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THE JANTZEN SUM FORMULA FOR CYCLOTOMIC q–SCHUR ALGEBRAS

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Abstract. The cyclotomic q-Schur algebra was introduced by Dipper, James and Mathas, in order to provide a new tool for studying the Ariki-Koike algebra. We here prove an analogue of Jantzen's sum formula for the cyclotomic q-Schur algebra. Among the applications is a criterion for certain Specht modules of the Ariki-Koike algebras to be irreducible.

1. Introduction

In [6] Richard Dipper and the authors introduced the cyclotomic q–Schur algebras and showed that they are quasi–hereditary cellular algebras. By definition, a cyclotomic q–Schur algebra is a certain endomorphism algebra attached to an Ariki–Koike algebra in much the same way as the q–Schur algebra [5] is defined as an endomorphism algebra of a particular module for the Iwahori–Hecke algebra of the symmetric group.

One of our motivations for defining the cyclotomic q–Schur algebras was to provide another tool for studying the Ariki–Koike algebras. In this paper we use the cyclotomic q–Schur algebras to prove a version of the Jantzen sum formula for the Ariki–Koike algebras. Most of the argument is devoted to first proving an analogue of Jantzen's sum formula for the Weyl modules of the cyclotomic q–Schur algebra. The result for the Ariki–Koike algebras is then deduced by a Schur functor argument. As a corollary of these results we obtain criteria for the Weyl modules of the cyclotomic q–Schur algebras, and for certain of the Specht modules of the Ariki–Koike algebras, to be irreducible.

We note that as a special case of our results we obtain, for the first time, an analogue of the Jantzen sum formula for Coxeter groups of type **B**.

In the case of the q–Schur algebra it is possible to give a geometric proof of Jantzen's sum formula [1]. As yet, in the cyclotomic case there is no algebra which plays the rôle of the quantum group of type **A**; consequently, an algebraic approach is necessary. The proof we give generalizes and extends the argument of [13].

2. The cyclotomic q–Schur algebra

We recall some definitions and results from [6].

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Fix integers r and n with $r \geq 1$ and $n \geq 1$ and let R be a commutative ring with 1 and let $q, Q_1, Q_2, \dots, Q_r$ be elements of R, with q invertible.

The Ariki–Koike algebra $\mathcal{H}$ [3] is the associative R–algebra with generators $T_0, T_1, \dots, T_{n-1}$ subject to the following relations:

$$\begin{aligned}(T_0 - Q_1)\cdots(T_0 - Q_r) &= 0,\\ T_0T_1T_0T_1 &= T_1T_0T_1T_0,\\ (T_i+1)(T_i-q) &= 0, && \text{for } 1 \leq i \leq n-1,\\ T_{i+1}T_iT_{i+1} &= T_iT_{i+1}T_i, && \text{for } 1 \leq i \leq n-2,\\ T_iT_j &= T_jT_i, && \text{for } 0 \leq i < j-1 \leq n-2.\end{aligned}$$

Let $\mathfrak{S}_n$ be the symmetric group on $\{1, 2, \dots, n\}$. Then $\mathfrak{S}_n$ is generated by $s_1, s_2, \dots, s_{n-1}$ where $s_i = (i, i+1)$ for $1 \leq i < n$. If $w = s_{i_1}s_{i_2}\dots s_{i_k}$ is a reduced expression for $w \in \mathfrak{S}_n$ (that is, k is minimal), we write $\ell(w) = k$ and define $T_w = T_{i_1}T_{i_2}\dots T_{i_k}$. Let $\mathcal{H}(\mathfrak{S}_n)$ be the subalgebra of $\mathcal{H}$ spanned by $\{\, T_w \mid w \in W \,\}$; then $\mathcal{H}(\mathfrak{S}_n)$ is the Iwahori–Hecke algebra of $\mathfrak{S}_n$.

Define elements $L_1, L_2, \dots, L_n$ of $\mathcal{H}$ by $L_i = q^{1-i}T_{i-1}\dots T_1T_0T_1\dots T_{i-1}$. We have the following well–known result (cf. [3, 3.3] and [4, (2.1), (2.2)]).

2.1. *Suppose that $1 \leq i \leq n-1$ and $1 \leq j \leq n$. Then*
(i) *L_i and L_j commute.*
(ii) *T_i and L_j commute if $i \neq j-1, j$.*
(iii) *T_i commutes with L_iL_{i+1} and with $L_i + L_{i+1}$.*
(iv) *If $a \in R$ and $i \neq j$, then T_i commutes with $(L_1 - a)(L_2 - a)\dots(L_j - a)$.*

A composition $\alpha = (\alpha_1, \alpha_2, \dots)$ is a finite sequence of non–negative integers; we denote by $|\alpha|$ the sum of the sequence. A multicomposition of n (into r components) is an ordered r–tuple $\mu = (\mu^{(1)}, \dots, \mu^{(r)})$ of compositions $\mu^{(k)}$ such that $\sum_{k=1}^{r} |\mu^{(k)}| = n$. We call $\mu^{(k)}$ the kth component of μ. A partition is a composition whose parts are non–increasing; a multicomposition is a multipartition if all of its components are partitions.

Definition 2.2. Suppose that μ is a multicomposition of n and let $\mathbf{a} = (a_1, \dots, a_r)$ be an r–tuple of integers a_k such that $0 \leq a_k \leq n$ for all k.
(i) Let $u_{\mathbf{a}}^+ = u_{\mathbf{a},1}u_{\mathbf{a},2}\cdots u_{\mathbf{a},r}$ where $u_{\mathbf{a},k} = \prod_{i=1}^{a_k}(L_i - Q_k)$ for $1 \leq k \leq r$.
(ii) Let $x_\mu = \sum_{w \in \mathfrak{S}_\mu} T_w$ where $\mathfrak{S}_\mu = \mathfrak{S}_{\mu^{(1)}} \times \mathfrak{S}_{\mu^{(2)}} \times \cdots \times \mathfrak{S}_{\mu^{(r)}}$.
(iii) Let $m_\mu = u_{\mathbf{a}}^+ x_\mu$ where $\mathbf{a} = (a_1, \dots, a_r)$ and $a_k = \sum_{i=1}^{k-1} |\mu^{(i)}|$, for $1 \leq k \leq r$.
(iv) Let $M^\mu = m_\mu\mathcal{H}$.

Note that $u_{\mathbf{a}}^+ x_\mu = x_\mu u_{\mathbf{a}}^+$ by 2.1(iv) and that $a_1 = 0$ in Definition 2.2(iii).

Given two multicompositions μ and ν write $\mu \trianglerighteq \nu$ if for all $i \geq 0$ and k with $1 \leq k \leq r$

$$\sum_{j=1}^{k-1} |\mu^{(j)}| + \sum_{j=1}^{i} \mu_j^{(k)} \geq \sum_{j=1}^{k-1} |\nu^{(j)}| + \sum_{j=1}^{i} \nu_j^{(k)}.$$

If $\mu \trianglerighteq \nu$ and $\mu \neq \nu$, then we write $\mu \triangleright \nu$.

Let Λ be a finite subset of the set of all multicompositions of n which have r components such that if $\mu \in \Lambda$ and $\lambda \trianglerighteq \mu$ for some multipartition λ, then $\lambda \in \Lambda$. Let Λ^+ be the set of multipartitions in Λ.

The main arena for the investigations of this paper is the cyclotomic q–Schur algebra, which we now define.

Definition 2.3 ([6])**.** The cyclotomic q–Schur algebra is the endomorphism algebra

$$\mathcal{S}(\Lambda) = \operatorname{End}_{\mathcal{H}}\Big(\bigoplus_{\mu\in\Lambda} M^\mu\Big).$$

Generally we omit Λ and simply write $\mathcal{S}$.

In order to describe a basis of $\mathcal{S}$ and its irreducible representations we next recall the combinatorics of semistandard tableaux from [6].

Suppose that ν is a multicomposition of n and let $\mathbf{r} = \{1, 2, \dots, r\}$. The diagram of ν is the set

$$[\nu] = \{\, (i, j, k) \in \mathbb{N}\times\mathbb{N}\times\mathbf{r} \mid 1 \le j \le \lambda_i^{(k)} \,\}.$$

The elements of $[\nu]$ are the nodes of ν; more generally, a node is any element of $\mathbb{N}\times\mathbb{N}\times\mathbf{r}$.

A ν–tableau $\mathtt{T}$ is a mapping from the diagram of ν into $\mathbb{N}\times\mathbf{r}$; informally, we shall regard $\mathtt{T}$ as an ordered r–tuple of labelled diagrams, as in the example below. In particular, we will write $\mathtt{T} = (\mathtt{T}^{(1)}, \dots, \mathtt{T}^{(r)})$ and will speak of the components of $\mathtt{T}$ and their rows and columns. We say that $\mathtt{T}$ is a tableau of type μ if the number of entries in $\mathtt{T}$ equal to (i, k) is $\mu_i^{(k)}$ for all $(i, k) \in \mathbb{N}\times\mathbf{r}$.

Below, and in all later examples, we write i_k in place of the ordered pair (i, k).

Example 2.4. (i) Suppose that ν is a multicomposition and let $\mathtt{T}^\nu$ be the ν–tableau of type ν such that $\mathtt{T}^\nu(i, j, k) = (i, k)$ for all $(i, j, k) \in [\nu]$.

(ii) Let $\lambda = \big((3, 2), (2, 1), (1^2)\big)$. Then two λ–tableaux are

$$\mathtt{T}^\lambda = \left(\begin{array}{|c|c|c|}\hline 1_1 & 1_1 & 1_1 \\ \hline 2_1 & 2_1 \\ \cline{1-2}\end{array},\ \begin{array}{|c|c|}\hline 1_2 & 1_2 \\ \hline 2_2 \\ \cline{1-1}\end{array},\ \begin{array}{|c|}\hline 1_3 \\ \hline 2_3 \\ \hline\end{array} \right) \text{and } \mathtt{S} = \left(\begin{array}{|c|c|c|}\hline 1_1 & 2_1 & 1_2 \\ \hline 2_2 & 3_3 \\ \cline{1-2}\end{array},\ \begin{array}{|c|c|}\hline 2_2 & 1_3 \\ \hline 2_3 \\ \cline{1-1}\end{array},\ \begin{array}{|c|}\hline 3_3 \\ \hline 4_3 \\ \hline\end{array} \right).$$

Here $\mathtt{S}$ is a λ–tableau of type $\big((1^2), (1, 2), (1, 1, 2, 1)\big)$.

Given (i, k) and (j, l) in $\mathbb{N}\times\mathbf{r}$, we say that $(i, k) < (j, l)$ if either $k < l$, or $k = l$ and $i < j$.

Definition 2.5 ([6])**.** A λ–tableau $\mathtt{T} = (\mathtt{T}^{(1)}, \dots, \mathtt{T}^{(r)})$ of type μ is semistandard if λ is a multipartition and

(i) the entries in each row of each component $\mathtt{T}^{(k)}$ are non–decreasing; and,

(ii) the entries in each column of each component $\mathtt{T}^{(k)}$ are strictly increasing; and,

(iii) if $(a, b, c) \in [\lambda]$ and $\mathtt{T}(a, b, c) = (i, k)$, then $k \ge c$.

Let $\mathcal{T}_0(\lambda, \mu)$ be the set of semistandard λ–tableaux of type μ.

For example, the λ–tableau $\mathtt{T}^\lambda$ defined in Example 2.4(i) is the unique semistandard λ–tableau of type λ. The λ–tableau $\mathtt{S}$ in Example 2.4(ii) is also semistandard.

Before we can describe how the semistandard tableaux index a basis for $\mathcal{S}$, we first need to single out special semistandard tableaux which index a basis of the Ariki–Koike algebra $\mathcal{H}$.

Definition 2.6. (i) Let $\omega = \big((0), \dots, (0), (1^n)\big)$, a multipartition of n.

(ii) Let λ be a multipartition. A standard λ–tableau is a semistandard λ–tableau of type ω.

(iii) Let $\operatorname{Std}(\lambda) = \mathcal{T}_0(\lambda, \omega)$ be the set of standard λ–tableaux.

Let $\mathtt{T}$ be a ν–tableau of type ω. Then, for all $x \in [\nu]$, we have $\mathtt{T}(x) = (i, r)$ for some $i \in \{1, 2, \dots, n\}$; we identify $\mathtt{T}$ with the map $\mathfrak{t}$ determined by $\mathtt{T}(x) = (\mathfrak{t}(x), r)$ for all $x \in [\nu]$. Then $\mathfrak{t}$ is a standard tableau if and only if ν is a multipartition and in each component $\mathfrak{t}^{(k)}$ the entries are strictly increasing along each row and down each column.

We will always denote tableaux of type ω by lower case letters in order to distinguish them from tableaux of other types.

Given a multicomposition ν, let $\mathfrak{t}^\nu$ be the tableau with the integers $1, 2, \dots, n$ entered in order along the rows of $[\nu]$. The symmetric group $\mathfrak{S}_n$ acts on the set of ν–tableaux of type ω by letter permutations; note that the Young subgroup $\mathfrak{S}_\nu$ is precisely the row stabilizer of $\mathfrak{t}^\nu$.

If λ is a multipartition and $\mathfrak{t}$ is a standard λ–tableau, then define $d(\mathfrak{t}) \in \mathfrak{S}_n$ by $\mathfrak{t} = \mathfrak{t}^\lambda d(\mathfrak{t})$. Then $d(\mathfrak{t})$ is a (distinguished) right coset representative of $\mathfrak{S}_\lambda$ in $\mathfrak{S}_n$.

Let $*\colon \mathcal{H} \to \mathcal{H}$ be the R–linear antiautomorphism of $\mathcal{H}$ determined by $T_i^* = T_i$ for all i with $0 \le i < n$. In particular, $T_w^* = T_{w^{-1}}$ for all $w \in \mathfrak{S}_n$.

Definition 2.7. Suppose that λ is a multipartition of n and that $\mathfrak{s}$ and $\mathfrak{t}$ are standard λ–tableaux. Let $m_{\mathfrak{s}\mathfrak{t}} = T_{d(\mathfrak{s})}^* m_\lambda T_{d(\mathfrak{t})}$.

The proof of the following result can be found in [6, 3.26].

Theorem 2.8 (The Standard Basis Theorem)**.** *The Ariki–Koike algebra $\mathcal{H}$ is a free R–module with cellular basis*

$$\left\{ m_{\mathfrak{s}\mathfrak{t}} \,\middle|\, \begin{matrix} \mathfrak{s} \text{ and } \mathfrak{t} \text{ are standard } \lambda\text{–tableaux for} \\ \text{some multipartition } \lambda \text{ of } n \end{matrix} \right\}.$$

We call this basis the standard basis of $\mathcal{H}$; it is a cellular basis in the sense of Graham and Lehrer [8]. Note that $m_{\mathfrak{s}\mathfrak{t}}^* = m_{\mathfrak{t}\mathfrak{s}}$.

Given a standard λ–tableau $\mathfrak{t}$ and a multicomposition μ let $\mu(\mathfrak{t})$ be the λ–tableau of type μ obtained by replacing each entry m in $\mathfrak{t}$ by (i, k) if m appears in row i of the kth component of $\mathfrak{t}^\mu$. For example, $\mathtt{T}^\lambda = \lambda(\mathfrak{t}^\lambda)$.

2.9 ([6, Proposition 6.3])**.** *Let μ and ν be multicompositions of n. Then $M^\mu \cap M^{\nu *}$ is a free R–module with basis*

$$\left\{ m_{\mathtt{S}\mathtt{T}} \,\middle|\, \begin{matrix} \mathtt{S} \in \mathcal{T}_0(\lambda, \mu) \text{ and } \mathtt{T} \in \mathcal{T}_0(\lambda, \nu) \text{ for some} \\ \text{multipartition } \lambda \text{ of } n \end{matrix} \right\}$$

where $m_{\mathtt{S}\mathtt{T}} = \sum_{\mathfrak{s},\mathfrak{t}} m_{\mathfrak{s}\mathfrak{t}}$ and $(\mathfrak{s}, \mathfrak{t})$ runs over all pairs of standard λ–tableaux such that $\mathtt{S} = \mu(\mathfrak{s})$ and $\mathtt{T} = \nu(\mathfrak{t})$.

Note that $m_\lambda = m_{\mathfrak{t}^\lambda \mathfrak{t}^\lambda} = m_{\mathtt{T}^\lambda \mathfrak{t}^\lambda} = m_{\mathfrak{t}^\lambda \mathtt{T}^\lambda} = m_{\mathtt{T}^\lambda \mathtt{T}^\lambda}$.

In particular, 2.9 shows that the maps $\varphi_{\mathtt{S}\mathtt{T}}$ below are well–defined elements of $\mathcal{S}$.

