boldsymbol{\beta}}$ are orthogonal, and of the same length $(\hat{L})^{1/2}$. From the last three of the relations (68), we see then that $\mathbf{e}=\hat{\boldsymbol{\alpha}}(\hat{L})^{-1/2}$, $\mathbf{f}=\boldsymbol{\beta}(\hat{L})^{-1/2}$, and $\mathbf{g}=\hat{\mathbf{L}}(\hat{L})^{-1}$ form a right-handed system of orthogonal unit vectors, of which the first two are time-dependent.

Classically, the motion is elliptical, in the plane perpendicular to $\hat{\mathbf{L}}$, i.e., in the plane determined by $\hat{\boldsymbol{\alpha}}$ and $\hat{\boldsymbol{\beta}}$. For any particular motion we can choose time-origin and space-axes such that the motion is anticlockwise in the XY-plane, with

$$\hat{\mathbf{x}}=(A\cos\omega t,\,B\sin\omega t,\,0),\quad A\geqslant B\geqslant 0\,.$$

Then

$$\begin{aligned}&\hat{\mathbf{p}}=M\omega(-A\sin\omega t,\,B\cos\omega t,\,0)\,,\\ &\hat{\mathbf{L}}=M\omega(0,0,AB)\,,\quad \hat{L}=M\omega AB\,,\\ &\hat{H}=\tfrac{1}{2}M\omega^2(A^2+B^2)\,,\quad \hat{K}=\tfrac{1}{4}M\omega(A-B)^2\,,\\ &\hat{v}=\tfrac{1}{2}\sqrt{M\omega}(A-B)e^{-2i\omega t}=\sqrt{\hat{K}}\,e^{-2i\omega t}\,,\\ &\hat{\boldsymbol{\lambda}}=\sqrt{\tfrac{1}{2}M\omega AB}\,e^{-i\omega t}(1,i,0)=\sqrt{\tfrac{1}{2}\hat{L}}\,e^{-i\omega t}(1,i,0)\,,\\ &\mathbf{e}=(\cos\omega t,\,\sin\omega t,\,0)\,,\quad \mathbf{f}=(-\sin\omega t,\,\cos\omega t,\,0)\,,\\ &\mathbf{g}=(0,0,1)\,.\end{aligned} \tag{69}$$

The periodic variables $\hat{\boldsymbol{\lambda}}$ and $\hat{\boldsymbol{\lambda}}^*$ have angular frequency ω, the natural frequency of the oscillator, but $\hat{v}$ and $\hat{v}^*$ have angular frequency 2ω. This is at first glance rather puzzling, but we can understand it as follows, and perhaps at the same time appreciate the geometrical significance of all these variables.

The elliptical motion can be regarded as arising from the superposition of two uniform circular motions, with angular frequencies 2ω and ω, respectively. In the particular coordinate system adopted above,

$$\begin{aligned}\hat{\mathbf{x}} &= (A\cos\omega t,\;B\sin\omega t,\;0)\\ &= \tfrac{1}{2}(A+B)\mathbf{e}+\tfrac{1}{2}(A-B)(\cos 2\omega t\,\mathbf{e}-\sin 2\omega t\,\mathbf{f})\,.\end{aligned} \tag{70}$$

Thus the particle can be regarded as moving uniformly *clockwise*, with angular frequency 2ω, in a circle which is fixed relative to the vectors $\mathbf{e}$ and $\mathbf{f}$ and has radius $\frac{1}{2}(A-B)$. This circle (epicycle) itself rotates *anticlockwise* (along with $\mathbf{e}$ and $\mathbf{f}$) about an exterior point 0, so that its center moves uniformly, with angular frequency ω, around the circumference of a larger circle (deferent) of radius $\frac{1}{2}(A+B)$, centered at 0. The point 0 is the center of the resultant anticlockwise elliptical motion. (See Fig. 1.)

The relevance of the variables $\hat{v}$, $\hat{v}^*$, $\hat{\boldsymbol{\lambda}}$, and $\hat{\boldsymbol{\lambda}}^*$ to this decomposition of the motion can be appreciated when one notes that Eq. (70) is [in the particular coordinate system of Eqs. (69)] just the real part of the formula

$$\hat{\mathbf{a}}=[(\hat{K}+\hat{L})/\hat{L}]^{1/2}\hat{\boldsymbol{\lambda}}+[\hat{L}]^{-1/2}\hat{v}\hat{\boldsymbol{\lambda}}^*\,, \tag{71}$$

which is the classical equivalent of the first of Eqs. (47).

The resolution of the harmonic motion into two circular motions can also be seen and understood in the following way. The equation of motion for the oscillator is

$$m\frac{d^2\hat{\mathbf{x}}}{dt^2}=-m\omega^2\hat{\mathbf{x}}\,. \tag{72}$$

Since the force on the particle is central, the motion is in a fixed plane perpendicular to the angular momentum $\hat{\mathbf{L}}$. We make a change of reference frame, to the frame rotating anticlockwise, with angular frequency ω, about a unit vector $\mathbf{n}$ which passes through the origin and which is parallel to $\hat{\mathbf{L}}$. In the rotating frame, the equation of motion for the particle at $\mathbf{r}$ is

$$m\frac{d^2\mathbf{r}}{dt^2}=-m\omega^2\mathbf{r}-2m\boldsymbol{\omega}\times\frac{d\mathbf{r}}{dt}-m\,\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})\,, \tag{73}$$

with $\boldsymbol{\omega}=\omega\mathbf{n}$. Here the second term is the Coriolis "force," and the third is the centrifugal "force" on the particle

Now $\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})=-\omega^2\mathbf{r}$,

since $\boldsymbol{\omega}$ is orthogonal to $\hat{\mathbf{x}}$, and hence to $\mathbf{r}$. Thus in this frame the centrifugal force exactly cancels the true force, and the particle moves under the Coriolis force alone, with

$$\frac{d^2\mathbf{r}}{dt^2}=-2\boldsymbol{\omega}\times\frac{d\mathbf{r}}{dt}\,. \tag{74}$$

Then

$$\begin{aligned}\frac{d^3\mathbf{r}}{dt^3}&=-2\boldsymbol{\omega}\times\frac{d^2\mathbf{r}}{dt^2}\\ &=-2\boldsymbol{\omega}\times\left(-2\boldsymbol{\omega}\times\frac{d\mathbf{r}}{dt}\right)\\ &=-4\omega^2\frac{d\mathbf{r}}{dt}\,.\end{aligned} \tag{75}$$

Integrating once we have

$$\frac{d^2\mathbf{R}}{dt^2}=-4\omega^2\mathbf{R}\,, \tag{76}$$

where $\mathbf{R}=\mathbf{r}-\mathbf{r}_0$, with $\mathbf{r}_0$ an arbitrary constant vector, which must be orthogonal to $\boldsymbol{\omega}$ in view of Eq. (74).

