

# The Julia sets of Chebyshev's method with small degrees

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## Abstract

Given a polynomial  $p$ , the degree of its Chebyshev's method  $C_p$  is determined. If  $p$  is cubic then the degree of  $C_p$  is found to be 4, 6 or 7 and we investigate the dynamics of  $C_p$  in these cases. If a cubic polynomial  $p$  is unicritical or non-generic then, it is proved that the Julia set of  $C_p$  is connected. The family of all rational maps arising as the Chebyshev's method applied to a cubic polynomial which is non-unicritical and generic is parametrized by the multiplier of one of its extraneous fixed points. Denoting a member of this family with an extraneous fixed point with multiplier  $\lambda$  by  $C_\lambda$ , we have shown that the Julia set of  $C_\lambda$  is connected whenever  $\lambda \in [-1, 1]$ .

*Keyword:* Chebyshev's method; Extraneous fixed points; Connected Julia sets.

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## 1 Introduction

Finding the roots of a given polynomial is a classical and widely studied topic. A root-finding method is a function that associates a polynomial  $p$  with a rational map  $F_p$  such that each root  $z$  of  $p$  is an attracting fixed point of  $F_p$ , i.e.,  $F_p(z) = z$  and  $|F_p'(z)| < 1$ . It is well-known that there is an open connected subset of the extended complex plane  $\widehat{\mathbb{C}}$  containing the attracting fixed point such that every point in this

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set converges to the attracting fixed point under the iteration of  $F_p$ . This is how a root of a polynomial can be approximated starting with a suitably chosen point.

The Fatou set of a rational map  $R$ , denoted by  $\mathcal{F}(R)$  is the set of all points where  $\{R^n\}_{n>0}$  is equicontinuous, and its complement in  $\widehat{\mathbb{C}}$  is called the Julia set of  $R$ . The Julia set of  $R$  is denoted by  $\mathcal{J}(R)$ . Complex dynamics is the study of the Fatou set and the Julia set of a given rational map. For an introduction to the subject, one may refer to a book by Beardon[1]. The root-finding methods present themselves as an interesting class of rational maps from a dynamical point of view.

A fixed point  $z_0 \in \mathbb{C}$  of a rational map  $R$  is called attracting, repelling or indifferent if the modulus of its multiplier  $|R'(z_0)|$  is less than, greater than or is equal to 1 respectively. The fixed point  $z_0$  is called superattracting if  $R'(z_0) = 0$ . If  $\infty$  is a fixed point of  $R$  then its multiplier is defined as  $|h'(0)|$  where  $h(z) = \frac{1}{R(\frac{1}{z})}$  and is called attracting, repelling or indifferent accordingly. The basin of attraction of an attracting fixed point  $z_0$ , denoted by  $\mathcal{A}_{z_0}$  is the set  $\{z \in \widehat{\mathbb{C}} : \lim_{n \rightarrow \infty} R^n(z) = z_0\}$ . This is an open set and is not necessarily connected. The component of  $\mathcal{A}_{z_0}$  containing  $z_0$  is called the immediate attracting basin of  $z_0$ . An indifferent fixed point is called rationally indifferent or parabolic if its multiplier is a root of unity. The basin of a parabolic fixed point  $z_0$  of a rational map  $R$  is the set  $\{z \in \widehat{\mathbb{C}} \setminus \mathcal{J}(R) : \lim_{n \rightarrow \infty} R^n(z) = z_0\}$ . Every component of this basin contains  $z_0$  on its boundary and is called an immediate parabolic basin. It is important to note that any point whose iterated image is  $z_0$  is not in the basin of the parabolic fixed point  $z_0$ . The basin of an attracting or a parabolic fixed point is in the Fatou set. The immediate attracting basin or the parabolic basin corresponding to a periodic point of period  $p$  can be defined accordingly and is a  $p$ -periodic Fatou component (maximally connected subset of the Fatou set which is invariant under  $R^p$ ). The only other possible periodic Fatou component is a Siegel disk or a Herman ring. The details can be found in [1].

One of the widely discussed root-finding methods is the Newton method and that is the first member of a family known as the Konig's method. A systematic study of Konig's method is done by Buff and Henriksen in [2]. A fixed point of a root-finding method  $F_p$  is called extraneous if it is not a root of  $p$ . An important aspect of Konig's method is that all its extraneous fixed points are repelling. This article is concerned with a root-finding method for which an extraneous fixed point can be non-repelling.

