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THE ITALIAN MARKET FOR ‘PREMIUM’ CONTRACTS
An Application of Option Pricing Theory

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SUMMARY

Despite their growing importance on Italian stock exchanges, premium contracts have not received very much attention in analytical studies. This paper starts with a description of the working of the market for premium contracts with the aim of highlighting its institutional peculiarities compared with foreign markets for stock options and then develops a formula for the determination of premia based on arbitrage methods similar to those used in option pricing theory. The empirical test of the correspondence between actual market premia and the theoretical values obtained provides some indication of the scope for arbitrage between premium and forward contracts, and hence of the efficiency of Italian stock exchange markets.

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1. INTRODUCTION

The market for premium contracts ('contratti a premio') is the Italian equivalent of an options market. It handles special contracts known as 'premium' or 'conditional forward' contracts, which, against payment of a 'premium' give one of the counterparts a special right of choice (option) regarding the execution of an order to buy or sell securities.

The right varies with the type of contract involved. A 'dont' contract (equivalent to a call option) gives its owner the right to buy a fixed number of shares of a specified common stock at a fixed price ('prezzo base') by a given date ('risposta premi' day). Conversely, a 'put' contract (equivalent to a put option) gives its owner the right to sell securities. In addition to dont and put contracts, which are 'single premium contracts, there are combinations known as 'double premium' contracts: 'stellage' contracts give premium payers the right to deliver or take delivery of a given quantity of securities at the striking price, 'strip' contracts entitle them to take delivery of a given quantity of securities or to deliver twice that amount, while 'strap' contracts entitle them to take delivery of the agreed quantity or to deliver half that amount.

If the rights acquired in this way are exercised, premium contracts turn into ordinary forward contracts and are liquidated in accordance with the normal procedure for such contracts, based on the Stock Exchange calendar. The options, in other words, are written on forward contracts, rather than on spot contracts. Furthermore, in Italy there is no exchange of money when options are bought since premia are paid on settlement day.

Despite their growing importance on Italian stock exchanges,[1] premium contracts have so far received little analytical attention. In particular, no serious attempt has yet been made to verify the applicability of option pricing theory to the Italian premium market, even though the structural analogy between premium contracts and options suggests that it could be usefully applied to the determination of premia.

The purpose of this article is to develop a formula for the valuation of premium contracts and the determination of equilibrium premia. The paper is arranged as follows: section 2 provides a preliminary description of the working of the premium market; section 3 starts the analysis of the determination of premia by deriving the no-arbitrage conditions between the premia of the different contracts on a given security. These equilibrium relations make it possible to reduce the problem of determining equilibrium premia to that of determining the premia for dont contracts. Section 4 shows how option pricing theory can be used to value these contracts, while the formula obtained is tested in section 5.

2. THE ITALIAN MARKET FOR PREMIUM CONTRACTS: SIZE AND OPERATIONAL ASPECTS

Even though premium contracts have always been written on Italian stock exchanges, the market for them used to be considered of marginal importance with just a few specialized operators and was left completely unregulated.

It was only after July 1968, with the concentration of premium contract trading on the Milan Stock Exchange at the 4th *Corbeille* (assigned to handle securities with a small turnover) and the subsequent institution of a special pit for premium contracts that the premium market began to provide the guarantees that derive from the existence of an official market with continuous trading, publication of prices and specialized intermediaries.

Nearly all premium contracts are written on the Milan Stock Exchange, which is the only one so far to have instituted a special pit. Though premium contracts can be written on the other Italian stock exchanges, this is rare and they are not usually recorded in the list of official quotations.

[1] In 1987, 108,997 premium contracts were written. The value of the shares involved (11,020 billion lire) was equal to 26% of the total turnover in shares on the Milan Stock Exchange (41,967 billion).

2.1 Size aspects

The development of premium contract trading on the Milan Stock Exchange is shown in Table 1.[2] The pronounced variability of the turnover is immediately evident.

The value of the securities involved in premium contracts increased by a factor of 15 in the 20 months from July 1984 to March 1986, halved in the last few months of that year as a result of the fall in stock prices and the resolution adopted by the Consob (the Italian SEC) on 2 April 1986,[3] and then remained basically stable during 1987.

In the three years 1985 - 1987 the share of turnover in premium contracts in the total turnover of the Milan Stock Exchange (in terms of the value of the securities involved) fluctuated on a monthly basis between 8% and 50%, and averaged 19.88%.

More than 95% of the premium contracts were donts, about 3% stellages and the remaining 2% consisted of puts, strips, and straps, which only began to be used to a significant extent after 1980.

2.2 Organization of the market

Method of trading

Premium contracts are traded continuously, i.e. without the determination of official opening and closing prices, using a two-sided open outcry system, the same method that is adopted for forward contracts concluded during the trading session (the 'durante') outside the official closing dealings, for which a call system is used instead. Premium contracts are recorded immediately, together with the time and date of each transaction, and are shown on the board of the Milan Stock Exchange.

Since there are no opening or closing premia, the list of official prices shows only the minimum and maximum premium recorded during the day for each contract traded.[4]

Striking prices

Until 15 April 1986 premium contracts could be concluded with a striking price equal to the price of the underlying security at the opening, in the 'durante' or at the closing. The last of these methods was by far the most common, which led to a remarkable anomaly: when premium contracts were written, the striking price was often still not known, with the result that premium trading was highly uncertain.[5]

This method of determining striking prices has been modified twice by the Consob with the aim of eliminating this anomaly.

Initially, Consob resolution no. 2077 of 2 April 1986 provided that from 15 April 1986 the striking prices of premium contracts was to be referred exclusively to the previous day's official closing price of the underlying security or, in the absence of this, to the latest official closing price.

Under Consob resolution no. 2265 of 26 June 1986 (and subsequent amendments) a system similar to that used for options was introduced, starting on 17 July 1986. Seven price ranges were defined for securities and, for each range, a 'reference value' and the permitted variations of the striking price were established, as shown in Table 2.

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[2] These figures, provided by the Milan Stock Exchange EDP Centre, underestimate the actual volume of premium contracts since off-market business (especially among banks) is not included.

[3] The resolution banned sales of dont contracts without ownership of the underlying securities.

[4] There is nonetheless a serious shortcoming in the official information. Even after the Consob resolution of 2 April 1986, which established the new system of eligible striking prices, the list of official prices on the Milan Stock Exchange continues to show the minimum and maximum premia recorded for each security independently of the striking price used. A breakdown of these prices by striking price is provided in the provisional list of quotations, which is based on the information stockbrokers transmit to the Stock Exchange EDP Centre (*CED Borsa*) and published in the financial pages of some daily newspapers. These figures are incomplete, however, since they exclude all the bargains for which there is any discrepancy between the buying and selling data.

[5] 1t should also be noted that this uncertainty with regard to the striking price is another factor precluding the application of Black and Scholes (1973) 'standard' formula to the data for the period prior to April 1986. For an extension of the formula to the case of stochastic striking prices, cf. Fischer (1978).

Table 1 Forward and premium contracts on the Milan Stock Exchange (July 1984 - December 1987; Lit mn).

Month	*Year*	*Value of premia*	*Value of shares*		*Proportion of premium contracts*
			Premium contracts (a)	*Forward contracts (b)*	*(a)/(b) * 100*
July	84	3,346	83,418	358,536	23.27
August	84	4,181	105,110	609,888	17.23
September	84	4,793	137,820	509,857	27.03
October	84	4,362	123,842	580,017	21.35
November	84	4,747	140,687	584,958	24.05
December	84	4,340	126,351	625,077	20.21
January	85	10,194	182,780	1,624,935	11.25
February	85	19,902	374,114	2,042,271	t8.32
March	85	17,356	304,404	1,383,640	22.00
April	85	10,891	257,114	875,558	29.37
May	85	11,184	270,191	1,664,933	16.23
June	85	17,667	392,520	2,038,034	19.26
July	85	26,264	451.824	2,355,842	19.18
August	85	18,093	345,913	1,824,088	18.96
September	85	22,101	477,731	2,911,620	16.41
October	85	31,992	625,721	3,314,411	18.88
November	85	29,320	593,432	3,228,282	18.38
December	85	48,144	931,814	3,051,273	30.54
January	86	34,470	666,138	3,996,373	16.67
February	86	43,832	891,448	5,280,830	16.88
March	86	111,382	1,503,198	7,412,156	20.28
April	86	139,806	1,353,245	7,200,615	18.79
May	86	79,834	827,277	9,577,351	8.64
June	86	76,156	769,439	5,459,389	14.09
July	86	24,014	320,378	3,858,106	8.30
August	86	35,930	609,623	4,386,229	13.90
September	86	42,992	803,363	5,717,704	14.05
October	86	38,097	848,283	6,677,413	12.70
November	86	49,141	1,321,725	4,172,488	31.68
December	86	24,555	688,137	2,922,190	23.55
January	87	21,126	611,985	3,668,380	16.68
February	87	24,083	860,872	3,065,563	28.08
March	87	18,643	684,379	3,906,179	17.52
April	87	38,392	1,255,425	5,158,249	24.34
May	87	40,790	1,304,265	3,687,723	35.37
June	87	36,852	1,295,981	2,592,848	49.98
July	87	19,697	754,382	2,512,469	30.03
August	87	15,541	592,031	2,947,450	20.09
September	87	19,854	766,956	3,459,319	22.17
October	87	40,164	1,263,571	5,701,429	22.16
November	87	41,496	1,124,380	3,331,885	33.75
December	87	20,348	505,340	1,935,505	26.11

[a] Source: The Stockbrokers' Council of the Milan Stock Exchange, 'II comportamento in Borsa dei valori azionari', Milan, 1988.

Table 2 Striking prices (lire).