Definition 2.10. Let λ be a multipartition of n and let μ and ν be multicompositions of n. Suppose that $\mathtt{S} \in \mathcal{T}_0(\lambda, \mu)$ and $\mathtt{T} \in \mathcal{T}_0(\lambda, \nu)$. Then $\varphi_{\mathtt{S}\mathtt{T}} \in \mathcal{S}$ is the $\mathcal{H}$–homomorphism such that

$$\varphi_{\mathtt{S}\mathtt{T}}(m_\alpha h) = \delta_{\alpha\nu} m_{\mathtt{S}\mathtt{T}} h$$

for all $\alpha \in \Lambda$ and all $h \in \mathcal{H}$.

Theorem 2.11 (The Semistandard Basis Theorem [6, 6.12])**.** *The cyclotomic q-Schur algebra $\mathcal{S}$ is free as an R–module with cellular basis*

$$q\left\{ \varphi_{\mathtt{S}\mathtt{T}} \,\middle|\, \begin{matrix} \mathtt{S} \in \mathcal{T}_0(\lambda, \mu), \mathtt{T} \in \mathcal{T}_0(\lambda, \nu) \text{ for some} \\ \mu, \nu \in \Lambda \text{ and } \lambda \in \Lambda^+ \end{matrix} \right\}.$$

We call the basis $\{\varphi_{\mathtt{ST}}\}$ the semistandard basis of $\mathcal{S}$. Because it is cellular, the R–linear map $*\colon \mathcal{S} \to \mathcal{S}$ determined by $\varphi_{\mathtt{ST}}^* = \varphi_{\mathtt{TS}}$ is an anti-automorphism of $\mathcal{S}$ (see [6, 6.9]).

For each multipartition λ in Λ^+ let $\bar{\mathcal{S}}^\lambda$ be the R–submodule of $\mathcal{S}$ with basis

$$\left\{ \varphi_{\mathtt{UV}} \middle| \begin{array}{c} \mathtt{U} \in \mathcal{T}_0(\alpha,\mu), \mathtt{V} \in \mathcal{T}_0(\alpha,\nu) \text{ for some } \mu,\nu \in \Lambda \text{ and} \\ \alpha \in \Lambda^+ \text{ with } \alpha \rhd \lambda \end{array} \right\}.$$

By [6, 6.11], $\bar{\mathcal{S}}^\lambda$ is a two–sided ideal of $\mathcal{S}$.

Recall the λ–tableau $\mathtt{T}^\lambda$ from Example 2.4(i). It is easy to see from the definitions that $\varphi_{\mathtt{T}^\lambda\mathtt{T}^\lambda}$ restricts to the identity map on M^λ.

Definition 2.12. Let $\lambda \in \Lambda^+$. The Weyl module W^λ is the submodule of $\mathcal{S}/\bar{\mathcal{S}}^\lambda$ given by $W^\lambda = (\bar{\mathcal{S}}^\lambda + \varphi_{\mathtt{T}^\lambda\mathtt{T}^\lambda})\mathcal{S}$.

We remark that in [6] we defined the Weyl module W^λ to be a left $\mathcal{S}$–module; however it is more convenient here to define it as a right module.

By Theorem 2.11 the Weyl module W^λ is a free R–module with basis

$$\{\, \varphi_{\mathtt{T}} \mid \mathtt{T} \in \mathcal{T}_0(\lambda,\mu), \mu \in \Lambda \,\}$$

where $\varphi_{\mathtt{T}} = \bar{\mathcal{S}}^\lambda + \varphi_{\mathtt{T}^\lambda\mathtt{T}}$. The cellular structure of $\mathcal{S}$ defines a natural symmetric bilinear form $\langle\ ,\ \rangle$ on W^λ which is determined by the requirement that

$$\langle \varphi_{\mathtt{S}}, \varphi_{\mathtt{T}} \rangle \varphi_{\mathtt{T}^\lambda\mathtt{T}^\lambda} \equiv \varphi_{\mathtt{T}^\lambda\mathtt{S}} \varphi_{\mathtt{T}\mathtt{T}^\lambda} \bmod \bar{\mathcal{S}}^\lambda$$

for all semistandard λ–tableaux $\mathtt{S}$ and $\mathtt{T}$. Note that $\langle \varphi_{\mathtt{S}}, \varphi_{\mathtt{T}} \rangle = 0$ unless $\mathtt{S}$ and $\mathtt{T}$ are tableaux of the same type. Also, by Theorem 2.11,

$$\langle x\varphi, y \rangle = \langle x, y\varphi^* \rangle \text{ for all } x, y \in W^\lambda \text{and all } \varphi \in \mathcal{S}. \tag{2.13}$$

Consequently, $\operatorname{rad} W^\lambda = \{\, x \in W^\lambda \mid \langle x, y \rangle = 0 \text{ for all } y \in W^\lambda \,\}$ is an $\mathcal{S}$–submodule of W^λ.

Because $\varphi_{\mathtt{T}^\lambda\mathtt{T}^\lambda}$ is the identity map on M^λ, one sees that $\langle \varphi_{\mathtt{T}^\lambda}, \varphi_{\mathtt{T}^\lambda} \rangle = 1$; together with Theorem 2.11 this implies the following result.

2.14 ([6, Theorem 6.16])**.** *For each $\lambda \in \Lambda^+$ let $F^\lambda = W^\lambda / \operatorname{rad} W^\lambda$. Then $F^\lambda \neq 0$; moreover, if R is a field, then F^λ is absolutely irreducible and $\{\, F^\lambda \mid \lambda \in \Lambda^+ \,\}$ is a complete set of non–isomorphic irreducible right $\mathcal{S}$–modules.*

In addition to the simple $\mathcal{S}$–modules we will be concerned with the simple $\mathcal{H}$–modules. Define $\bar{N}^\lambda$ to be the R–submodule of $\mathcal{H}$ with basis

$$\left\{ m_{\mathfrak{st}} \middle| \begin{array}{c} \mathfrak{s} \text{ and } \mathfrak{t} \text{ are standard } \mu\text{–tableaux for} \\ \text{some multipartition } \mu \text{ of } n \text{ with } \mu \rhd \lambda \end{array} \right\}$$

(cf. the definition of $\bar{\mathcal{S}}^\lambda$). It follows from Theorem 2.8 that $\bar{N}^\lambda$ is a two–sided ideal of $\mathcal{H}$.

Definition 2.15. Let λ be a multipartition. The Specht module S^λ is the right $\mathcal{H}$–module $(\bar{N}^\lambda + m_\lambda)\mathcal{H}$, a submodule of $\mathcal{H}/\bar{N}^\lambda$.

For $\mathfrak{t} \in \operatorname{Std}(\lambda)$ let $m_{\mathfrak{t}} = \bar{N}^\lambda + m_{\mathfrak{t}^\lambda\mathfrak{t}}$. Then, by Theorem 2.8, S^λ is free as an R–module with basis $\{\, m_{\mathfrak{t}} \mid \mathfrak{t} \in \operatorname{Std}(\lambda) \,\}$.

Define a bilinear form on S^λ by requiring that $\langle m_{\mathfrak{s}}, m_{\mathfrak{t}} \rangle m_\lambda \equiv m_{\mathfrak{t}^\lambda\mathfrak{s}} m_{\mathfrak{t}\mathfrak{t}^\lambda} \bmod \bar{N}^\lambda$. As before, $\operatorname{rad} S^\lambda$ is an $\mathcal{H}$–module and we set $D^\lambda = S^\lambda / \operatorname{rad} S^\lambda$.

2.16 ([6, Theorem 3.30]). *Suppose that R is a field. Then*

$$\{\, D^\lambda \neq 0 \mid \lambda \textit{ is a multipartition of } n \,\}$$

is a complete set of non–isomorphic irreducible $\mathcal{H}$–modules. Moreover, each D^λ is either absolutely irreducible or zero.

We remark that the bilinear forms defined on the modules W^λ and S^λ, and the results relating to them, fit into the general framework of cellular algebras, as devised by Graham and Lehrer [8].

Now suppose that $\omega \in \Lambda^+$; equivalently, assume that Λ^+ is the set of all multipartitions of n. Then $\varphi_{\mathtt{T}^\omega\mathtt{T}^\omega}$ is idempotent in $\mathcal{S}$; indeed, if ν is a multicomposition in Λ, then $\varphi_{\mathtt{T}^\nu\mathtt{T}^\nu}$, the identity map on M^ν, is idempotent and the identity element of $\mathcal{S}$ is $\sum_{\nu\in\Lambda}\varphi_{\mathtt{T}^\nu\mathtt{T}^\nu}$.

By identifying $h \in \mathcal{H}$ with the homomorphism $\rho_h \in \operatorname{Hom}_{\mathcal{H}}(\mathcal{H},\mathcal{H})$ which is given by $\rho_h(h') = hh'$ for all $h' \in \mathcal{H}$, we see that $\mathcal{H} \cong \varphi_{\mathtt{T}^\omega\mathtt{T}^\omega}\mathcal{S}\varphi_{\mathtt{T}^\omega\mathtt{T}^\omega}$. Let $(W^\nu : F^\lambda)$ denote the composition multiplicity of the simple module F^λ in W^ν and similarly for $(S^\nu : D^\lambda)$. Then, by general arguments (see, for example, [9, §6]), we have the following.

Proposition 2.17. *Assume that every multipartition of n belongs to Λ^+. Let U be a right $\mathcal{S}$–module. Then $U\varphi_{\mathtt{T}^\omega\mathtt{T}^\omega}$ is a right $\mathcal{H}$–module. In particular, if λ is a multipartition of n, then $W^\lambda\varphi_{\mathtt{T}^\omega\mathtt{T}^\omega} \cong S^\lambda$ and $F^\lambda\varphi_{\mathtt{T}^\omega\mathtt{T}^\omega} \cong D^\lambda$. Furthermore, if R is a field and $D^\lambda \neq (0)$, then $(S^\nu : D^\lambda) = (W^\nu : F^\lambda)$ for all multipartitions ν.*

Thus, the decomposition matrix of $\mathcal{H}$ embeds into the decomposition matrix of $\mathcal{S}$.

Finally, we also require a better understanding of the basis elements $\varphi_{\mathtt{T}}$ of the Weyl module W^λ. As for the Specht modules, if $\mathtt{T}$ is a semistandard λ–tableau of type μ we let $m_{\mathtt{T}} = \bar{N}^\lambda + m_{\mathtt{T}^\lambda\mathtt{T}}$. We claim that $\varphi_{\mathtt{T}}$ can be identified with the map (also denoted $\varphi_{\mathtt{T}}$)

$$\varphi_{\mathtt{T}}\colon M^\mu \to (\bar{N}^\lambda + M^\lambda)/\bar{N}^\lambda \text{ given by } m_\mu h \longmapsto m_{\mathtt{T}} h \tag{2.18}$$

for all $h \in \mathcal{H}$. In order to see this let $M = \bigoplus_{\nu\in\Lambda} M^\nu$ and $\bar{M}^\lambda = \bigoplus_{\nu\in\Lambda} \bar{N}^\lambda \cap M^\nu$. Then each φ in $\bar{\mathcal{S}}^\lambda$ maps M into $\bar{M}^\lambda$; so we may regard $\mathcal{S}/\bar{\mathcal{S}}^\lambda$ as a set of maps from M into $M/\bar{M}^\lambda$. Consequently, $\varphi_{\mathtt{T}} = \bar{\mathcal{S}}^\lambda + \varphi_{\mathtt{T}^\lambda\mathtt{T}}$ can be identified with the map from M^μ into $(\bar{M}^\lambda + M^\lambda)/\bar{M}^\lambda$ which sends m_μ to $\bar{M}^\lambda + m_{\mathtt{T}^\lambda\mathtt{T}}$ (since $\varphi_{\mathtt{T}^\lambda\mathtt{T}}(m_\mu) = m_{\mathtt{T}^\lambda\mathtt{T}}$). Then, by the third isomorphism theorem,

$$(\bar{M}^\lambda + M^\lambda)/\bar{M}^\lambda \cong M^\lambda/(\bar{M}^\lambda \cap M^\lambda) = M^\lambda/(\bar{N}^\lambda \cap M^\lambda) \cong (\bar{N}^\lambda + M^\lambda)/\bar{N}^\lambda$$

justifying our claim.

3. The Gram determinant of W_μ^λ

Throughout this section, fix a multipartition $\lambda \in \Lambda^+$ and a multicomposition $\mu \in \Lambda$. The μ–weight space of W^λ is the R–submodule $W_\mu^\lambda = W^\lambda\varphi_{\mathtt{T}^\mu\mathtt{T}^\mu}$; thus W_μ^λ is R–free with basis $\{\, \varphi_{\mathtt{T}} \mid \mathtt{T} \in \mathcal{T}_0(\lambda,\mu) \,\}$.

Definition 3.1. The Gram determinant of W_μ^λ with respect to the semistandard basis is $G_\mu(\lambda) = \det\big(\langle\varphi_{\mathtt{S}},\varphi_{\mathtt{T}}\rangle\big)$, where $\mathtt{S}$ and $\mathtt{T}$ run over the elements of $\mathcal{T}_0(\lambda,\mu)$. If $\mathcal{T}_0(\lambda,\mu)$ is empty, we set $G_\mu(\lambda) = 1$.

(We determine the sign of $G_\mu(\lambda)$ by fixing a total ordering of $\mathcal{T}_0(\lambda,\mu)$ which is compatible with the partial ordering of Definition 3.6 below.)

The purpose of this section is to compute $G_\mu(\lambda)$; we do this by first constructing an orthogonal basis for W^λ_μ when $R = \mathbb{F}(q, Q_1, \dots, Q_r)$ and q, $Q_1, \dots, Q_r$ are independent transcendental elements over a field $\mathbb{F}$.

Definition 3.2. Given $i \geq 1$ and $k \in \mathbf{r}$ let $y, y+1, \dots, z$ be the entries in row i of $\mathfrak{t}^{\mu^{(k)}}$. Then $L^\mu_{i,k}$ is the element of $\operatorname{Hom}_{\mathcal{H}}\big(M^\mu, M^\mu\big)$ given by

$$L^\mu_{i,k}(m_\mu h) = (L_y + L_{y+1} + \cdots + L_z)m_\mu h,$$

for all $h \in \mathcal{H}$.

The homomorphism $L^\mu_{i,k}$ maps into M^μ because 2.1(ii) and (iii) imply the following result.

Lemma 3.3. *Suppose that $(i,k) \in \mathbb{N} \times \mathbf{r}$ and let $y, y+1, \dots, z$ be the entries in row i of $\mathfrak{t}^{\mu^{(k)}}$. Then $L_y + L_{y+1} + \cdots + L_z$ commutes with every element of $\mathcal{H}(\mathfrak{S}_\mu)$. In particular, $(L_y + L_{y+1} + \cdots + L_z)m_\mu = m_\mu(L_y + L_{y+1} + \cdots + L_z)$.*

Note that $L^\mu_{i,k} = 0$ if $\mu^{(k)}_i = 0$. Furthermore, using 2.1 again, the homomorphisms $L^\mu_{i,k}$ and $L^\mu_{j,l}$ commute for all (i,k) and (j,l) in $\mathbb{N} \times \mathbf{r}$.

Below, often without mention, we will identify an element h of $\mathcal{H}$ with the homomorphism $\rho_h \in \operatorname{Hom}_{\mathcal{H}}(\mathcal{H}, \mathcal{H})$ given by $\rho_h(h') = hh'$ for all $h' \in \mathcal{H}$. Under this identification we have that $L_i = L^\omega_{i,r}$ for $i = 1, 2, \dots, n$.

Definition 3.4. (i) Let $x = (a,b,c) \in [\lambda]$. Then the residue of x is

$$\operatorname{res}(x) = q^{b-a}Q_c.$$

(ii) Let $(i,k) \in \mathbb{N} \times \mathbf{r}$ and suppose that $\mathtt{T} \in \mathcal{T}_0(\lambda,\mu)$. Then

$$\operatorname{res}_{\mathtt{T}}(i,k) = \sum_{x \in [\lambda] : \mathtt{T}(x) = (i,k)} \operatorname{res}(x).$$

Similarly, given a standard tableau $\mathfrak{t} \in \operatorname{Std}(\lambda)$ we write $\operatorname{res}_{\mathfrak{t}}(i) = \operatorname{res}(x)$ where x is the unique node in $[\lambda]$ such that $\mathfrak{t}(x) = i$.

(iii) Let $\mathbf{a} = (a_1, \dots, a_r)$, with $0 \leq a_k \leq n$ for all k, and suppose that $\mathfrak{t} \in \operatorname{Std}(\lambda)$. Then

$$\operatorname{res}_{\mathfrak{t}}(\mathbf{a}) = \prod_{k=1}^{r} \prod_{i=1}^{a_k} \big(\operatorname{res}_{\mathfrak{t}}(i) - Q_k\big)$$

(cf. Definition 2.2(i)).