For a non-constant, non-linear polynomial  $p$ , its Chebyshev's method is defined as the rational map

$$C_p(z) = z - \left(1 + \frac{1}{2}L_p(z)\right)\frac{p(z)}{p'(z)},$$

where

$$L_p(z) = \frac{p(z)p''(z)}{p'(z)^2}.$$

This is a third order convergent method, i.e., its local degree (made precise in the following paragraph) is three at each simple root of the polynomial  $p$ . Note that for a monomial or for a linear polynomial  $p$ ,  $C_p$  is a linear polynomial. We do not consider this trivial situation and are concerned with polynomials with degree at least two and which are not monomials.

The degree of a rational map is the first thing one needs to know for investigating its dynamics. While discussing a number of basic properties of  $C_p$ , the authors in [5] mention that the degree of  $C_p$  is at most  $3d-2$  where  $d$  is the degree of the polynomial  $p$ . We found that the exact degree of  $C_p$  depends not only on the number of distinct roots of  $p$  but on certain types of its critical points also. We determine the exact degree of  $C_p$  for every  $p$ . Some discussion and definitions are required to state this result. If a rational map  $R$  is analytic at a point  $z_0$  and its Taylor series about  $z_0$  is  $a_k(z-z_0)^k + a_{k+1}(z-z_0)^{k+1} + \dots$  for some  $k > 0$  where  $a_k \neq 0$  then we say the local degree of  $R$  at  $z_0$ , denoted by  $\deg(R, z_0)$  is  $k$ . The map  $R$  is *like*  $z \mapsto z^k$  near  $z_0$ . The local degree of  $R$  at  $\infty$  or at a pole is defined by a change of coordinate using  $z \mapsto \frac{1}{z}$ . More precisely, if  $R(\infty)$  is finite then  $\deg(R, \infty)$  is defined as the degree of  $R(\frac{1}{z})$  at 0. If  $R(\infty) = \infty$  then  $\deg(R, \infty)$  is defined as  $\deg(\frac{1}{R(\frac{1}{z})}, 0)$ . Similarly, if  $z_0$  is a pole of  $R$  then its local degree at  $z_0$  is defined to be  $\deg(\frac{1}{R(z)}, z_0)$ . A root  $\tilde{z}$  of a rational map  $R$  is said to have multiplicity  $k$  if  $R(\tilde{z}) = R'(\tilde{z}) = \dots = R^{(k-1)}(\tilde{z}) = 0$ , but  $R^{(k)}(\tilde{z}) \neq 0$  i.e., the local degree of  $R$  at  $\tilde{z}$  is  $k$ . A root is called simple if its multiplicity is 1. It is called multiple if its multiplicity is at least two. A root with multiplicity exactly equal to two is called a double root. A point  $z \in \widehat{\mathbb{C}}$  is called a critical point of a rational map  $R$  if  $\deg(R, z) \geq 2$ . In particular, multiple roots and multiple poles are critical points. By definition, the multiplicity of a critical point  $z$  of  $R$  is  $\deg(R, z) - 1$ . A critical point is called simple if its multiplicity is one.

**Definition 1.1.** (*Special critical point*) For a polynomial  $p$ , a critical point  $c \in \mathbb{C}$  is called special if  $p(c) \neq 0$  but  $p''(c) = 0$ .

A finite critical point with multiplicity at least 2 and which is not a root is a special critical point. For example, 0 is a special critical point of  $p(z) = z^d + b$  whenever  $d \geq 3$  and  $b \neq 0$ . But it is not so for  $b = 0$ . We now present the first result of this article.

**Theorem 1.1.** (*Degree of  $C_p$* ) Let  $p$  be a polynomial of degree  $d$ . Let  $m, n$  and  $r$  denote the number of its distinct simple roots, double roots and roots of multiplicity bigger than 2 respectively. If  $p$  has  $s$  number of distinct special critical points then

$$\deg(C_p) = 3(m + n + r) - 2 - B + s$$

where  $B$  is the sum of multiplicities of all the special critical points. If  $p$  has no special critical point then  $\deg(C_p) = 3(m + n + r) - 2$ .

If  $p$  is generic then  $m = d$  and  $n = r = 0$ , and we have an immediate consequence.

**Corollary 1.1.** *If  $p$  is generic then*

$$\deg(C_p) = \begin{cases} 3d - 2 - B + s & \text{if } p \text{ has } s \text{ many special critical points with total multiplicity } B \\ 3d - 2 & \text{if } p \text{ has no special critical point.} \end{cases}$$

The following corollary deals with some other special situations.