Price range				*Reference value*	*Striking price variation*
	up	to	500	0	10
from	501	to	1,000	500	25
from	1,001	to	2,500	1,000	50
from	2,501	to	7,500	2,500	100
from	7,501	to	15,000	7,500	250
from	15,001	to	50,000	15,000	500
	more than		50,001	50,000	1,000

The parties to a premium contract choose the striking price among those obtained by applying the permitted variations to the 'reference value' corresponding to the price range. The only condition imposed by the Consob is that the striking price should be 'close to the market price of the underlying security'. For example, assuming that the market value of a security is Lit. 11,700, the striking price will probably be between Lit. 11,000 and Lit. 12,500, with steps of Lit. 250.[6]

Maturities

The stock exchange calendar establishes the last day in each monthly account[7] (called 'risposta premi' day) on which the party with the right to buy or sell shares can do so and inform the counterpart. 'Risposta premi' day is normally in the middle of each calendar month. It precedes 'riporti' day (the last day for 'fine corrente' trades) in order to allow counterparts with short positions to close them.

There are no legal restrictions on Italian stock exchanges to the maturity of a premium contract, which can therefore be stipulated for the 'risposta premi' day of the same monthly account or for any subsequent one. As a general rule, however, premium contracts are stipulated to fall due within two accounts after the current one (i.e. with a maximum maturity of three months). The most commonly used maturity is 'fine prossimo' (the 'risposta premi' day of the following account; 60 - 30 days), while little use is made of 'fine corrente' (30 - 0 days), normally adopted during the first two weeks of an account), or 'fine secondo mese' (90-60 days), normally adopted in the last two weeks of an account.

The limitation of premium contract maturities, which used to be explained partly on tax grounds (the rate at which tax was levied on stock exchange contracts doubled for maturities of more than 135 days) is primarily due to the lack of liquidity in the market, which is an obstacle to the early closing of positions, and to the difficulty of coordinating premium transactions with forward contracts for maturities beyond the end of the current account ('fine corrente').

[6] If the market price of a security is close to the corresponding 'reference value', the striking price can also be the 'reference value' itself or a value obtained by subtracting multiples of the permitted variation. For example, the striking price of a security whose market price is Lit. 1,030 can be Lit. 950, Lit. 1,000, Lit. 1,050 or Lit. 1,100.

[7] The Italian stock market is a forward market on which the payment for and delivery of shares traded in each monthly account is delayed until settlement day, which is fixed by the stock exchange calendar and normally coincides with the last trading day of the calendar month. The account ('mese borsistico'), during which are traded forward contracts with the same settlement day ('contratti per fine corrente'), lasts from 'compensi' day to 'riporti' day, i.e. from the middle of one calendar month to the middle of the next. For example, the contracts settled at the end of March are those made between the middle of February and the middle of March. The maturities of contracts made at the beginning and end of an account thus differ by around 30 days, with the first normally being settled after 45 days and the latter after 15 days.

2.3 Risposta premi

A Consob resolution of 13 May 1987 requires the owner of a premium contract to exercise his option, by declaring if and how he intends to execute the contract, by 10:00 on 'risposta premi' day.[8]

In operational practice, however, an explicit exercising of options is only found if on 'risposta premi' day the price of the underlying security is close to the striking price. Otherwise the 'premium contract itself responds'. This procedure is based on Article 39 of the Milan Stock Exchange's Rulebook, which lays down that if the owner of a premium contract does not declare his intention the writer shall proceed in the best interests of the counterpart and notify the Stockbroker's Council accordingly.[9]

Article 38 of the Milan Stock Exchange Rulebook confirms the right of the buyer of a premium contract to exercise it before 'risposta premi' day, without this changing the maturity of the forward contract resulting from the exercise of the premium contract.

As a rule no advantage is gained by exercising a premium contract in advance of the deadline fixed by the stock exchange calendar. In practice, such action is only found in connection with the distribution of cash and stock dividends and the issue of subscription rights. When an option to buy is exercised at least one day before the stock starts trading ex rights, the contract becomes a forward contract and the buyer is entitled to the rights; otherwise, the striking price the buyer has to pay is reduced by an amount corresponding to the average price of the right (determined by the Stockbrokers' Council; cf. section 2.4). It will be shown below (section 3.2) that this adjustment of the terms of the contract does not fully protect the owner of a premium contract against the detachment of the foregoing rights. The detachment of a large coupon is therefore a good reason for the early exercise of a premium contract.[10]

When the option is exercised, the premium contract is either terminated or it turns into an ordinary forward contract maturing at the end-of-account settlement, together with all the other forward contracts entered into during the account.

2.4 Regulation of ancillary rights

The Milan Stock Exchange Rulebook regulates the ancillary rights of buyers and writers of premium contracts in accordance with the provisions of the Italian Civil Code on forward transactions (Articles 1531 and 1532).

Since the exercise of a premium contract (whether early or on 'risposta premi' day) results in its turning into an ordinary forward contract, from that day on all the rights attaching to the underlying securities pertain to the forward buyer, with the sole exception of the voting rights, which continue to pertain to the seller until delivery is actually made.

Prior to the exercise of a premium contract, the buyer's obligations are only contingent (and, in the case of a 'double premium' contract, indeterminate, since it is not known which of the two parties will be the buyer and which the seller) and all the rights pertain to the owner of the securities.

The Stock Exchange Rulebook explicitly foresees the application of this general rule to the case of subscription rights, providing that the rights pertain to the owner of a premium contract (who decides to buy the underlying securities) only if the exercise of the premium contract precedes the first day for the exercise of the subscription rights (Article 38 of the Milan Stock Exchange Rulebook). Otherwise, the striking price is reduced by the value of the rights (determined on the basis of the so-called 'average offset' especially calculated by the Stockbrokers' Council). The premium remains unchanged, however, even though it refers to a 'devalued' quantity of securities (Article 41 of the Rulebook).

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[8] Prior to this resolution, it was stock exchange practice that options had to be declared by 11:15.

[9] The recent Consob resolution anticipating the deadline for exercising options to 10.00 (the start of trading) results in the reference price for automatic responses being the official price of the previous day. This was done to overcome some shortcomings of the earlier system, which often led to 'suspicious' price developments in the early trading on 'risposta premi' day.

[10] Another reason for the early exercise of premium contracts is the Consob's current regulation of the margins on forward transactions. When securities that can be bought under a premium contract are resold before the premium contract is actually exercised, the Consob requires a margin for this position as if it were a short sale.

Table 3 Margin requirements in respect of premium contracts (Consob resolution 2077/1986 as amended).[a]

Single premium contracts (dont/put)		
DONT	*Buyer:*	must deposit the amount of the premium
	Seller:	must demonstrate availability of securities
PUT	*Buyer:*	must deposit the amount of the premium
	Seller:	100% of the maximum contractual obligation (less the amount of the premium)
Double premium contracts (stellage/strip/strap)		
	Buyer:	must deposit the amount of the premium plus a cash margin equal to the difference with respect to 50% (100% for 'strip' contracts) of the maximum contractual obligation or: must deposit the underlying securities
	Seller:	must demonstrate the availability of the underlying securities and: must deposit 1000. of the maximum contractual obligation or: must be long in a forward contract (in the case of 'strip' contracts must also deposit 100% of the maximum contractual obligation)

NB.: The notice of the deposit has to be recorded on the contract note. When premium contracts are exercised (whether early or on 'risposta premi' day), the parties become subject to the requirements applying to ordinary forward contracts (100% of the presumed value or, for sellers, delivery of the securities within three trading days) and the deposits made in respect of the premium contracts are released.

[a] Source: Milan Stock Exchange Stockbrokers' Council.

2.5 Ancillary costs

Commissions

Special arrangements exist for brokers' commissions on premium contracts: they are based on the striking price of the underlying securities (and not of the premium) and are normally fixed at 3.5 per thousand, half the rate for ordinary forward contracts. This rate is also usually applied to simultaneous forward contracts (of the opposite sign) involving the same securities.

To ensure the intermediary's right to the commission, this is always included on the contract note as an increase (or decrease) in the premium, rather than in the striking price (in view of the possibility of withdrawing from the contract). No extra commission is charged if the premium contract is exercised and the underlying securities change hands.

Taxes

Like all other contracts involving securities, premium contracts are liable to a special tax called the 'tax on stock exchange contracts', which takes the place of the ordinary stamp taxes. The rate applicable to premium contracts written on a stock exchange through a stockbroker is currently 0.15 per thousand.

At the time a contract is written, the tax is applied only to the amount of the premium (Law 947/1964). When the premium contract is exercised, a new contract note is prepared on the basis of the amount of the forward transaction that results.

2.6 Margin requirements

Consob resolution 929/1981 introduced a margin requirement for forward and premium contracts, without prejudice to the right of stockbrokers and commission dealers to demand additional guarantees.[11] The margin requirements in respect of premium contracts on the Milan Stock Exchange are currently as shown in Table 3.

[11] Prior to this resolution, stockbrokers adhered to a professional rule, whereby buyers of premium contracts were required to deposit an amount equal to the value of the premium (the maximum loss they could incur) and writers to deposit a percentage of the market value of the underlying securities that varied with the margin demanded by banks on 'riporto' contracts (repurchase agreements).

The deposit can be made with the intermediary (a stockbroker, a bank or a commission dealer who is a member of the clearing system managed by the Banca d'Italia) in cash or securities, provided the latter are identified, accompanied by an irrevocable order to sell and linked to a specific operation.

3. PARITIES BETWEEN PREMIA

In this section we derive the no-arbitrage conditions between the premia of the different contracts on a given security. In section 3.1 we show that double premium contracts are structurally equivalent to combinations of single premium contracts and consequently that the equilibrium premia for stellage, strip and strap contracts can be expressed in terms of the equilibrium premia for dont and put contracts. In section 3.2 we show that purchasing a put premium contract is equivalent to purchasing a dont contract, selling forward the underlying security and investing or borrowing at a fixed rate. These results make it possible to determine the equilibrium relationship between dont and put premia and reduce the problem of determining premia to that of determining dont premium.