Note that if $L^\mu_{i,k} = 0$, then $\operatorname{res}_{\mathtt{T}}(i,k) = 0$ for all $\mathtt{T} \in \mathcal{T}_0(\lambda,\mu)$.

Example 3.5. Let $\lambda = \big((3,1),(1)\big)$, $\mu = \big((2),(2,1)\big)$, and let

$$\mathfrak{t} = \left(\begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 5 \\ \cline{1-1} \end{array}\,, \ \begin{array}{|c|} \hline 3 \\ \hline \end{array} \right) \quad \text{and} \quad \mathtt{T} = \left(\begin{array}{|c|c|c|} \hline 1_1 & 1_1 & 1_2 \\ \hline 2_2 \\ \cline{1-1} \end{array}\,, \ \begin{array}{|c|} \hline 1_2 \\ \hline \end{array} \right).$$

Then $\mathfrak{t} \in \operatorname{Std}(\lambda)$, $\mathtt{T} \in \mathcal{T}_0(\lambda,\mu)$ and $\mathtt{T} = \mu(\mathfrak{t})$. The residues in the diagram of λ are

$$\left(\begin{array}{|c|c|c|} \hline Q_1 & qQ_1 & q^2Q_1 \\ \hline q^{-1}Q_1 \\ \cline{1-1} \end{array}\,, \quad \begin{array}{|c|} \hline Q_2 \\ \hline \end{array} \right).$$

Thus, $\mathrm{res}_{\mathtt{T}}(1_1) = \mathrm{res}_{\mathfrak{t}}(1) + \mathrm{res}_{\mathfrak{t}}(2) = Q_1 + qQ_1$, $\mathrm{res}_{\mathtt{T}}(1_2) = \mathrm{res}_{\mathfrak{t}}(4) + \mathrm{res}_{\mathfrak{t}}(3) = q^2Q_1 + Q_2$ and $\mathrm{res}_{\mathtt{T}}(2_2) = \mathrm{res}_{\mathfrak{t}}(5) = q^{-1}Q_1$. Let $\mathbf{a} = (0, 2)$; then $m_\mu = u_{\mathbf{a}}^+ x_\mu$ and $\mathrm{res}_{\mathfrak{t}}(\mathbf{a}) = (Q_1 - Q_2)(qQ_1 - Q_2)$.

Definition 3.6. (i) Suppose that $\mathtt{T} \in \mathcal{T}_0(\lambda, \mu)$ and $(i, k) \in \mathbb{N} \times \mathbf{r}$. Let $\mathtt{T}_{i,k}$ denote the subtableau of $\mathtt{T}$ consisting of all entries $(j, l) \le (i, k)$, and let $\mathtt{T}^\#_{i,k}$ be the multipartition whose diagram is determined by $\mathtt{T}_{i,k}$.

(ii) Given $\mathtt{S}$ and $\mathtt{T} \in \mathcal{T}_0(\lambda, \mu)$ write $\mathtt{S} \trianglerighteq \mathtt{T}$ if $\mathtt{S}^\#_{i,k} \trianglerighteq \mathtt{T}^\#_{i,k}$ for all $(i, k) \in \mathbb{N} \times \mathbf{r}$.

(iii) If $\mathtt{S} \trianglerighteq \mathtt{T}$ and $\mathtt{S} \ne \mathtt{T}$, we write $\mathtt{S} \vartriangleright \mathtt{T}$.

Proposition 3.7. *Suppose that $\mathfrak{t}$ is a standard λ–tableau and let i be an integer with $1 \le i \le n$. Then for each $\mathfrak{s} \in \mathrm{Std}(\lambda)$ there exists $a_{\mathfrak{s}} \in R$ such that*

$$m_{\mathfrak{t}} L_i = \mathrm{res}_{\mathfrak{t}}(i) m_{\mathfrak{t}} + \sum_{\mathfrak{s} \vartriangleright \mathfrak{t}} a_{\mathfrak{s}} m_{\mathfrak{s}}.$$

Proof. First consider the case where $\mathfrak{t} = \mathfrak{t}^\lambda$; then $m_{\mathfrak{t}} = \bar{N}^\lambda + m_\lambda$. Suppose that i appears in row a and column b of the cth component of $\mathfrak{t}^\lambda$ and let j be the smallest integer appearing in $\mathfrak{t}^{\lambda^{(c)}}$; then $j \le i$. Write $m_\lambda = x_\lambda u_{\mathbf{a}}^+$ as in Definition 2.2 and let $T_{j,i} = T_j T_{j+1} \dots T_{i-1}$ (set $T_{j,i} = 1$ if $j = i$) and $T_{i,j} = T^*_{j,i}$.

Working modulo $\bar{N}^\lambda$ and using 2.1(ii) and 2.1(iii), we find that

$$\begin{aligned} m_{\mathfrak{t}^\lambda} L_i \equiv m_\lambda L_i = x_\lambda u_{\mathbf{a}}^+ L_i &= q^{j-i} x_\lambda u_{\mathbf{a}}^+ T_{i,j} L_j T_{j,i} = q^{j-i} x_\lambda T_{i,j} u_{\mathbf{a}}^+ L_j T_{j,i} \\ &= q^{j-i} Q_c x_\lambda T_{i,j} u_{\mathbf{a}}^+ T_{j,i} + q^{j-i} x_\lambda T_{i,j} u_{\mathbf{a}}^+ (L_j - Q_c) T_{j,i} \\ &= q^{j-i} Q_c u_{\mathbf{a}}^+ x_\lambda T_{i,j} T_{j,i} + q^{j-i} x_\lambda T_{i,j} u_{\mathbf{b}}^+ T_{j,i}, \end{aligned}$$

where $\mathbf{b} = (a_1, \dots, a_{c-1}, a_c + 1, a_{c+1}, \dots, a_r)$. Therefore,

$$m_{\mathfrak{t}^\lambda} L_i \equiv q^{j-i} Q_c u_{\mathbf{a}}^+ x_\lambda T_{i,j} T_{j,i} \mod \bar{N}^\lambda$$

because $x_\lambda T_{i,j} u_{\mathbf{b}}^+ \in \mathcal{H} m_\lambda \mathcal{H} \cap \mathcal{H} u_{\mathbf{b}}^+ \mathcal{H} \subseteq \bar{N}^\lambda$ by Theorem 2.8. Now $q^{j-i} T_{i,j} T_{j,i}$ is a q–Murphy operator in the Iwahori–Hecke algebra of the symmetric group on $\{j, j+1, \dots, i\}$; consequently, by [17, Theorem 4.6],

$$m_{\mathfrak{t}^\lambda} L_i \equiv q^{b-a} Q_c u_{\mathbf{a}}^+ x_\lambda = \mathrm{res}_{\mathfrak{t}^\lambda}(i) u_{\mathbf{a}}^+ x_\lambda \equiv \mathrm{res}_{\mathfrak{t}^\lambda}(i) m_{\mathfrak{t}^\lambda} \mod \bar{N}^\lambda.$$

This completes the proof when $\mathfrak{t} = \mathfrak{t}^\lambda$. If $\mathfrak{t} \ne \mathfrak{t}^\lambda$, then there exists an integer k such that $\mathfrak{s} = \mathfrak{t}(k, k+1) \vartriangleright \mathfrak{t}$ (and $1 \le k < n$). Then $m_{\mathfrak{t}} = m_{\mathfrak{s}} T_k$ and the result follows by the argument of [4, Theorem 3.15]. □

Recalling the definitions of $u_{\mathbf{a}}^+$ and $\mathrm{res}_{\mathfrak{t}}(\mathbf{a})$ from (2.2) and (3.4) respectively, we obtain the next result.

Corollary 3.8. *Let $\mathfrak{t}$ be a standard λ–tableau. Then for each $\mathfrak{s} \in \mathrm{Std}(\lambda)$ there exists $a_{\mathfrak{s}} \in R$ such that*

$$m_{\mathfrak{t}} u_{\mathbf{a}}^+ = \mathrm{res}_{\mathfrak{t}}(\mathbf{a}) m_{\mathfrak{t}} + \sum_{\mathfrak{s} \vartriangleright \mathfrak{t}} a_{\mathfrak{s}} m_{\mathfrak{s}}.$$

Lemma 3.9. *Let $\mathtt{T} \in \mathcal{T}_0(\lambda, \mu)$. There exist unique standard λ–tableaux* $\mathrm{first}(\mathtt{T})$ *and* $\mathrm{last}(\mathtt{T})$ *such that*

(i) $\mu(\mathrm{first}(\mathtt{T})) = \mu(\mathrm{last}(\mathtt{T})) = \mathtt{T}$*; and,*

(ii) $\mathrm{first}(\mathtt{T}) \trianglerighteq \mathfrak{t} \trianglerighteq \mathrm{last}(\mathtt{T})$ *for all* $\mathfrak{t} \in \mathrm{Std}(\lambda)$ *such that* $\mu(\mathfrak{t}) = \mathtt{T}$.

Furthermore, if $d = d(\text{first}(\mathtt{T}))$ *and* $m_\lambda = u_{\mathbf{b}}^+ x_\lambda$*, then*

$$m_{\mathtt{T}^\lambda \mathtt{T}} = \sum_{w \in \mathfrak{S}_\lambda d^{-1} \mathfrak{S}_\mu} u_{\mathbf{b}}^+ T_w$$

and $\mathfrak{S}_\lambda \cap d\mathfrak{S}_\mu d^{-1} = \mathfrak{S}_{\nu_\mathtt{T}}$ *for some multicomposition* $\nu_\mathtt{T}$ *of* n*.*

Proof. Parts (i) and (ii) follow easily from the definitions; see [6, 4.7]. The final statements are a consequence of the definition of $m_{\mathtt{T}^\lambda \mathtt{T}}$ and well–known properties of distinguished double coset representatives (see, for example, [4, 1.6]). □

We remark that if $\mathfrak{t}$ is a standard λ–tableau, then $\mathfrak{t} = \text{first}(\mathfrak{t}) = \text{last}(\mathfrak{t})$ and $\nu_\mathfrak{t} = \omega$.

Theorem 3.10. *Suppose that* $\mathtt{T}$ *is a semistandard* λ*–tableau of type* μ *and let* $(i,k) \in \mathbb{N} \times \mathbf{r}$*. Then for each* $\mathtt{S} \in \mathcal{T}_0(\lambda,\mu)$ *there exists* $a_\mathtt{S} \in R$ *such that*

$$\varphi_\mathtt{T} L^\mu_{i,k} = \text{res}_\mathtt{T}(i,k)\varphi_\mathtt{T} + \sum_{\mathtt{S} \rhd \mathtt{T}} a_\mathtt{S} \varphi_\mathtt{S}.$$

Proof. Recalling our conventions for $\varphi_\mathtt{T}$ from 2.18, and using Lemma 3.3, we have

$$\varphi_\mathtt{T} L^\mu_{i,k}(m_\mu) = \varphi_\mathtt{T}\big(m_\mu(L_y + L_{y+1} + \cdots + L_z)\big) = m_\mathtt{T}(L_y + L_{y+1} + \cdots + L_z)$$

where $y, y+1, \ldots, z$ are the entries in row i of $\mathfrak{t}^{\mu^{(k)}}$. Let $\mathfrak{t} = \text{first}(\mathtt{T})$; then, by Lemma 3.9, $m_\mathtt{T} = m_\mathfrak{t} h$ for some $h \in \mathcal{H}(\mathfrak{S}_\mu)$. Therefore, by Proposition 3.7,

$$\begin{aligned}\varphi_\mathtt{T} L^\mu_{i,k}(m_\mu) &= m_\mathfrak{t}(L_y + L_{y+1} + \cdots + L_z)h \\ &= \text{res}_\mathtt{T}(i,k) m_\mathfrak{t} h + \sum_{\mathfrak{s} \rhd \mathfrak{t}} a_\mathfrak{s} m_\mathfrak{s} h \\ &= \text{res}_\mathtt{T}(i,k) m_\mathtt{T} + \sum_{\mathtt{S} \in \mathcal{T}_0(\lambda,\mu)} a_\mathtt{S} m_\mathtt{S}\end{aligned}$$

for some $a_\mathfrak{s}, a_\mathtt{S} \in R$, by 2.9. Since $\mathfrak{t} = \text{first}(\mathtt{T})$, we deduce that $a_\mathtt{S} = 0$ unless $\mathtt{S} \rhd \mathtt{T}$. Therefore,

$$\varphi_\mathtt{T} L^\mu_{i,k} = \text{res}_\mathtt{T}(i,k)\varphi_\mathtt{T} + \sum_{\mathtt{S} \rhd \mathtt{T}} a_\mathtt{S} \varphi_\mathtt{S}$$

as required. □

Until further notice we assume that R is the rational function field $\mathbb{F}(q, Q_1, \ldots, Q_r)$ for some field $\mathbb{F}$. We will compute the Gram determinant of W^λ_μ as an element of this field and derive the general case from this.

Definition 3.11 (cf. [13, 3.18])**.** Let $\mathtt{T} \in \mathcal{T}_0(\lambda,\mu)$.

(i) Let $E_\mathtt{T} = \displaystyle\prod_{(i,k) \in \mathbb{N} \times \mathbf{r}} \prod_{\substack{\mathtt{S} \in \mathcal{T}_0(\lambda,\mu) \\ \text{res}_\mathtt{S}(i,k) \neq \text{res}_\mathtt{T}(i,k)}} \frac{L^\mu_{i,k} - \text{res}_\mathtt{S}(i,k)}{\text{res}_\mathtt{T}(i,k) - \text{res}_\mathtt{S}(i,k)}$.

(ii) Let $\psi_\mathtt{T} = \varphi_\mathtt{T} E_\mathtt{T}$.

In the above definition we adopt the convention that empty products are 1. In particular, only finitely many terms are non–trivial in the definition of $E_\mathtt{T}$ because the second product is empty whenever $L^\mu_{i,k} = 0$. We also do not need to specify the order of the terms in the product since all of the terms commute.

The main reason why we have assumed that $R = \mathbb{F}(q, Q_1, \dots, Q_r)$ is because of the following crucial lemma. The lemma is false for general R.

Lemma 3.12. *Suppose that $R = \mathbb{F}(q, Q_1, \dots, Q_r)$ and let $\mathtt{S}$ and $\mathtt{T}$ be distinct semistandard tableaux in $\mathcal{T}_0(\lambda, \mu)$. Then $\operatorname{res}_{\mathtt{S}}(i,k) \neq \operatorname{res}_{\mathtt{T}}(i,k)$ for some $(i,k) \in \mathbb{N} \times \mathbf{r}$.*

Proof. Since $\mathtt{S} \neq \mathtt{T}$ we may choose $(i,k) \in \mathbb{N} \times \mathbf{r}$ minimal such that $\mathtt{S}_{i,k} \neq \mathtt{T}_{i,k}$. Then $\operatorname{res}_{\mathtt{S}}(i,k) \neq \operatorname{res}_{\mathtt{T}}(i,k)$. □

Standard arguments using Theorem 3.10 and Lemma 3.12 now prove the following (cf. [16, (3.4)–(3.11)] or [15, Prop. 3.35]).

Theorem 3.13. *Suppose that $R = \mathbb{F}(q, Q_1, Q_2, \dots, Q_r)$ and let $\mathtt{S}, \mathtt{T} \in \mathcal{T}_0(\lambda, \mu)$ and $(i,k) \in \mathbb{N} \times \mathbf{r}$. Then the following hold:*

(i) $\psi_{\mathtt{T}} L^{\mu}_{i,k} = \operatorname{res}_{\mathtt{T}}(i,k)\psi_{\mathtt{T}}$,
(ii) $E_{\mathtt{T}} L^{\mu}_{i,k} = \operatorname{res}_{\mathtt{T}}(i,k) E_{\mathtt{T}}$,
(iii) $\psi_{\mathtt{T}} = \varphi_{\mathtt{T}} + \sum_{\mathtt{S} \rhd \mathtt{T}} a_{\mathtt{S}} \varphi_{\mathtt{S}}$ *for some* $a_{\mathtt{S}} \in R$,
(iv) $\varphi_{\mathtt{S}} E_{\mathtt{T}} = 0$ *if* $\mathtt{S} \rhd \mathtt{T}$,
(v) $\psi_{\mathtt{S}} E_{\mathtt{T}} = \delta_{\mathtt{ST}} \psi_{\mathtt{T}}$,
(vi) $\{\, \psi_{\mathtt{T}} \mid \mathtt{T} \in \mathcal{T}_0(\lambda, \mu) \,\}$ *is an orthogonal basis of* W^{λ}_{μ}.