**Corollary 1.2.** *1. If  $p$  has two distinct roots then  $\deg(C_p) = 4$ . In all other cases,  $\deg(C_p) \geq 6$ . In particular, there is no polynomial  $p$  such that  $\deg(C_p) = 5$ .*

*2. If  $\deg(p) = d$  and all its critical points are special then  $\deg(C_p) = 2d + s - 1$  where  $s$  is the number of distinct special critical points of  $p$ . Further, if  $p$  is unicritical then  $\deg(C_p) = 2d$ .*

The degree of the Chebyshev's method applied to a cubic polynomial is found to be 4, 6 or 7. The remaining part of this article focusses on the dynamics of the Chebyshev's method in these cases.

The dynamics of the Chebyshev's method for quadratic polynomials has been investigated by Kneisl in [7] who calls this as Super-Newton method. The author gives examples of cubic polynomial whose Chebyshev's method has a superattracting extraneous fixed point. Olivo et al. [4] consider a one parameter family of cubic polynomials and study their Chebyshev's method. The case of cubic polynomials are also studied in [5] where the authors have discussed the cubic polynomials whose Chebyshev's method has attracting extraneous fixed points and attracting periodic points. In all these results, the connectedness of the Julia set remains unexplored.

For a polynomial  $p$ , the Chebyshev-Halley method of order  $\sigma$  is given by  $H_p^\sigma(z) = z - \left[ 1 + \frac{1}{2} \frac{p(z)p''(z)}{(p'(z))^2 - \sigma p(z)p''(z)} \right] \frac{p(z)}{p'(z)}$  where  $\sigma \in \mathbb{C}$ . The Chebyshev's method is a special member of the Chebyshev-Halley family. In fact,  $C_p = H_p^0$ . The dynamics of the Chebyshev-Halley family applied to unicritical polynomials  $z \mapsto z^n - 1$ ,  $n \in \mathbb{N}$  is investigated in [3]. A necessary and sufficient condition for disconnected Julia sets is found. More precisely, it is proved that the Julia set of these root-finding methods is disconnected if and only if the immediate basin of 1 (which is a superattracting fixed point corresponding to a root of the polynomial) contains a critical point but no pre-image of 1 other than itself. The numerical study done in this paper suggests the existence of disconnected Julia set for the Chebyshev-Halley method  $H_p^\sigma$  for several

values of  $\sigma$  when  $p$  is a cubic or higher degree unicritical polynomial. However, the paper does not contain any theoretical proof for this statement.

Though the Julia set of Newton method (applied to a polynomial) is always connected [9], there are other members of König's methods with a disconnected Julia set [6]. The Chebyshev's method applied to non-generic cubic polynomials is dealt in [4], where the connectivity question of their Julia sets remain to be answered.

We prove that the Julia set of Chebyshev's method of each cubic polynomial that are either unicritical or non-generic is connected. This is Proposition 4.4 of this article. In fact, this proposition shows that the Fatou set of  $C_p$  is the union of the immediate superattracting basins (and their pre-images) corresponding to the three roots of  $p$  when  $p$  is unicritical whereas for non-generic  $p$ , the Fatou set of  $C_p$  is the union of the two immediate attracting basins (and their pre-images) corresponding to the two roots of  $p$ .

As noted earlier, the existence of an attracting or a rationally indifferent extraneous fixed point is a new feature of the Chebyshev's method. Its dynamical relevance is revealed by parametrizing all the cubic polynomials in terms of the multiplier of an extraneous fixed point of its Chebyshev's method. This is done in Lemma 4.5. More precisely, it is shown that for every  $\lambda \in \mathbb{C} \setminus \{5, 6\}$ , if the Chebyshev's method of a cubic, generic and non-unicritical polynomial has an extraneous fixed point with multiplier  $\lambda$  then it is conjugate to the Chebyshev's method  $C_\lambda$  of  $p_\lambda(z) = z^3 + 3z + \frac{3\lambda^2 - 39\lambda + 124}{(5-\lambda)\sqrt{5-\lambda}}$  where the principal branch of the square root  $\sqrt{5-\lambda}$  is considered. The case  $\lambda = 5$  is possible when  $p$  is unicritical whereas there is no cubic polynomial which has an extraneous fixed point with multiplier equal to 6 (Remark 4.3(2)). Then we study the dynamics of  $C_\lambda$  for  $\lambda \in [-1, 1]$  and prove the following.