In order to focus attention on the basic aspects of the problem, it will be assumed throughout this section that there are no transaction costs and that it is possible to invest and borrow at the same rate.[12]

3.1 Double premium contracts

Let

T = date of 'risposta premi' day
$T+\tau$ = date of settlement of the premium contract
S_t = price at time t of the underlying security
K = striking price of the premium contract
P_D = amount of the dont premium
P_P = amount of the put premium
P_{Se} = amount of the stellage premium
P_{Si} = amount of the strip premium
P_{Sa} = amount of the strap premium
D_t = value at time t ($t \in [0, T]$) of a dont contract
P_t = value at time t ($t \in [0, T]$) of a put contract
Se_t = value at time t ($t \in [0, T]$) of a stellage contract
Si_t = value at time t ($t \in [0, T]$) of a strip contract
Sa_t = value at time t ($t \in [0, T]$) of a strap contract
$B_t(T+\tau)$ = price at time t of a discount bond maturing at time $T+\tau$.

Accordingly, the following equations define the final value of a premium contract at maturity:

$$D_T = \{\max[0,(S_T - K)] - P_D\} \cdot B_T(T+\tau). \tag{1}$$

$$P_T = \{\max[(K - S_T),0] - P_P\} \cdot B_T(T+\tau). \tag{2}$$

$$Se_T = \{\max[(K - S_T),(S_T - K)] - P_{Se}\} \cdot B_T(T+\tau). \tag{3}$$

[12] It is worth noting that for the purpose of examining arbitrage conditions these assumptions are not only convenient but also appropriate. Given an adequate period of adjustment, the scope for arbitrage in an efficient market will be eliminated, so that there will no longer be any profit opportunities for operators with lower transaction costs and a narrower spread between borrowing and lending rates. The equations derived in this section can nonetheless easily be extended to take account of both transaction costs and interest rate differentials.

$$Si_T = \{\max[2(K - S_T),(S_T - K)] - P_{Si}\}\cdot B_T(T+\tau). \quad (4)$$

$$Sa_T = \{\max[½(K - S_T),(S_T - K)] - P_{Sa}\}\cdot B_T(T+\tau). \quad (5)$$

For example, the owner of a dont premium contract will renounce the underlying securities if on 'risposta premi' day their price is below the striking price, otherwise he will take delivery. In the first case, the dont contract will have a negative final value at maturity, equal to the present value of the premium to be paid at the settlement. In the second case, the value of the dont contract will be equal to the current value of a forward contract with a forward price equal to the striking price of the premium contract, less the present value of the premium. Since the value of a forward contract is equal to the present value of the difference between the current price of the security and the forward price (insofar as it would be possible to close the position by selling the securities and obtaining the difference between the two prices at the settlement), we have:

$$D_T = \begin{cases} -P_D \cdot B_T(T+\tau) & \text{if} \quad S_T \le K \\ (S_T - K - P_D)\cdot B_T(T+\tau) & \text{if} \quad S_T > K \end{cases}$$

from which (1) can be derived.

A portfolio acquired by buying a dont contract and a put contract with the same maturity and striking price and buying (or selling) fixed rate securities for an amount equal to $(P_D + P_P - P_{Se})\, B_0\,(T + \tau)$ involves a current outlay (or receipt) equal to $(P_D + P_P - P_{Se})\, B_0\,(T + \tau)$ and a final value on 'risposta premi' day equal to:

$$\begin{aligned} & D_T + P_T + (P_D + P_P - P_{Se})\cdot B_T(T+\tau) \\ & = [\max(0, S_T - K) + \max(0, K - S_T) - P_{Se}]\cdot B_T(T+\tau) \\ & = \{\max[(K - S_T),(S_T - K)] - P_{Se}\}\cdot B_T(T+\tau) = Se_T. \end{aligned}$$

A stellage premium contract is therefore exactly equivalent to a portfolio acquired by buying a dont and a put contract and buying (selling) fixed rate securities. Since the purchase of a stellage contract involves a zero investment, in the absence of arbitrage opportunities the investment required to acquire the portfolio must also be zero. Hence:

$$P_{Se} = P_D + P_P. \quad (6)$$

In other words, the stellage premium must be equal to the sum of the dont and put premia. If $P_{Se} > P_D + P_P$, the acquisition of the portfolio and the simultaneous sale of a stellage contract would produce an immediate gain equal to $(P_{Se} - P_D - P_P) \times B_T(T + \tau)$ and a zero balance at maturity. Similarly, if $P_{Se} < P_D + P_P$, the purchase of a stellage premium contract and the simultaneous sale of the portfolio would result in an immediate gain equal to $-(P_{Se} - P_D - P_P) \times B_T(T + \tau)$ at the moment the contracts were entered into and a zero balance at maturity.

A stellage premium contract is thus equivalent in every way to the combination of a dont and a put contract.[13]

Analogously, it can be demonstrated that:

$$P_{Si} = P_D + 2P_P \quad (7)$$

[13] Unlike the combination of a dont and a put contract, the stellage contract does not permit withdrawal from the contract. However, this additional right has a negligible value so that the equivalence is valid.

$$P_{Sa} = P_D + \frac{1}{2}P_P \qquad (8)$$

and that the purchase of a strip (of a strap) premium contract on a given quantity is equivalent to the acquisition of a portfolio comprising a dont contract on the same quantity of underlying securities and a put contract on double (half) that quantity.[14]

3.2 The dont - put parity

The problem of determining the equilibrium premia can be further simplified by demonstrating the existence of an important arbitrage relationship linking the premia of dont and put contracts on the same security and having the same maturity and striking price.

In addition to the assumptions made earlier (zero transaction costs and equal lending and borrowing rates), it will be assumed that the forward price of the security for the settlement of the month in which the premium contract matures is always observable. In other words, it is assumed that it is always possible to underwrite a 'riporto' contract (repurchase agreement) by means of which to extend the position from the settlement of the current month to that of the month in which the premium contract matures. If F_t is the forward price, r' the 'riporto' rate on the security at time t and δ the interval between the settlement date of the premium contract and the current settlement date, we have:

$$F_t = S_t \cdot e^{r'\delta} = S_t + I_R \qquad (9)$$

where I_R is the 'riporto' interest.[15]

In the arbitrage relationships derived in this section and the next, F_t, and not S_t, is considered the relevant variable, in order to take account of the possibly different maturities of premium and forward contracts. Since a premium contract can be written for a settlement after the current one, the forward operation may have to be prolonged by means of a 'riporto' contract in order to refer to the same maturity. Accordingly, F_t is the cost that has to be borne to have the security available on the settlement date of the premium contract. Obviously, if the premium contract is written for the end of the current month ('fine corrente') (so that $\delta = 0$), $F_t - S_t$. It should be noted, moreover, that on 'risposta premi' day we will always have $F_T - S_T$.

Finally, it is assumed that no dividends or other rights are detached from the security during the period considered. This implies that it will never be advantageous to exercise the premium contract early and that this possibility can be ignored in the rest of the treatment. The following lemma thus applies:

Lemma 1. If no dividends or other rights are detached from the underlying security in the period considered, early exercise of the premium contract is never advantageous.

Proof: The effect of the exercise of a premium contract is to transform it into an ordinary forward contract for settlement on the settlement day of the premium contract. The current value of a dont premium contract that has been exercised is therefore $(F_t - K - P_D) \times B_t(T + \tau)$. It is nonetheless easy to prove that this value is always less than that of a dont contract that has not been exercised, unless coupons are detached before the expiration of the option. In fact, if $D_t < (F_t - K - P_D) \times B_t(T + \tau)$, the portfolio acquired by buying the dont premium contract, selling the underlying security forward and borrowing an amount equal to $(F_t - K - P_D) \times B_t(T + \tau)$ would give an immediate gain and a non-negative balance at maturity (see Table 4). Accordingly, in the absence of opportunities for arbi-

[14] The possibility of withdrawing from the contract is also lacking in the case of strip and strap contracts, but the consideration of footnote 13 regarding stellage contracts applies again here.

[15] A 'riporto' contract can be considered as a double loan: one party lends the shares while borrowing money, the other party borrows the shares while lending money. Therefore, the 'riporto' rate is the difference between the interest rate on money and the 'interest rate on the security' (i.e. the opportunity cost of the temporary renunciation to the availability of the security): this implies that in some particular circumstances the 'riporto' rate can be negative. Cf. Williams and Barone (1989).

Table 4 Demonstration of the relationship $D_t \geq (F_t - K - P_D)B_t(T + \tau)$.

Composition of the portfolio	*Current receipt/outlay*	*Final value*	
		$F_T \leq K$	$F_T > K$
Buy dont contract	$-D_t$	$-P_D B_T(T+\tau)$	$(F_T - K - P_D)B_T(T+\tau)$
Sell forward contract		$-(F_T - F_t)B_T(T+\tau)$	$-(F_T - F_t)B_T(T+\tau)$
Borrow	$(F_t - K - P_D)B_t(T+\tau)$	$-(F_T - K - P_D)B_T(T+\tau)$	$-(F_T - K - P_D)B_T(T+\tau)$
Total	$(F_t - K - P_D)B_t(T+\tau) - D_t$	$(K - F_T)B_T(T+\tau)$	0

trage, $D_t \geq (F_t - K - P_D) \times B_t(T + \tau)$ must hold: early exercise of the premium contract is therefore never advantageous with the assumptions adopted.

