The Jordan Structure of Two Dimensional Loop Models

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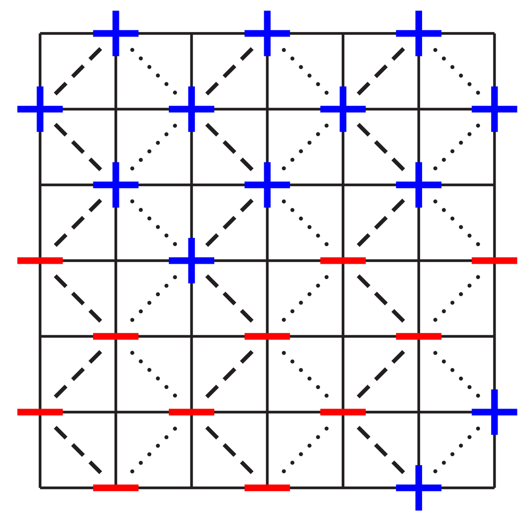
Introduction

We show how to use the link representation of the transfer matrix D_N of loop models on the lattice to calculate partition functions, at criticality, of the Q-Potts spin models. To probe the Jordan structure of the Hamiltonian, we study C_{2N}, the top Fourier coefficient of D_N. The eigenvalues and eigenvectors of C_{2N} are determined. Studying singularities of the eigenvectors, we show that C_{2N} and D_N have non trivial Jordan blocks for particular values of the spectral parameter, λ.

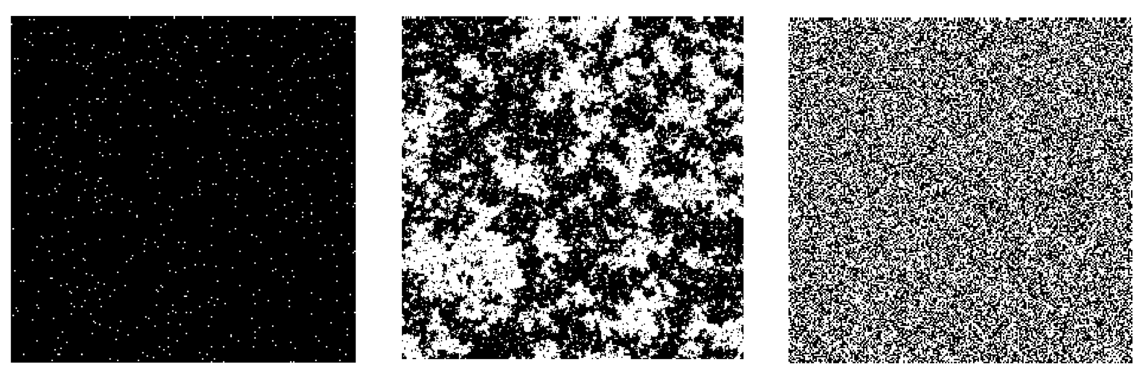
Q-Potts spin model

In the Potts spin model, spins on a lattice take Q different values and interact solely with nearest neighbors. The energy of a spin configuration is $E_\sigma = -J\sum_{\langle i,j\rangle}\delta_{\sigma_i,\sigma_j}$ where J is the interaction constant and $\langle i,j\rangle$ denotes all pairs of neighboring spins i and j. Spins on left and right boundaries are free.



Ising configuration, $N=3$, $M=3$

The model exhibits second order *phase transition* at a finite temperature, $kT_c = J(\log(1+\sqrt{Q}))^{-1}$. For $Q=2$ (Ising model), the partition function and two point function have been calculated exactly, for various choices of boundary conditions. Monte-Carlo simulations give the following:

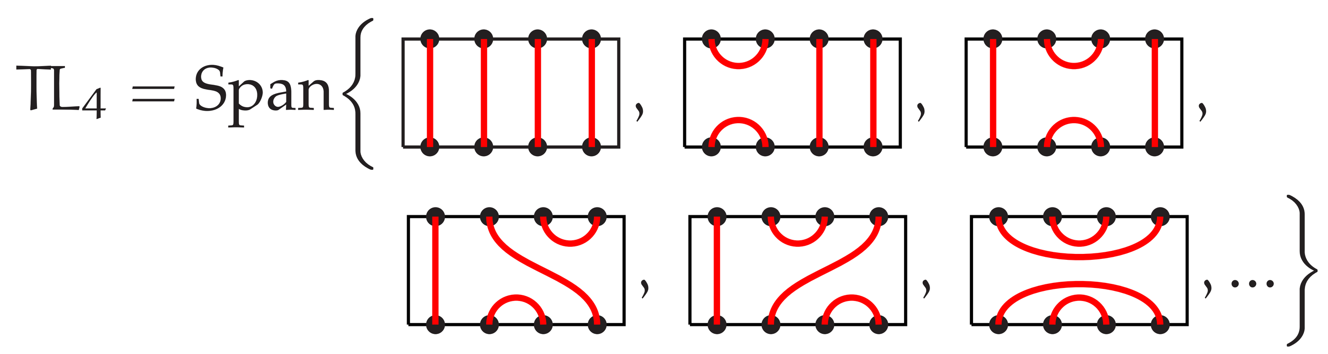


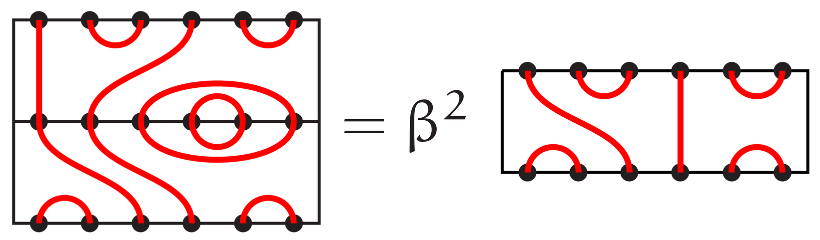
$T<Tc$ $\quad$ $T=Tc$ $\quad$ $T>Tc$

These models are conformally invariant and, in the continuum limit, described by rational *conformal field theories* (CFT)!

Temperley-Lieb algebra and link representation

Let N be a positive integer and draw a rectangle with $2N$ marked points on it, N on its upper side, N on the bottom. A *connectivity* is a pairwise pairing of all points by non-crossing curves drawn within the rectangular box. The Temperley-Lieb algebra $TL_N(\beta)$ is the set of linear combination of connectivities endowed with the following β-product:

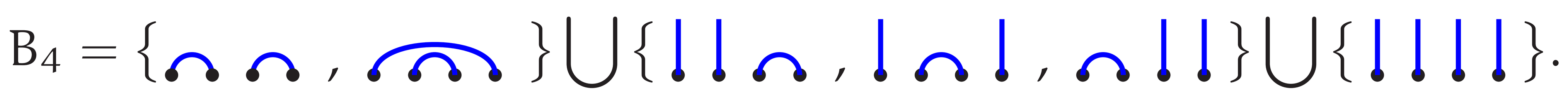




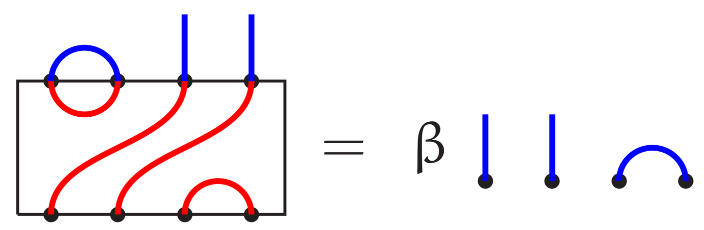
The β-product

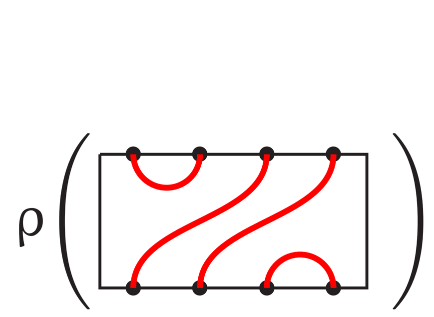
A multiplicative factor of β is added for every closed loop.