**Theorem 1.2.** *For  $-1 \leq \lambda \leq 1$ , the Julia set of  $C_\lambda$  is connected.*

The proof of the theorem in fact describes the dynamics of  $C_\lambda$  completely. The Fatou set of  $C_\lambda$  is the union of the superattracting immediate basins corresponding to the three roots of  $p_\lambda$  and the immediate basin of the extraneous fixed point with multiplier  $\lambda$  - this is attracting if  $\lambda \in (-1, 1)$  and rationally indifferent if  $\lambda = \pm 1$ .

Though both  $C_{-1}$  and  $C_1$  have a rationally indifferent extraneous fixed point they differ in terms of the number of extraneous fixed points. The number of extraneous fixed point of  $C_{-1}$  is four whereas it is three for  $C_1$ . A fixed point of a rational map  $R$  is called multiple, with multiplicity  $k \geq 2$  if it is a multiple root of  $R(z) - z = 0$  with multiplicity  $k$ . Otherwise, it is called simple. Here  $C_1$  has a multiple fixed point whereas all the fixed points of  $C_{-1}$  are simple.

The images of the Julia set of  $C_\lambda$  for  $\lambda = -1, 0$  and  $1$  are given in Figure 1, Figure 2 and Figure 3 respectively. In each figure, the three immediate basins

corresponding to the roots of the polynomial are given in blue, green and pink. The immediate basin of the extraneous attracting (for  $C_0$ ) and parabolic (for  $C_{-1}$  and  $C_1$ ) fixed point is indicated in red whose zoomed version is also given in each figure.

