# The Iterated Extended Kalman Particle Filter

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***Abstract*— Particle filtering shows great promise in addressing a wide variety of non-linear and /or non-Gaussian problem. A crucial issue in particle filtering is the selection of the importance proposal distribution. In this paper, the iterated extended kalman filter (IEKF) is used to generate the proposal distribution. The proposal distribution integrates the latest observation into system state transition density, so it can match the posteriori density well. The simulation results show that the new particle filter superiors to the standard particle filter and the other filters such as the unscented particle filter (UPF), the extended kalman particle filter (PF -EKF), the EKF.**

## 1. INTRODUCTION

Nonlinear filtering problems arise in many fields including statistical signal processing, economics, statistics, biostatistics, and engineering such as communications, radar tracking, sonar ranging, target tracking, and satellite navigation [1-7]. The best-known algorithm to solve the problem of nonlinear filtering is the extended Kalman filter. This filter is based upon the principle of linearizing the measurements and evolution models using Taylor series expansions. The series approximations in the EKF algorithm can, however, lead to poor representations of the nonlinear functions and probability distribution of interest. As a result, this filter can diverge. Another popular solution strategy for the general nonlinear filtering problem is to use sequential Monte Carlo methods, also known as particle filters.

Particle filtering is a class of methods for filtering, smoothing etc. in non-linear and/or non-Gaussian state space models that may perform significantly better than traditional methods like the extended Kalman filter. A key issue in particle filtering is the selection of the proposal distribution function. In general, it is hard to design such proposals. Now there have many proposed distributions have been proposed in the literature. For example, the prior, the EKF Garssian approximation and the UKF proposal are used as the proposal distribution for particle filter [2,3,4]. In this paper, we follow the same approach, but replace the EKF proposal by an IEKF proposal. Because the IEKF compute the updated state not as an approximate conditional mean-that is, a linear combination of the prediction and the innovation, but as a maximum a posteriori (MAP) estimate [1]. So the IEKF can be used to generate proposal distributions with more precise that are close to the true mean of the target distribution.

This work is organized as follows. The generic particle filter is formulated in Section 2.The proposed iterated extended kalman particle filter is presented in Section 3. Simulation results are given in Section 5.Conclusions are presented in Section 6.

## 2. PARTICLE FILTER

### *2.1 Modeling Assumption*

Consider the nonlinear discrete time dynamic system:

$$x_k = f_k\left(x_{k-1}, v_{k-1}\right) \tag{1}$$

$$z_k = h_k\left(x_k, e_k\right) \tag{2}$$

where $f_k : \Re^{n_x} \times \Re^{n_v} \to \Re^{n_x}$ and $h_k : \Re^{n_x} \times \Re^{n_e} \to \Re^{n_z}$ represent the system evolution function and the measurement model function. $x_k \in \Re^{n_x}$ is the system state at time $k$, $z_k \in \Re^{n_z}$ is the measurement vector at time $k$. $v_{k-1} \in R^n$ and $e_k \in R^m$ represent the process noise and the measurement noise respectively.

### *2.2 Particle Filter*

The sequential importance-sampling (SIS) algorithm is a Monte Carlo method that forms the basis for most sequential MC filters developed over the past decades; see [3]. The sequential MC approach is known variously as bootstrap filtering, the condensation algorithm, particle filtering etc. The key idea is to represent the required posterior density function by a set of random samples with associated weights and to compute estimates based on these samples and weights.

Let $\left\{x_{0:k}^i, w_k^i\right\}_{i=1}^{N_s}$ denote a random measure that characterizes the posterior pdf $p(x_{0:k} \mid z_{1:k})$, where $\left\{x_{0:k}^i, i = 0,..., N_s\right\}$ is a set of support points with associated weights $\left\{w_k^i, i = 0,..., N_s\right\}$ and $x_{0:k} = \left\{x_j, j = 0,..., k\right\}$ is the set of all states up to time $k$. The weights are normalized such that $\sum_i w_k^i = 1$. Then, the posterior density at $k$ can be approximated as

$$p(\mathbf{x}_{0:k} \mid \mathbf{z}_{1:k}) \approx \sum_{i=1}^{N_s} \omega_k^i \delta(\mathbf{x}_{0:k} - \mathbf{x}_{0:k}^i) \tag{3}$$

where

$$w_k^i \propto w_{k-1}^i \frac{p(z_k \mid x_k^i)\, p(x_k^i \mid x_{k-1}^i)}{q(x_k^i \mid x_{k-1}^i, z_k)} \tag{4}$$

$p(z_k \mid x_k^i)$ is the likelihood function of the measurements $z_k$, $q(x_k^i \mid x_{k-1}^i, z_k)$ is the importance density. It can be

shown that as $N_s \to \infty$, the approximation (1) approaches the true posterior density $p(x_{0:k} \mid z_{1:k})$.