We see that in this rotating frame the motion is harmonic with angular frequency 2ω, about an arbitrary fixed point $\mathbf{r}_0$ in the plane of the motion. That this motion must actually be circular (it is the motion around the smaller circle in Fig. 1.) follows from the fact that we also have, from Eq. (74)

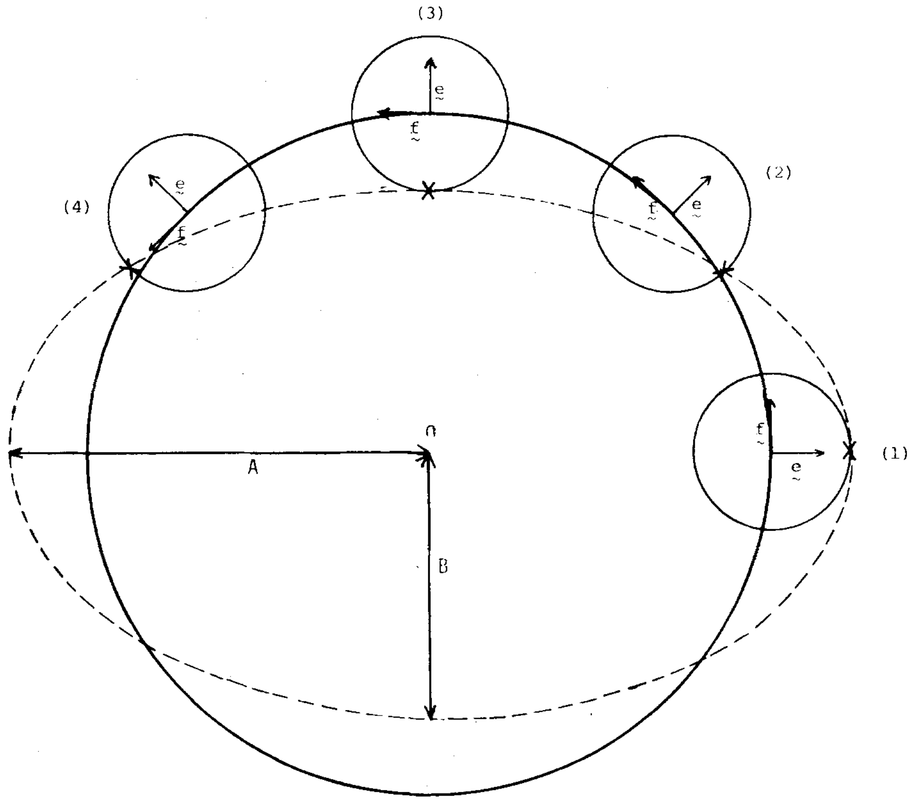

FIG. 1. Resolution of elliptical motion into two circular motions. The particle, whose position is marked X, rotates clockwise with angular frequency 2ω around the smaller circle, whose center moves anticlockwise with angular frequency ω around the larger circle. Positions are shown at (1) $\omega t = 0$; (2) $\omega t = \pi/4$; (3) $\omega t = \pi/2$; (4) $\omega t = 3\pi/4$.

$$\frac{d^2\mathbf{R}}{dt^2} = -2\boldsymbol{\omega}\times\frac{d\mathbf{R}}{dt}, \tag{77}$$

so that

$$-4\omega^2\mathbf{R} = -2\boldsymbol{\omega}\times\frac{d\mathbf{R}}{dt}, \tag{78}$$

implying that $\mathbf{R}\cdot(d\mathbf{R}/dt) = 0$. and hence that $\mathbf{R}^2$ is constant. We also note from Eq. (78) that

$$\mathbf{R}\times\frac{d\mathbf{R}}{dt} = \frac{1}{2\omega^2}\left(\boldsymbol{\omega}\times\frac{d\mathbf{R}}{dt}\right)\times\frac{d\mathbf{R}}{dt} = -\frac{1}{2\omega^2}\left(\frac{d\mathbf{R}}{dt}\right)^2\boldsymbol{\omega}, \tag{79}$$

so that the circular motion is in the opposite sense to $\boldsymbol{\omega}$, i.e., it is clockwise about an axis which passes through $\mathbf{R} = \mathbf{0}$ and which is parallel to $\mathbf{n}$.

5. A NEW SPECTRUM-GENERATING ALGEBRA FOR THE OSCILLATOR

For the treatment of the common eigenvalue problem for the operators N_1, N_2, and N_3, a spectrum-generating (Lie) algebra is the 21-dimensional Lie algebra sp(6,R), with Hermitian basis

$$\begin{aligned}&(a_ia_j + a_i^\dagger a_j^\dagger), \quad i(a_ia_j - a_i^\dagger a_j^\dagger),\\ &(a_i^\dagger a_j + a_ia_j^\dagger), \quad i(a_i^\dagger a_j - a_j^\dagger a_i).\end{aligned} \tag{80}$$

The vectors $|n_1, n_2, n_3\rangle$ with odd $(n_1 + n_2 + n_3)$ span one irreducible representation of this algebra, and those with even $(n_1 + n_2 + n_3)$ span another.[12]

For the common eigenvalue problem for N, $\mathbf{L}^2$, and L_3 another Lie algebra is more relevant. Define the Hermitian operators

$$\begin{aligned}\boldsymbol{\Lambda} &= (2L+1)^{1/2}\boldsymbol{\lambda} = \boldsymbol{\lambda}(2L-1)^{1/2},\\ \boldsymbol{\Lambda}^\dagger &= \boldsymbol{\lambda}^\dagger(2L+1)^{1/2} = (2L-1)^{1/2}\boldsymbol{\lambda}^\dagger,\end{aligned} \tag{81}$$

and note that, as well as commuting with ν and $\nu^\dagger$, and having the same shifting properties for L as $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}^\dagger$, they satisfy

$$\begin{aligned}&[\Lambda_i, \Lambda_j] = 0 = [\Lambda_i^\dagger, \Lambda_j^\dagger],\\ &[\Lambda_i, \Lambda_j^\dagger] = (2L+1)\delta_{ij} - 2il_{ij},\\ &\boldsymbol{\Lambda}\cdot\boldsymbol{\Lambda} = 0 = \boldsymbol{\Lambda}^\dagger\cdot\boldsymbol{\Lambda}^\dagger,\\ &\boldsymbol{\Lambda}^\dagger\cdot\boldsymbol{\Lambda} = L(2L-1),\\ &\Lambda_i^\dagger\Lambda_j - \Lambda_j^\dagger\Lambda_i = il_{ij}(2L-1).\end{aligned} \tag{82}$$

The proof of the results (82) is elementary, with the use of Eqs. (41), (45), and (46).

Now define

$$\begin{aligned}&A_1 = \tfrac{1}{4}(\nu\nu + \nu^\dagger\nu^\dagger), \quad A_2 = \tfrac{1}{4}i(\nu\nu - \nu^\dagger\nu^\dagger),\\ &A_3 = \tfrac{1}{2}(\nu^\dagger\nu + \tfrac{1}{2}),\\ &B_{4i} = \tfrac{1}{2}(\Lambda_i + \Lambda_i^\dagger) = -B_{i4},\\ &B_{5i} = \tfrac{1}{2}i(\Lambda_i - \Lambda_i^\dagger) = -B_{i5},\\ &B_{ij} = l_{ij}, \quad B_{54} = (L + \tfrac{1}{2}) = -B_{45}.\end{aligned} \tag{83}$$

It is easily checked that these operators span an Hermitian representation of the Lie algebra so(2,1) $\oplus$ so(3,2) [$\simeq$sp(2,R) $\oplus$ sp(4,R)], with the only nontrivial commutation relations being

$$[A_1, A_2] = -iA_3, \quad [A_2, A_3] = iA_1, \quad [A_3, A_1] = iA_2, \tag{84}$$

$$i[B_{\mu\nu}, B_{\rho\sigma}] = g_{\nu\rho}B_{\mu\sigma} + g_{\mu\sigma}B_{\nu\rho} - g_{\mu\rho}B_{\nu\sigma} - g_{\nu\sigma}B_{\mu\rho},$$

where μ, ν, ρ, and σ run over 1,2,3,4,5, and the metric tensor $g_{\mu\nu}$ is diagonal, with $g_{11} = g_{22} = g_{33} = -g_{44} = -g_{55} = 1$.