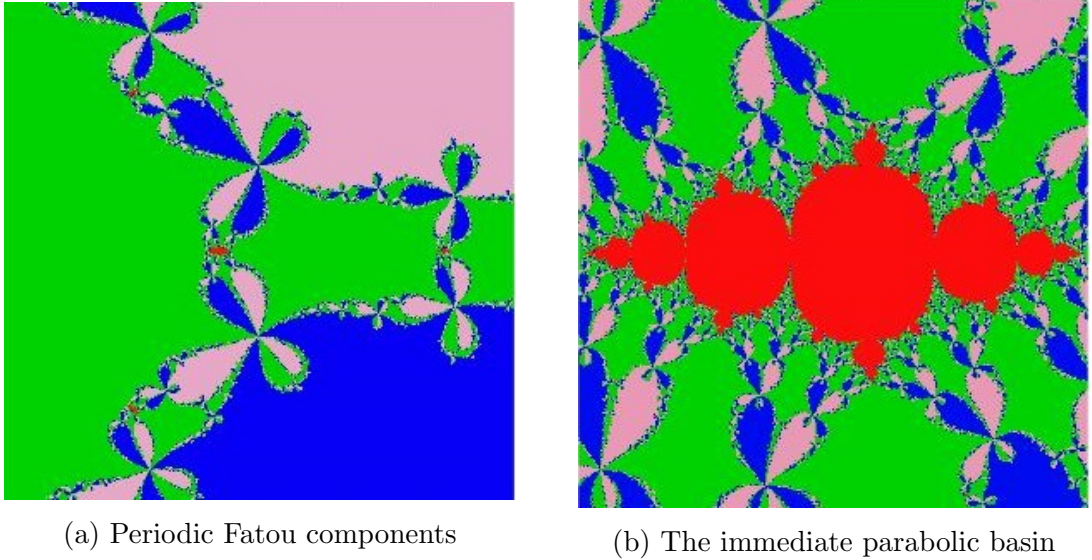


Figure 1: The Julia set of  $C_{-1}$

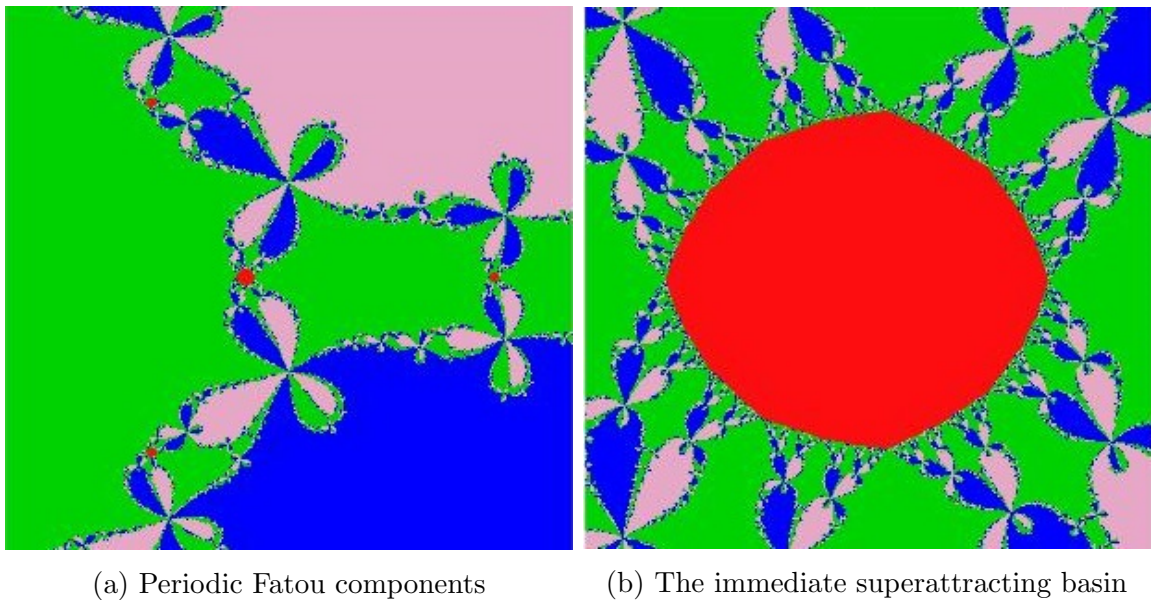
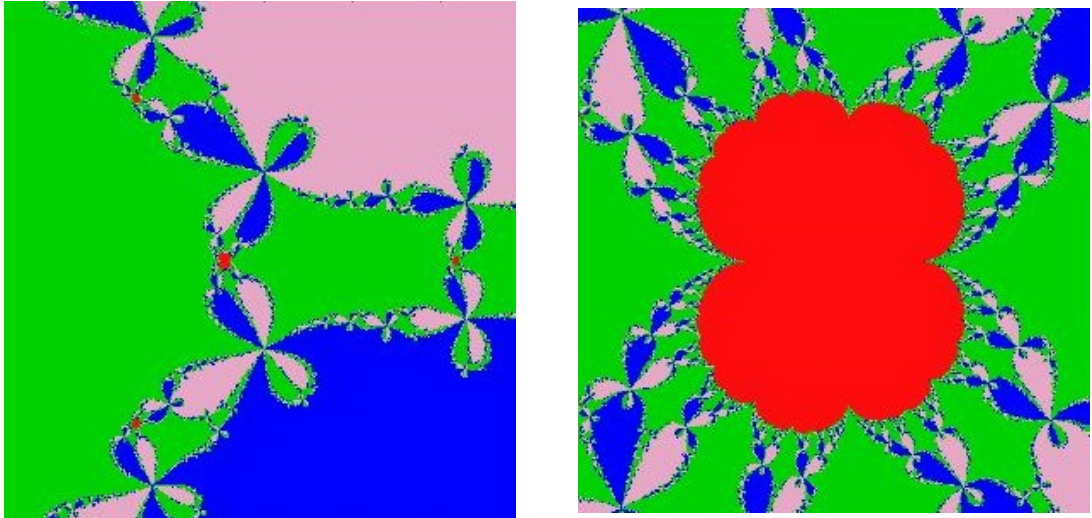


Figure 2: The Julia set of  $C_0$

Section 2 describes some useful properties of the Chebyshev's method. The degree of this method is determined in Section 3. The dynamics of  $C_\lambda$ ,  $-1 \leq \lambda \leq 1$  is investigated in Section 4.



(a) Periodic Fatou components

(b) The immediate parabolic basin

Figure 3: The Julia set of  $C_1$

## 2 Preliminaries

### 2.1 Some properties of the Chebyshev's method

Recall that, for a polynomial  $p$ ,

$$C_p(z) = z - \left(1 + \frac{1}{2}L_p(z)\right)\frac{p(z)}{p'(z)},$$

and the derivative of the Chebyshev's method is given by

$$C'_p(z) = \frac{L_p(z)^2}{2}(3 - L_{p'}(z)) \quad (1)$$

where  $L_p(z) = \frac{p(z)p''(z)}{p'(z)^2}$  and  $L_{p'}(z) = \frac{p'(z)p'''(z)}{p''(z)^2}$ .