Corollary 3.14. *Let $\mathfrak{t} \in \operatorname{Std}(\lambda)$ and suppose that there exists $\mathfrak{s} \in \operatorname{Std}(\lambda)$ such that $\mathfrak{t} = \mathfrak{s}(i, i+1)$ and $\mathfrak{s} \rhd \mathfrak{t}$ for some i with $1 \leq i < n$. Then $\psi_{\mathfrak{t}} = \psi_{\mathfrak{s}}(T_i + \alpha)$ where $\alpha = \frac{(q-1)\operatorname{res}_{\mathfrak{t}}(i)}{\operatorname{res}_{\mathfrak{s}}(i) - \operatorname{res}_{\mathfrak{t}}(i)}$.*

Proof. By definition $\psi_{\mathfrak{t}} = \varphi_{\mathfrak{t}} E_{\mathfrak{t}}$ and $\varphi_{\mathfrak{t}} = \varphi_{\mathfrak{s}} T_i$. Furthermore, by assumption $\operatorname{res}_{\mathfrak{s}}(i) = \operatorname{res}_{\mathfrak{t}}(i+1)$ and $\operatorname{res}_{\mathfrak{s}}(i+1) = \operatorname{res}_{\mathfrak{t}}(i)$; consequently, $E_{\mathfrak{s}} + E_{\mathfrak{t}}$ is symmetric in L_i and L_{i+1} and so T_i commutes with $E_{\mathfrak{s}} + E_{\mathfrak{t}}$ by 2.1. Therefore,

$$\begin{aligned}
\psi_{\mathfrak{t}}\big(\operatorname{res}_{\mathfrak{s}}(i) - \operatorname{res}_{\mathfrak{t}}(i)\big) &= \varphi_{\mathfrak{t}} E_{\mathfrak{t}}\big(L_{i+1} - \operatorname{res}_{\mathfrak{t}}(i)\big), && \text{by (Theorem 3.13)(i),}\\
&= \varphi_{\mathfrak{t}}(E_{\mathfrak{s}} + E_{\mathfrak{t}})\big(L_{i+1} - \operatorname{res}_{\mathfrak{t}}(i)\big), && \text{by (Theorem 3.13)(ii),}\\
&= \varphi_{\mathfrak{s}} T_i (E_{\mathfrak{s}} + E_{\mathfrak{t}})\big(L_{i+1} - \operatorname{res}_{\mathfrak{t}}(i)\big) &&\\
&= \varphi_{\mathfrak{s}}(E_{\mathfrak{s}} + E_{\mathfrak{t}}) T_i \big(L_{i+1} - \operatorname{res}_{\mathfrak{t}}(i)\big), && \text{by 2.1(iii),}\\
&= \psi_{\mathfrak{s}} T_i \big(L_{i+1} - \operatorname{res}_{\mathfrak{t}}(i)\big), && \text{by (Theorem 3.13)(iv).}
\end{aligned}$$

Now $T_i L_{i+1} = (q-1)L_{i+1} + L_i T_i$, so

$$\psi_{\mathfrak{t}}\big(\operatorname{res}_{\mathfrak{s}}(i) - \operatorname{res}_{\mathfrak{t}}(i)\big) = (q-1)\operatorname{res}_{\mathfrak{t}}(i)\psi_{\mathfrak{s}} + \big(\operatorname{res}_{\mathfrak{s}}(i) - \operatorname{res}_{\mathfrak{t}}(i)\big)\psi_{\mathfrak{s}} T_i$$

and the result follows. □

Our next aim is to compute the inner products $\langle \psi_{\mathtt{T}}, \psi_{\mathtt{T}} \rangle$. This will require a considerable amount of combinatorial machinery.

Recall the definition of the tableau $\mathtt{T}_{i,k}$ from (Definition 3.6)(i). We say that a node $y \notin [\mathtt{T}^{\#}_{i,k}]$ is an addable node of $\mathtt{T}_{i,k}$ if $[\mathtt{T}^{\#}_{i,k}] \cup \{y\}$ is the diagram of a multipartition. Similarly, a node $y \in [\mathtt{T}^{\#}_{i,k}]$ is removable from $\mathtt{T}_{i,k}$ if $[\mathtt{T}^{\#}_{i,k}] \setminus \{y\}$ is the diagram of a multipartition.

Given nodes $x = (i,j,k)$ and $y = (a,b,c)$ define $x < y$ if $k < c$ or $k = c$ and $j > b$.

Definition 3.15 (cf. [13, 2.8] and [7, 4.1])**.** Let $\mathtt{T} \in \mathcal{T}_0(\lambda, \mu)$ and for $x \in [\lambda]$ suppose that $\mathtt{T}(x) = (i,k)$ and let $(j,l) \in \mathbb{N} \times \mathbf{r}$ be maximal such that $(j,l) < (i,k)$.

(i) Let $A_{\mathtt{T}}(x)=\prod_y\big(\operatorname{res}(x)-\operatorname{res}(y)\big)$ where the product is over the addable nodes y of $\mathtt{T}_{i,k}$ such that $x<y$ and an (i,k) can be added to $\mathtt{T}_{i,k}$ at y to give a semistandard tableau.

(ii) Let $R_{\mathtt{T}}(x)=\prod_y\big(\operatorname{res}(x)-\operatorname{res}(y)\big)$ where the product is over the removable nodes y of $\mathtt{T}_{j,l}$ such that $x<y$ and (i,k) does not appear in the column of $\mathtt{T}$ containing y.

(iii) Let $\gamma_{\mathtt{T}}=\displaystyle\prod_{x\in[\lambda]}\frac{A_{\mathtt{T}}(x)}{R_{\mathtt{T}}(x)}$.

Example 3.16. As in Example 3.5, let $\lambda=\big((3,1),(1)\big)$ and $\mu=\big((2),(2,1)\big)$. Then $\mathcal{T}_0(\lambda,\mu)$ consists of the three tableaux

$$\mathtt{T}_1=\left(\begin{array}{|c|c|c|}\hline 1_1&1_1&1_2\\\hline 1_2\\\cline{1-1}\end{array},\ \begin{array}{|c|}\hline 2_2\\\hline\end{array}\right),\quad \mathtt{T}_2=\left(\begin{array}{|c|c|c|}\hline 1_1&1_1&1_2\\\hline 2_2\\\cline{1-1}\end{array},\ \begin{array}{|c|}\hline 1_2\\\hline\end{array}\right)$$

$$\text{and}\quad \mathtt{T}_3=\left(\begin{array}{|c|c|c|}\hline 1_1&1_1&2_2\\\hline 1_2\\\cline{1-1}\end{array},\ \begin{array}{|c|}\hline 1_2\\\hline\end{array}\right).$$

Recall that we are currently assuming that $q,Q_1,\dots,Q_r$ are indeterminates. We find that

$$\gamma_{\mathtt{T}_1}=\left\{\frac{q^2Q_1-Q_1}{q^2Q_1-qQ_1}\,\frac{q^2Q_1-Q_2}{}\right\}\left\{\frac{q^{-1}Q_1-Q_2}{}\right\},$$

$$\gamma_{\mathtt{T}_2}=\left\{\frac{q^2Q_1-q^{-1}Q_1}{q^2Q_1-qQ_1}\,\frac{q^2Q_1-qQ_2}{}\right\}\left\{\frac{q^{-1}Q_1-qQ_2}{}\,\frac{q^{-1}Q_1-q^{-1}Q_2}{q^{-1}Q_1-Q_2}\right\},$$

and

$$\gamma_{\mathtt{T}_3}=\left\{\frac{q^2Q_1-Q_1}{q^2Q_1-qQ_1}\,\frac{q^2Q_1-q^{-2}Q_1}{q^2Q_1-q^{-1}Q_1}\,\frac{q^2Q_1-qQ_2}{}\,\frac{q^2Q_1-q^{-1}Q_2}{q^2Q_1-Q_2}\right\}\left\{\frac{q^{-1}Q_1-qQ_2}{}\right\}.$$

Inspection shows that $\gamma_{\mathtt{T}_1}\gamma_{\mathtt{T}_2}\gamma_{\mathtt{T}_3}\in\mathbb{F}[q,q^{-1},Q_1,\dots,Q_r]$; *cf.* Corollary 3.29.

Lemma 3.17. *Let $\mathfrak{t}\in\operatorname{Std}(\lambda)$ and suppose that there exists a tableau $\mathfrak{s}\in\operatorname{Std}(\lambda)$ such that $\mathfrak{s}\rhd\mathfrak{t}$ and $\mathfrak{t}=\mathfrak{s}(i,i+1)$ for some i with $1\le i<n$. Then*

$$\gamma_{\mathfrak{t}}=\frac{\big(\operatorname{res}_{\mathfrak{s}}(i)-q\operatorname{res}_{\mathfrak{t}}(i)\big)\big(q\operatorname{res}_{\mathfrak{s}}(i)-\operatorname{res}_{\mathfrak{t}}(i)\big)}{\big(\operatorname{res}_{\mathfrak{s}}(i)-\operatorname{res}_{\mathfrak{t}}(i)\big)^2}\gamma_{\mathfrak{s}}.$$

Proof. The proof of this result is straightforward and essentially identical to [13, 2.11]. We leave the details to the reader. □

If m is an integer, let $[m]_q=1+q+\dots+q^{m-1}$ and $\{m\}_q=[1]_q[2]_q\dots[m]_q$. For a multicomposition ν, let $\{\nu\}_q=\prod_{k\in\mathbf{r}}\prod_{i\ge1}\{\nu_i^{(k)}\}_q$.

Given two rational functions f and g in R we write $f\simeq g$ if $f=q^zg$ for some integer z. Since q^z is always a unit in the rings we consider, there is no loss in restricting our attention to $R/\simeq$. We extend this relation to elements of $\mathcal{H}$ and $\mathcal{S}$ in the obvious way.

Lemma 3.18. *Suppose that $m_\lambda=u_{\mathbf{b}}^+x_\lambda$. Then $\gamma_{\mathfrak{t}^\lambda}\simeq\operatorname{res}_{\mathfrak{t}^\lambda}(\mathbf{b})\{\lambda\}_q$.*

Proof. Remove n from $\mathfrak{t}^\lambda$ and apply induction (cf. [13, 2.10]). □

Proposition 3.19. *Suppose that $\mathfrak{t}$ is a standard λ–tableau. Then $\langle\psi_{\mathfrak{t}},\psi_{\mathfrak{t}}\rangle\simeq\gamma_{\mathfrak{t}}$.*

Proof. First suppose that $\mathfrak{t} = \mathfrak{t}^\lambda$ and write $m_\lambda = u_{\mathbf{b}}^+ x_\lambda$ as in Definition 2.2. Then $\psi_\mathfrak{t} = \varphi_{\mathfrak{t}^\lambda}$ by Theorem 3.13(iii), so $\langle \psi_\mathfrak{t}, \psi_\mathfrak{t} \rangle = \langle \varphi_{\mathfrak{t}^\lambda}, \varphi_{\mathfrak{t}^\lambda} \rangle$. By definition, $\langle \varphi_{\mathfrak{t}^\lambda}, \varphi_{\mathfrak{t}^\lambda} \rangle \varphi_{\mathtt{T}^\lambda \mathtt{T}^\lambda} \equiv \varphi_{\mathtt{T}^\lambda \mathfrak{t}^\lambda} \varphi_{\mathfrak{t}^\lambda \mathtt{T}^\lambda} \bmod \bar{\mathcal{S}}^\lambda$ and

$$\varphi_{\mathtt{T}^\lambda \mathfrak{t}^\lambda} \varphi_{\mathfrak{t}^\lambda \mathtt{T}^\lambda}(m_\lambda) = \varphi_{\mathtt{T}^\lambda \mathfrak{t}^\lambda}(m_\lambda) = m_\lambda m_\lambda = m_\lambda x_\lambda u_{\mathbf{b}}^+ = \{\lambda\}_q m_\lambda u_{\mathbf{b}}^+ \equiv \gamma_{\mathfrak{t}^\lambda} m_\lambda \bmod \bar{N}^\lambda$$

by Corollary 3.8 and Lemma 3.18. Thus, $\langle \varphi_{\mathfrak{t}^\lambda}, \varphi_{\mathfrak{t}^\lambda} \rangle \simeq \gamma_{\mathfrak{t}^\lambda}$ as required.

Now assume that $\mathfrak{t} \neq \mathfrak{t}^\lambda$. Then there exists a standard λ–tableau $\mathfrak{s}$ such that $\mathfrak{s} \rhd \mathfrak{t}$ and $\mathfrak{s} = \mathfrak{t}(i, i+1)$ where $1 \le i < n$. By Corollary 3.14, $\psi_\mathfrak{t} = \psi_\mathfrak{s}(T_i + \alpha)$ where $\alpha = \frac{(q-1)\operatorname{res}_\mathfrak{t}(i)}{\operatorname{res}_\mathfrak{s}(i) - \operatorname{res}_\mathfrak{t}(i)}$. Therefore, using the facts that $T_i^2 = (q-1)T_i + q$ and that $\psi_\mathfrak{t}$ and $\psi_\mathfrak{s}$ are orthogonal, we see that

$$\begin{aligned}
\langle \psi_\mathfrak{t}, \psi_\mathfrak{t} \rangle &= \langle \psi_\mathfrak{s} T_i + \alpha \psi_\mathfrak{s}, \psi_\mathfrak{s} T_i + \alpha \psi_\mathfrak{s} \rangle \\
&= \langle \psi_\mathfrak{s} T_i, \psi_\mathfrak{s} T_i \rangle + 2\alpha \langle \psi_\mathfrak{s} T_i, \psi_\mathfrak{s} \rangle + \alpha^2 \langle \psi_\mathfrak{s}, \psi_\mathfrak{s} \rangle \\
&= \langle \psi_\mathfrak{s} T_i^2, \psi_\mathfrak{s} \rangle + 2\alpha \langle \psi_\mathfrak{s} T_i, \psi_\mathfrak{s} \rangle + \alpha^2 \langle \psi_\mathfrak{s}, \psi_\mathfrak{s} \rangle \\
&= (2\alpha + q - 1) \langle \psi_\mathfrak{s} T_i, \psi_\mathfrak{s} \rangle + (\alpha^2 + q) \langle \psi_\mathfrak{s}, \psi_\mathfrak{s} \rangle \\
&= (2\alpha + q - 1) \langle \psi_\mathfrak{t} - \alpha \psi_\mathfrak{s}, \psi_\mathfrak{s} \rangle + (\alpha^2 + q) \langle \psi_\mathfrak{s}, \psi_\mathfrak{s} \rangle \\
&= (q + \alpha)(1 - \alpha) \langle \psi_\mathfrak{s}, \psi_\mathfrak{s} \rangle.
\end{aligned}$$

The reader will easily verify that $(q - \alpha)(1 + \alpha) = \frac{(\operatorname{res}_\mathfrak{s}(i) - q \operatorname{res}_\mathfrak{t}(i))(q \operatorname{res}_\mathfrak{s}(i) - \operatorname{res}_\mathfrak{t}(i))}{(\operatorname{res}_\mathfrak{s}(i) - \operatorname{res}_\mathfrak{t}(i))^2}$; so induction and Lemma 3.17 complete the proof. □

Remark 3.20. The Proposition is really a statement about the orthogonal basis of the Specht module S^λ. Indeed, if we let $f_\mathfrak{t} = \psi_\mathfrak{t}(m_\lambda)$, then $\{ f_\mathfrak{t} \mid \mathfrak{t} \in \operatorname{Std}(\lambda) \}$ is an orthogonal basis of S^λ (see Proposition 2.17) and $\langle f_\mathfrak{t}, f_\mathfrak{t} \rangle \simeq \gamma_\mathfrak{t}$ where $\langle\ ,\ \rangle$ now denotes the standard inner product on S^λ. It follows that, up to a power of q, the determinant of the Gram matrix of S^λ is $\prod_{\mathfrak{t} \in \operatorname{Std}(\lambda)} \gamma_\mathfrak{t}$.

In order to compute the inner products $\langle \psi_\mathtt{T}, \psi_\mathtt{T} \rangle$, for arbitrary semistandard tableaux $\mathtt{T}$, we compare the homomorphisms $\psi_\mathfrak{t}$ and $\psi_\mathtt{T}$.