The quadratic invariant of the so(2,1) algebra has the value

$$(A_1)^2 + (A_2)^2 - (A_3)^2 = \tfrac{3}{16}. \tag{85}$$

There are two irreducible Hermitian representations of so(2,1), labelled $\mathscr{D}^{(+)}(-1/4)$ and $\mathscr{D}^{(+)}(-3/4)$ by Barut and Fronsdal,[16] for which the invariant has this value, and in which the spectrum of A_3 is bounded below (as it evidently is in the present situation). In the representation $\mathscr{D}^{(+)}(-1/4)$, A_3 has eigenvalues 1/4, 5/4, 9/4, ⋯; and in the representation $\mathscr{D}^{(+)}(-3/4)$ it has eigenvalues 3/4, 7/4, 11/4, ⋯. It can be seen that representations of both types are involved in the problem under discussion—the former associated with even-integral eigenvalues of $K\ (=\nu^\dagger\nu)$, the latter with odd-integral eigenvalues.

A simple calculation shows that the quadratic invariant of the so(3,2) algebra has the value

$$\tfrac{1}{2}B_{\mu\nu}B^{\mu\nu} = -\tfrac{5}{4}. \tag{86}$$

Moreover, the two invariants of the so(3,1) subalgebra spanned by the B_{ij} and B_{4i}, have the values

$$\begin{aligned} &\tfrac{1}{2}B_{ij}B_{ij} - B_{4i}B_{4i} = -\tfrac{3}{4}, \\ &\tfrac{1}{2}\epsilon_{ijk}B_{ij}B_{4k} = 0, \end{aligned} \tag{87}$$

indicating that any irreducible representation of so(3,2) which appears here, remains irreducible when restricted to the so(3,1) subalgebra. In the commonly used[17] $[k_0, c]$ labelling of the irreducible representation of so(3,1), these two invariants have values $(k_0^2 + c^2 - 1)$ and ik_0c, respectively. Thus the irreducible representations of so(3,1) appearing here can only be $[\frac{1}{2}, 0]$ or $[0, \frac{1}{2}]$; and since the eigenvalues of B_{12} are integral, only the representation $[0, \frac{1}{2}]$ can be involved. It is known (see for example Böhm,[18]) that this representation of so(3,1) extends to either of two irreducible Hermitian representations (two of the four Majorana representations) of so(3,2), each consistent with Eq. (86). But in only one of these—let us call it $\mathscr{T}$—is the spectrum of B_{54} bounded below, as it evidently is in the present situation. In this representation $\mathscr{T}$, B_{54} has eigenvalues 1/2, 3/2, 5/2⋯.

The representation of so(2,1) ⊕ so(3,2) associated with the harmonic oscillator in an angular momentum basis can now be identified, in view of the nondegeneracy of the eigenvectors $|k\,l\,m\rangle$, as simply

$$(\mathscr{D}^{(+)}(-1/4), \mathscr{T}) \oplus (\mathscr{D}^{(+)}(-3/4), \mathscr{T}). \tag{88}$$

The Hamiltonian operator appears in the form

$$H = \hbar\omega(4A_3 + B_{54}), \tag{89}$$

and its eigenvalues are immediately deducible from the known spectra of A_3 in the representations $\mathscr{D}^{(+)}(-1/4)$, $\mathscr{D}^{(+)}(-3/4)$, and of B_{54} in $\mathscr{T}$.

The reader may wonder why we did not, in Sec. 2, choose to work with the operators Λ and $\Lambda^\dagger$ rather than λ and $\lambda^\dagger$. A simple change of the function f in Eqs. (44) {to $f(K,L) = [(4L^2-1)/(2K+2L+1)]^{1/2}$} would have accomplished such a substitution. The commutation relations satisfied by Λ and $\Lambda^\dagger$ are simpler than those satisfied by λ and $\lambda^\dagger$, and the connection with the spectrum-generating algebra is more immediate. For these reasons it may be argued that the operators Λ and $\Lambda^\dagger$ are more suitable for the algebraic treatment of the eigenvalue problem.

Our preference for the operators λ and $\lambda^\dagger$ is mainly determined by our intention to define in a subsequent publication, "coherent angular momentum states" for the oscillator as eigenvectors of the lowering operators. The expectation values of the important operators H, K, L, and l_{ij} will be very simple in such states, if we diagonalize the operators λ and ν, because

$$K = \nu^\dagger\nu, \quad L = \lambda^\dagger\cdot\lambda, \quad il_{ij} = \lambda_i^\dagger\lambda_j - \lambda_j^\dagger\lambda_i. \tag{90}$$

On the other hand, if we diagonalize the operators Λ, we shall need to work with the expressions

$$\begin{aligned} L &= \tfrac{1}{4} + \tfrac{1}{4}(1 + 8\underline{\Lambda}^\dagger\cdot\underline{\Lambda})^{1/2}, \\ il_{ij} &= (2L-1)^{-1}(\Lambda_i^\dagger\Lambda_j - \Lambda_j^\dagger\Lambda_i), \end{aligned} \tag{91}$$

whose expectation values will not be simple.

ACKNOWLEDGMENTS

We thank Professor L. Bass for very helpful discussions, and for a critical reading of the manuscript. A.J.B. wishes to thank Professor H.S. Green for impressing on him the possibility of a purely algebraic approach to this and other problems in quantum mechanics.

APPENDIX A

Here and in Appendix B we present the derivations of some of the results stated above. The ends of proofs are indicated thus: □

From the definition of $\mathbf{a}^{\dagger(+)}$ in the first of Eqs. (29), it follows that

$$(a_j^{\dagger(+)})^\dagger = (2L+1)^{-1}[(L+1)\delta_{ij} + il_{ij}]a_i. \tag{A1}$$

Using Eqs. (21) and then Eqs. (23) we see that

$$\begin{aligned} il_{ij}a_i^{(\pm)} &= ia_i^{(\pm)}l_{ij} - 2a_j^{(\pm)} \\ &= -a_j^{(\pm)}[3/2 \pm (L + \tfrac{1}{2})]. \end{aligned} \tag{A2}$$

Now using Eq. (27) in Eq. (A1), we have

$$\begin{aligned} (a_j^{\dagger(+)})^\dagger &= (2L+1)^{-1}(L+1)(a_j^{(+)} + a_j^{(-)}) \\ &\quad + (2L+1)^{-1}(il_{ij}a_i^{(+)} + il_{ij}a_i^{(-)}) \\ &= (2L+1)^{-1}[a_j^{(+)}(L+2) + a_j^{(-)}L \\ &\quad - a_j^{(+)}(L+2) + a_j^{(-)}(L-1)] \\ &\hspace{6cm}[\text{using Eq. (A2)}] \\ &= (2L+1)^{-1}a_j^{(-)}(2L-1) \\ &= a_j^{(-)} \quad [\text{using Eq. (26)}]. \end{aligned}$$