Two rational maps  $R, S$  are conformally conjugate, in short conjugate if there is a mobius map  $\phi$  such that  $S = \phi \circ R \circ \phi^{-1}$ . Here  $\circ$  denotes the composition of functions. Since  $S^n = \phi \circ R^n \circ \phi^{-1}$  for all  $n$ , the iterative behaviour of  $R$  and  $S$  are essentially the same. More precisely, we have the following.

**Lemma 2.1** (Theorem 3.1.4 [1]). *If  $S$  and  $R$  are two rational maps such that  $S = \phi \circ R \circ \phi^{-1}$  for a mobious map  $\phi$  then  $\mathcal{J}(S) = \phi(\mathcal{J}(R))$*

There are different polynomials giving rise to the same Chebyshev's method up to conjugacy. The so called scaling theorem, which is also true for the Chebyshev-Halley method makes it precise. For brevity we use  $H_p$  instead of  $H_p^\sigma$  for denoting the Chebyshev-Halley method of order  $\sigma$  applied to  $p$ .

**Theorem 2.2** (Scaling Theorem). *Let  $p$  be a polynomial of degree at least two. Then  $H_p = H_{\lambda p}$  for all  $\lambda \neq 0$ . If  $T(z) = \alpha z + \beta$ , with  $\alpha, \beta \in \mathbb{C}, \alpha \neq 0$  and  $g = p \circ T$  then  $T \circ H_g \circ T^{-1} = H_p$ .*

*Proof.* For  $g(z) = \lambda p(T(z))$ ,  $g'(z) = \lambda p'(T(z))T'(z) = \lambda \alpha p'(T(z))$  and  $g''(z) = \lambda \alpha^2 p''(T(z))$ . Then

$$\begin{aligned} H_g(z) &= z - \left[ 1 + \frac{1}{2} \frac{g(z)g''(z)}{[g'(z)]^2 - \sigma g(z)g''(z)} \right] \frac{g(z)}{g'(z)} \\ &= z - \left[ 1 + \frac{1}{2} \frac{\lambda^2 \alpha^2 p(T(z))p''(T(z))}{[\lambda \alpha p'(T(z))]^2 - \sigma \lambda^2 \alpha^2 p(T(z))p''(T(z))} \right] \frac{\lambda p(T(z))}{\lambda \alpha p'(T(z))} \\ &= z - \frac{1}{\alpha} \left[ 1 + \frac{1}{2} \frac{p(T(z))p''(T(z))}{[p'(T(z))]^2 - \sigma p(T(z))p''(T(z))} \right] \frac{p(T(z))}{p'(T(z))}. \end{aligned}$$

This implies that

$$T \circ H_g(z) = \alpha H_g(z) + \beta = T(z) - \left[ 1 + \frac{1}{2} \frac{p(T(z))p''(T(z))}{[p'(T(z))]^2 - \sigma p(T(z))p''(T(z))} \right] \frac{p(T(z))}{p'(T(z))}.$$

This is nothing but  $H_p(T(z))$ . Now putting  $T(z) = z$  we get  $H_p = H_{\lambda p}$ . Similarly, for  $\lambda = 1$ , we have  $T \circ H_g \circ T^{-1} = T_p$ . □

**Remark 2.1.** 1. *For every polynomial  $p(z) = a_d z^d + a_{d-1} z^{d-1} + \dots + a_1 z + a_0$ , there is an affine map  $T(z) = \alpha z + \beta$  where  $\alpha^d = \frac{1}{a_d}$  and  $\beta = -\frac{a_{d-1}}{d a_d}$  so that the coefficients of  $z^d$  and  $z^{d-1}$  in  $(p \circ T)(z)$  are 1 and 0 respectively. It follows from the Scaling theorem that  $C_p$  is conjugate to  $C_{p \circ T}$  leading to a considerable amount of simplification. We can assume without loss of generality that  $p$  is monic or centered, or both as long as the dynamics of  $C_p$  is concerned.*

2. *In view of the previous remark, for a cubic polynomial  $p$ , we assume without loss of generality that  $p(z) = z^3 + az + b$  for some  $a, b \in \mathbb{C}$ . Then  $p'(z) = 3z^2 + a$ ,  $p''(z) = 6z$  and  $p'''(z) = 6$ . In this case,*

$$L_p(z) = \frac{(z^3 + az + b)6z}{(3z^2 + a)^2} \quad \text{and} \quad L_{p'}(z) = \frac{3z^2 + a}{6z^2}.$$

Hence

$$C_p(z) = \frac{15z^7 + 6az^5 - 15bz^4 - a^2z^3 - 12abz^2 - 3b^2z - a^2b}{(3z^2 + a)^3},$$

and

$$C'_p(z) = \frac{L_p(z)^2}{2} (3 - L_{p'}(z)) = \frac{3(z^3 + az + b)^2 (15z^2 - a)}{(3z^2 + a)^4}.$$



As stated earlier, the roots of  $p$  are fixed points of  $C_p$ . But every fixed points of  $C_p$  is not necessarily a root of  $p$ .