From the equation (3) and (4), it is show that the particle filter consists of recursive propagation of the weights and support points as each measurement is received sequentially. Furthermore, a common problem with the particle filter is the degeneracy phenomenon. In order to avoid the degeneracy phenomenon of the particles, the resampling scheme is important to the particle filter. To the resampling scheme, there are many selections such as sampling importance resampling, residual resamping and minimum variance sampling. In this paper, the residual resampling is used in all of the experiments. So a generic particle filter is then as described by Algorithm 1. The detail derivation of the particle filter can be founded in [3].

Algorithm 1: Generic Particle Filter

1. Initialization: $k = 0$
   - For $i = 0, \ldots, N_s$, draw the states $x_0^{(i)}$ from the prior $p(x_0)$
2. FOR $k = 1, 2 \ldots .$
   - FOR $i = 1 : N_s$

     Draw $x_k^i \sim q(x_k^i \mid x_{k-1}^i, z_k)$

     Assign the particle a weight, $w_k^i$, according to (4)

     END FOR
   - FOR $i = 1 : N_s$

     Normalize the weights:

     $w_k^i = w_k^i \Big/ \sum_{j=1:N_s} w_k^j$
   - END FOR
   - Resample:

     Multiply /Suppress samples $x_{0:k}^{(i)}$ with high/low importance weights $w_k^i$, respectively, to obtain $N_s$ random samples $x_{0:k}^{(i)}$ approximately distributed according to $p(x_{0:k} \mid z_{1:k})$

     For $i = 1 : N_s$, set $w_k^i = 1/N_s$

## 3. THE ITERATED EXTENDED KALMAN PARTICLE FILTER

The choice of proposal function is one of the most critical design issues in importance sampling algorithm. Doucet, etc [7] proved the optimal proposal distribution $q(x_k \mid x_{0:k-1}, z_{1:k}) = p(x_k \mid x_{0:k-1}, z_{1:k})$ can minimize the variance of the importance weights conditional on $x_{0:k-1}$ and $z_{1:k}$. This choice of proposal distribution has also advocated by other researchers. However, This optimal importance density suffers two major drawbacks. It requires the ability to sample from $p(x_k \mid x_{0:k-1}, z_{1:k})$ and to evaluate the integral over the new state. In the general case, it may not be straightforward to do either of these things. Therefore, the distribution $p(x_k \mid x_{0:k-1})$ is the most popular choice of proposal function. Compared with the optimal proposal distribution, it is attractive due to its simplicity in sampling from the prior densities and the calculation of weights. But because it is not incorporating the most recent observations, the proposal distribution is very inefficient sometimes, and the estimation result is poor. In order to solve this problem, several techniques have been proposed. For example, the EKF or UKF approximation is used as the proposal distribution for a particle filter. In this section, we will use the IEKF to generate proposal distributions with a better performance than the EKF or UKF.

### *3.1 The Iterated Extended Kalman Filter*

The conditional probability density function of $x(k+1)$ given $Z^{k+1}$ can be written, assuming all the pertinent random variables to be gaussian, as

$$\begin{aligned} p\left[x(k+1) \mid Z^{k+1}\right] &= p\left[x(k+1) \mid z(k+1), Z^{k+1}\right] \\ &= \frac{1}{c}\, p\left[z(k+1) \mid x(k+1)\right] p\left[x(k+1) \mid Z^{k}\right] \\ &= \frac{1}{c}\, N\left[z(k+1); h\left[k+1, x(k+1)\right], R(k+1)\right] \\ &\quad .N\left[x(k+1); \hat{x}(k+1 \mid k), P(k+1 \mid k)\right] \end{aligned} \tag{5}$$

Maximizing the above with respect to $x(k+1)$ is equivalent to minimizing

$$\begin{aligned} J\left[x(k+1)\right] = &\frac{1}{2}\left\{z(k+1) - h\left[k+1, x(k+1)\right]\right\}' R(k+1)^{-1} \left\{z(k+1) - h\left[k+1, x(k+1)\right]\right\} \\ &+ \frac{1}{2}\left[x(k+1) - \hat{x}(k+1 \mid k)\right]' P(k+1 \mid k)^{-1} \left[x(k+1) - \hat{x}(k+1 \mid k)\right] \end{aligned} \tag{6}$$

The iterative minimization of (6), say, using a Newton-Raphson algorithm, will yield an approximate MAP estimate of $x(k+1)$. This is done by expanding $J$ in a taylor series up to second order about the $i$ th iterated value of the estimate of $x(k+1)$, denoted (without time argument) as $x^i$,

$$J = J^i + J^{i'}\left(x - x^i\right) + \frac{1}{2}\left(x - x^i\right)' J_{xx}^i \left(x - x^i\right) \tag{7}$$

where, using abbreviated notation,

$$J' = J\,|_{x = x^i}$$

and

$$J_x^i = \nabla_x J\,|_{x = x^i}$$

$$J_{xx}^i = \nabla_x \nabla_x' J\,|_{x = x^i}$$

are the gradient and Hessian of $J$ with respect to $x(k+1)$.