In a similar way we show that $(\mathbf{a}^{\dagger(-)})^\dagger = \mathbf{a}^{(+)}$, so completing the verification of Eqs. (31). □

From the definition in Eqs. (20) we have that

$$\begin{aligned} a_j^{(-)}(2L+1) &= a_jL + ia_il_{ij} \\ &= a_jL - a_i(a_ia_j^\dagger - a_ja_i^\dagger) \quad [\text{using Eq. (9)}], \end{aligned}$$

$$= -a_j^\dagger(\mathbf{a}\cdot\mathbf{a}) + a_j(N+L+1) \quad \text{[using Eqs. (3)]}. \qquad \text{(A3)}$$

Similarly, we find

$$\begin{aligned} a_j^{(+)}(2L+1) &= a_j^\dagger(\mathbf{a}\cdot\mathbf{a}) - a_j(N-L), \\ a_j^{\dagger(-)}(2L+1) &= a_j(\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger) - a_j^\dagger(N-L+2), \qquad \text{(A4)} \\ a_j^{\dagger(+)}(2L+1) &= -a_j(\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger) + a_j^\dagger(N+L+3). \end{aligned}$$

It is easily deduced from the definitions (9) and the relations (3) that

$$\begin{aligned} \tfrac{1}{2}l_{ij}l_{ij} &= N^2 + N - (\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger)(\mathbf{a}\cdot\mathbf{a}) \\ &= N^2 + 5N + 6 - (\mathbf{a}\cdot\mathbf{a})(\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger). \end{aligned}$$

Since $\frac{1}{2}l_{ij}l_{ij} = L(L+1)$, it follows that

$$\begin{aligned} (\mathbf{a}\cdot\mathbf{a})(\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger) &= (N-L+2)(N+L+3), \\ (\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger)(\mathbf{a}\cdot\mathbf{a}) &= (N-L)(N+L+1). \end{aligned} \qquad \text{(A5)}$$

Multiplying on the right by $(\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger)$ in Eq. (A3), we get

$$\begin{aligned} a_j^{(-)}&(\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger)(2L+1) \\ &= -a_j^\dagger(\mathbf{a}\cdot\mathbf{a})(\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger) + a_j(N+L+1)(\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger) \\ &= -a_j^\dagger(N-L+2)(N+L+3) + a_j(\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger) \\ &\quad \times(N+L+3) \text{ [using Eqs. (A5) and (3)]} \\ &= a_j^{\dagger(-)}(2L+1)(N+L+3) \text{ [using Eqs. (A4)]}. \end{aligned}$$

Thus

$$\mathbf{a}^{(-)}(\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger) = \mathbf{a}^{\dagger(-)}(N+L+3),$$

and in a similar way we show that

$$\mathbf{a}^{\dagger(+)}(\mathbf{a}\cdot\mathbf{a}) = \mathbf{a}^{(+)}(N+L+1),$$

establishing Eqs. (38). □

Consider the product $\lambda_i\lambda_j$, with $\boldsymbol{\lambda}$ having the general form given in Eqs. (39):

$$\begin{aligned} \lambda_i\lambda_j &= a_i^{(-)}f(K,L)a_j^{(-)}f(K,L) \\ &= f(K,L+1)f(K,L+2)a_i^{(-)}a_j^{(-)}. \end{aligned} \qquad \text{(A6)}$$

From Eq. (A3) we have (recalling that $N = 2K + L$)

$$\begin{aligned} a_i^{(-)}&(2L+1)a_j^{(-)}(2L+1) \\ &= [-a_i^\dagger(\mathbf{a}\cdot\mathbf{a}) + a_i(2K+2L+1)]a_j^{(-)}(2L+1) \\ &= -a_i^\dagger(\mathbf{a}\cdot\mathbf{a})a_j^{(-)}(2L+1) + a_ia_j^{(-)}(2L+1) \\ &\quad \times(2K+2L-1) \quad \text{[using Eqs. (26) and (34)]} \\ &= -a_i^\dagger(\mathbf{a}\cdot\mathbf{a})[-a_j^\dagger(\mathbf{a}\cdot\mathbf{a}) + a_j(2K+2L+1)] \\ &\quad + a_i[-a_j^\dagger(\mathbf{a}\cdot\mathbf{a}) + a_j(2K+2L+1)] \\ &\quad \times(2K+2L-1) \quad \text{[using Eq. (A3) again]} \\ &= a_i^\dagger a_j^\dagger(\mathbf{a}\cdot\mathbf{a})^2 + a_ia_j(2K+2L+1)(2K+2L-1) \\ &\quad -(a_i^\dagger a_j + a_ia_j^\dagger)(\mathbf{a}\cdot\mathbf{a})(2K+2L-1) \\ &\qquad \text{[using Eqs. (3)]}. \end{aligned} \qquad \text{(A7)}$$

The right-hand side of this equation is symmetric in i and j. Thus

$$a_i^{(-)}(2L+1)a_j^{(-)}(2L+1) = a_j^{(-)}(2L+1)a_i^{(-)}(2L+1),$$

that is,

$$(2L+3)(2L+5)[a_i^{(-)}, a_j^{(-)}] = 0,$$

which implies that $[a_i^{(-)}, a_j^{(-)}] = 0$. It follows at once from Eq. (A6) that $[\lambda_i, \lambda_j] = 0$; and in a similar way we deduce that $[\lambda_i^\dagger, \lambda_j^\dagger] = 0$.

We see also from Eq. (A7) that

$$\begin{aligned} (2L+3)&(2L+5)\,\mathbf{a}^{(-)}\cdot\mathbf{a}^{(-)} \\ &= (\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger)(\mathbf{a}\cdot\mathbf{a})^2 + (\mathbf{a}\cdot\mathbf{a})(N+L+1)(N+L-1) \\ &\quad -(2N+3)(\mathbf{a}\cdot\mathbf{a})(N+L-1) \\ &= [(N-L)(N+L+1) + (N+L+3) \\ &\quad \times(N+L+1) - (2N+3)(N+L+1)](\mathbf{a}\cdot\mathbf{a}) \\ &= 0. \end{aligned}$$

Thus $\mathbf{a}^{(-)}\cdot\mathbf{a}^{(-)} = 0$, and it follows from Eq. (A6) that $\boldsymbol{\lambda}\cdot\boldsymbol{\lambda} = 0$. In a similar way, we deduce that $\boldsymbol{\lambda}^\dagger\cdot\boldsymbol{\lambda}^\dagger = 0$.