**Definition 2.1.** *A fixed point of  $C_p$  is called extraneous if it is not a root of  $p$ .*

The extraneous fixed points of  $C_p$  can be attracting, repelling or indifferent. This is where the Chebyshev's method stands out from the comparatively well-studied Konig's methods, where all the extraneous fixed points are repelling. Now we deal with all the fixed points of  $C_p$ . Though the following is well-known, we choose to provide a proof for the sake of completeness.

**Proposition 2.3** (Fixed points of  $C_p$ ). *Let  $C_p$  be the Chebyshev's method applied to a polynomial  $p$  with degree  $d$  where  $d \geq 2$ .*

1. *Every root of  $p$  with multiplicity  $k$  is a fixed point of  $C_p$  with multiplier  $\frac{(k-1)(2k-1)}{2k^2}$ . In particular, every (simple) root of  $p$  is an attracting (superattracting) fixed point of  $C_p$ .*
2. *The point at  $\infty$  is a fixed point of  $C_p$  with multiplier  $\frac{2d^2}{2d^2-3d+1}$ . In particular, it is repelling.*
3. *A finite extraneous fixed point  $\zeta$  of  $C_p$  is precisely a root of  $L_p(z) = -2$ , and is attracting, repelling or indifferent if  $2|3 - L_p(\zeta)|$  is less than, greater than or is equal to 1 respectively.*

*Proof.* 1. Let  $\alpha$  be a root of  $p$  with multiplicity  $k$ . Then  $p(z) = (z - \alpha)^k g(z)$  for some polynomial  $g$  with  $g(\alpha) \neq 0$ . Further,

$$p'(z) = k(z - \alpha)^{k-1} g(z) + (z - \alpha)^k g'(z) = (z - \alpha)^{k-1} [k g(z) + (z - \alpha) g'(z)],$$

$$\begin{aligned} p''(z) &= k(k-1)(z - \alpha)^{k-2} g(z) + 2k(z - \alpha)^{k-1} g'(z) + (z - \alpha)^k g''(z) \\ &= (z - \alpha)^{k-2} [k(k-1)g(z) + 2k(z - \alpha)g'(z) + (z - \alpha)^2 g''(z)] \end{aligned}$$

and

$$\begin{aligned} p'''(z) &= k(k-1)(k-2)(z - \alpha)^{k-3} g(z) + 3k(k-1)(z - \alpha)^{k-2} g'(z) \\ &\quad + 3k(z - \alpha)^{k-1} g''(z) + (z - \alpha)^k g'''(z) \\ &= (z - \alpha)^{k-3} [k(k-1)(k-2)g(z) \\ &\quad + 3k(k-1)(z - \alpha)g'(z) + 3k(z - \alpha)^2 g''(z) + (z - \alpha)^3 g'''(z)]. \end{aligned}$$

Note that for  $k = 2$ , the first term in the expression of  $p'''$  vanishes and we have  $p'''(z) = 6g'(z) + 6(z - \alpha)g''(z) + (z - \alpha)^2 g'''(z)$ . Hence we get

$$L_p(z) = \frac{p(z)p''(z)}{(p'(z))^2} = \frac{g(z)[k(k-1)g(z) + 2k(z-\alpha)g'(z) + (z-\alpha)^2g''(z)]}{[kg(z) + (z-\alpha)g'(z)]^2}.$$

This gives

$$L_p(\alpha) = \frac{k-1}{k}. \quad (2)$$

Similarly it is found that  $L_{p'}(\alpha) = \frac{k-2}{k-1}$ . Hence

$$C'_p(\alpha) = \frac{[L_p(\alpha)]^2}{2} [3 - L_{p'}(\alpha)] = \frac{(k-1)^2}{2k^2} \left[ 3 - \frac{k-2}{k-1} \right] = \frac{(k-1)(2k-1)}{2k^2}.$$

The rest is straightforward.