A *link state* is a set of non-crossing curves drawn above a horizontal segment pairing N points among themselves or to infinity (point connected to infinity are called *defects*). B_N is the set of all link states :



The definition of the action of connectivities on link states is analogous to the β-product (every closed loop gives a power of β). This gives the ρ representation of TL_N:

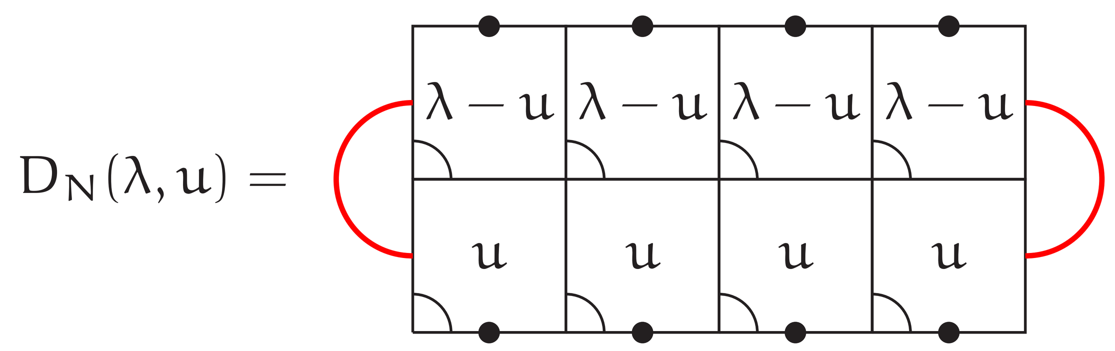




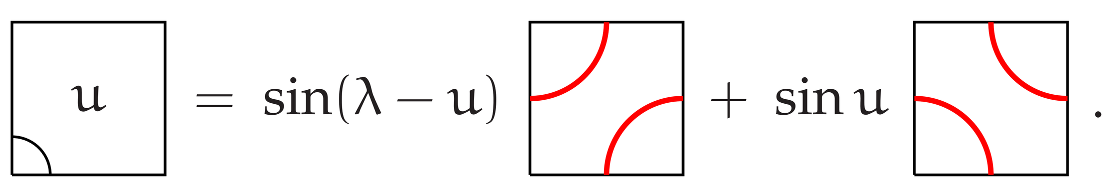
$$\rho\left(\cdot\right) = \begin{pmatrix} \beta & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \beta & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Double-row transfer matrix...

We're interested in only one element of TL_N, the *double-row transfer matrix*:



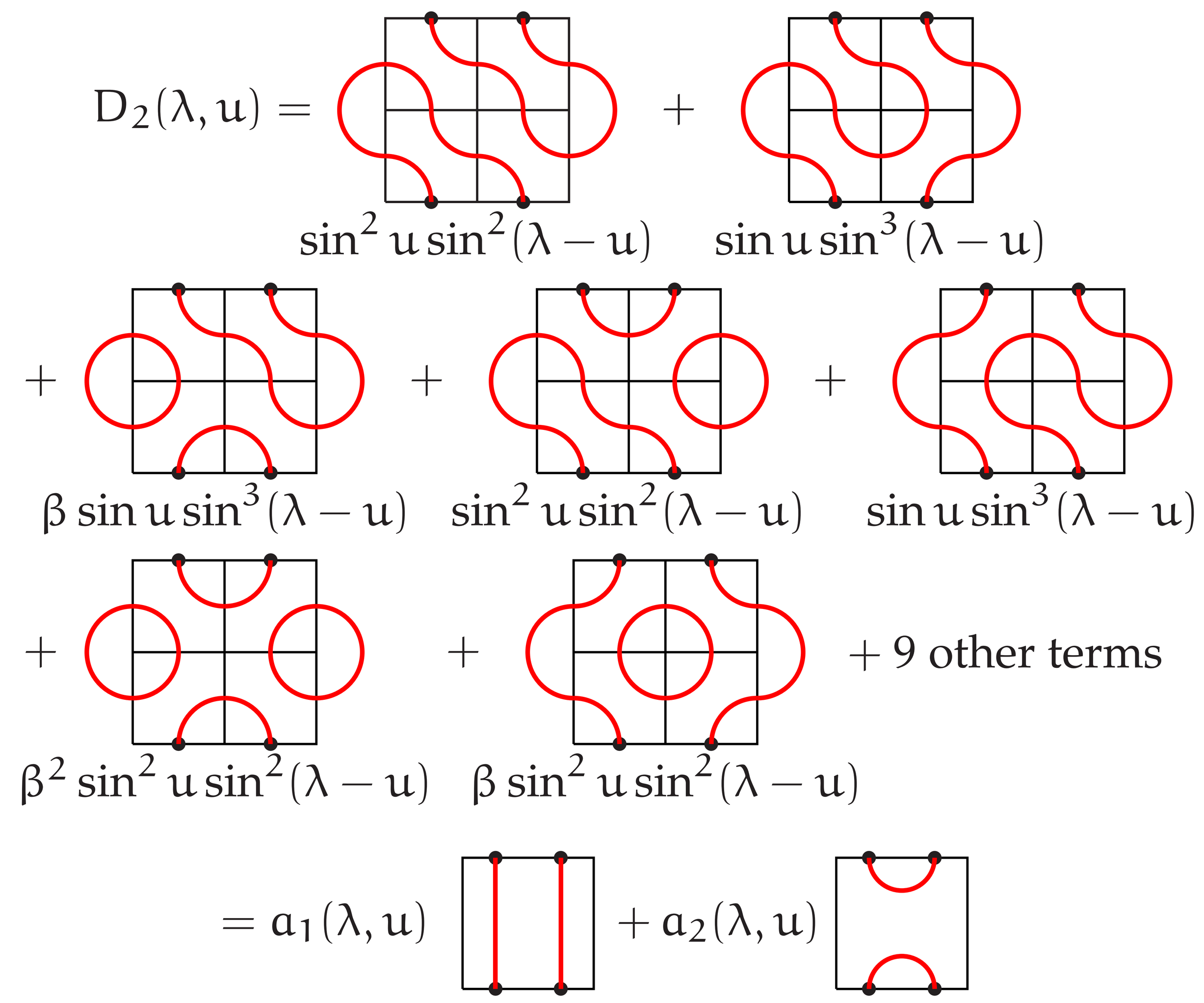
where each box stands for the sum



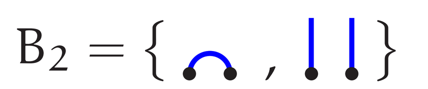
$u \in [0,\lambda]$ is the anisotropy and $\lambda \in [0,\pi/2]$, the spectral parameter, related to β by $\beta = 2\cos\lambda$. Again, a β is added for every closed loop.

D_N is the Hamiltonian of our loop model! It satisfies the Yang-Baxter equation, $[D_N(\lambda,u), D_N(\lambda,v)] = 0\ \forall u,v$, a key element for integrability.

... and an example for $N=2$



and the ρ -representation of D_2 is:



$$\rightarrow \quad \rho(D_2) = \begin{pmatrix} \beta a_2 + a_1 & a_2 \\ 0 & a_1 \end{pmatrix}.$$

Spins and loops

Partition functions of the Q-Potts model at T_c can be calculated using $\rho(D_N)$ with $\beta=\sqrt{Q}$. For instance, with cylindrical boundary conditions:

Let $W : B_N \to B_N$ be the linear transformation that acts as a multiple of the identity on elements of B_N with d defects, with $W|_d = \frac{\sin\lambda(d+1)}{\sin\lambda}\mathrm{id}$. Then

$$Z_{N,M} = \mathrm{tr}(\rho(D_N)^M W), \qquad \text{for all } M.$$

	d	0	2	4	6	8	...
Ising	$W\|_d$	1	1	−1	−1	1	...
3-Potts	$W\|_d$	1	2	1	−1	−2	...

Jordan blocks

Since $\rho(D_N)$ is not hermitian, it may not be diagonalizable. This happens, for instance, when $N=2$ and $\beta=0$ ($\lambda=\pi/2$). For example, a simple matrix:

$$m(x) = \begin{pmatrix} x & 1 \\ 0 & 0 \end{pmatrix}.$$

Its eigenvectors, when $x\neq 0$, are

$$v_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad v_x = \begin{pmatrix} -1/x \\ 1 \end{pmatrix}.$$

When $x=0$, the matrix can't be diagonalized: it a has *Jordan block*. Jordan blocks can be studied by looking at singularities in the eigenvectors! Since eigenvectors of $\rho(D_N)$ are generally unknown, to probe its Jordan structure, we expand

$$D_N(\lambda,u) = \sum_{i=0}^{N} C_{2i}(\lambda)\cos(2iv),$$

and find the eigenvectors of C_{2N}, the top Fourier coefficient. Their singularities give the values of λ where Jordan blocks appear, and an understanding of the pattern of Jordan blocks $\rho(D_N)$!

Conclusion

The Jordan blocks of $\rho(D_N)$ result in two points functions behaving logarithmically, a strong indicator that the theory is described, in the continuum, by *logarithmic conformal field theories* (LCFT's)

Funding and references

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[1] A. Morin-Duchesne, Y. Saint-Aubin. The Jordan Structure of Two Dimensional Loop Models. arXiv:math-ph/1101.2885v3