Equations (41) have now been confirmed. Their validity can be seen also from more general arguments. Since $\boldsymbol{\lambda}$ shifts the value of L down by one unit, the vector operator

$$\theta_i = \epsilon_{ijk}[\lambda_j, \lambda_k]$$

shifts the value of L down by two units. But a vector operator can only have components which commute with L, or shift its value up or down by one unit. Thus $\boldsymbol{\theta}$, and hence $[\lambda_i, \lambda_j]$ must vanish. Similarly, the scalar $\boldsymbol{\lambda}\cdot\boldsymbol{\lambda}$ shifts the value of L down by two units. But a scalar operator commutes with L; and therefore $\boldsymbol{\lambda}\cdot\boldsymbol{\lambda} = 0$. □

Equations (42) follow trivially from Eqs. (23) and (29), since $f(K,L)$ is a scalar operator, commuting with the l_{ki}; and Eqs. (43) follow at once from the fact that $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}^\dagger$ are vector operators by the manner of their construction. □

Consider now

$$\begin{aligned} \nu\nu^\dagger &= (\mathbf{a}\cdot\mathbf{a})g^2(K,L)(\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger) \\ &= (\mathbf{a}\cdot\mathbf{a})(\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger)g^2(K+1,L) \\ &= 2(K+1)(2K+2L+3)\,g^2(K+1,L) \\ &\qquad \text{[using Eqs. (15) and (A5)]}. \end{aligned}$$

Similarly we find

$$\begin{aligned} \nu^\dagger\nu &= (\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger)(\mathbf{a}\cdot\mathbf{a})g^2(K,L) \\ &= 2K(2K+2L+1)\,g^2(K,L). \end{aligned}$$

With g as in Eqs. (44), these two equations reduce to

$$\nu\nu^\dagger = K+1, \quad \nu^\dagger\nu = K,$$

establishing the first of Eqs. (45) and the first of Eqs. (46). □

Next consider the products

$$\begin{aligned} \lambda_i\nu &= a_i^{(-)}f(K,L)(\mathbf{a}\cdot\mathbf{a})g(K,L) \\ &= a_i^{(-)}(\mathbf{a}\cdot\mathbf{a})f(K-1,L)g(K,L), \\ \nu\lambda_i &= (\mathbf{a}\cdot\mathbf{a})g(K,L)a_i^{(-)}f(K,L) \\ &= (\mathbf{a}\cdot\mathbf{a})a_i^{(-)}g(K,L-1)f(K,L) \\ &= a_i^{(-)}(\mathbf{a}\cdot\mathbf{a})g(K,L-1)f(K,L) \\ &\qquad \text{[using Eqs. (3) and (20)]}. \end{aligned}$$

With f and g as in Eqs. (44) we have

$$g(K,L-1)f(K,L) = f(K-1,L)g(K,L)$$

and it then follows that $[\lambda_i, \nu] = 0$. Taking the Hermitian conjugate of this equation, we deduce that $[\lambda_i^\dagger, \nu^\dagger] = 0$, and the second set of Eqs. (45) is verified. □

Consider next the products

$$\lambda_i \nu^\dagger = a_i^{(-)} f(K,L)\,(\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger) g(K+1,L)$$
$$= a_i^{(-)}(\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger) f(K+1,L) g(K+1,L),$$
$$\nu^\dagger\lambda_i = (\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger) a_i^{(-)} g(K+1,L-1) f(K,L)$$
$$= [a_i^{(-)}(\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger) - 2a_i^{\dagger(-)}] g(K+1,L-1) \times f(K,L) \quad [\text{using Eqs. (3) and (20)}],$$
$$= [a_i^{(-)}(\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger) - 2a_i^{(-)}(\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger)(2K+2L+3)^{-1}] \times g(K+1,L-1) f(K,L) \quad [\text{using Eq. (38b)}].$$

Thus

$$[\lambda_i, \nu^\dagger] = a_i^{(-)}(\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger)(2K+2L+3)^{-1} \times [(2K+2L+3) g(K+1,L) f(K+1,L) - (2K+2L+1) g(K+1,L-1) f(K,L)]$$
$$= 0 \tag{A8}$$

because of the form of f and g in Eqs. (44). Taking the Hermitian conjugate of Eq. (A8) we deduce also that $[\lambda_i^\dagger, \nu] = 0$, so that the third set of Eqs. (45) is confirmed. □

Now consider the product

$$\lambda_i^\dagger\lambda_j = f(K,L) a_i^{\dagger(+)} a_j^{(-)} f(K,L)$$
$$= a_i^{\dagger(+)}(2L+1) a_j^{(-)} f^2(K,L)(2L-1)^{-1}$$
$$= [-a_i(\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger) + a_i^\dagger(N+L+3)] a_j^{(-)} \times f^2(K,L)(2L-1)^{-1} \quad [\text{using Eqs. (A4)}]$$
$$= [-a_i(\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger) a_j^{(-)}(2L+1) + a_i^\dagger a_j^{(-)}(2L+1) \times (N+L+1)] f^2(K,L)(4L^2-1)^{-1}$$
$$= \{-a_i(\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger)[-a_j^\dagger(\mathbf{a}\cdot\mathbf{a}) + a_j(N+L+1)] + a_i^\dagger[-a_j^\dagger(\mathbf{a}\cdot\mathbf{a}) + a_j(N+L+1)](N+L+1)\} \times f^2(K,L)(4L^2-1)^{-1} \quad [\text{using Eq. (A3)}]$$
$$= \{a_i a_j^\dagger(\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger)(\mathbf{a}\cdot\mathbf{a}) - a_i a_j(\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger)(N+L+1) + 2a_i a_j^\dagger(N+L+1) - a_i^\dagger a_j^\dagger(\mathbf{a}\cdot\mathbf{a})(N+L+1) + a_i^\dagger a_j(N+L+1)^2\} f^2(K,L)(4L^2-1)^{-1}$$
$$= \{2a_i a_j^\dagger(K+1) + a_i^\dagger a_j(2K+2L+1) - a_i a_j(\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger) - a_i^\dagger a_j^\dagger(\mathbf{a}\cdot\mathbf{a})\} \times f^2(K,L)(2K+2L+1)(4L^2-1)^{-1} \tag{A9}$$

[using Eqs. (A5) and (15)].

In a similar way, we show that

$$\lambda_i\lambda_j^\dagger = \{2a_i^\dagger a_j K + a_i a_j^\dagger(2K+2L+3) - a_i a_j(\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger) - a_i^\dagger a_j^\dagger(\mathbf{a}\cdot\mathbf{a})\} f^2(K,L+1)(2K+2L+3) \times (2L+1)^{-1}(2L+3)^{-1}. \tag{A10}$$

It follows from Eqs. (A9) and (A5) that

$$\boldsymbol{\lambda}^\dagger\cdot\boldsymbol{\lambda} = \{2(2K+L+3)(K+1) + (2K+L) \times (2K+2L+1) - 2(K+1)(2K+2L+3) - 2K(2K+2L+1)\} \times f^2(K,L)(2K+2L+1)(4L^2-1)^{-1}$$
$$= L f^2(K,L)(2K+2L+1)(2L+1)^{-1}.$$

Thus $\boldsymbol{\lambda}^\dagger\cdot\boldsymbol{\lambda} = L$ for the choice of f in Eqs. (44), verifying the second of Eqs. (46). □