However, if a coupon is detached from the underlying security before 'risposta premi' day, the final balance of the portfolio will not be negative only if the dont premium contract is exercised early. To demonstrate this, suppose that a coupon for an amount X is detached on day t' ($0 < t' < T$) and that the coupon is paid immediately. Since the forward buyer is entitled to the amount of the coupon, while the buyer of the premium contract is entitled (in the absence of early exercise of the premium contract) only to a reduction in the striking price of the premium contract, the final value of the foregoing portfolio will be:

$$\begin{aligned}&\{\max[0, F_T - (K - X)] - P_D - [F_T + X / B_t(T+\tau) - F_t] - (F_t - K - P_D)\}\ B_T(T+\tau)\\&= \{\max(0, F_T - K + X) - [F_T - K + X / B_t(T+\tau)]\} B_T(T+\tau)\end{aligned}$$

which is not necessarily non-negative. A coupon for a large amount may therefore justify early exercise of a premium contract. Q.E.D.

Given these hypotheses, the following theorem can be proved:

Theorem 1. The premium of a put contract is equal to that of a dont contract with the same striking price and maturity, less the difference between the forward price of the underlying security and the striking price:

$$P_P = P_D - (F_0 - K). \tag{10}$$

Proof: Consider the portfolio acquired by buying a dont contract, selling the underlying security forward and buying (selling) fixed rate securities for an amount equal to $(K - F_0 + P_D - P_P)B_0(T + \tau)$. This portfolio involves a current outlay (receipt) equal to $(K - F_0 + P_D - P_P)B_0(T + \tau)$ and a final value on 'risposta premi' day equal to:

$$\begin{aligned}&\{[\max(0, F_T - K) - P_D] - (F_T - F_0) + (K - F_0 + P_D - P_P)\}\ B_T(T+\tau)\\&= [\max(0, K - F_T) - P_P] B_T(T+\tau) = P_T.\end{aligned}$$

A put premium contract is thus equivalent to a portfolio acquired by buying a dont contract, selling the underlying security forward and buying (selling) fixed rate securities. Since the purchase of a put contract involves a zero outlay at the time it is written, in the absence of arbitrage opportunities, the current value of the portfolio must also be zero: $K - F_0 + P_D - P_P = 0$. Q.E.D.

Eq. (10) expresses the *dont - put parity.* If the market were not to respect this relationship, it would be possible to make an immediate profit at zero risk by selling the overvalued premium contract and buying the undervalued one.

In view of (9), the dont - put parity can be rewritten as:

$$P_P = P_D - (S_0 - K) - I_R.$$

Table 5 Factors for the transformation of any premium contract into a dont premium contract and a forward sale.

	Forward	*Dont*
Dont	-	1
Put	-1	1
Stellage	-1	3
Strip	-2	3
Strap	-½	3/2

Accordingly, if the two premium contracts are written for settlement at the end of the current month ('fine corrente'), so that $I_R = 0$, and the striking price of the two contracts is equal to the current price of the underlying security, the put premium is equal to the dont premium. If $F_0 > K$, $P_P < P_D$, whereas if $F_0 < K$, $P_P > P_D$.

The dont - put parity makes it possible to express the equilibrium value of every premium in terms of the equilibrium value of the dont premium. Substituting eq. (10) in eqs. (6), (7) and (8) gives:

$$P_{Se} = 2P_D - (F_0 - K) \tag{11}$$

$$P_{Si} = 3P_D - 2(F_0 - K) \tag{12}$$

$$P_{Sa} = \frac{3}{2}P_D - \frac{1}{2}(F_0 - K). \tag{13}$$

The arbitrage relationships that have been demonstrated imply that the purchase of put or double premium contracts can be reproduced by appropriate portfolios acquired by buying dont contracts, selling the underlying securities forward and buying (selling) fixed rate securities. These equivalencies enable intermediaries to offer their customers put, stellage, strip and strap premium contracts by setting up risk-free combinations through the purchase of the replicating portfolio and the sale of the premium or vice versa.

Table 5 shows the factors for the transformation of the various premium contracts into the equivalent combined transactions. The numerical values indicate the number of shares involved in the premium and forward contracts. Positive values indicate purchases and negative values sales.

4. THE VALUATION OF DONT PREMIUM CONTRACTS

In this section Option Pricing Theory is used to determine the value of dont premium contracts. The resulting valuation formula provides not only a solution to the problem of determining equilibrium premia but also a method for identifying overvalued and undervalued premium contracts or, in other words, those that offer an opportunity to make immediate arbitrage profits at no risk.

The derivation of the formula in section 4.1 is based on the binomial model by Cox, Ross and Rubinstein.[16] The assumption of a simple process with a discrete parameter for the price of the security underlying the premium contract makes it unnecessary to use differential calculus and allows the arbitrage processes upon which the valuation model is based to be clarified. The resulting formula can also be easily modified to take account of transaction costs and other market imperfections. In section 4.2 it is shown that the hypothesis of a continuous parameter process identical to that originally assumed by Black and Scholes[17] is a special case of the model.

[16] Cf. Cox, Ross and Rubinstein (1979).

[17] Cf. Black and Scholes (1973).

In section 4.3 the formula is used to obtain the equilibrium value of the dont premium. An empirical test of the formula is reported in section 5.

4.1 The binomial model

As in section 3, it is assumed that there are no transaction costs, that it is possible to invest and borrow at the same (constant) rate of interest, that the forward price of the underlying security for the settlement day of the month in which the premium contract matures is always observable and that dividends are not distributed during the period considered. It is also assumed that the forward price of the underlying security follows a simple multiplicative binomial process. In each period there can only be two possible values of the return on the security: u, with a probability of q, and d, with a probability of $1 - q$. If the current forward price of the underlying security is F, the price at the end of the period will therefore be uF or dF:

$$F = \begin{cases} uF & \text{with a probability of } q \\ dF & \text{with a probability of } 1-q. \end{cases}$$

To ensure equilibrium, it is also assumed that: $u > 1 > d$.[18]

The basic idea

To illustrate the technique adopted to value a dont premium contract on this security, we shall consider the case of contract with only one residual period before the 'risposta premi' day. Let $D(F, K, 1, P_D)$ be the current value of the dont contract, D_u^* its final value if the price of the share is uF, and D_d^* its final value if the price of the share is dF. Using $\hat{r}$ to indicate the interest rate over one period and $\hat{\tau}$ the number of periods between the 'risposta premi' day and the settlement day, we have:

$$D(F,K,1,P_D) = \begin{cases} D_u^* = & \{\max[0,(uF-K)] - P_D\}(1+\hat{r})^{-\hat{\tau}} \\ D_d^* = & \{\max[0,(dF-K)] - P_D\}(1+\hat{r})^{-\hat{\tau}}. \end{cases}$$

A portfolio P_1 can now be set up by buying forward α shares and buying (selling) β fixed rate securities, so that the final value of the portfolio P_1 exactly duplicates the final value of the premium contract. It is therefore necessary for:

$$\alpha(uF - F)(1+\hat{r})^{-\hat{\tau}} + \beta(1+\hat{r}) = D_u^* \text{ and}$$
$$\alpha(dF - F)(1+\hat{r})^{-\hat{\tau}} + \beta(1+\hat{r}) = D_d^*.$$

Solving these two equations, we have:

$$\alpha = (D_u^* - D_d^*)/[(u-d)F(1+\hat{r})^{-\hat{\tau}}] \text{ and}$$
$$\beta = \{[(1-d)/(u-d)]D_u^* + [(u-1)/(u-d)]D_d^*\}/(1+\hat{r}).$$

The last equation can be simplified by putting $p = (1 - d)/(u - d)$, and rearranging:

$$\beta = [pD_u^* + (1-p)D_d^*]/(1+\hat{r}).$$

[18] If this were not so, the forward purchase of a share at price F would entail a zero investment and a certain gain or loss at maturity, a situation that is obviously incompatible with equilibrium.

Choosing α and β in this way gives the 'equivalent portfolio' or, in other words, a portfolio that exactly duplicates the value at maturity of a dont premium contract for every possible price of the underlying security.

Since the dont contract and the portfolio P_1 are equivalent assets (since their final values are the same in all possible circumstances),[19] we can define the current value of the premium contract $D(F, K, 1, P_D)$ as the current value of the portfolio P_1:

$$D(F,K,1,P_D) = P_1 = \left[pD_u^* + (1-p)D_d^*\right] / (1+\hat{r}). \tag{14}$$

The binomial formula

We shall now consider the case of a dont contract with two periods to run before the 'risposta premi' day.

In view of the assumptions made regarding the forward price, there are three possible values of F at the end of the second period:

$$\begin{array}{ccc} & & u^2F \\ & uF & \\ F & & udF \\ & dF & \\ & & d^2F. \end{array}$$

Analogously, for the dont contract we have:

$$\begin{array}{lll} & & D_{uu}^* = \left\{\max\left[0,\left(u^2F-K\right)\right]-P_D\right\}(1+\hat{r})^{-\hat{\tau}} \\ & D_u = D(uF,K,1,P_D) & \\ D(F,K,2,P_D) & & D_{ud}^* = \left\{\max\left[0,\left(udF-K\right)\right]-P_D\right\}(1+\hat{r})^{-\hat{\tau}} \\ & D_d = D(dF,K,1,P_D) & \\ & & D_{dd}^* = \left\{\max\left[0,\left(d^2F-K\right)\right]-P_D\right\}(1+\hat{r})^{-\hat{\tau}} \end{array}$$

In order to identify a portfolio P_2 whose final value will duplicate that of the premium contract it is necessary to choose the quantities α and β so that the value of the portfolio at the end of the first period is equal to the sum needed to buy the new portfolio P_1, equivalent to a dont contract with a residual life of only one period. Using (14):

$$P_2 = \begin{cases} D_u = \left[pD_{uu}^* + (1-p)D_{ud}^*\right] / (1+\hat{r}) & \text{if the price of the underlying security is } uF \\ D_d = \left[pD_{du}^* + (1-p)D_{dd}^*\right] / (1+\hat{r}) & \text{if the price of the underlying security is } dF \end{cases}$$

The portfolio P_2 is therefore chosen to make its value at the end of the first period equal to D_u if the forward price of the underlying security is uF and D_d if the forward price is dF.

