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The Iterative Transition Phenomenon between Periodic and Turbulent States in a Dissipative Dynamical System

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Recently, examples[1] have been presented by several authors of dissipative dynamical systems which exhibit erratic motions. The situation, however, seems unsatisfactory concerning the stability of these motions under deformations of the vector fields.

Employing the Lyapunov number described below, we want to show in this paper that erratic motions (called to be in turbulent states) do not, in general, persist over the whole space of bifurcation parameter, but iterative transition between turbulent and periodic states may occur in dissipative dynamical systems.

Here we take, as an example, the Lorenz system defined by $\dot{X}=-\sigma X+\sigma Y$, $\dot{Y}=-XZ+\gamma X-Y$ and $\dot{Z}=XY-bZ$, where γ, σ and b are parameters. It is known that this system has a high-dimensional attractor (called the Lorenz attractor) beyond $\gamma=\gamma_T=\sigma(\sigma+b+3)/(\sigma-b-1)$. It seems natural to ask what happens to the Lorenz attractor, when γ takes sufficiently large value than γ_T. In order to answer this question, we employ the Lyapunov number estimated by Eq. (1), which measures the asymptotic orbital separation between two trajectories initially located very close in the state space. This quantity has been introduced successfully in a previous paper[2] to evaluate the degrees of stochasticity of erratic motions.

The Lyapunov number is estimated by the following quantity:

$$k=\lim_{n\to\infty}\frac{1}{n\tau}\sum_{i=1}^{n}\ln\left(\frac{\boldsymbol{d}_i}{\boldsymbol{d}_0}\right), \tag{1}$$

where, roughly speaking, $\boldsymbol{d}_0$ and $\boldsymbol{d}_i$ correspond respectively to the relative distance vectors between two orbits at an initial time and at time $i\cdot\tau$ (with τ a const.). The meaning of the symbols and the detailed properties of the quantity given by Eq. (1) are found in Ref. 2). The dependence of the Lyapunov number on the parameter γ (σ and b are assumed to be const., i.e., $\sigma=16.0$ and $b=4.0$) is calculated numerically for the Lorenz system and is shown in Fig. 1.

According to the definition of Eq. (1), it may be understood that three types of the motion are distinguishable from each other, i.e., (a) if $k>0$, the motion is asymptotically non-periodic and, furthermore, shows the exponential orbital instability, (b) if $k=0$, the motion is asymptotically periodic and/or quasi-periodic, and (c) if $k<0$, the motion is asymptotically attracted to a fixed point.

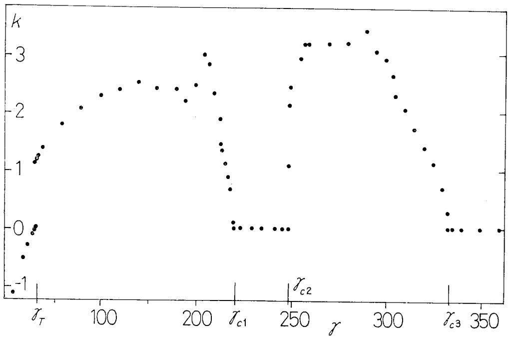

Fig. 1.

Figure 1 shows that there exist different kinds of attractors in the Lorenz system which appear successively depending on the parameter:

(0) $\gamma<\gamma_T$: the fixed point,

(1) $\gamma_T<\gamma<\gamma_{c_1}$: the Lorenz attractor (type L_1) cited above,

(2) $\gamma_{c_1}<\gamma<\gamma_{c_2}$: the periodic attractor (type P_1),

(3) $\gamma_{c_2}<\gamma<\gamma_{c_3}$: the high-dimensional attractor (type L_2) being different from the Lorenz attractor,

(4) $\gamma>\gamma_{c_3}$: the periodic attractor (type P_2) being different from the type P_1. (The existence of the high-dimensional attractor of the type L_2 was shown by Rössler.[3])

The structure, on the Poincaré surface, of the high-dimensional attractor of the type L_2 and the orbits of the periodic attractors are shown in Fig. 2 and Fig. 3, respectively.

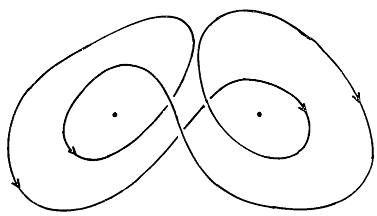
Fig. 2(a).