It can also be seen from Eqs. (A9) that

$$\lambda_i^\dagger\lambda_j - \lambda_j^\dagger\lambda_i = \{2(a_i a_j^\dagger - a_j a_i^\dagger)(K+1) + (a_i^\dagger a_j - a_j^\dagger a_i)(2K+2L+1)\} \times f^2(K,L)(2K+2L+1) \times (4L^2-1)^{-1}$$
$$= i l_{ij} f^2(K,L)(2K+2L+1)(2L+1)^{-1},$$

so that, again with f as in Eqs. (44), we confirm the last of Eqs. (46). □

From Eqs. (A9) and (A10), we see that with this choice of f,

$$(2L+1)[\lambda_i, \lambda_j^\dagger] + 2\lambda_i^\dagger\lambda_j$$
$$= (2L+1)\lambda_i\lambda_j^\dagger - (2L-1)\lambda_j^\dagger\lambda_i + 2(\lambda_i^\dagger\lambda_j - \lambda_j^\dagger\lambda_i)$$
$$= 2a_i^\dagger a_j K + a_i a_j^\dagger(2K+2L+3) - a_i a_j(\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger) - a_i^\dagger a_j^\dagger(\mathbf{a}\cdot\mathbf{a}) - 2a_j a_i^\dagger(K+1) - a_j^\dagger a_i(2K+2L+1) + a_j a_i(\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger) + a_j^\dagger a_i^\dagger(\mathbf{a}\cdot\mathbf{a}) + 2i l_{ij}$$
$$= -2a_j a_i^\dagger - 2K\delta_{ij} + 2a_i a_j^\dagger + \delta_{ij}(2K+2L+1) + 2i\, l_{ij}$$
$$= (2L+1)\delta_{ij},$$

thus confirming the last of Eqs. (45). □

APPENDIX B

In order to derive Eq. (57) from Eq. (55), it is necessary to calculate the effect of $\lambda_{-\epsilon}$ on the vector

$$(\lambda_\epsilon^\dagger)^{|m|}(\lambda_3^\dagger)^{l-|m|}|0\rangle.$$

We note that

$$\lambda_{-\epsilon}\lambda_\epsilon^\dagger = (\lambda_1 - i\epsilon\lambda_2)(\lambda_1^\dagger + i\epsilon\lambda_2^\dagger)$$
$$= \boldsymbol{\lambda}\cdot\boldsymbol{\lambda}^\dagger - \lambda_3\lambda_3^\dagger + i\epsilon(\lambda_1\lambda_2^\dagger - \lambda_2\lambda_1^\dagger).$$

It follows from Eqs. (45) and (46) that

$$\boldsymbol{\lambda}\cdot\boldsymbol{\lambda}^\dagger = (2L+3)(L+1)(2L+1)^{-1},$$
$$\lambda_1\lambda_2^\dagger - \lambda_2\lambda_1^\dagger = -i(2L+3)(2L+1)^{-1} l_{12},$$

so that

$$\lambda_{-\epsilon}\lambda_\epsilon^\dagger = (L+1+\epsilon l_{12})(2L+3)(2L+1)^{-1} - \lambda_3\lambda_3^\dagger. \tag{B1}$$

We now see that if $|m| \geqslant 1$ (which requires $l \geqslant 1$),

$$\lambda_{-\epsilon}(\lambda_\epsilon^\dagger)^{|m|}(\lambda_3^\dagger)^{l-|m|}|0\rangle$$
$$= (l+|m|-1)(2l+1)(2l-1)^{-1} \times (\lambda_\epsilon^\dagger)^{|m|-1}(\lambda_\epsilon^\dagger)^{l-|m|}|0\rangle - \lambda_3(\lambda_\epsilon^\dagger)^{|m|-1} \times (\lambda_3^\dagger)^{l-|m|+1}|0\rangle. \tag{B2}$$

Working from Eq. (56), we deduce that if r, s, and t are nonnegative integers, then

$$\lambda_3(\lambda_1^\dagger)^r(\lambda_2^\dagger)^s(\lambda_3^\dagger)^t|0\rangle$$
$$= [2(r+s+t)-1]^{-1}\{t(t+2r+2s)(\lambda_1^\dagger)^r(\lambda_2^\dagger)^s \times (\lambda_3^\dagger)^{t-1} - r(r-1)(\lambda_1^\dagger)^{r-2}(\lambda_2^\dagger)^s(\lambda_3^\dagger)^{t+1} - s(s-1)(\lambda_1)^r(\lambda_2)^{s-2}(\lambda_3^\dagger)^{t+1}\}|0\rangle. \tag{B3}$$

Now

$$(\lambda_\epsilon^\dagger)^{|m|}(\lambda_3^\dagger)^{l-|m|}|0\rangle = \sum_{r=0}^{|m|} \frac{|m|!}{r!(|m|-r)!} \times (\lambda_1^\dagger)^r(i\epsilon\lambda_2^\dagger)^{|m|-r}(\lambda_3^\dagger)^{l-|m|}|0\rangle, \tag{B4}$$

and so

$$\lambda_3(\lambda_\epsilon^\dagger)^{|m|}(\lambda_3^\dagger)^{l-|m|}|0\rangle$$

$$= (2l-1)^{-1}\sum_{r=0}^{|m|}\frac{|m|!}{r!(|m|-r)!}\{(l^2-m^2)(\lambda_1^\dagger)^r \times(i\epsilon\lambda_2^\dagger)^{|m|-r}(\lambda_3^\dagger)^{l-|m|-1} - r(r-1)(\lambda_1^\dagger)^{r-2} \times(i\epsilon\lambda_2^\dagger)^{|m|-r}(\lambda_3^\dagger)^{l-|m|+1} + (|m|-r) \times(|m|-r-1)(\lambda_1^\dagger)^r(i\epsilon\lambda_2^\dagger)^{|m|-r-2} \times(\lambda_3^\dagger)^{l-|m|+1}\}|0\rangle$$

$$= \{(l^2-m^2)(2l-1)^{-1}(\lambda_\epsilon^\dagger)^{|m|}(\lambda_3^\dagger)^{l-|m|-1} - (2l-1)^{-1}\sum_{r=2}^{|m|}\frac{|m|!}{(r-2)!(|m|-r)!}(\lambda_1^\dagger)^{r-2} \times(i\epsilon\lambda_2^\dagger)^{|m|-r}(\lambda_3^\dagger)^{l-|m|+1} + (2l-1)^{-1}\sum_{r=0}^{|m|-2}\frac{|m|!}{r!(|m|-r-2)!} \times(\lambda_1^\dagger)^r(i\epsilon\lambda_2^\dagger)^{|m|-r-2}(\lambda_3^\dagger)^{l-|m|+1}\}|0\rangle$$

$$= (l^2-m^2)(2l-1)^{-1}(\lambda_\epsilon^\dagger)^{|m|}(\lambda_3^\dagger)^{l-|m|-1}|0\rangle, \quad \text{(B5)}$$

since the two sums cancel.