Setting the gradient of (7) with respect to $x$ to zero yields the next value of $x$ in the iteration to minimize as

$$x^{i+1} = x^i - \left(J_{xx}^i\right)^{-1} J_x^i \tag{8}$$

The gradient $J$ is, using now the full notation,

$$J_x^i = -h_x^i\left[k+1, \hat{x}^i(k+1|k+1)'\right] R(k+1)^{-1} \cdot\left\{z(k+1) - h\left[k+1, \hat{x}^i(k+1|k+1)\right]\right\} + P(k+1|k)^{-1}\left[\hat{x}^i(k+1|k+1) - \hat{x}(k+1|k)\right] \quad (9)$$

The Hessian of $J$, retaining only up to the first derivative of $h$, is

$$J_{xx}^i = h_x^i\left[k+1, \hat{x}^i(k+1|k+1)\right]^{-1} R(k+1)^{-1} h_x\left[k+1, \hat{x}^i(k+1|k+1)\right] + P(k+1|k)^{-1} = H^i(k+1)' R(k+1)^{-1} H^i(k+1) + P(k+1|k)^{-1} \quad (10)$$

where

$$H^i(k+1) \triangleq h_x\left[k+1, \hat{x}^i(k+1|k+1)\right]$$

is the Jacobian of the relinearized measurement equation.

Using the matrix inversion lemma[1], one has

$$\left(J_{xx}^i\right)^{-1} = P(k+1|k)^{-1} - P(k+1|k)H^i(k+1)' \cdot\left[H^i(k+1)P(k+1|k)H^i(k+1)' + R(k+1)\right]^{-1} H^i(k+1)P(k+1|k) = P^i(k+1|k+1) \quad (11)$$

Substituting (9) and (11) into (8) yields

$$\hat{x}^{i+1}(k+1|k+1) = \hat{x}^i(k+1|k+1) + P^i(k+1|k+1)H^i(k+1)' \cdot R(k+1)^{-1}\left\{z(k+1) - h\left[k+1, \hat{x}^i(k+1|k+1)\right]\right\} - P^i(k+1|k+1)P(k+1|k)^{-1}\left[\hat{x}^i(k+1|k+1) - \hat{x}(k+1|k)\right] \quad (12)$$

which is the iterated extended Kalman filter.

Starting the iteration for $i = 0$ with

$$\hat{x}^0(k+1|k+1) \triangleq \hat{x}(k+1|k) \quad (13)$$

causes the last term in ()to be zero and yields after the first iteration $\hat{x}^1(k+1|k+1)$,that is ,the same as the first-order (noniterated)EKF

The covariance associated with $\hat{x}^i(k+1|k+1)$is, from (11), given by

$$P^i(k+1|k+1) = P(k+1|k) - P(k+1|k)H^i(k+1)' \cdot\left[H^i(k+1)P(k+1|k)H^i(k+1)' + R(k+1)\right]^{-1} H^i(k+1)P(k+1|k) \quad (14)$$

### *3.2 The Iterated Extended Kalman Particle Filter*

As shows in section above, an approximate MAP estimate can be obtained by an iteration that amounts to relinearization of the measurement equation. So the iterated extended kalman filter is able to more accurate than the EKF. Distribution generated by the IEKF generally has a bigger support overlap with the true posterior distribution than the overlap achieved by the EKF estimates. So this makes the IEKF a better candidate for more accurate proposal distribution generation within the particle filter framework.

The new filter that results from using a IEKF for proposal distribution generation within a particle filter framework is called the iterated extended kalman particle filter. The algorithm of the IEKF is as follows

Algorithm 2: The Iterated Extended Kalman Particle Filter

1. Initialization: $k = 0$
   - For $i = 0, ..., N_s$, draw the states $x_0^{(i)}$ from the prior $p(x_0)$
2. FOR $k = 1, 2....$
   - FOR $i = 1: N_s$

     --Compute the Jacobians $F_k^i$ & $G_k^i$ of the process model

     --Update the particles with the IEKF:

     $\hat{x}_{k+1/k}^i = f(\hat{x}_{k/k}^i)$

     $P(k+1|k) = F_k^i P(k|k) F_k^{i'} + G_k^i$

     FOR $j = 1: c$ (c is the number of iteration)