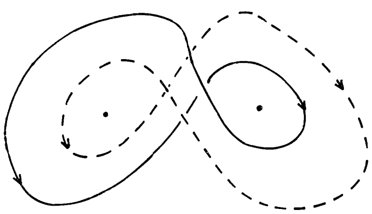
Fig. 2(b).

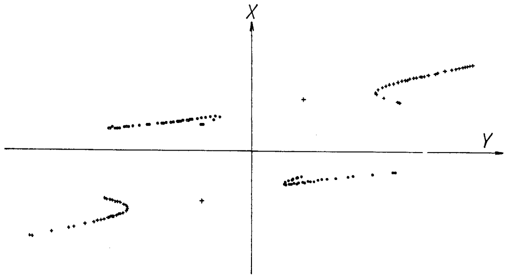

Fig. 3.

It may be noteworthy that the structure of the periodic orbits in the regions $\gamma_{c_1}<\gamma<\gamma_{c_2}$ and $\gamma>\gamma_{c_3}$ is different from each other. While the orbit shown in Fig. 2(a) is self-symmetric under the transformation $(X, Y, Z)\to(-X, -Y, Z)$, the other shown in Fig. 2(b) is not symmetric, but maps into the mirror image of itself under the same transformation. This difference of the structure of two orbits is intimately connected with the topologically distinct structures of two high-dimensional attractors which emerge from the periodic attractors, when the parameter γ is changed to smaller values. If γ is chosen as the bifurcation parameter ($\sigma=16.0$ and $b=4.0$), it seems to exist in the Lorenz system, at the least, four bifurcation points between the simple attractor such as the fixed point or the closed orbit, and the more complicated high-dimensional attractor. The scheme of these bifurcations is the following:

(1) The 1st bifurcation occurs at γ_T and this bifurcation is characterized as the inverted bifurcation.

(2) The 2nd (4-th) bifurcation occurs at γ_{c_1} (γ_{c_3}), and this bifurcation involves similar feature to the bifurcation studied by Brunovsky,[4] i.e., if the parameter γ is gradually decreased from a comparatively large value through γ_{c_1} (γ_{c_3}), the periodic motion with the period 2^n ($n=0, 1, 2, \cdots$) bifurcates successively, and then, forming an enormous number (perhaps, an infinite number) of disconnected sub-region on the Poincaré surface, the non-periodic motion occurs at the critical value γ_{c_1} (γ_{c_3}), below which disconnected sub-regions unite successively in a pairwise manner leading to a single connected region. A half of the whole succession described above was analyzed by May[5] in case of 1-dimensional difference equations.

(3) The 3rd bifurcation occurs at γ_{c_2} and this bifurcation is of the different type from (1) or (2), and has very complicated structures.

The details concerning these bifurcations will be reported in a forthcoming paper.

The Lyapunov number which is employed in this paper is, if the system has the orbital instability, connected to the holding property of the Poincaré mapping and, if there exists an invariant measure on the attractor, gives the Kolmogorov-entropy with respect to this invariant measure. Therefore, there is a possibility that the values k calculated in our numerical work become the Kolmogorov-entropy.

Finally, we want to note an implication of this work that arises if it would be allowed to consider our result as a thermodynamic phenomenon. The phenomenon being similar to the one, found in the present work, that turbulent state does not persist over the whole range of the bifurcation parameter, but ends up with a periodic state at some finite value has been known in the experiment of the Gunn instability.[6] However, complete theoretical explanation of the Gunn instability has not been given yet. Of course our result cannot explain the mechanism of the Gunn instability itself either, but the iterative transition phenomenon between periodic and turbulent states obtained, as an example, for the Lorenz system may give a new picture in the study of the dissipative structure far from thermal-equilibrium.

1) See, for example, O. E. Rössler, Z. Naturforsch. **31a** (1976), 1664.
2) T. Nagashima and I. Shimada, Prog. Theor. Phys. **58** (1977), 1318.
3) O. E. Rössler, Phys. Letters **60A** (1977), 392.
4) P. Burnovsky, *Symposium on Differential Equation and Dynamical Systems* (Warwick, 1968-69).
5) R. M. May, J. Theor. Biol. **51** (1975), 511.
6) S. Kabashima, H. Yamazaki and T. Kawakubo, J. Phys. Soc. Japan **40** (1976), 921.