Lemma 3.21. *Suppose that* $\mathtt{T} \in \mathcal{T}_0(\lambda, \mu)$ *and let* $\mathfrak{t} = \operatorname{last}(\mathtt{T})$. *Then there exist* $b_\mathfrak{s} \in R$ *such that*

$$\psi_\mathtt{T} \varphi_{\mathtt{T}^\mu \mathtt{T}^\omega} = \psi_\mathfrak{t} + \sum_{\mathfrak{s} \rhd \mathfrak{t}} b_\mathfrak{s} \psi_\mathfrak{s}.$$

Proof. By Theorem 3.13(iii), there exist $a_\mathtt{S} \in R$ such that

$$\begin{aligned}
\psi_\mathtt{T} \varphi_{\mathtt{T}^\mu \mathtt{T}^\omega}(1) = \psi_\mathtt{T}(m_\mu) &= \varphi_\mathtt{T}(m_\mu) + \sum_{\mathtt{S} \rhd \mathtt{T}} a_\mathtt{S} \varphi_\mathtt{S}(m_\mu) \\
&= m_\mathtt{T} + \sum_{\mathtt{S} \rhd \mathtt{T}} a_\mathtt{S} m_\mathtt{S} \\
&= m_\mathfrak{t} + \sum_{\mathfrak{s} \rhd \mathfrak{t}} a_\mathfrak{s} m_\mathfrak{s}
\end{aligned}$$

for some $a_\mathfrak{s} \in R$ since $\mathfrak{t} = \operatorname{last}(\mathtt{T})$. The lemma now follows from parts (iii) and (vi) of Theorem 3.13. □

Given a semistandard tableau $\mathtt{T}$ recall the multicomposition $\nu_\mathtt{T}$ from Lemma 3.9.

Lemma 3.22. *Suppose that $\mathfrak{t}$ is a standard λ–tableau and let $\mathtt{T} = \mu(\mathfrak{t})$. Write $m_\mu = u_{\mathbf{a}}^+ x_\mu$. Then there exist $c_{\mathtt{S}} \in R$ such that*

$$\varphi_{\mathfrak{t}}\varphi_{\mathtt{T}^\omega\mathtt{T}^\mu} = c_{\mathtt{T}}\varphi_{\mathtt{T}} + \sum_{\mathtt{S}\rhd\mathtt{T}} c_{\mathtt{S}}\varphi_{\mathtt{S}}$$

and where $c_{\mathtt{T}} \simeq \operatorname{res}_{\mathfrak{t}}(\mathbf{a})\{\nu_{\mathtt{T}}\}_q$ if $\mathtt{T}$ is semistandard and $c_{\mathtt{T}} = 0$ otherwise.

Proof. We have

$$\varphi_{\mathfrak{t}}\varphi_{\mathtt{T}^\omega\mathtt{T}^\mu}(m_\mu) = \varphi_{\mathfrak{t}}(m_\mu) = \varphi_{\mathfrak{t}}(1)m_\mu = m_{\mathfrak{t}}u_{\mathbf{a}}^+x_\mu = \operatorname{res}_{\mathfrak{t}}(\mathbf{a})m_{\mathfrak{t}}x_\mu + \sum_{\mathfrak{s}\rhd\mathfrak{t}} a_{\mathfrak{s}}m_{\mathfrak{s}}x_\mu$$

for some $a_{\mathfrak{s}} \in R$ by Corollary 3.8. If $\mathtt{T}$ is semistandard, then $m_{\mathtt{T}^\lambda\mathfrak{t}}x_\mu \simeq \{\nu_{\mathtt{T}}\}_q m_{\mathtt{T}^\lambda\mathtt{T}}$ by Lemma 3.9; *cf.* [13, 3.9]. On the other hand, if $\mathtt{T}$ is not semistandard, then $m_{\mathtt{T}^\lambda\mathfrak{t}}x_\mu$ is a linear combination of terms $m_{\mathtt{T}^\lambda\mathtt{S}}$ where $\mathtt{S} \rhd \mathtt{T}$ by 2.9. Either way, if we define $c_{\mathtt{T}}$ as in the statement of the lemma, then

$$\varphi_{\mathfrak{t}}\varphi_{\mathtt{T}^\omega\mathtt{T}^\mu}(m_\mu) = c_{\mathtt{T}}m_{\mathtt{T}} + \sum_{\mathtt{S}\rhd\mathtt{T}} c_{\mathtt{S}}m_{\mathtt{S}}$$

for some $c_{\mathtt{S}} \in R$ since $\varphi_{\mathtt{T}^\lambda\mathfrak{t}}\varphi_{\mathtt{T}^\omega\mathtt{T}^\mu}(m_\mu) \in M^\lambda \cap M^{\mu*}$. Hence, $\varphi_{\mathfrak{t}}\varphi_{\mathtt{T}^\omega\mathtt{T}^\mu} = c_{\mathtt{T}}\varphi_{\mathtt{T}} + \sum_{\mathtt{S}\rhd\mathtt{T}} c_{\mathtt{S}}\varphi_{\mathtt{S}}$ as required. □

Definition 3.23. Suppose that $\mathfrak{t}$ is a standard λ–tableau such that $\mathtt{T} = \mu(\mathfrak{t})$ is semistandard. We define

$$P_{\mathfrak{t}}^\mu = \{\, (x,y) \mid x < y \text{ and } \mathfrak{t}(x) < \mathfrak{t}(y) \text{ and } \mathtt{T}(x) = \mathtt{T}(y) \,\}$$

and

$$\pi_{\mathfrak{t}}^\mu = \operatorname{res}_{\mathfrak{t}}(\mathbf{a}) \prod_{(x,y)\in P_{\mathfrak{t}}^\mu} \frac{q\operatorname{res}(x) - \operatorname{res}(y)}{\operatorname{res}(x) - \operatorname{res}(y)}$$

where $m_\mu = u_{\mathbf{a}}^+ x_\mu$.

Example 3.24. Let λ, μ and $\mathtt{T}$ be as in Example 3.5 and let $\mathfrak{t}_1 = \operatorname{first}(\mathtt{T})$ and $\mathfrak{t}_2 = \operatorname{last}(\mathtt{T})$. Then

$$\mathfrak{t}_1 = \left(\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 5 \\ \cline{1-1}\end{array},\ \begin{array}{|c|}\hline 4 \\ \hline\end{array}\right), \quad \mathfrak{t}_2 = \left(\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 5 \\ \cline{1-1}\end{array},\ \begin{array}{|c|}\hline 3 \\ \hline\end{array}\right),$$

$\operatorname{res}_{\mathfrak{t}_1}(\mathbf{a}) = \operatorname{res}_{\mathfrak{t}_2}(\mathbf{a}) = (Q_1 - Q_2)(qQ_1 - Q_2)$ and

$$\pi_{\mathfrak{t}_1}^\mu = \operatorname{res}_{\mathfrak{t}_1}(\mathbf{a})\frac{q^2Q_1 - Q_1}{qQ_1 - Q_1} \quad \text{and} \quad \pi_{\mathfrak{t}_2}^\mu = \operatorname{res}_{\mathfrak{t}_2}(\mathbf{a})\frac{q^2Q_1 - Q_1}{qQ_1 - Q_1}\frac{q^3Q_1 - Q_2}{q^2Q_1 - Q_2}.$$

Recall that $q, Q_1, \dots, Q_r$ are currently indeterminates. The reader may check that $\gamma_{\mathfrak{t}_2} \simeq \pi_{\mathfrak{t}_2}^\mu\gamma_{\mathtt{T}}$ (use Example 3.16 where $\mathtt{T} = \mathtt{T}_2$).

Lemma 3.25. *Let $\mathtt{T}$ be a semistandard λ–tableau of type μ and write $m_\mu = u_{\mathbf{a}}^+ x_\mu$.*

(i) *If $\mathfrak{t} = \operatorname{first}(\mathtt{T})$, then $\pi_{\mathfrak{t}}^\mu = \operatorname{res}_{\mathfrak{t}}(\mathbf{a})\{\nu_{\mathtt{T}}\}_q$.*

(ii) *If $\mathfrak{t} = \operatorname{last}(\mathtt{T})$, then $\gamma_{\mathfrak{t}} \simeq \pi_{\mathfrak{t}}^\mu\gamma_{\mathtt{T}}$.*

Proof. (i) Since $\mathfrak{t} = \operatorname{first}(\mathtt{T})$, the elements of $P_{\mathfrak{t}}^\mu$ are ordered pairs (x,y) such that x and y are in the same row of $[\lambda]$ and $\mathtt{T}(x) = \mathtt{T}(y)$. Therefore, the nodes in $P_{\mathfrak{t}}^\mu$ contribute a factor of $\{\nu_{\mathtt{T}}\}_q$ to $\pi_{\mathfrak{t}}^\mu$ (cf. [13, 2.15]).

(ii) This is a routine exercise in induction (cf. [13, 2.16]).

□

Lemma 3.26. *Let* $\mathfrak{t} \in \operatorname{Std}(\lambda)$ *and suppose that there exists an integer* i *with* $1 \le i < n$ *and* $\mathfrak{s} \in \operatorname{Std}(\lambda)$ *such that* $\mathfrak{s} = \mathfrak{t}(i, i+1)$, $\mathfrak{s} \rhd \mathfrak{t}$ *and* $\mu(\mathfrak{s}) = \mu(\mathfrak{t}) \in \mathcal{T}_0(\lambda, \mu)$. *Then*

$$\pi_{\mathfrak{t}}^{\mu} = \frac{q \operatorname{res}_{\mathfrak{s}}(i) - \operatorname{res}_{\mathfrak{t}}(i)}{\operatorname{res}_{\mathfrak{s}}(i) - \operatorname{res}_{\mathfrak{t}}(i)} \pi_{\mathfrak{s}}^{\mu}.$$

Proof. Let x and y be the nodes in the diagram of λ such that $\mathfrak{t}(x) = i$ and $\mathfrak{t}(y) = i+1$. Then $\mathfrak{s}(x) = i+1$ and $\mathfrak{s}(y) = i$ and, since $\mathfrak{s} \rhd \mathfrak{t}$, we have $x < y$. Therefore, $P_{\mathfrak{t}}^{\mu} = P_{\mathfrak{s}}^{\mu} \cup \{(x,y)\}$ and the result follows. □

Lemma 3.27. *Suppose that* $\mathfrak{t}$ *is a standard* λ*–tableau such that* $\mathtt{T} = \mu(\mathfrak{t})$ *is semistandard. Then* $\psi_{\mathfrak{t}} \varphi_{\mathtt{T}^\omega \mathtt{T}^\mu} \simeq \pi_{\mathfrak{t}}^{\mu} \psi_{\mathtt{T}}$.

Proof. We first show that $\psi_{\mathfrak{t}} \varphi_{\mathtt{T}^\omega \mathtt{T}^\mu}$ is an R–multiple of $\psi_{\mathtt{T}}$. Let $(i,k) \in \mathbb{N} \times \mathbf{r}$ and suppose that $y, y+1, \dots, z$ are the entries in row i of $\mathfrak{t}^{\mu^{(k)}}$. Then, by Lemma 3.3 and Theorem 3.13(i),

$$\psi_{\mathfrak{t}} \varphi_{\mathtt{T}^\omega \mathtt{T}^\mu} L_{i,k}^{\mu} = \psi_{\mathfrak{t}}(L_y + \dots + L_z) \varphi_{\mathtt{T}^\omega \mathtt{T}^\mu} = \operatorname{res}_{\mathtt{T}}(i,k) \psi_{\mathfrak{t}} \varphi_{\mathtt{T}^\omega \mathtt{T}^\mu}.$$

Hence, by Lemma 3.12, if $\mathtt{S} \in \mathcal{T}_0(\lambda, \mu)$ and $\mathtt{S} \neq \mathtt{T}$, then $\psi_{\mathfrak{t}} \varphi_{\mathtt{T}^\omega \mathtt{T}^\mu} E_{\mathtt{S}} = 0$. However, by Theorem 3.13(vi), $\psi_{\mathfrak{t}} \varphi_{\mathtt{T}^\omega \mathtt{T}^\mu} = \sum_{\mathtt{S} \in \mathcal{T}_0(\lambda,\mu)} a_{\mathfrak{t}\mathtt{S}} \psi_{\mathtt{S}}$ for some $a_{\mathfrak{t}\mathtt{S}} \in R$; so $\psi_{\mathfrak{t}} \varphi_{\mathtt{T}^\omega \mathtt{T}^\mu} = a_{\mathfrak{t}\mathtt{T}} \psi_{\mathtt{T}}$ by Theorem 3.13(v). Hence, $\psi_{\mathfrak{t}} \varphi_{\mathtt{T}^\omega \mathtt{T}^\mu}$ is an R–multiple of $\psi_{\mathtt{T}}$ as claimed and it remains to show that $a_{\mathfrak{t}\mathtt{T}} \simeq \pi_{\mathfrak{t}}^{\mu}$ for all $\mathfrak{t}$.

Suppose first that $\mathfrak{t} = \operatorname{first}(\mathtt{T})$. By Theorem 3.13(iii) there exist $b_{\mathfrak{s}} \in R$ such that $\psi_{\mathfrak{t}} = \varphi_{\mathfrak{t}} + \sum_{\mathfrak{s} \rhd \mathfrak{t}} b_{\mathfrak{s}} \varphi_{\mathfrak{s}}$. Write $m_\mu = u_{\mathbf{a}}^{+} x_\mu$. Then, by Lemma 3.22 there exist $c_{\mathtt{S}} \in R$ such that

$$\psi_{\mathfrak{t}} \varphi_{\mathtt{T}^\omega \mathtt{T}^\mu} = \varphi_{\mathfrak{t}} \varphi_{\mathtt{T}^\omega \mathtt{T}^\mu} + \sum_{\mathfrak{s} \rhd \mathfrak{t}} b_{\mathfrak{s}} \varphi_{\mathfrak{s}} \varphi_{\mathtt{T}^\omega \mathtt{T}^\mu} \simeq \operatorname{res}_{\mathfrak{t}}(\mathbf{a}) \{\nu_{\mathtt{T}}\}_q \varphi_{\mathtt{T}} + \sum_{\mathtt{S} \rhd \mathtt{T}} c_{\mathtt{S}} \varphi_{\mathtt{S}}.$$

However, $\pi_{\mathfrak{t}}^{\mu} \simeq \operatorname{res}_{\mathfrak{t}}(\mathbf{a}) \{\nu_{\mathtt{T}}\}_q$ by Lemma 3.25(i); so $\psi_{\mathfrak{t}} \varphi_{\mathtt{T}^\omega \mathtt{T}^\mu} \simeq \pi_{\mathfrak{t}}^{\mu} \psi_{\mathtt{T}}$ by Theorem 3.13(iii) as required.

Finally, suppose that $\mathfrak{t} \neq \operatorname{first}(\mathtt{T})$. Then by Lemma 3.9 there exists a standard λ–tableau $\mathfrak{s}$ such that $\mathfrak{s} = \mathfrak{t}(i, i+1)$, $\mathfrak{s} \rhd \mathfrak{t}$ and $\mu(\mathfrak{s}) = \mu(\mathfrak{t})$. Now $\varphi_{\mathfrak{t}} = \varphi_{\mathfrak{s}} T_i$ and $\varphi_{\mathtt{T}^\omega \mathtt{T}^\mu} T_i = q \varphi_{\mathtt{T}^\omega \mathtt{T}^\mu}$ since $(i, i+1) \in \mathfrak{S}_\mu$. Once again by Corollary 3.14, $\psi_{\mathfrak{t}} = \psi_{\mathfrak{s}}(T_i + \alpha)$ where $\alpha = \frac{(q-1)\operatorname{res}_{\mathfrak{t}}(i)}{\operatorname{res}_{\mathfrak{s}}(i) - \operatorname{res}_{\mathfrak{t}}(i)}$. Therefore,

$$\psi_{\mathfrak{t}} \varphi_{\mathtt{T}^\omega \mathtt{T}^\mu} = \psi_{\mathfrak{s}}(T_i + \alpha) \varphi_{\mathtt{T}^\omega \mathtt{T}^\mu} = (q + \alpha) \psi_{\mathfrak{s}} \varphi_{\mathtt{T}^\omega \mathtt{T}^\mu} \simeq \Big(\frac{q \operatorname{res}_{\mathfrak{s}}(i) - \operatorname{res}_{\mathfrak{t}}(i)}{\operatorname{res}_{\mathfrak{s}}(i) - \operatorname{res}_{\mathfrak{t}}(i)} \Big) \pi_{\mathfrak{s}}^{\mu} \psi_{\mathtt{T}}$$

since $\psi_{\mathfrak{s}} \varphi_{\mathtt{T}^\omega \mathtt{T}^\mu} \simeq \pi_{\mathfrak{s}}^{\mu} \psi_{\mathtt{T}}$ by induction. To complete the proof it remains to apply Lemma 3.26. □

Theorem 3.28. *Let* $\mathtt{T}$ *be a semistandard* λ*–tableau of type* μ. *Then* $\langle \psi_{\mathtt{T}}, \psi_{\mathtt{T}} \rangle \simeq \gamma_{\mathtt{T}}$.