Combining this result with Eq. (B2), we see that if $|m|\geqslant 1$,

$$\lambda_{-\epsilon}(\lambda_\epsilon^\dagger)^{|m|}(\lambda_3^\dagger)^{l-|m|}|0\rangle = [(l+|m|-1)(2l+1)(2l-1)^{-1} - (l-|m|+1) \times(l+|m|+1)(2l-1)^{-1}](\lambda_\epsilon^\dagger)^{|m|-1}(\lambda_3^\dagger)^{l-|m|}|0\rangle = (l+|m|)(l+|m|-1)(2l-1)^{-1} \times(\lambda_\epsilon^\dagger)^{|m|-1}(\lambda_3^\dagger)^{l-|m|}|0\rangle. \quad \text{(B6)}$$

It follows that, if $|m|\geqslant 1$,

$$\langle 0|(\lambda_3)^{l-|m|}(\lambda_{-\epsilon})^{|m|}(\lambda_\epsilon^\dagger)^{|m|}(\lambda_3^\dagger)^{l-|m|}|0\rangle = (l+|m|)(l+|m|-1)(2l-1)^{-1}\langle 0|(\lambda_3)^{l-|m|} \times(\lambda_{-\epsilon})^{|m|-1}(\lambda_\epsilon^\dagger)^{|m|-1}(\lambda_3^\dagger)^{l-|m|}|0\rangle = \left[\prod_{i=0}^{|m|-1}\left(\frac{(l-2i+|m|)(l-2i+|m|-1)}{(2l-2i-1)}\right)\right] \times\langle 0|(\lambda_3)^{l-|m|}(\lambda_3^\dagger)^{l-|m|}|0\rangle. \quad \text{(B7)}$$

We see from Eq. (B3) that

$$\lambda_3(\lambda_3^\dagger)^{l-|m|}|0\rangle = [2(l-|m|)-1]^{-1}(l-|m|)^2 \times(\lambda_3^\dagger)^{l-|m|-1}|0\rangle,$$

so that

$$\langle 0|(\lambda_3)^{l-|m|}(\lambda_3^\dagger)^{l-|m|}|0\rangle = [2(l-|m|-1)]^{-1}(l-|m|)^2 \times\langle 0|(\lambda_3)^{l-|m|-1}(\lambda_3^\dagger)^{l-|m|-1}|0\rangle = \left[\prod_{j=0}^{l-|m|-1}\left(\frac{(l-|m|-j)^2}{(2l-2|m|-2j-1)}\right)\right]\langle 0|0\rangle. \quad \text{(B8)}$$

We now combine Eqs. (B7) and (B8) to obtain (for $|m|\geqslant 1$)

$$\langle 0|(\lambda_3)^{l-|m|}(\lambda_{-\epsilon})^{|m|}(\lambda_\epsilon^\dagger)^{|m|}(\lambda_3^\dagger)^{l-|m|}|0\rangle = \frac{2^l l!(l+|m|)!(l-|m|)!}{(2l)!}. \quad \text{(B9)}$$

It is seen from Eq. (B8) that this result is valid also if $m=0$. Combining Eqs. (B9) and (55), we obtain Eq. (57). □

The first two of Eqs. (59) are well known in the boson calculus, and require no derivation here. Consider

$$\lambda_3^\dagger|k\,l\,m\rangle = \lambda_3^\dagger c_{klm}(v^\dagger)^k(\lambda_\epsilon^\dagger)^{|m|}(\lambda_3^\dagger)^{l-|m|}|0\rangle = \frac{c_{klm}}{c_{kl+1m}}|k\,l+1\,m\rangle = \left(\frac{(l+1-m)(l+1+m)}{(2l+1)}\right)^{1/2}|k\,l+1\,m\rangle,$$

verifying the third of Eqs. (59). □

Next consider, for $m\neq 0$,

$$\lambda_\epsilon^\dagger|k\,l\,m\rangle = \lambda_\epsilon^\dagger c_{klm}(v^\dagger)^k(\lambda_\epsilon^\dagger)^{|m|}(\lambda_3^\dagger)^{l-|m|}|0\rangle = \frac{c_{klm}}{c_{kl+1m+\epsilon}}|k\;l+1\;m+\epsilon\rangle = (-\epsilon)^\epsilon\left(\frac{(l+|m|+2)(l+|m|+1)}{(2l+1)}\right)^{1/2} \times|k\;l+1\;m+\epsilon\rangle.$$

From this equation we have

$$\lambda_+^\dagger|k\,l\,m\rangle = -\left(\frac{(l+m+2)(l+m+1)}{(2l+1)}\right)^{1/2} \times|k\;l+1\;m+1\rangle, \text{ for } m>0,$$

$$\lambda_-^\dagger|k\,l\,m\rangle = +\left[\frac{(l-m+2)(l-m+1)}{(2l+1)}\right]^{1/2} \times|k\;l+1\;m-1\rangle, \text{ for } m<0. \quad \text{(B10)}$$

Now consider, also for $m\neq 0$,

$$\lambda_{-\epsilon}^\dagger|k\,l\,m\rangle = \lambda_{-\epsilon}^\dagger c_{klm}(v^\dagger)^k(\lambda_\epsilon^\dagger)^{|m|}(\lambda_3^\dagger)^{l-|m|}|0\rangle = (\lambda_{-\epsilon}^\dagger\lambda_\epsilon^\dagger)c_{klm}(v^\dagger)^k(\lambda_\epsilon^\dagger)^{|m|-1}(\lambda_3^\dagger)^{l-|m|}|0\rangle = -(\lambda_3^\dagger)^2 c_{klm}(v^\dagger)^k(\lambda_\epsilon^\dagger)^{|m|-1}(\lambda_3^\dagger)^{l-|m|}|0\rangle$$

[using the last of Eqs. (41)]

$$= -\frac{c_{klm}}{c_{kl+1m-\epsilon}}|k\;l+1\;m-\epsilon\rangle = -(-\epsilon)^\epsilon\left(\frac{(l-|m|+2)(l-|m|+1)}{(2l+1)}\right)^{1/2} \times|k\;l+1\;m-\epsilon\rangle.$$

From this equation we have

$$\lambda_+^\dagger|k\,l\,m\rangle = -\left(\frac{(l+m+2)(l+m+1)}{(2l+1)}\right)^{1/2} \times|k\;l+1\;m+1\rangle, \text{ for } m<0,$$

$$\lambda_-^\dagger|k\,l\,m\rangle = \left(\frac{(l-m+2)(l-m+1)}{(2l+1)}\right)^{1/2} \times|k\;l+1\;m-1\rangle, \text{ for } m>0. \quad \text{(B11)}$$

Next consider (with $\epsilon=\pm 1$)

$$\lambda_\epsilon^\dagger|k\,l\,0\rangle = \lambda_\epsilon^\dagger c_{kl0}(v^\dagger)^k(\lambda_3^\dagger)^l|0\rangle = \frac{c_{kl0}}{c_{kl+1\epsilon}}|k\;l+1\;\epsilon\rangle = (-\epsilon)^\epsilon\left(\frac{(l+2)(l+1)}{(2l+1)}\right)^{1/2}|k\;l+1\;\epsilon\rangle. \quad \text{(B12)}$$

Combining Eqs. (B10), (B11) and (B12), we arrive at the fourth set of Eqs. (59). □

From Eq. (B5) we have

$$\lambda_3|k\,l\,m\rangle = c_{klm}\frac{(l^2-m^2)}{(2l-1)}(\nu^\dagger)^k(\lambda_\epsilon^\dagger)^{|m|}(\lambda_3^\dagger)^{l-|m|-1}|0\rangle$$
$$= \frac{(l^2-m^2)}{(2l-1)}\frac{c_{klm}}{c_{kl-1m}}|k\;l-1\;\;m\rangle$$
$$= \left(\frac{(l-m)(l+m)}{(2l-1)}\right)^{1/2}|k\;l-1\;\;m\rangle\,,$$

as in the fifth of Eqs. (59). □

Using Eq. (B6), we see that, for $|m|\geqslant 1$,

$$\lambda_{-\epsilon}|k\,l\,m\rangle = \lambda_{-\epsilon}c_{klm}(\nu^\dagger)^k(\lambda_\epsilon^\dagger)^{|m|}(\lambda_3^\dagger)^{l-|m|}|0\rangle$$
$$= \frac{(l+|m|)(l+|m|-1)}{(2l-1)}\frac{c_{klm}}{c_{kl-1m-\epsilon}}\times|k\;l-1\;\;m-\epsilon\rangle$$
$$= (-\epsilon)^\epsilon\left(\frac{(l+|m|)(l+|m|-1)}{(2l-1)}\right)^{1/2}\times|k\;l-1\;\;m-\epsilon\rangle\,.$$