Putting:

[19] It is worth noting (cf. section 3.2) that if no dividends or other rights are detached from the underlying security, early exercise of a premium contract is never advantageous.

$$\alpha(uF-F)(1+\hat{r})^{-(1+\hat{\tau})}+\beta(1+\hat{r})=D_u$$

and:

$$\alpha(dF-F)(1+\hat{r})^{-(1+\hat{\tau})}+\beta(1+\hat{r})=D_d,$$

we have:

$$\alpha=(D_u-D_d)\left[(u-d)F(1+\hat{r})^{-(1+\hat{\tau})}\right]$$

and:

$$\beta=[pD_u+(1-p)D_d]/(1+\hat{r}).$$

We thus have a strategy that, in all possible circumstances, can duplicate the final value of a dont premium contract with a residual life of two periods. The only difference compared with the previous case of a dont contract with a residual life of one period is. that at the end of the current period it will be necessary to liquidate the portfolio P_2 and acquire the portfolio P_1, which is then held during the second period. In no circumstances will the adjustment of the portfolio involve an outlay of cash: the value of the portfolio P_2 at the end of the first period will always be exactly equal to the amount needed to acquire the new portfolio P_1.

We can therefore write:

$$\begin{aligned}D(F,K,2,P_D)&=P_2=[pD_u+(1-p)D_d]/(1+\hat{r})\\&=\left[p^2D^*_{uu}+2p(1-p)D^*_{ud}+(1-p)^2D^*_{dd}\right]/(1+\hat{r})^2\\&=\left\{p^2\max\left[0,\left(u^2F-K\right)\right]+2p(1-p)\max[0,(udF-K)]\right.\\&\quad\left.+(1-p)^2\max\left[0,\left(d^2F-K\right)\right]-P_D\right\}/(1+\hat{r})^{2+\hat{\tau}}.\end{aligned}$$

There thus exists an iterative procedure for determining the value of a dont premium contract with an arbitrary number of periods to maturity. Starting from the date of the 'risposta premi' day and working backwards, the formula for the valuation of a dont premium contract with an arbitrary maturity of n periods can be written as:

$$D(F,K,n,P_D)=\left\{\sum_{j=1}^{n}\binom{n}{j}p^j(1-p)^{n-j}\max\left[0,\left(u^jd^{n-j}F-K\right)\right]-P_D\right\}/(1+\hat{r})^{(n+\hat{\tau})}\qquad(15)$$

where $\binom{n}{j}=n!/[j!(n-j)!]$ is the binomial coefficient.

The composition of the equivalent portfolio is given by:

$$\begin{aligned}\alpha&=(D_u-D_d)/\left[(u-d)F(1+\hat{r})^{-(n-1+\hat{\tau})}\right]\\\beta&=[pD_u+(1-p)D_d]/(1+\hat{r}).\end{aligned}\qquad(16)$$

Eq. (15) is the formula for the valuation of a dont premium contract. It can be rewritten as follows (see appendix):

$$D(F,K,n,P_D)=\{F\Phi[a;n,p']-K\Phi[a;n,p]-P_D\}/(1+\hat{r})^{n+\hat{\tau}}\qquad(17)$$

where:

p = $(1-d)/(u-d)$, $p' = up$
a is the smallest non-negative integer greater than $\ln[K/(Fd^n)]/\ln(u/d)$
$\Phi[.]$ is the binomial complementary distribution function.

4.2 The valuation formula in continuous time

The assumption of a binomial process for the price of the underlying security made it possible to derive a formula for the valuation of a dont premium contract and to illustrate the arbitrage restrictions upon which it is based.

However, since trading in a security proceeds more or less continuously rather than at discrete intervals, it might appear more appropriate to assume a continuous rather than a discrete parameter process. This can be considered a limiting case of the binomial model, corresponding to the situation when the interval between two successive changes in the price of the underlying security becomes increasingly small and tends towards zero.

Let h be the length of the interval between two successive changes in price. Using the notation introduced above, if $(T-t)$ is the time to maturity and n is the number of intervals of length h until maturity, then $h = (T-t)/n$. It is shown in the appendix that when trades are made with increasing frequency, so that h becomes smaller and smaller and n tends towards infinity, (17) tends towards the following valuation formula in continuous time:

$$\begin{aligned} D(F_t, K, T-t, P_D) &= e^{-r(T+\tau-t)}\left[F_t \cdot N(x) - K \cdot N\left(x - \sigma\sqrt{T-t}\right) - P_D\right] \\ x &= \frac{\ln(F_t / K) + ½\sigma^2 (T-t)}{\sigma\sqrt{T-t}}. \end{aligned} \qquad (18)$$

where r is the instantaneous interest rate.

4.3 The equilibrium premia

The valuation formula can be used to determine the equilibrium premia. It needs to be remembered that, as for forward contracts, there is no exchange of cash between the parties at the time a premium contract is written: premia are therefore fixed on the stock exchange so as to make the value of the contract when it is written equal to zero. Imposing this equilibrium constraint, we have:

$$D\left(F_0, K, T, \hat{P}_D\right) = 0,$$

and solving (18) for $\hat{P}_D$, we obtain:

$$\begin{aligned} \hat{P}_D &= F_0 \cdot N(x) - K \cdot N\left(x - \sigma\sqrt{T}\right) \\ x &= \frac{\ln(F_0 / K) + ½\sigma^2 T}{\sigma\sqrt{T}}. \end{aligned} \qquad (19)$$

Eq. (19) gives the equilibrium premium for a dont contract. Whenever the market premium diverges from this equilibrium value, it becomes possible, under the assumptions made, to make an immediate profit at no risk by selling the dont premium contract and acquiring the equivalent portfolio (if $D < 0$) or buying the dont contract and disposing of the equivalent portfolio (if $D > 0$). In such circumstances this course results in a profit equal to the value of the portfolio and a zero balance when the position is closed.

The equilibrium premium given by (19) is an increasing function of the forward price of the underlying security (F) and of its volatility (σ), and a decreasing function of the striking price (K). On the other hand, the effect on the premium of lengthening the maturity of the contract is generally indeterminate; a sufficient condition for $(\partial\hat{P}_D/\partial T) > 0$ is $r' > -\sigma N'(x)/[2\sqrt{T}N(x)]$.

Table 6 Equilibrium values of the premium on a dont contract.

S = 1,000		'Riporto' rate (r')														
		-10%			-5%			5%			10%			15%		
		Maturity (T × 365)														
σ	K	30	60	90	30	60	90	30	60	90	30	60	90	30	60	90
	900	93	89	87	97	96	97	105	111	118	109	119	129	113	127	141
0,20	1,000	19	25	28	21	28	34	25	37	46	27	42	54	30	47	52
	1,100	1	3	6	1	4	7	1	6	12	2	8	15	2	9	18
	900	104	112	118	107	118	127	114	130	145	118	137	154	121	144	164
0,40	1,000	42	56	67	44	60	73	48	69	86	50	74	93	52	79	101
	1,100	12	25	34	13	27	38	15	32	47	16	35	52	17	38	57
	900	121	139	153	124	145	161	130	156	178	133	162	187	136	169	196
0,60	1,000	64	88	105	66	92	112	71	101	125	73	106	133	75	111	140
	1,100	30	53	70	31	56	75	34	63	86	35	66	92	37	70	98

Table 6 exemplifies the values obtained with the formula (19), for a given price of the underlying security and various assumptions regarding the striking price, the number of days to 'risposta premi' day, the volatility of the underlying security and the 'riporto' interest rate. The values given in the table were obtained assuming $\delta = T$ in (9).

5. AN EMPIRICAL TEST

In this section the formula (19) for the determination of equilibrium premia is tested empirically.

Section 5.1 contains a description of the data used for the test, while section 5.2 explains the method used of calculate the volatility of the underlying security σ, the only parameter of the model that is not directly observable.

In section 5.3 a comparison is made between the premia obtained with the model and those fixed by the market. Specifically, tests are made to see if there are any systematic relationships between the single variables of the valuation model and the percentage differences between the market premia and those given by the valuation model.

In section 5.4 the efficiency of the market for premium contracts is tested by verifying the possibility of using the model to make arbitrage profits. Tests of this type are based on the construction of portfolios by buying 'undervalued' premium contracts (or selling 'overvalued' ones) and selling (or buying) the underlying security forward in an amount that will make the combined position without risk. Since such portfolios require a zero initial investment, they should earn a zero profit. The analysis of the profits obtainable thus provides a test of the efficiency of the premium market on the Milan Stock Exchange. Finally, section 5.5 assesses the results obtained.

5.1 The data

The data used in this section are those published by 'Il Sole-24 Ore' and refer to the dont premium contracts written on the Milan Stock Exchange the last day of each week on the three securities with the largest market (Fiat, Generali and Montedison). The period of observation lasted for 8 monthly accounts, from August 1986 to March 1987.

The sample comprised 316 premium contracts, of which 114 on Fiat, 110 on Generali and 92 on Montedison.

Since there is no closing price for premium contracts, in the following tests an average premium, calculated as the arithmetic mean of the highest and lowest premia, was used.

Table 7 shows some of the features of the sample. On average, the striking price of the premium contracts tended to be slightly above the current price of the underlying security, while the average maturity of the contracts was about 33 calendar days.

Table 7 Distribution of the parameters of the 316 dont contracts of the sample.

		Fiat				*Generali*				*Montedison*			
		P_D	S	K	$T\times365$	P_D	S	K	$T\times365$	P_D	S	K	$T\times365$
Average		541	14,350	14,575	33	4,846	137,179	138,686	32	157	3,169	3,215	33
Std devn		291	1,021	1,161	12	3,037	14,788	15,709	13	112	322	362	14
Std/Avg		0.538	0.071	0.08	-	0.627	0.108	0.113	-	0.716	0.102	0.113	-
Quartiles	0.25	300	13,600	13,500	24	2,500	130,800	130,000	24	65	2,876	3,000	21
	0.50	518	14,303	14,500	33	4,125	134,000	135,000	33	128	3,085	3,000	32
	0.75	725	15,405	15,500	42	6,500	139,200	145,000	42	210	3,492	3,500	42

The 'riporto' rates on securities were taken to be equal to the current interest rate. This procedure was appropriate since during the period considered the 'riporto' rates fixed for the shares in the sample at the special meetings held on the Stock Exchange never differed from the current interest rate.