Proof. Let $\mathfrak{t} = \operatorname{last}(\mathtt{T})$. By 3.27, $\pi_{\mathfrak{t}}^{\mu} \psi_{\mathtt{T}} \simeq \psi_{\mathfrak{t}} \varphi_{\mathtt{T}^\omega \mathtt{T}^\mu}$ and, by Lemma 3.21, there exist $b_{\mathfrak{s}} \in R$ such that $\psi_{\mathtt{T}} \varphi_{\mathtt{T}^\mu \mathtt{T}^\omega} = \psi_{\mathfrak{t}} + \sum_{\mathfrak{s} \rhd \mathfrak{t}} b_{\mathfrak{s}} \psi_{\mathfrak{s}}$. Therefore, using 2.13 and Theorem 3.13(vi),

$$\begin{aligned} \pi_{\mathfrak{t}}^{\mu} \langle \psi_{\mathtt{T}}, \psi_{\mathtt{T}} \rangle &\simeq \langle \psi_{\mathtt{T}}, \psi_{\mathfrak{t}} \varphi_{\mathtt{T}^\omega \mathtt{T}^\mu} \rangle = \langle \psi_{\mathtt{T}} \varphi_{\mathtt{T}^\omega \mathtt{T}^\mu}^{*}, \psi_{\mathfrak{t}} \rangle = \langle \psi_{\mathtt{T}} \varphi_{\mathtt{T}^\mu \mathtt{T}^\omega}, \psi_{\mathfrak{t}} \rangle \\ &= \langle \psi_{\mathfrak{t}}, \psi_{\mathfrak{t}} \rangle + \sum_{\mathfrak{s} \rhd \mathfrak{t}} b_{\mathfrak{s}} \langle \psi_{\mathfrak{s}}, \psi_{\mathfrak{t}} \rangle = \langle \psi_{\mathfrak{t}}, \psi_{\mathfrak{t}} \rangle. \end{aligned}$$

However, $\langle \psi_{\mathfrak{t}}, \psi_{\mathfrak{t}} \rangle \simeq \gamma_{\mathfrak{t}}$ by Proposition 3.19 and $\gamma_{\mathfrak{t}} \simeq \pi_{\mathfrak{t}}^{\mu} \gamma_{\mathtt{T}}$ by Lemma 3.25(ii), so the theorem follows. □

Corollary 3.29. *Suppose that* $\lambda\in\Lambda^+$ *and* $\mu\in\Lambda$. *Then* $G_\mu(\lambda)\simeq\prod_{\mathtt{T}\in\mathcal{T}_0(\lambda,\mu)}\gamma_{\mathtt{T}}$.

Proof. By definition, $G_\mu(\lambda)=\det\big(\langle\varphi_{\mathtt{S}},\varphi_{\mathtt{T}}\rangle\big)$, where $\mathtt{S}$ and $\mathtt{T}$ run over $\mathcal{T}_0(\lambda,\mu)$. By Theorem 3.13, $\{\psi_{\mathtt{T}}\}_{\mathtt{T}\in\mathcal{T}_0(\lambda,\mu)}$ is also a basis of W^λ_μ and the transition matrix between this basis and the semistandard basis is unitriangular. □

Remark 3.30. By definition, $G_\mu(\lambda)\in\mathbb{F}[q,q^{-1},Q_1\dots,Q_r]$; however, *a priori*, there is no reason why the rational function $\prod_{\mathtt{T}}\gamma_{\mathtt{T}}$ should belong to this ring (cf. Example 3.16).

Now that we have computed $G_\mu(\lambda)$ we rewrite it in a more usable form. To do this we introduce beta numbers for multipartitions.

Definition 3.31. Suppose that ν is a multipartition of n and let $c\in\{1,2,\dots,rn\}$ and write $c=(r-k)n+j$ where $j\in\{1,2\dots,n\}$.

(i) Define column c of ν to be column j of $\nu^{(k)}$.
(ii) Let $s(c)=k$.
(iii) Let $\beta_c=n-j+\nu^{(k)'}_j$. Then the sequence

$$\beta=(\beta_1,\dots,\beta_n,\beta_{n+1},\dots,\beta_{2n},\dots,\beta_{(r-1)n+1},\dots,\beta_{rn}),$$

is the sequence of beta numbers for ν.

Note that, in the sense of [11, p. 77], $(\beta_1\dots,\beta_n)$ are beta numbers for $\nu^{(r)'}$, the partition which is conjugate to $\nu^{(r)}$, and $(\beta_{n+1},\dots,\beta_{2n})$ are beta numbers for $\nu^{(r-1)'}$ and so on.

Definition 3.32. Given a sequence of integers $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_{rn})$ and a multicomposition τ of n, define the integer $d_\tau(\alpha)$ as follows.

(i) If $\alpha_b=\alpha_c$ for some $b\neq c$ with $s(b)=s(c)$, or $\alpha_c<0$ for any c, then $d_\tau(\alpha)=0$.
(ii) If α does not satisfy (i), then there exists a unique multipartition ν and unique elements $w_1,\dots,w_r$ of $\mathfrak{S}_n$ such that if β is the sequence of beta numbers for ν, then

$$\beta=(\alpha_{1w_1},\dots,\alpha_{nw_1},\alpha_{n+1w_2},\dots,\alpha_{n+nw_2},\dots,\alpha_{(r-1)n+nw_r}).$$

Define $d_\tau(\alpha)=(-1)^{\ell(w_1)+\dots+\ell(w_r)}|\mathcal{T}_0(\nu,\tau)|$.

Given a multipartition $\nu\in\Lambda^+$ we also define $d_\tau(\nu)=|\mathcal{T}_0(\nu,\tau)|$.

Now fix $\lambda\in\Lambda^+$ and $\mu\in\Lambda$ and let $\beta=(\beta_1,\dots,\beta_{rn})$ be the sequence of beta numbers for λ. Then $d_\mu(\beta)=|\mathcal{T}_0(\lambda,\mu)|$.

Let $(i,k)\in\mathbb{N}\times\mathbf{r}$ be maximal such that $\mu^{(k)}_i\neq0$ and set $z=\mu^{(k)}_i$. Then every semistandard tableau $\mathtt{T}\in\mathcal{T}_0(\lambda,\mu)$ has precisely z entries equal to (i,k) and these entries are at the feet of distinct columns of $\mathtt{T}$. Indexing the columns of λ by $1,2\dots,rn$ as in (3.31)(ii), let the columns which contain an entry (i,k) be labelled by $C=\{c_1<c_2<\dots<c_z\}$ and let λ^C be the multipartition of $n-z$ whose sequence of beta numbers is

$$\beta^C=(\beta_1,\dots,\beta_{c_1}-1,\dots,\beta_{c_2}-1,\dots,\beta_{c_z}-1,\dots,\beta_{rn}).$$

Then the tableau $\bar{\mathtt{T}}$ obtained from $\mathtt{T}$ by deleting all of the entries (i,k) is a semistandard λ^C–tableau of weight $\bar\mu$, where $\bar\mu$ is the multicomposition of $n-z$ with $\bar\mu^{(l)}_j=\mu^{(l)}_j$, if $(j,l)\neq(i,k)$, and $\bar\mu^{(k)}_i=0$.

Since $d_\mu(\beta) = |\mathcal{T}_0(\lambda,\mu)|$, by letting $\mathtt{T}$ range over the elements of $\mathcal{T}_0(\lambda,\mu)$, the above argument shows that $d_\mu(\beta) = \sum_C d_{\bar\mu}(\beta^C)$ where the sum is over all subsets of $\{1,2,\dots,rn\}$ with z elements. Similarly, with the help of Definition 3.32, for any sequence of integers $\alpha = (\alpha_1,\dots,\alpha_{rn})$ we have that

$$d_\mu(\alpha) = \sum_{\substack{C\subseteq\{1,2,\dots,rn\}\\ |C|=z}} d_{\bar\mu}(\alpha^C), \tag{3.33}$$

where the sequence α^C is defined in the same way as β^C.

Now consider the Gram determinant $G_\mu(\lambda)$. Applying Corollary 3.29 and Definition 3.15,

$$\begin{aligned} G_\mu(\lambda) &\simeq \prod_{\mathtt{T}\in\mathcal{T}_0(\lambda,\mu)} \gamma_{\mathtt{T}} = \prod_{\mathtt{T}\in\mathcal{T}_0(\lambda,\mu)} \prod_{x\in[\lambda]} \frac{A_{\mathtt{T}}(x)}{R_{\mathtt{T}}(x)} \\ &= \prod_{\mathtt{T}\in\mathcal{T}_0(\lambda,\mu)} \Bigg\{ \prod_{\substack{x\in[\lambda]\\ \mathtt{T}(x)\neq(i,k)}} \frac{A_{\mathtt{T}}(x)}{R_{\mathtt{T}}(x)} \Bigg\}\Bigg\{ \prod_{\substack{x\in[\lambda]\\ \mathtt{T}(x)=(i,k)}} \frac{A_{\mathtt{T}}(x)}{R_{\mathtt{T}}(x)} \Bigg\} \\ &\simeq \prod_{\substack{C\subseteq\{1,2,\dots,rn\}\\ |C|=z}} G_{\bar\mu}(\lambda^C) \quad\times \prod_{\mathtt{T}\in\mathcal{T}_0(\lambda,\mu)} \prod_{\substack{x\in[\lambda]\\ \mathtt{T}(x)=(i,k)}} \frac{A_{\mathtt{T}}(x)}{R_{\mathtt{T}}(x)}. \end{aligned}$$

Let $\mathtt{T}\in\mathcal{T}_0(\lambda,\mu)$ and fix a node $x\in[\lambda]$ with $\mathtt{T}(x)=(i,k)$. By definition, both $R_{\mathtt{T}}(x)$ and $A_{\mathtt{T}}(x)$ are products of the form $\prod_y \big(\operatorname{res}(x)-\operatorname{res}(y)\big)$. For each node y appearing in the product $R_{\mathtt{T}}(x)$ there exists a unique y' in $A_{\mathtt{T}}(x)$ such that y' is in the row below y; moreover, this gives a one–to–one correspondence between the factors of $R_{\mathtt{T}}(x)$ and the factors of $A_{\mathtt{T}}(x)$ which are indexed by nodes which are not in the first row of some component. Given such a triple x,y,y', suppose that x is in column c of λ and that y and y' are in columns b and b' respectively. Then $c > b \ge b'$, $s(b) = s(b')$ and $A_{\mathtt{T}}(x)/R_{\mathtt{T}}(x)$ contains the factor

$$\frac{\operatorname{res}(x)-\operatorname{res}(y')}{\operatorname{res}(x)-\operatorname{res}(y)} = \frac{q^{-\beta_c}Q_{s(c)} - q^{-\beta_{b'}-1}Q_{s(b)}}{q^{-\beta_c}Q_{s(c)} - q^{-\beta_b}Q_{s(b)}} \simeq \prod_{j=b'}^{b} \frac{q^{-\beta_c+1}Q_{s(c)} - q^{-\beta_j}Q_{s(b)}}{q^{-\beta_c}Q_{s(c)} - q^{-\beta_j}Q_{s(b)}}.$$

In this way, each column $b < c$ which does not contain an entry (i,k), and such that $n \nmid b$, contributes a factor to $A_{\mathtt{T}}(x)/R_{\mathtt{T}}(x)$. The restriction that b is not divisible by n is due to the fact that, if $n \mid b$ and $b < c$, then column b cannot contain a removable node. The columns $b < c$ with n dividing b correspond to the as yet unaccounted for factors of $A_{\mathtt{T}}(x)$; thus such columns contribute a factor only to the numerator of $\gamma_{\mathtt{T}}$. Hence, we obtain the following "branching rule" for $G_\mu(\lambda)$.

(3.34)

$$G_\mu(\lambda) \simeq \prod_{\substack{C\subseteq\{1,2,\dots,rn\}\\ |C|=z}} G_{\bar\mu}(\lambda^C) \prod_{c\in C} \left\{ \frac{\displaystyle\prod_{\substack{b<c\\ b\notin C}} (q^{-\beta_c+1}Q_{s(c)} - q^{-\beta_b}Q_{s(b)})}{\displaystyle\prod_{\substack{b<c\\ n\nmid b\notin C}} (q^{-\beta_c}Q_{s(c)} - q^{-\beta_b}Q_{s(b)})} \right\}^{d_{\bar\mu}(\beta^C)}.$$

We are now ready to give our first reformulation of Corollary 3.29. For convenience, we let $\mathbf{rn} = \{1,2\dots,rn\}$ and given integers $h \ge 0$ and $c \in \mathbf{rn}$ define

$$S(h,c) = \{\, b\in\mathbf{rn} \mid s(b) > s(c) \text{ or } s(b) = s(c) \text{ and } \beta_b \ge \beta_c + h \,\}.$$

Theorem 3.35. *Let $\lambda \in \Lambda^+$, $\mu \in \Lambda$ and let β be the sequence of beta numbers for λ. Then*

$$G_\mu(\lambda) \simeq \prod_{h\geq 1}\prod_{c=1}^{rn}\prod_{b\in S(h,c)} \left(q^{-\beta_c+h}Q_{s(c)} - q^{-\beta_b}Q_{s(b)}\right)^{d_\mu(\beta_1,\dots,\beta_b+h,\dots,\beta_c-h,\dots,\beta_{rn})}.$$

Proof. If $n=0$, then $G_\mu(\lambda)=1$ and, by virtue of Definition 3.32, the right hand side is also 1. Consequently, we may assume by induction that $\prod_C G_{\bar\mu}(\lambda^C) \simeq G_1G_2G_3G_4$ where

$$G_1 = \prod_{h\geq 1}\prod_{\substack{C\subseteq \mathbf{rn}\\ |C|=z}}\prod_{\substack{c\in\mathbf{rn}\\ c\notin C}}\prod_{\substack{b\in S(h,c)\\ b\notin C}} (q^{-\beta_c+h}Q_{s(c)}-q^{-\beta_b}Q_{s(b)})^{d_{\bar\mu}(\beta_1^C,\dots,\beta_b^C+h,\dots,\beta_c^C-h,\dots,\beta_{rn}^C)},$$

$$G_2 = \prod_{h\geq 1}\prod_{\substack{C\subseteq \mathbf{rn}\\ |C|=z-1}}\prod_{\substack{c\in\mathbf{rn}\\ c\notin C}}\prod_{\substack{b\in S(h,c)\\ b\notin C}} (q^{-\beta_c+1+h}Q_{s(c)}-q^{-\beta_b}Q_{s(b)})^{d_{\bar\mu}(\beta_1^C,\dots,\beta_b^C+h,\dots,\beta_c^C-1-h,\dots,\beta_{rn}^C)}$$

$$= \prod_{h\geq 2}\prod_{\substack{C\subseteq \mathbf{rn}\\ |C|=z-1}}\prod_{\substack{c\in\mathbf{rn}\\ c\notin C}}\prod_{\substack{b\in S(h,c)\\ b\notin C}} (q^{-\beta_c+h}Q_{s(c)}-q^{-\beta_b}Q_{s(b)})^{d_{\bar\mu}(\beta_1^C,\dots,\beta_b^C+h-1,\dots,\beta_c^C-h,\dots,\beta_{rn}^C)},$$

$$G_3 = \prod_{h\geq 1}\prod_{\substack{C\subseteq \mathbf{rn}\\ |C|=z-1}}\prod_{\substack{c\in\mathbf{rn}\\ c\notin C}}\prod_{\substack{b\in S(h,c)\\ b\notin C}} (q^{-\beta_c+h}Q_{s(c)}-q^{-\beta_b+1}Q_{s(b)})^{d_{\bar\mu}(\beta_1^C,\dots,\beta_b^C-1+h,\dots,\beta_c^C-h,\dots,\beta_{rn}^C)}$$

$$\simeq \prod_{h\geq 0}\prod_{\substack{C\subseteq \mathbf{rn}\\ |C|=z-1}}\prod_{\substack{c\in\mathbf{rn}\\ c\notin C}}\prod_{\substack{b\in S(h,c)\\ b\notin C}} (q^{-\beta_c+h}Q_{s(c)}-q^{-\beta_b}Q_{s(b)})^{d_{\bar\mu}(\beta_1^C,\dots,\beta_b^C+h,\dots,\beta_c^C-h-1,\dots,\beta_{rn}^C)},$$

and

$$G_4 = \prod_{h\geq 1}\prod_{\substack{C\subseteq \mathbf{rn}\\ |C|=z-2}}\prod_{\substack{c\in\mathbf{rn}\\ c\notin C}}\prod_{\substack{b\in S(h,c)\\ b\notin C}} (q^{-\beta_c+h}Q_{s(c)}-q^{-\beta_b}Q_{s(b)})^{d_{\bar\mu}(\beta_1^C,\dots,\beta_b^C-1+h,\dots,\beta_c^C-1-h,\dots,\beta_{rn}^C)}.$$

Note, that in all of the products above, the sets $S(h,c)$ should really depend upon β^C; however, the reader may check that the additional factors that this introduces (in G_3 and G_4) all have exponent 0 and hence are trivial.