     Compute the Jacobians $H_{k_j}^i$ &$U_{k_j}^i$ of the measurement model

     Update the covariance $P_{k_j}^i$ with the equation (14)

     Update the state estimate $\hat{x}_{k_j}^i$ with the equation (12)

     END FOR

     --Draw $x_k^i \sim q(x_k^i | x_{k-1}^i, z_k) = N\left(\hat{x}_{k_j}^i, P_{k_j}^i\right)$

     --Assign the particle a weight, $w_k^i$, according to (4)

     END FOR
   - FOR $i = 1: N_s$

     Normalize the weights: $w_k^i = w_k^i \Big/ \sum_{j=1:N_s} w_k^j$
   - END FOR
   - Resample:

     Multiply /Suppress samples $x_{0:k}^{(i)}$ with high/low importance weights $w_k^i$, respectively, to obtain $N_s$ random samples $x_{0:k}^{(i)}$ approximately distributed according to $p(x_{0:k} | z_{1:k})$

     For $i = 1: N_s$, set $w_k^i = 1/N_s$

## 4. SIMULATION

This section presents the simulation result of the proposed algorithm described above. In order to compare the performance with these of the convention methods, the system models were taken from [4] as following

$$x_k = 1 + \sin(0.4\pi k) + 0.5 x_{k-1} + v_{k-1}$$
$$y_k = \begin{cases} 0.2 x_k^2 + e_k & k \le 30 \\ 0.5 x_k - 2 + e_k & k > 30 \end{cases} \quad (15)$$

where $v_k$ is a Gamma $\varsigma_a(3,2)$ random variable modeling the process noise, The measurement noise $e_k$ is drawn from a Gaussian distribution $N(0, 0.0001)$ .The experiment was repeated 100 times with random re-initialization for each run. All of the particle filters used 200 particles and residual resampling. The output of the algorithm is the mean of

samples set that can be computed. $\hat{x}_k = \frac{1}{N}\sum_{j=1}^{N_s} x_t^j$ . The mean square errors of each run is defined as

$$MSE = \left( \frac{1}{T}\sum_{k=1}^{T} \left( \hat{x}_k - x_k \right)^2 \right)^{1/2} \quad (16)$$

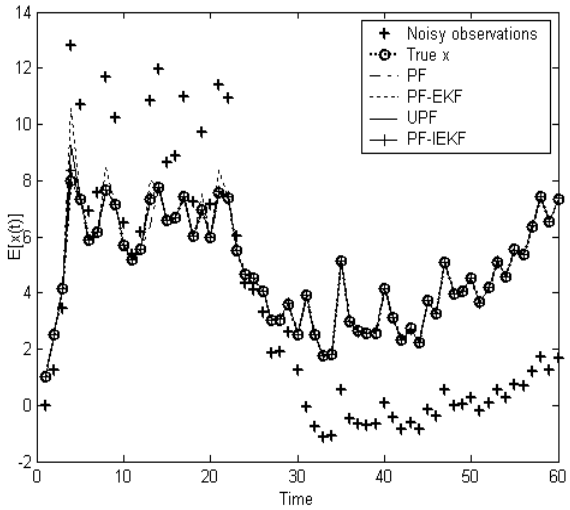


Fig.1. Estimate of the system state

Fig.1 compares the estimate of the system state generated from a single run of the different particle filter. The results show that the estimate of the PF and the PF-EKF sometimes biased the true state very large. Because the UKF and IEKF calculates the posterior covariance accurately to the 2rd order, it make the UPF and the PF-IEKF provided improvement over the results of the PF and that PF-EKF clearly achieved the better performance. Table1 summarizes the performance of the different filters. The table shows the means and variances of the mean-square-error (MSE) of the state estimates. To non-linear and /or non-Gaussian problem, the performance of the PF-IEKF is superior to the other solutions, and the EKF is the worst.

Table 1: The mean and variance of the MSE

| Algorithm | MSE | |
|---|---|---|
| | Mean | Var |
| EKF | 0.3933 | 0.0182 |
| UKF | 0.2969 | 0.0127 |
| IEKF | 0.1504 | 0.0098 |
| PF | 0.2205 | 0.0477 |
| PF-EKF | 0.3231 | 0.0183 |
| PF-UKF | 0.0675 | 0.0062 |
| PF-IEKF | 0.0495 | 0.0012 |

## 5. CONCLUSION

In this paper, a new particle filter that uses the IEKF to generate the proposal distribution is proposed. Because the IEKF can generate an approximate MAP estimate of the system state, the proposal distribution based on the IEKF is closer to the true posterior distribution than the other such as the EKF and UKF. The simulation result shows the proposed particle filter is evidently better than the standard particle filter. At the same time, because the performance of the IEKF is superior to the EKF and UKF, the precise of the proposed particle filter is higher the PF-EKF and the UPF.

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