The interest rates were obtained by observing at weekly intervals the yields on the Treasury bills whose residual maturity was closest to one month (the average maturity of the premium contracts in the sample) and then converting them into equivalent instantaneous rates.

One important limitation of the data used needs to be mentioned. The valuation formula requires that the price of the underlying security be known at the moment the premium contract in question is written. For the test recourse was made, instead, to the closing prices. To the extent that these differed from the prices current when the premium contracts were written, the data suffer from an error of asynchronism.

5.2 The estimation of volatility

The expected volatility of the forward price *(F)* can be estimated on the basis of the prices observed in a given period.[20] Black and Scholes[21] used an estimate of this type to test their valuation model, but concluded that a significant proportion of the deviations of their theoretical prices from market prices was due to errors in the estimation of this parameter.

An alternative method of estimating the expected volatility is to use the current market premia: the implied standard deviation (ISD) is the value of the volatility that makes the theoretical premium equal to the current market premium. Even though eq. (19) cannot be solved directly in terms of σ, a numerical procedure can be used to approximate its value.

Prices on an efficient market reflect all the relevant information available. Hence, the volatility implied in market premia should reflect not only the information provided by past prices but also all the other information available. It therefore appears reasonable to expect the implied volatility to be more accurate than the historical one in predicting future volatility.[22]

If the assumptions on which the model is based were completely valid and the market perfectly efficient, all the premia on a given security would be determined at every moment on the basis of the same implied volatility. Since this is unlikely to happen in practice, there is the problem of choosing an average of the various ISDs observable at any given moment.

We adopted the following procedure:

[20] When estimating volatility on the basis of historical data, account has to be taken of the discontinuity in stock market prices at the beginning of each monthly account (cf. the appendix).

[21] Cf. Black and Scholes (1972).

[22] Empirical tests on US markets have shown implied volatilities to be more accurate than those based on historical data. Latané and Rendleman (1976) and Chiras and Manaster (1978) studied the correlation between historical and implied volatility on the one hand and actual volatility on the other (observed during the life of the option) and concluded that the implied volatility gave better forecasts. A preliminary analysis of the stocks considered in the empirical part of this paper confirmed that the weighted implied volatility was a better predictor than the historical volatility also in the Italian market.

Table 8 Distribution of the WISDs calculated on the sample.

		Fiat	*Generali*	*Montedison*
Average		0.3290	0.2827	0.3988
Standard deviation		0.0843	0.0942	0.0947
	0.25	0.2514	0.1967	0.3185
Quartiles	0.50	0.3109	0.2744	0.4368
	0.75	0.3999	0.3432	0.4788
Average no. of ISDs used		3.26	3.11	2.63

- first, the implied volatility was determined for each of the 316 premia in the sample by using a Newtonian algorithm to obtain the value of σ (with a tolerance of 0.0001) that made the theoretical premium given by eq. (19) equal to the market premium;[23]
- second, for each observation day, the ISD calculated on the various premium contracts written on each underlying security were used to obtain a weighted implied standard deviation (WISD), calculated as the average of the ISDs with weights equal to the elasticity of the premia with respect to the volatility:

$$WISD = \frac{\sum_{j=1}^{m} \left(\sigma_j \cdot \frac{\partial P_{Dj}}{\partial \sigma_j} \cdot \frac{\sigma_j}{P_{Dj}} \right)}{\sum_{j=1}^{m} \left(\frac{\partial P_{Dj}}{\partial \sigma_j} \cdot \frac{\sigma_j}{P_{Dj}} \right)},$$

where m is the number of premium contracts written on the same underlying security.

In this way greater weight is given to the volatilities implied in the premia theoretically most sensitive to the value of σ. This is because the premium contracts whose values are little affected by volatility (primarily dont contracts with striking prices well below the current price of the underlying security and with a short residual maturity), are probably less representative of the market's expectations.[24]

Some statistics of the distribution of the WISDs obtained for each underlying security are shown in Table 8.

In the subsequent analyses we used the WISDs calculated on the basis of the premium contracts written on the last day of the week *prior* to that in which the valuation is made. Using the implied volatilities calculated on the basis of the premia observed on the same day as the valuation would not only have eliminated from the sample, on any given day, the securities on which just one premium contract had been written (since this would have implied the theoretical premium being equal to the actual premium), but would also have undermined the analysis of the market's efficiency by resulting in the use of information that was not available when overvalued and undervalued contracts were to be selected.

[23] The research of the implied volatility is without a solution when the market premium falls outside the theoretical boundaries or, in other words, when the dont premium is greater than the forward price of the underlying stock or smaller than the difference between the forward price and the exercise price. With our sample the procedure failed to converge only in one instance, and even then the adoption of a premium slightly above the average (and in any case less than the maximum observed) would have ensured convergence, thus reflecting the non-simultaneity of the variables.

[24] This weighting system was originally proposed by D. P. Chiras and S. Manaster (1978).

Table 9 Comparison between market and theoretical premia.

		Fiat			*Generali*			*Montedison*			*Total sample*		
		P_D	$\hat{P}_D$	*Dev (%)*	P_D	$\hat{P}_D$	*Dev (%)*	P_D	$\hat{P}_D$	*Dev (%)*	P_D	$\hat{P}_D$	*Dev (%)*
Average		541	528	2.37	4,846	4,816	0.22	157	155	0.19	1,928	1,912	0.99
Std devn		292	250	29.15	3,051	2,758	19.59	113	322	28.01	2,800	2,688	25.79
Quartiles	0.25	300	345	-13.55	2,500	2,943	-13.42	65	81	-16.62	205	221	-15.30
	0.50	518	500	-0.92	4,125	4,297	-2.24	128	142	-4.04	550	543	-1.90
	0.75	725	663	14.89	6,500	6,191	10.97	210	218	9.97	2,950	3,083	13.04

5.3 Comparison between model and market premia

Using the implied volatilities calculated with the method described in the previous section, the theoretical premium $\hat{P}_D$ given by the formula (19) was calculated for every dont contract in the sample. The results are summarized in Table 9, which shows, for each stock and for the whole sample, the average, the standard deviation and the quartiles of the distribution of the market premia (P_D), of the theoretical values ($\hat{P}_D$) and of the percentage deviation of the market premium from its theoretical value [Devn % = $(P_D - \hat{P}_D)/\hat{P}_D \times 100$].

The valuation formula thus gives values that, on average, are very close to the market premia, with an average percentage error that is generally less than 1%. In 50% of the cases the percentage deviation between market and theoretical premia was less than 15% in absolute terms.[25]

In order to analyze the differences between the market premia and the theoretical values in greater detail, the following regressions were estimated for each underlying security and for the sample as a whole:

$$\text{Test 1:} \quad \frac{P_j}{K_j} = b_0 + b_1 \frac{\hat{P}_j}{K_j} + u_j,$$

$$\text{Test 2:} \quad \frac{P_j - \hat{P}_j}{\hat{P}_j} = b_0 + b_1 \frac{S_j - K_j}{K_j} + u_j,$$

$$\text{Test 3:} \quad \frac{P_j - \hat{P}_j}{\hat{P}_j} = b_0 + b_1 (T_j * 365) + u_j,$$

$$\text{Test 4:} \quad \frac{P_j - \hat{P}_j}{\hat{P}_j} = b_0 + b_1 \sigma_j + u_j,$$

$$\text{Test 5:} \quad \frac{P_j - \hat{P}_j}{\hat{P}_j} = b_0 + b_1 l_j + u_j.$$

The regression of the standardized market premia on the standardized theoretical values (Test 1) serves to assess the correspondence of the model. with the valuations of the market.[26] If the market premia were exactly equal to the theoretical ones, b_0 and b_1 should not be different from 0 and 1

[25] It is of interest to compare these results with those obtained by Whaley (1982) for the US market using a large sample of 15,582 options on 91 securities quoted on the Chicago Board Options Exchange during the 160 weeks between 17.1.1975 and 3.2.1978. The average percentage deviation between the market price and the theoretical price amounted to 2.15% using Black and Scholes' standard formula and to 1.08% using the formula developed by Whaley to take account of the possibility of early exercise of options on the CBOE. The standard deviation of the percentage error was equal to 25.24% in the first case and to 23.82% in the second.

[26] Premia were standardized (in relation to the striking price) to avoid problems of heteroscedasticity in the regression.

Table 10 Regression parameters.