By our branching rule 3.34, $G_\mu(\lambda) \simeq B_2B_3\prod_C G_{\bar\mu}(\lambda^C)$ where

$$B_2 = \prod_{\substack{C\subseteq \mathbf{rn}\\ |C|=z}}\prod_{\substack{b<c\\ b\notin C}} (q^{-\beta_c+1}Q_{s(c)} - q^{-\beta_b}Q_{s(b)})^{d_{\bar\mu}(\beta^C)}$$

and

$$B_3 = \prod_{\substack{C\subseteq \mathbf{rn}\\ |C|=z}}\prod_{\substack{b<c\\ n\nmid b\notin C}} (q^{-\beta_c}Q_{s(c)} - q^{-\beta_b}Q_{s(b)})^{-d_{\bar\mu}(\beta^C)}.$$

Now if $s(b)=s(c)$, then $b<c$ if and only if $\beta_b>\beta_c$ and this is if and only if $\beta_b\geq\beta_c+1$. Therefore,

$$B_2G_2 = \prod_{h\geq 1}\prod_{\substack{C\subseteq \mathbf{rn}\\ |C|=z-1}}\prod_{\substack{c\in\mathbf{rn}\\ c\notin C}}\prod_{\substack{b\in S(h,c)\\ b\notin C}} (q^{-\beta_c+h}Q_{s(c)}-q^{-\beta_b}Q_{s(b)})^{d_{\bar\mu}(\beta_1^C,\dots,\beta_b^C+h-1,\dots,\beta_c^C-h\dots\beta_{rn}^C)},$$

Similarly, we find that

$$B_3G_3 \simeq \prod_{h\geq 1}\prod_{\substack{C\subseteq \mathbf{rn}\\ |C|=z-1}}\prod_{\substack{c\in\mathbf{rn}\\ c\notin C}}\prod_{\substack{b\in S(h,c)\\ b\notin C}} (q^{-\beta_c+h}Q_{s(c)}-q^{-\beta_b}Q_{s(b)})^{d_{\bar\mu}(\beta_1^C,\dots,\beta_b^C+h,\dots,\beta_c^C-h-1,\dots,\beta_{rn}^C)}.$$

Note that the restriction that $n \nmid b$ in B_3 is not important because if $n \mid b$ and $b \in S(0,c)$, then $\beta_b = 0$ and so $\beta_c - 1 < 0$ and $d_{\bar\mu}(\beta^C) = 0$; consequently, neither B_3 nor G_3 has a non–trivial factor indexed by column b.

Now $G_\mu(\lambda) \simeq G_1(B_2G_2)(B_3G_3)G_4$; so using 3.33 we deduce the result. □

Finally, we reinterpret Theorem 3.35 in terms of moving nodes in the diagram of λ. For each node $x = (i,j,k)$ in $[\lambda]$ let r_x denote the corresponding rim hook (in $\lambda^{(k)}$); see [10, §18]. Let $\ell\ell(r_x) = \lambda_i^{(k)'} - i$ be the leg length of r_x and $\mathrm{res}(r_x) = \mathrm{res}(f_x)$ where f_x is the foot node of r_x (that is, f_x is the last node in the column of $\lambda^{(k)}$ which contains x).

Definition 3.36. Suppose that λ and ν are multipartitions in Λ^+. If $\lambda \ntriangleright \nu$ let $g_{\lambda\nu} = 1$; otherwise let $g_{\lambda\nu}$ be the element of $\mathbb{Q}(q, Q_1, \dots, Q_r)$ given by

$$g_{\lambda\nu} = \prod_{x\in[\lambda]} \prod_{\substack{y\in[\nu] \\ [\nu]\setminus r_y = [\lambda]\setminus r_x}} \big(\mathrm{res}(r_x) - \mathrm{res}(r_y)\big)^{\varepsilon_{xy}},$$

where $\varepsilon_{xy} = (-1)^{\ell\ell(r_x)+\ell\ell(r_y)}$.

Remark 3.37. These functions are not as complicated as their definition suggests. First, note that $g_{\lambda\nu} = 1$ unless $\lambda^{(k)} = \nu^{(k)}$ for all but at most two $k \in \mathbf{r}$. If $g_{\lambda\nu} \neq 1$ and $\lambda^{(k)} \neq \nu^{(k)}$ and $\lambda^{(l)} \neq \nu^{(l)}$ for $k \neq l$, then $g_{\lambda\nu} \simeq (q^dQ_k - Q_l)^{\pm 1}$ for some integer d with $-n < d < n$. If λ and ν differ only on the kth component, then $g_{\lambda\nu} \simeq (q^aQ_k - Q_k)/(q^bQ_k - Q_k) \simeq [a]_q/[b]_q$ for some integers a and b in $\{1, 2, \dots, n\}$; *cf.* [13, 2.30].

Given an integral domain R containing parameters $\hat q, \widehat{Q}_1, \dots, \widehat{Q}_r$ we can consider $g_{\lambda\nu}$ as an element of the field of fractions of R by evaluating the indeterminates in $g_{\lambda\nu}$ appropriately and replacing 1 by the identity of R.

Corollary 3.38. *Let R be an integral domain. Then*

$$G_\mu(\lambda) \simeq \prod_{\nu\in\Lambda^+} (g_{\lambda\nu})^{d_\mu(\nu)},$$

considered as an element of R.

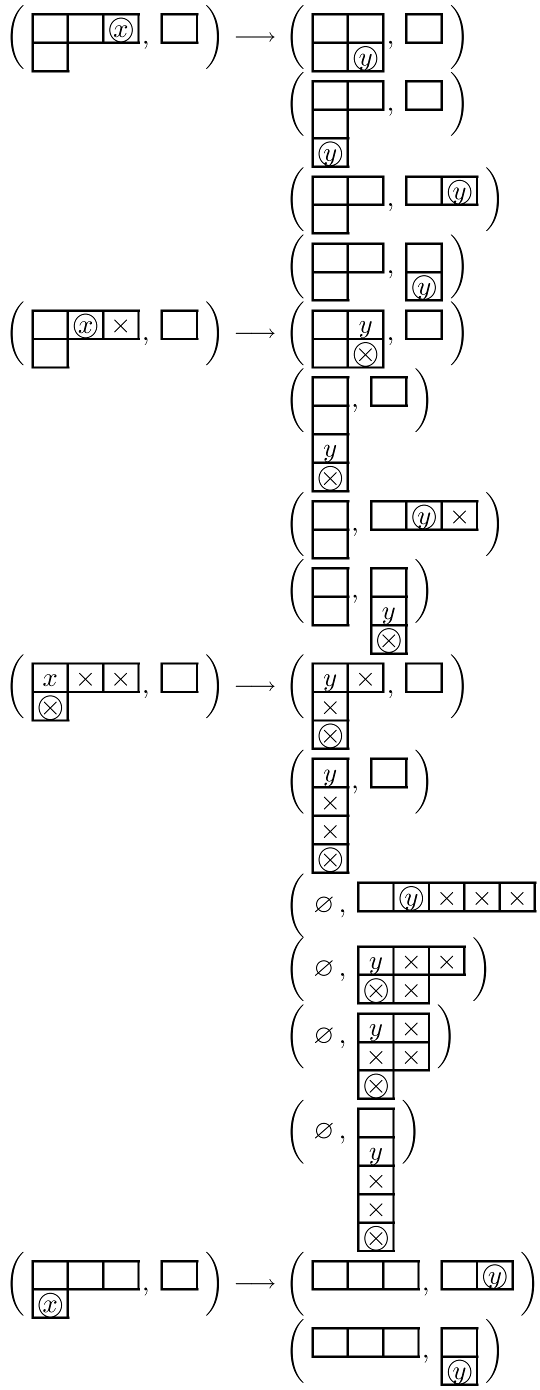
Proof. By general arguments, if R is an integral domain, then the Gram determinant $G_\mu(\lambda)$ can be computed by evaluating the polynomial $\prod_{\mathsf{T}} \gamma_{\mathsf{T}}$ at appropriate values of the indeterminates. Thus it suffices to consider the case where $R = \mathbb{F}[q, q^{-1}, Q_1, \dots, Q_r]$.

Consider the expression we have obtained for $G_\mu(\lambda)$ in Theorem 3.35. As in the proof of [13, 2.30], the only time that $d_\mu(\beta_1, \dots, \beta_b + h, \dots, \beta_c - h, \dots, \beta_{rn})$ is non–zero is when a rim hook with foot node in column c of λ can be moved to a rim hook with foot node in column b. Hence $G_\mu(\lambda)$ can be expressed as in the statement of the corollary. □

This formula for $G_\mu(\lambda)$ is the nicest of the three we have obtained. It is not hard to apply; for each node $x \in [\lambda]$ we have to move the rim hook r_x down in the diagram in all possible ways.

Example 3.39. Consider once more the case where $\lambda = \big((3,1),(1)\big)$ and $\mu = \big((2),(2,1)\big)$. In the diagrams below, we label the nodes x and y, their rim hooks, and we have circled the corresponding foot nodes. We list the factors $\mathrm{res}(r_x) -$

$\operatorname{res}(r_y)$ rather than the functions $(g_{\lambda\nu})^{d_\mu(\nu)}$. In the last column we have given the numbers $d_\mu(\nu) = |\mathcal{T}_0(\nu, \mu)|$.



$(q^2Q_1 - Q_1)^{d_\mu((2^2),(1))}$	2
$(q^2Q_1 - q^{-2}Q_1)^{d_\mu((2,1^2),(1))}$	1
$(q^2Q_1 - qQ_2)^{d_\mu((2,1),(2))}$	2
$(q^2Q_1 - q^{-1}Q_2)^{d_\mu((2,1),(1^2))}$	1
$(qQ_1 - Q_1)^{-d_\mu((2^2),(1))}$	2
$(qQ_1 - q^{-3}Q_1)^{-d_\mu((1^4),(1))}$	0
$(qQ_1 - qQ_2)^{d_\mu((1^2),(3))}$	0
$(qQ_1 - q^{-2}Q_2)^{-d_\mu((1^2),(1^3))}$	0
$(q^{-1}Q_1 - q^{-2}Q_1)^{-d_\mu((2,1^2),(1))}$	1
$(q^{-1}Q_1 - q^{-3}Q_1)^{d_\mu((1^4),(1))}$	0
$(q^{-1}Q_1 - qQ_2)^{-d_\mu((0),(5))}$	0
$(q^{-1}Q_1 - q^{-1}Q_2)^{d_\mu((0),(3,2))}$	0
$(q^{-1}Q_1 - q^{-2}Q_2)^{-d_\mu((0),(2^2,1))}$	0
$(q^{-1}Q_1 - q^{-4}Q_2)^{d_\mu((0),(1^5))}$	0
$(q^{-1}Q_1 - qQ_2)^{d_\mu((3),(2))}$	2
$(q^{-1}Q_1 - q^{-1}Q_2)^{d_\mu((3),(1^2))}$	1

If it is raining, the reader might like to check that the answer this gives for $G_\mu(\lambda)$ agrees with that of Example 3.16 (it does).

Motivated by the theory of Coxeter groups, we make our next definition.

Definition 3.40. The Poincaré polynomial of the generic Ariki–Koike algebra $\mathcal{H}$ is the element of $\mathbb{Z}[q, q^{-1}, Q_1, \dots, Q_r]$ given by

$$P(q; Q_1, \dots, Q_r) = \{n\}_q \prod_{1 \le k < l \le r} \prod_{-n < d < n} (q^d Q_k - Q_l).$$

If R is an integral domain, let $P_R(q; Q_1, \dots, Q_r)$ denote the corresponding specialization of $P(q; Q_1, \dots, Q_r)$.

Note that it is possible that $P_R(q; Q_1, \dots, Q_r) \neq 0$ even if $Q_k = 0$ for some $k \in \mathbf{r}$. It is for this reason that we had to introduce the rational functions $g_{\lambda\nu}$ above.

Corollary 3.41. *Suppose R is an integral domain such that $P_R(q; Q_1, \dots, Q_r)$ is a non–zero element of R. Then $G_\mu(\lambda) \neq 0$ for all $\lambda \in \Lambda^+$ and all $\mu \in \Lambda$.*

Proof. By 3.37, for all λ and μ, each factor of the polynomial $g_{\lambda\nu}$ divides $P(q; Q_1, \dots, Q_r)$; hence the result. □

Suppose that R is a field and that Λ^+ is the set of all multipartitions of n. Then it is easy to see that the converse of Corollary 3.41 is true. Furthermore, $G_\mu(\lambda) \neq 0$ for all λ and μ if and only if each Weyl module is irreducible, so this is equivalent to $\mathcal{S}$ being semisimple by [8, 3.8]. However, $\mathcal{S}$ is semisimple if and only if $\mathcal{H}$ is semisimple by Proposition 2.17; so we see that $\mathcal{H}$ is semisimple if and only if $P_R(q; Q_1, \dots, Q_r) \neq 0$. Thus we recover the main result of [2].

We also note that with a little more care the proof of Lemma 3.12, and hence Theorem 3.13, goes through for any field R provided that $q \neq 1$, $P_R(q; Q_1, \dots, Q_r) \neq 0$ and $Q_k \neq 0$ for all $k \in \mathbf{r}$.

4. The sum formula and irreducibility

We now apply the results of the previous section to describe the Jantzen filtration of the Weyl modules W^λ and the Specht modules S^λ. First we need some preparation.

Throughout this section we assume that R is a principal ideal domain and that $\mathfrak{p}$ is a prime in R. Let $\mathbb{F} = R/\mathfrak{p}R$. Then given an R–module U_R its reduction modulo $\mathfrak{p}$ is the $\mathbb{F}$–module $U_\mathbb{F} = \big(U_R + \mathfrak{p}U_R\big)/\mathfrak{p}U_R \cong U_R \otimes_R \mathbb{F}$.

Suppose that U_R is a free R–module of finite rank equipped with a symmetric bilinear form $\langle\ ,\ \rangle$. For each $i \geq 0$ let

$$U_R(i) = \{\, u \in U_R \mid \mathfrak{p}^i \text{ divides } \langle u, u' \rangle \text{ for all } u' \in U_R \,\}.$$

The Jantzen filtration of the $\mathbb{F}$–module $U_\mathbb{F}$ is

$$U_\mathbb{F} = U_\mathbb{F}(0) \supseteq U_\mathbb{F}(1) \supseteq \cdots$$

where $U_\mathbb{F}(i) = U_R(i) \otimes_R \mathbb{F}$ for all $i \geq 0$. Since $U_\mathbb{F}$ is finite dimensional, $U_\mathbb{F}(i) = 0$ for all sufficiently large i.

Let $e_1, e_2, \dots, e_N$ be an R–basis of U_R and let $G = \det\big(\langle e_i, e_j \rangle\big)$ be the determinant of the associated Gram matrix (an element of R).

Let $\nu_\mathfrak{p} \colon R^\times \to \mathbb{N}$ be the $\mathfrak{p}$–adic valuation map. Then an argument due to Jantzen proves the following.

4.1 ([14, Lemma 3]). *Suppose that the bilinear form $\langle\ ,\ \rangle$ is non–degenerate. Then*

$$\nu_{\mathfrak{p}}(G)=\sum_{i>0} dim_{\mathbb{F}} U_{\mathbb{F}}(i).$$

Fix $\hat{q},\widehat{Q}_1,\dots,\widehat{Q}_r\in R$ and let $q=\hat{q}+\mathfrak{p}R$, $Q_1=\widehat{Q}_1+\mathfrak{p}R,\dots,Q_r=\widehat{Q}_r+\mathfrak{p}R$ be their canonical images in $\mathbb{F}$. Let $\mathcal{S}_R$ be the associated cyclotomic $\hat{q}$–Schur algebra over R with parameters $\hat{q},\widehat{Q}_1,\dots,\widehat{Q}_r$. Then its reduction modulo $\mathfrak{p}$, $\mathcal{S}_{\mathbb{F}}=\mathcal{S}_R\otimes_R\mathbb{F}$, is the cyclotomic q–Schur algebra with parameters $q,Q_1,\dots,Q_r$. If $\lambda\in\Lambda^+$, then W_R^λ will denote the corresponding Weyl module of $\mathcal{S}_R$ and $W_{\mathbb{F}}^\lambda\cong W_R^\lambda\otimes_R\mathbb{F}$ the Weyl module of $\mathcal{S}_{\mathbb{F}}$.

As we did in [13, §3], given a right $\mathcal{S}_{\mathbb{F}}$–module $U_{\mathbb{F}}$ and integers $a_\lambda\in\mathbb{Z}$ we write $U_{\mathbb{F}}\longleftrightarrow\displaystyle\sum_{\lambda\in\Lambda^+}a_\lambda W_{\mathbb{F}}^\lambda$ if

$$U_{\mathbb{F}}\oplus\bigoplus_{\substack{\lambda\in\Lambda^+\\ a_\lambda<0}}(-a_\lambda)W_{\mathbb{F}}^\lambda\quad\text{and}\quad\bigoplus_{\substack{\lambda\in\Lambda^+\\ a_\lambda>0}}a_\lambda W_{\mathbb{F}}^\lambda$$

have the same composition factors. Using the cellular structure of $\mathcal{S}_{\mathbb{F}}$ we obtain the following result.