From this equation we have

$$\lambda_-|k\,l\,m\rangle = -\left(\frac{(l+m)(l+m-1)}{(2l-1)}\right)^{1/2}\times|k\;l-1\;\;m-1\rangle\,,\quad\text{for } m>0\,,$$
$$\lambda_+|k\,l\,m\rangle = \left(\frac{(l-m)(l-m-1)}{(2l-1)}\right)^{1/2}\times|k\;l-1\;\;m+1\rangle\,,\quad\text{for } m<0\,. \tag{B13}$$

Now consider the vector

$$\lambda_\epsilon|k\,l\,m\rangle = \lambda_\epsilon c_{klm}(\nu^\dagger)^k(\lambda_\epsilon^\dagger)^{|m|}(\lambda_3^\dagger)^{l-|m|}|0\rangle\,,$$

on which L has the value $l-1$, and L_3 the value $\hbar(m+\epsilon)$. Since $\hbar^2L(L+1)=(L_1)^2+(L_2)^2+(L_3)^2$, the value of L_3 cannot be greater in modulus than that of $\hbar L$. Therefore, this vector vanishes unless $l-1\geqslant|m+\epsilon|=|m|+1$; i.e., unless $l\geqslant|m|+2$. Supposing this inequality is satisfied, we write

$$\lambda_\epsilon|k\,l\,m\rangle = c_{klm}\lambda_\epsilon(\lambda_3^\dagger)^2(\nu^\dagger)^k(\lambda_\epsilon^\dagger)^{|m|}(\lambda_3^\dagger)^{l-|m|-2}|0\rangle$$
$$= -c_{klm}\lambda_\epsilon\lambda_{-\epsilon}^\dagger(\nu^\dagger)^k(\lambda_\epsilon^\dagger)^{|m|+1}(\lambda_3^\dagger)^{l-|m|-2}|0\rangle$$
$$[\text{using the last of Eqs. (41)}]\,.$$

Now using Eq. (B1) (with ϵ replaced by $-\epsilon$), we have

$$\lambda_\epsilon|k\,l\,m\rangle = -c_{klm}\{(L+1-\epsilon l_{12})(2L+3)\times(2L+1)^{-1}(\nu^\dagger)^k(\lambda_\epsilon^\dagger)^{|m|+1}(\lambda_3^\dagger)^{l-|m|-2}$$
$$-\lambda_3(\nu^\dagger)^k(\lambda_\epsilon^\dagger)^{|m|+1}(\lambda_3^\dagger)^{l-|m|-1}\}|0\rangle$$
$$= -c_{klm}\{(l-|m|-1)(2l+1)(2l-1)^{-1}-[l^2-(|m|+1)^2](2l-1)^{-1}\}$$
$$\times(\nu^\dagger)^k(\lambda_\epsilon^\dagger)^{|m|+1}(\lambda_3^\dagger)^{l-|m|-2}|0\rangle$$
$$[\text{using Eq. (B5)}]$$
$$= -\frac{(l-|m|)(l-|m|-1)}{(2l-1)}\frac{c_{klm}}{c_{kl-1m+\epsilon}}\times|k\;l-1\;\;m+\epsilon\rangle$$
$$= -(-\epsilon)^\epsilon\left(\frac{(l-|m|)(l-|m|-1)}{(2l-1)}\right)^{1/2}\times|k\;l-1\;\;m+\epsilon\rangle\,.$$

From this equation we see that

$$\lambda_+|k\,l\,m\rangle = \left(\frac{(l-m)(l-m-1)}{(2l-1)}\right)^{1/2}\times|k\;l-1\;\;m+1\rangle\,,\quad\text{for } m>0\,,$$
$$\lambda_-|k\,l\,m\rangle = -\left(\frac{(l+m)(l+m-1)}{(2l-1)}\right)^{1/2}\times|k\;l-1\;\;m-1\rangle\,,\quad\text{for } m<0\,. \tag{B14}$$

It is easily seen from Eq. (56) that

$$\lambda_\pm(\lambda_3^\dagger)^l|0\rangle = -\frac{l(l-1)}{(2l-1)}\lambda_\pm^\dagger(\lambda_3^\dagger)^{l-2}|0\rangle\,,$$

so that

$$\lambda_\pm|k\,l\,0\rangle = \lambda_\pm c_{kl0}(\nu^\dagger)^k(\lambda_3^\dagger)^l|0\rangle$$
$$= -\frac{l(l-l)}{(2l-1)}\frac{c_{kl0}}{c_{kl-1\pm1}}|k\;l-1\pm1\rangle\,.$$

From this equation we see that

$$\lambda_\pm|k\,l\,0\rangle = \pm\left(\frac{l(l-1)}{(2l-1)}\right)^{1/2}|k\;l-1\pm1\rangle\,, \tag{B15}$$

and combining Eqs. (B13), (B14), and (B15) we obtain the last of Eqs. (59). □

[1]Philip Stehle, *Quantum Mechanics* (Holden-Day, San Francisco, 1966).
[2]A.S. Davydov, *Quantum Mechanics* (Pergamon, Oxford, 1965).
[3]P. Kramer and M. Moshinsky, *Group Theory and its Applications*, edited by E. M. Loebl, Vol. I (Academic, New York, 1968).
[4]Harold V. McIntosh, *ibid.*, Vol. II (Academic, New York, 1971).
[5]M.E. Rose in *Spectroscopic and Group Theoretical Methods in Physics* edited by F. Bloch *et al.* (North-Holland, Amsterdam, 1968).
[6]H.S. Green, *Matrix Mechanics* (Noordhoff, Groningen, 1965).
[7]L. Infeld and T.E. Hull, Rev. Mod. Phys. **23**, 21 (1951); Willard Miller, Jr., *Lie Theory and Special Functions* (Academic, New York, 1968).
[8]P.W. Atkins and J.C. Dobson, Proc. R. Soc. London Ser. A **321**, 321 (1971).
[9]R. Delbourgo, J. Phys. A: Math. Gen. **10**, 1837 (1977).
[10]A.J. Bracken and H.S. Green, J. Math. Phys. **12**, 2099 (1971).
[11]A.J. Bracken, Ph.D. thesis, The University of Adelaide, 1970 (unpublished).
[12]R.C. Hwa and J. Nuyts, Phys. Rev. **145**, 1188 (1966).
[13]M.A. Lohe and C.A. Hurst, J. Math. Phys. **12**, 1882 (1971).
[14]Milton Abramowitz and Irene A. Stegun (Editors), *Handbook of Mathematical Functions* (Dover, New York, 1965).
[15]A.R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton U.P., Princeton, N.J., 1960).
[16]A.O. Barut and C. Fronsdal, Proc. R. Soc. London Ser. A **287**, 532 (1965).
[17]M.A. Naimark, *Linear Representations of the Lorentz Group* (Pergamon, Oxford, 1964).
[18]A. Böhm, Phys. Rev. **175**, 1767 (1968).