Test		b_0	$t(b_0)$	$P[t(b_0)]$	b_1	$t(b_1)$	$P[t(b_1)]$	R^2
(1) $P/K = b_0 + b_1\hat{P}/K$	Fiat	-0.002	-1.378	0.17	1.092	2.153	0.03	0.85
	Generali	-0.001	-0.910	0.37	1.032	1.111	0.27	0.92
	Montedison	-0.008	-3.870	0.00	1.182	4.969	0.00	0.92
	Total	-0.004	-4.351	0.00	1.117	5.801	0.00	0.91
(2) $(P - \hat{P})/\hat{P} = b_0 + b_1(S - K)/K$	Fiat	0.038	1.236	0.22	0.970	1.032	0.30	0.00
	Generali	-0.001	-0.060	0.95	-0.352	-0.554	0.58	0.00
	Montedison	0.012	0.383	0.70	0.796	1.186	0.24	0.00
	Total	0.016	1.051	0.29	0.523	1.229	0.22	0.00
(3) $(P - \hat{P})/\hat{P} = b_0 + b_1(T\times 365)$	Fiat	-0.050	-0.621	0.54	0.002	0.972	0.33	0.00
	Generali	-0.060	-1.183	0.24	0.002	1.316	0.19	0.01
	Montedison	0.015	0.193	0.85	0.000	-0.182	0.86	0.01
	Total	-0.031	-0.766	0.44	0.000	1.087	0.28	0.00
(4) $(P - \hat{P})/\hat{P} = b_0 + b_1\sigma$	Fiat	0.333	3.012	0.00	-0.920	-2.882	0.01	0.06
	Generali	0.050	0.767	0.44	-0.158	-0.767	0.45	0.00
	Montedison	0.221	1.646	0.10	-0.520	-1.671	0.10	0.02
	Total	0.151	2.931	0.00	-0.404	-2.852	0.01	0.02
(5) $(P - \hat{P})/\hat{P} = b_0 + b_1 l$	Fiat	0.222	2.294	0.02	-0.007	-2.133	0.04	0.04
	Generali	0.128	1.954	0.05	-0.004	-2.001	0.05	0.04
	Montedison	0.243	2.426	0.02	-0.008	-2.510	0.01	0.07
	Total	0.196	3.869	0.00	-0.006	-3.828	0.00	0.05

respectively. Conversely, a positive (negative) intercept and a coefficient greater (less) than 1 indicate a tendency for the model to systematically underestimate (overestimate) the premia.

Tests 2, 3 and 4 verify the existence of systematic relationships between the percentage deviation of the market premia from their theoretical values and the variables of the model (F, K, T and σ). Test 5, instead, is designed to determine whether there is a significant relationship between the percentage deviation and the stage of the monthly account (l is the number of days to the settlement day of the current month).

The results of these tests are summarized in Table 10. For each regression and each underlying security, the table shows the estimate of the parameters b_0 and b_1, the values of Student's t, $t(b)$ and the two-tail probability $P[t(b)]$ that a Student's t random variable will be greater than the absolute value of $t(b)$. All the t-ratios are constructed to verify the null hypothesis that $b_0 = 0$ and $b_1 = 0$; those of Test 1 are designed to verify the null hypothesis that $b_0 = 0$ and $b_1 = 1$. The value of R^2 is also shown for each regression.

The results of Test 1 shows that the valuation model explains more than 90% of the variance of the market premia. The model nonetheless tends to overvalue the smaller premia and to undervalue the larger ones ($b_0 < 0$ and $b_1 > 1$).[27]

Test 2 verifies whether the model systematically overvalues or undervalues in-the-money or out-of-the-money contracts. The results obtained provide sufficient evidence of the absence of such misspecification.

The same conclusion also holds for Test 3, designed to verify the existence of a systematic relationship between the percentage deviation between the market premium and the theoretical value, on the one hand, and the maturity of the premium contract, on the other. Here again, the values of the t ratios indicate that there is no significant relationship.

By contrast, there is a significant relationship (at least for the sample as a whole) between the percentage deviation between the market premium and the theoretical premium, on the one hand,

[27] A similar tendency also existed in Whaley's (1982) analysis of the US market.

and the estimate of the volatility of the underlying security, on the other. The model undervalues the premia on less volatile securities and overvalues those on more volatile securities.[28]

Test 5 also provides evidence of a significant relationship: the model undervalues the premia written in the last part of the monthly account (low value of l) and overvalues those written in the early part (high value of l).

5.4 Test of market efficiency

Methodology

The empirical evidence of the previous section shows that, on average, the theoretical values obtained with the valuation model are extremely close to the actual market values. Nonetheless, the existence of sometimes significant deviations can give rise to opportunities for arbitrage profits.

In fact, the derivation of the valuation formula was based on the possibility of an operator free from relevant transaction costs making (risk-free) profits from every deviation of market premia from the theoretical values.

The arbitrage procedure described in the previous section involved the purchase of undervalued premium contracts (or the sale of overvalued contracts) and the simultaneous sale (purchase) of $\alpha = N(x)$ shares.[29] If the formula determines the no-arbitrage premia correctly and the position is continuously updated in order to keep the number of shares in the portfolio constantly equal to $N(x)$, the purchase of undervalued premium contracts and the sale of overvalued ones must give a profit that is significantly more than zero. On the other hand, if the market is efficient and determines premia on the basis of more information than is incorporated in the model, the strategy will not be profitable.

An empirical test of the profits that can be made through such arbitrage operations thus provides a test of the efficiency of the market for premium contracts and of the validity of the valuation formula as a means of selecting overvalued and undervalued contracts.

The empirical test of the existence of opportunities for arbitrage was carried out as follows.

On the day a premium contract was written, it was considered to have been sold if it was overvalued compared with the model ($P_D > \hat{P}_D$) or bought if it was undervalued ($P_D < \hat{P}_D$). At the same time as the premium contract was bought (sold), $N(x_{j,t})$ shares were sold (bought), with $N(x_{j,t})$ denoting the value of α for the j^{th} premium contract at time t.

Since it would not be possible to adjust the portfolio continuously, the position was reviewed once a day on the basis of the new value of $N(x_{j,t})$. Every trading day the previous forward position was liquidated and the new one immediately acquired.

Let $D_{j,t}$ denote the value of the j^{th} premium contract at time t, as calculated on the basis of formula (17), and $V_{j,t} = V_j(t-1, t) = (F_{j,t} - F_{j,t-1})e^{-r(T+\tau-t)}$ the current value of a forward contract for the security underlying the j^{th} premium contract written at time $t-1$ for delivery at time $T+\tau$. Then the profit or loss on day t ($1 \le t \le T$) is given for each portfolio by:

$$\begin{aligned} R_{j.t} &= \pm\left[\left(D_{j,t} - D_{j,t-1}\right) - N\left(x_{j,t-1}\right)\left(V_{j,t} - V_{j,t-1}\right)\right] = \\ &= \pm\left[\left(D_{j,t} - D_{j,t-1}\right) - N\left(x_{j,t-1}\right)\left(F_{j,t} - F_{j,t-1}\right)e^{-r(T+\tau-t)}\right], \end{aligned} \quad (20)$$

given that V_{t-1} (the value at time $t-1$ of a forward contract written at time $t-1$) is equal to zero.

Since the valuation model was used to calculate $D_{j,t}$ and $D_{j,t-1}$ in eq. (20), it was necessary to make an adjustment to take account of the difference between the initial price of the contract given by the model, $D_{j,0}$, and the market price, which is always zero. To avoid heteroscedastic disturbances in the R_t series, this difference was distributed over the life of the contract by calculating the equivalent daily value and adding it to each $R_{j,t}$.[30]

[28] This result is also in line with those obtained in several studies of the US market.

[29] It is shown in the appendix, that, as n tends toward infinity, the number of shares in the equivalent portfolio tends towards $N(x)$.

[30] This method was originally proposed by Black and Scholes (1972).

Table 11 Results of the regressions.

Test		b_0	$t(b_0)$	$P[t(b_0)]$	b_1	$t(b_1)$	$P[t(b_1)]$	R^2
	Fiat	5.629	7.682	0.00	-0.087	-0.889	0.37	0.00
(1) $R_P = b_0 + b_1R_M$	Generali	34.414	5.551	0.00	-0.988	-1.177	0.24	0.01
(Ordinary least squares)	Montedison	1.037	6.369	0.00	-0.027	-1.215	0.23	0.01
	Total	14.274	6.414	0.00	-0.403	-1.356	0.18	0.01
	Fiat	4.835	7.304	0.00	-0.144	-1.558	0.12	0.25
(2) $R_P = b_0 + b_1R_M$	Generali	41.519	6.289	0.00	-1.066	-1.305	0.19	0.17
(Generalized least squares)	Montedison	1.089	6.241	0.00	-0.040	-1.818	0.07	0.17
	Total	14.702	6.526	0.00	-0.448	-1.528	0.13	0.18

The profit or loss made on each portfolio were aggregated in order to calculate the daily net profit of the whole portfolio. This profit was then divided by the number of portfolios in existence on day t to generate the series of the average daily profit per contract, R_{pt}.

Results

The test was applied to the dont premium contracts in the sample, assuming that the purchase and sales were triggered when the percentage difference between the market premium and the theoretical value was greater in absolute terms than 15%.

The daily profit series was calculated for the 202 trading days between 21.7.1986 and 11.5.1987 (the May ‘risposta premi’ day), with 55 positions being taken on Fiat, 49 on Generali and 44 on Montedison.

To test that the profits of the aggregate portfolio were really free from systematic risk (i.e. that they were not correlated with changes in the value of the market portfolio), the following regression was estimated:

$$R_{P_t} = b_0 + b_1 R_{M_t} + u_t,$$

where R_{M_t}, is the daily variation in the COMIT index at time t.[31]

The estimated intercept, b_0, was used as an indicator of the profitability of the portfolio and b_1, the coefficient of the regression of R_P on R_M, as an indicator of the systematic risk.

The results of the regression are shown in Table 11, for each stock and for the sample as a whole. Since the average daily profit per contract, R_P, is based on a variable number of arbitrage portfolios, the estimation of the regression could suffer from heteroscedasticity. Table 11 therefore also shows the results of the regression made using generalized least squares, in which all the variables were weighted with the square root of the number of arbitrage operations outstanding at time t. In all the eight regressions estimated, the intercept b_0 was always significant (at the 5% level), in contrast with the coefficient b_1.

The results of Table 11 imply that the strategy described would have produced a *daily* profit for each share involved in a premium contract of about 5 lire for Fiat, of 41 lire for Generali and 1 lira for Montedison. Since the average maturity of the contracts in the sample was 32.5 days, this corresponds to an average profit of 162 lire per share involved in a premium contract in the case of Fiat, of 1,332 lire for Generali and 32 lire for Montedison.

These results powerfully support the validity of the valuation formula as an instrument for distinguishing between overvalued and undervalued premium contracts.