Lemma 4.2. *Suppose that $U_{\mathbb{F}}$ is an $\mathcal{S}_{\mathbb{F}}$–module such that for all $\mu\in\Lambda^+$ we have*

$$\dim U_{\mathbb{F}}\varphi_{\mathtt{T}^\mu\mathtt{T}^\mu}=\sum_{\lambda\in\Lambda^+}a_\lambda\dim W_{\mathbb{F}}^\lambda\varphi_{\mathtt{T}^\mu\mathtt{T}^\mu}$$

for some $a_\lambda\in\mathbb{Z}$. Then $U_{\mathbb{F}}\longleftrightarrow\sum_{\lambda\in\Lambda^+}a_\lambda W_{\mathbb{F}}^\lambda$.

Let R_f be the field of fractions of R and extend $\nu_{\mathfrak{p}}$ to a map $R_f^\times\longrightarrow\mathbb{Z}$ in the natural way. Recall the rational functions $g_{\lambda\nu}\in R_f$ and $P_R(\hat{q};\widehat{Q}_1,\dots,\widehat{Q}_r)$ from the end of the previous section. We can now state one of our main results.

Theorem 4.3. *Let $\lambda\in\Lambda^+$ and suppose that $P_R(\hat{q};\widehat{Q}_1,\dots,\widehat{Q}_r)\neq0$. Then*

$$\sum_{i>0}W_{\mathbb{F}}^\lambda(i)\longleftrightarrow\sum_{\substack{\nu\\ \lambda\rhd\nu}}\nu_{\mathfrak{p}}(g_{\lambda\nu})W_{\mathbb{F}}^\nu.$$

Remark 4.4. Note that we may omit the condition that λ dominates ν from the second sum, since $\nu_{\mathfrak{p}}(g_{\lambda\nu})=0$ unless $\lambda\rhd\nu$; however we have included this condition to emphasize that only these multipartitions matter.

Proof. Suppose that $\mu\in\Lambda^+$ with $\lambda\trianglerighteq\mu$ and recall that $G_\mu(\lambda)$ is the Gram determinant of the μ–weight space $W_R^\lambda\varphi_{\mathtt{T}^\mu\mathtt{T}^\mu}$ with respect to the semistandard basis. Because $P_R(\hat{q};\widehat{Q}_1,\dots,\widehat{Q}_r)\neq0$ we know by Corollary 3.41 that $G_\mu(\lambda)\neq0$ in R. Therefore, we can apply 4.1 to $W_R^\lambda\varphi_{\mathtt{T}^\mu\mathtt{T}^\mu}$ to deduce that

$$\nu_{\mathfrak{p}}\big(G_\mu(\lambda)\big)=\sum_{i>0}\dim_{\mathbb{F}}W_{\mathbb{F}}^\lambda\varphi_{\mathtt{T}^\mu\mathtt{T}^\mu}(i).$$

Since $d_\mu(\lambda)=\dim_{\mathbb{F}}W_{\mathbb{F}}^\lambda\varphi_{\mathtt{T}^\mu\mathtt{T}^\mu}$, the result follows by Corollary 3.38 and Lemma 4.2. □

As our first application of this result we describe the irreducible Weyl modules. Recall that if $\lambda\in\Lambda^+$, then $F_{\mathbb{F}}^\lambda=W_{\mathbb{F}}^\lambda/\operatorname{rad}W_{\mathbb{F}}^\lambda$.

Corollary 4.5. *Let $\lambda \in \Lambda^+$ and suppose that $P_R(\hat{q};\widehat{Q}_1,\dots,\widehat{Q}_r) \neq 0$. Then $W_{\mathbb{F}}^\lambda$ is irreducible if and only if $\nu_{\mathfrak{p}}(g_{\lambda\nu}) = 0$ for all multipartitions $\nu \in \Lambda^+$ such that $\lambda \rhd \nu$.*

Proof. Because $\operatorname{rad} W_{\mathbb{F}}^\lambda = W_{\mathbb{F}}^\lambda(1)$, the Weyl module $W_{\mathbb{F}}^\lambda$ is irreducible if and only if $W_{\mathbb{F}}^\lambda(1) = (0)$. If there exists a multipartition $\nu \in \Lambda^+$ with $\nu_{\mathfrak{p}}(g_{\lambda\nu}) \neq 0$, then we may assume that ν is minimal in the dominance ordering with this property. Then $F_{\mathbb{F}}^\nu$ is a composition factor of the right hand side of Theorem 4.3 (in particular, we must have $\nu_{\mathfrak{p}}(g_{\lambda\nu}) > 0$); consequently, $W_{\mathbb{F}}^\lambda(1) \neq (0)$ and so $W_{\mathbb{F}}^\lambda$ is reducible.

Conversely, if $\nu_{\mathfrak{p}}(g_{\lambda\nu}) = 0$ for all $\nu \in \Lambda^+$ such that $\lambda \rhd \nu$, then $\operatorname{rad} W_{\mathbb{F}}^\lambda = (0)$ by Theorem 4.3; hence, $W_{\mathbb{F}}^\lambda$ is irreducible. ∎

We extend the relation $\longleftrightarrow$ to $\mathcal{H}_{\mathbb{F}}$–modules in the obvious way. By considering the case where Λ^+ is the set of all multipartitions and using Proposition 2.17 we obtain the following.

Theorem 4.6. *Let λ be a multipartition of n and suppose that $P_R(\hat{q};\widehat{Q}_1,\dots,\widehat{Q}_r)$ is a non–zero element of R. Then*

$$\sum_{i>0} S_{\mathbb{F}}^\lambda(i) \longleftrightarrow \sum_{\substack{\nu \\ \lambda \rhd \nu}} \nu_{\mathfrak{p}}(g_{\lambda\nu}) S_{\mathbb{F}}^\nu.$$

As in Corollary 4.5, Theorem 4.6 automatically gives sufficient conditions for $S_{\mathbb{F}}^\lambda = D_{\mathbb{F}}^\lambda$. In order to obtain necessary conditions we need to work a little harder.

Theorem 4.7. *Suppose that $\lambda \in \Lambda^+$.*

(i) *If $S_{\mathbb{F}}^\lambda = D_{\mathbb{F}}^\lambda$, then $W_{\mathbb{F}}^\lambda$ is irreducible.*

(ii) *Suppose $W_{\mathbb{F}}^\lambda$ is irreducible, $P_R(\hat{q};\widehat{Q}_1,\dots,\widehat{Q}_r) \neq 0$ and that every multipartition of n is contained in Λ^+. Then $S_{\mathbb{F}}^\lambda = D_{\mathbb{F}}^\lambda$.*

(iii) *Suppose that $P_R(\hat{q};\widehat{Q}_1,\dots,\widehat{Q}_r) \neq 0$. Then $S_{\mathbb{F}}^\lambda = D_{\mathbb{F}}^\lambda$ if and only if $\nu_{\mathfrak{p}}(g_{\lambda\nu}) = 0$ for all multipartitions ν of n such that $\lambda \rhd \nu$.*

Remark 4.8. Notice that when $D_{\mathbb{F}}^\lambda \neq (0)$ part (iii) gives necessary and sufficient conditions for $S_{\mathbb{F}}^\lambda$ to be irreducible; however, it can happen that $S_{\mathbb{F}}^\lambda$ is irreducible when $D_{\mathbb{F}}^\lambda = (0)$. Even for the symmetric group (that is, the case where $r = 1$ and $q = 1$), the complete classification of the irreducible Specht modules $S_{\mathbb{F}}^\lambda$ is an open problem (except in characteristic 2; see [12]). If $\omega \notin \Lambda^+$, then there also exist examples where $W_{\mathbb{F}}^\lambda$ is irreducible and $S_{\mathbb{F}}^\lambda \neq D_{\mathbb{F}}^\lambda$.

Proof. (i) Suppose that $S_{\mathbb{F}}^\lambda = D_{\mathbb{F}}^\lambda$. Then $\operatorname{rad} S_{\mathbb{F}}^\lambda = (0)$, so the Gram determinant of $S_{\mathbb{F}}^\lambda$, with respect to its standard basis $\{\, m_{\mathfrak{t}} \mid \mathfrak{t} \in \operatorname{Std}(\lambda) \,\}$ must be a non–zero element of $\mathbb{F}$. However, by Remark 3.20 and Corollary 3.38 this determinant is equal to

$$G_\omega(\lambda) + \mathfrak{p}R \simeq \prod_{\nu\in\Lambda^+} (g_{\lambda\nu})^{d_\omega(\nu)} + \mathfrak{p}R.$$

Hence, $S_{\mathbb{F}}^\lambda = D_{\mathbb{F}}^\lambda$ if and only if $(g_{\lambda\nu})^{d_\omega(\nu)} \notin \mathfrak{p}R$ for all multipartitions ν of n. On the other hand, the Gram determinant of $W_{\mathbb{F}}^\lambda$, with respect to its semistandard basis, is equal to

$$\prod_{\mu\in\Lambda} G_\mu(\lambda) + \mathfrak{p}R \simeq \prod_{\nu\in\Lambda^+}\prod_{\mu\in\Lambda} (g_{\lambda\nu})^{d_\mu(\nu)} + \mathfrak{p}R.$$

Therefore, $W_{\mathbb{F}}^{\lambda} = F_{\mathbb{F}}^{\lambda}$ if and only if $(g_{\lambda\nu})^{d_\mu(\nu)} \notin \mathfrak{p}R$ for all $\nu \in \Lambda^+$ and all $\mu \in \Lambda$. However, it is easy to see that if $\nu \in \Lambda^+$ and $\mu \in \Lambda$, then $d_\mu(\nu) \neq 0$ only if $d_\omega(\nu) \neq 0$ so the result follows.

(ii) Suppose that $W_{\mathbb{F}}^{\lambda}$ is irreducible and let ν be a multipartition of n such that $D_{\mathbb{F}}^{\nu} \neq (0)$ and $(S_{\mathbb{F}}^{\lambda} : D_{\mathbb{F}}^{\nu}) > 0$. By Proposition 2.17, $(W_{\mathbb{F}}^{\lambda} : F_{\mathbb{F}}^{\nu}) = (S_{\mathbb{F}}^{\lambda} : D_{\mathbb{F}}^{\nu}) > 0$. However, $W_{\mathbb{F}}^{\lambda}$ is irreducible, so $\nu = \lambda$ and $S_{\mathbb{F}}^{\lambda} = D_{\mathbb{F}}^{\lambda}$ as claimed.

Part (iii) now follows from parts (i), (ii) and Corollary 4.5. □

The last four results apply to fields $\mathbb{F}$ of the form $R/\mathfrak{p}$ where R is a principal ideal domain, $\mathfrak{p}$ is a prime in R and $P_R(\hat{q}; \widehat{Q}_1, \dots, \widehat{Q}_r) \neq 0$. At first sight the requirement that $P_R(\hat{q}; \widehat{Q}_1, \dots, \widehat{Q}_r) \neq 0$ appears to be very restrictive; however, given any field $\mathbb{F}$ containing $0 \neq q, Q_1, \dots, Q_r$ we can always find suitable R and $\mathfrak{p}$.

Notation 4.9. Fix a field $\mathbb{F}$ containing elements $q, Q_1, \dots, Q_r$, with $q \neq 0$, and let $\mathcal{S}_{\mathbb{F}}$ be the cyclotomic q–Schur algebra over $\mathbb{F}$ with these parameters. Let $R = \mathbb{F}[\,\hat{q}\,]$, where $\hat{q}$ is an indeterminate over $\mathbb{F}$, and $\mathfrak{p} = \hat{q} - q$. Define $\mathcal{S}_R$ to be the cyclotomic $\hat{q}$–Schur algebra over R with parameters $\hat{q}, \widehat{Q}_1, \dots, \widehat{Q}_r$ where

$$\widehat{Q}_k = \begin{cases} Q_k(\hat{q} - q + 1)^{kn}, & \text{if } Q_k \neq 0, \\ (\hat{q} - q)^{kn}, & \text{if } Q_k = 0. \end{cases}$$

Let $\mathcal{H}_R$ and $\mathcal{H}_{\mathbb{F}}$ be the associated Ariki–Koike algebras.

Notice that R is a principal ideal domain, $\mathfrak{p}$ is prime in R and $\mathbb{F} \cong R/\mathfrak{p}$. Moreover, if $\pi \colon R \to \mathbb{F}$ is the canonical projection, then $\pi(\hat{q}\,) = q$ and $\pi(\widehat{Q}_k) = Q_k$ for $k = 1, 2 \dots, r$. Therefore, $\mathcal{S}_{\mathbb{F}} \cong \mathcal{S}_R \otimes_R \mathbb{F}$ and $\mathcal{H}_{\mathbb{F}} \cong \mathcal{H}_R \otimes_R \mathbb{F}$. Finally, $P_R(\hat{q}; \widehat{Q}_1, \dots, \widehat{Q}_r) \neq 0$ because by construction every factor of $P_R(\hat{q}; \widehat{Q}_1, \dots, \widehat{Q}_r)$ is non–zero. Thus we are in a situation where we can apply Theorem 4.3 and its corollaries. Note that the Jantzen filtrations of the Weyl modules and Specht modules depend upon R and $\mathfrak{p}$ rather than on $\mathbb{F}$.

In [13, Theorem 4.19] we gave a purely combinatorial classification of those partitions λ such that $W_{\mathbb{F}}^{\lambda}$ is irreducible (this is the case $r = 1$). We build upon this to give such a criterion for the general case.

In addition to using the notation of (4.9), we write res_R and $\operatorname{res}_{\mathbb{F}}$ for residues in the rings R and $\mathbb{F}$ respectively. We also write $\mathcal{S}_{\mathbb{F}}^{(1)}(n)$ for the q–Schur algebra $\mathcal{S}_{\mathbb{F}}(\Lambda)$ where Λ is the set of all partitions of n and $Q_1 = 1$ (that is, the case $r = 1$).

Theorem 4.10. *Let $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)}) \in \Lambda^+$ be a multipartition of n. Then $W_{\mathbb{F}}^{\lambda}$ is reducible if and only if*

(i) *for some $k \in \mathbf{r}$ the $\mathcal{S}_{\mathbb{F}}^{(1)}(n_k)$–module $W_{\mathbb{F}}^{\lambda^{(k)}}$ is reducible where $n_k = |\lambda^{(k)}|$; or,*
(ii) *for some ν in Λ^+ there exist $x = (i, j, k) \in [\lambda]$ and $y = (a, b, c) \in [\nu]$ such that $c > k$, $[\lambda] \setminus r_x = [\nu] \setminus r_y$ and $\operatorname{res}_{\mathbb{F}}(r_x) = \operatorname{res}_{\mathbb{F}}(r_y)$.*

Remark 4.11. Less formally, condition (ii) says that it is possible to unwrap a rim hook from $[\lambda]$ and wrap it back on to a later component without changing the residue of the foot node.

Proof. By Corollary 4.5, $W_{\mathbb{F}}^{\lambda}$ is reducible if and only if there exists a multipartition $\nu \in \Lambda^+$ such that $\nu_{\mathfrak{p}}(g_{\lambda\nu}) \neq 0$. By Remark 3.37, $\nu_{\mathfrak{p}}(g_{\lambda\nu}) \neq 0$ if and only if $\lambda \rhd \nu$ and there exist $x = (i, j, k) \in [\lambda]$ and $y = (a, b, c) \in [\nu]$ such that $c \geq k$, $[\lambda] \setminus r_x = [\nu] \setminus r_y$ and $\mathfrak{p}$ divides $\operatorname{res}_R(r_x) - \operatorname{res}_R(r_y)$. Now, $\mathfrak{p} = \hat{q} - q$, so $\mathfrak{p}$ divides $\operatorname{res}_R(r_x) - \operatorname{res}_R(r_y)$

if and only if $\mathrm{res}_{\mathbb{F}}(r_x) = \mathrm{res}_{\mathbb{F}}(r_y)$. By [13, Theorem 4.15], $W_{\mathbb{F}}^{\lambda^{(k)}}$ is reducible if and only if there exists a partition μ of n_k (where $n_k = |\lambda^{(k)}|$) such that $\lambda^{(k)}$ dominates μ and there exists $x \in [\lambda^{(k)}]$ and $y \in [\mu]$ with $[\lambda^{(k)}] \setminus r_k = [\mu] \setminus r_y$ and $\mathrm{res}_{\mathbb{F}}(r_x) = \mathrm{res}_{\mathbb{F}}(r_y)$. The theorem now follows. □

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