[31] Since the profits of the arbitrage operations were measured in absolute rather than the percentage terms (in view of the fact that the initial investment is zero), the changes in the value of the market portfolio were also expressed in absolute terms.

5.5 Conclusions

The market values of the dont premium contracts are very close, on average, to their equilibrium values given by formula (19). The average percentage difference between the market premia and their theoretical values was less than 1% and not significantly different from zero; its standard deviation was equal to 25.8%, only slightly above that found using the same methodology for the US options market.[32]

The average daily profits obtained with the arbitrage strategy described in section 5.4 were significantly positive and amounted to around 0.30 - 0.35 per thousand of the average value of the underlying securities. These results prove the validity of the formula in identifying overvalued and undervalued premium contracts but do not appear to justify the rejection of the null hypothesis of an efficient market for premium contracts, since generally the transaction costs involved in the daily adjustment of the portfolio would offset this arbitrage profit. Furthermore, the limits on the information available and the use of average values of the premia in the analysis may sometimes have led to the identification of arbitrage opportunities as a result of the premia and the prices of the underlying securities not being recorded simultaneously.

It should also be noted that the three stocks considered in the analysis account for by far the greater part of the premium contracts written on the Milan Stock Exchange. It remains to be seen whether the foregoing conclusion also applies to stocks for which there is a smaller volume of premium contract business.

[32] Cf. Whaley (1982).

APPENDIX

A.1. Convergence of the binomial formula in continuous time

Consider the binomial formula (15):

$$D(F,K,n,P_D)=\left[\sum_{j=1}^{n}\binom{n}{j}p^j(1-p)^{n-j}\max\left(0,u^jd^{n-j}F-K\right)-P_D\right]/(1+\hat{r})^{(n+\hat{\tau})}.$$

Let a be the smallest non-negative integer for which $u^a d^{n-a}F > K$. For every $j < a$, max[0, $(u^j d^{n-j}F - K)$] = 0, while for every $j \geq a$, max[0, $(u^j d^{n-j}F - K)$] = $u^j d^{n-j}F - K$. Accordingly:

$$D(F,K,n,P_D)=\left[\sum_{j=a}^{n}\binom{n}{j}p^j(1-p)^{n-j}\left(u^jd^{n-j}F-K\right)-P_D\right]/(1+\hat{r})^{(n+\hat{\tau})}.$$

If $a > n$ at maturity, it will obviously never be advantageous to exercise the dont contract and the sum will be zero.

Dividing the foregoing expression in two terms, we can write:

$$D(F,K,n,P_D)=\left\{F\left[\sum_{j=a}^{n}\binom{n}{j}p^j(1-p)^{n-j}u^jd^{n-j}\right]-K\left[\sum_{j=a}^{n}\binom{n}{j}p^j(1-p)^{n-j}\right]-P_D\right\}/(1+\hat{r})^{n+\hat{\tau}}$$

The second expression in square brackets is the complementary binomial distribution function, which can be denoted by $\Phi[a; n, p]$. In turn, the first expression in square brackets can be interpreted as $\Phi[a; n, p']$, with $p' = up$, since $u > 1 > d \Rightarrow 0 \leq p' \leq 1$.

The valuation formula can therefore be rewritten as follows:

$$D(F,K,n,P_D)=\left\{F\Phi[a;n,p']-K\Phi[a;n,p]-P_D\right\}/(1+\hat{r})^{n+\hat{\tau}}. \qquad \text{(A.1)}$$

To see how this formula changes when transactions occur at shorter and shorter intervals and n tends towards infinity, it is first necessary to express as functions of n the variables $\hat{r}$, $\hat{\tau}$, u, d and q, which depend on the size of the interval considered.

As for the interest rate $\hat{r}$, it should be remembered that it refers to a period of length $h = (T - t)/n$. Consequently, on the assumption that the interest rate in the period $(T - t)$ remains unchanged, we have:

$$(1+\hat{r})^n = e^{r(T-t)},$$

where r is the instantaneous interest rate.

As for $\hat{\tau}$, which is the number of periods (of length h) between 'risposta premi' day and the settlement day, we have: $\hat{\tau} = \tau/h = n\tau/(T - t)$.

To specify the dependence of u, d and q on n, it is necessary to remember that the price at maturity of the security underlying the premium contract, F_T, is a random variable defined by:

$$F_T = u^j d^{n-j} F,$$

where j is the (random) number of periods in which the price of the underlying security rises during the n periods remaining to maturity.

Taking the logarithms of both expressions gives:

$$\ln(F_T / F) = j\ln(u) + (n-j)\ln(d) = j\ln(u/d) + n\ln(d).$$

So that:

$$E[\ln(F_T / F)] = E[j]\ln(u/d) + n\ln(d)$$

and:

$$\mathrm{var}[\ln(F_T / F)] = \mathrm{var}[j]\ln^2(u/d).$$

In each individual period we can have $j = 1$ (with a probability of q) or $j = 0$ (with a probability of $1 - q$), so that for n periods we have:

$$E[j] = n \cdot [1 \cdot q + 0 \cdot (1-q)] = nq$$

$$\mathrm{var}[j] = n \cdot \left[(1-q)^2 q + (0-q)^2 (1-q)\right] = nq(1-q).$$

So that:

$$E[\ln(F_T / F)] = [q\ln(u/d) + \ln(d)]n = \hat{\mu}\, n$$

and:

$$\mathrm{var}[\ln(F_T / F)] = q(1-q)\ln^2(u/d)n = \hat{\sigma}^2 n.$$

If $\mu(T-t)$ and $\sigma^2(T-t)$ are the parameters of the corresponding distribution in continuous time, and choosing u, d and q so that, as n tends towards infinity, $\hat{\mu}n$ tends towards $\mu(T-t)$ and $\hat{\sigma}^2 n$ tends towards $\sigma^2(T-t)$, or by putting:

$$u = e^{\sigma\sqrt{(T-t)/n}}$$
$$d = e^{-\sigma\sqrt{(T-t)/n}}$$
$$q = ½ + ½\frac{\mu}{\sigma}\sqrt{(T-t)/n},$$

it can be demonstrated (by means of the central limit theorem) that when $n \to \infty$, the binomial multiplicative distribution of the price F_T tends towards the lognormal distribution.[33] In this case, moreover:

$$\Phi[a; n, p'] \to N(x) \qquad \text{and} \qquad \Phi[a; n, p] \to N\left(x - \sigma\sqrt{T-t}\right) \quad \text{for} \quad n \to \infty$$

with $x = [\ln(F/K) + ½(T-t)]/\sigma\sqrt{T-t}$. The binomial valuation formula (A.1) therefore converges to the following valuation formula in continuous time:[34]

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[33] This is equivalent to assuming that in continuous time the forward price F follows a geometric Brownian motion.

[34] The formula (A.2) is analogous to that derived by Black to value a call option on a futures, which was to be expected in view of the nature of premium contracts.

$$D(F_t, K, T-t, P_D) = e^{-r(T+\tau-t)}\left[F_t \cdot N(x) - K \cdot N\left(x - \sigma\sqrt{T-t}\right) - P_D\right] \qquad \text{(A.2)}$$

$$x = \frac{\ln(F_t / K) + ½\sigma^2(T-t)}{\sigma\sqrt{T-t}}.$$

It should also be noted that, as $n \rightarrow \infty$, the number α of shares in the equivalent portfolio tends towards $N(x)$. In fact (16) gives:

$$\begin{aligned}
\lim_{n\to\infty} \alpha &= \lim_{n\to\infty}\left\{(D_u - D_d)/\left[(uF - dF)\cdot(1+\hat{r})^{-(n+\hat{\tau}-1)}\right]\right\} \\
&= \lim_{h\to\infty}\left\{\left[D(uF, T-t-h) - D(dF, T-t-h)\right]/(uF - dF)\right\} \times \lim_{n\to\infty}\left[e^{r(T-t)/n}\right]^{n+n\tau/(T-t)-1} \\
&= \frac{\partial D}{\partial F} e^{r(T+\tau-t)} = N(x).
\end{aligned}$$

A.2. Characteristics of the stochastic process of the price *S*

In deriving the valuation formula in continuous time (A.2) it was assumed that the forward price F followed a geometric Brownian motion:

$$\frac{dF}{F} = \mu dt + \sigma dz. \qquad \text{(A.3)}$$

This hypothesis is decidedly preferable to that whereby the price for settlement at the end of the current account S is made to follow a continuous Brownian-type process. During the life of the premium contract (in the normal case that this is written for settlement at the end of a subsequent account) the price S refers to a maturity that draws closer as 'riporti day' draws nearer and then suddenly lengthens with the start of the new monthly account. This makes it likely that there will be jumps in the price of the underlying security in correspondence with the start of the new monthly account, undermining the validity of the hypothesis of a continuous process.[35] The hypothesis (A.3) implies that the price S will behave exactly in this way.

On the other hand, it should be noted that *within the stock exchange account* the hypothesis of a geometric Brownian motion for S is valid. In fact, applying Ito's lemma to the equation

$$S = Fe^{-r'\delta} \qquad \text{(A.4)}$$

one obtains (since δ is constant):

$$\begin{aligned}
dS &= S_F dF + \left(S_t + ½F^2\sigma^2 S_{FF}\right)dt \\
&= e^{-r'\delta}\left(\mu F dt + \sigma F dz\right) \\
&= \mu S dt + \sigma S dz.
\end{aligned}$$

Within an account the price for settlement at the end of the current account is thus found to conform to a stochastic process with the same parameters as price F. This result is important for the calculation of the volatility, σ, which is an input in the valuation formula. It implies that the volatility of

[35] For an empirical test of the existence of these jumps in correspondence with the start of the new monthly account, cf. Banca d'Italia (1987).

price F can be estimated directly on the basis of the stock exchange prices S *within each account*, rather than by artificially generating the time series of the prices F, using the stock exchange prices S and the foregoing equation.

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