2. Since  $C_p = C_{\lambda p}$  for each  $\lambda \in \mathbb{C} \setminus \{0\}$  and every polynomial  $p$ , without loss of generality we assume that  $p$  is a monic polynomial. If  $\deg(p) = d \geq 2$  then  $p(z) = z^d + a_1z^{d-1} + \dots + a_d$ ,  $p'(z) = dz^{d-1} + (d-1)a_1z^{d-2} + \dots + a_{d-1}$  and  $p''(z) = d(d-1)z^{d-2} + (d-1)(d-2)a_1z^{d-3} + \dots + 2a_{d-2}$ . Now

$$C_p(z) = z - \left( 1 + \frac{1}{2} \frac{p(z)p''(z)}{[p'(z)]^2} \right) \frac{p(z)}{p'(z)} = \frac{2z[p'(z)]^3 - 2p(z)[p'(z)]^2 - [p(z)]^2p''(z)}{2[p'(z)]^3}.$$

Here  $2z[p'(z)]^3$ ,  $2p(z)[p'(z)]^2$  and  $[p(z)]^2p''(z)$  are all polynomials of the same degree  $3d-2$  with the leading coefficients  $2d^3$ ,  $2d^2$  and  $d(d-1)$  respectively. However  $2[p'(z)]^3$  is a polynomial with degree  $3d-3$  and its leading coefficient is  $2d^3$ . Therefore

$$C_p(z) = \frac{(2d^3 - 3d^2 + d)z^{3d-2} + \alpha_{3d-1}z^{3d-1} + \dots + \alpha_0}{2d^3z^{3d-3} + \beta_{3d-2}z^{3d-2} + \dots + \beta_0}, \quad (3)$$

for some  $\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_{3d-1}, \beta_0, \beta_1, \beta_2, \dots, \beta_{3d-2} \in \mathbb{C}$ . Hence  $C_p(\infty) = \infty$  and its multiplier is  $\frac{2d^3}{2d^3-3d^2+d} = \frac{2d^2}{2d^2-3d+1}$  (See page 41,[1]).

3. Each solution of  $L_p(z) = -2$  is a fixed point of  $C_p$  but is not a root of  $p$ . This is because the value of  $L_p$  at each root of  $p$  is in  $(0, 1)$  by Equation (2). Thus, the extraneous fixed points of  $C_p$  are precisely the roots of  $L_p(z) = -2$ . It now follows from Equation (1) that the multiplier of an extraneous fixed point  $\zeta$  is  $|2(3 - L_{p'}(\zeta))|$ . The rest is obvious.  $\square$

**Remark 2.2.** 1. It is possible that the numerator and the denominator of  $C_p$  in Equation (3) have a common factor making the degree of  $C_p$  strictly less than  $2d^3 - 3d^2 + d$ . For example, if  $p(z) = z^3 + c$ ,  $c \neq 0$  then  $\deg(C_p) = 6$ .

(See Proposition 4.4 in Section 4) In this case, the leading coefficients of the numerator and the denominator of  $C_p$  changes after cancelling the common factors. However, their ratio remains unchanged giving that the multiplier of infinity is well-defined.

2. Every fixed point of  $C_p$  which is not attracting is extraneous. But an extraneous fixed point of  $C_p$  can be attracting, repelling or indifferent depending on the nature of  $p$ .

### 3 Degree of the Chebyshev's method

A fixed point is multiple if and only if it is rationally indifferent with multiplier equal to 1 (See Page 142, [8]). This fact is used in the following proof.

*Proof of Theorem 1.1.* Let  $p$  be a monic polynomial with simple root at  $\alpha_i$ ,  $i = 1, 2, \dots, m$ ; double root at  $\beta_j$ ,  $j = 1, 2, \dots, n$  and root  $\gamma_k$ ,  $k = 1, 2, \dots, r$  with multiplicity  $a_k \geq 3$ . Then

$$p(z) = \prod_{i=1}^m (z - \alpha_i) \prod_{j=1}^n (z - \beta_j)^2 \prod_{k=1}^r (z - \gamma_k)^{a_k}$$

and  $\deg(p) = d = m + 2n + M$  where  $M = \sum_{k=1}^r a_k$ .

If  $C_p(z) = \frac{F(z)}{G(z)}$  then  $\deg(F) = \deg(G) + 1$  by Equation (3) and therefore, the sum of all the roots of  $F(z) - zG(z) = 0$  counting multiplicities is nothing but  $\deg(C_p)$ . This is because the leading coefficients of  $F$  and  $G$  are different,  $\infty$  is a simple fixed point of  $C_p$  and the number of fixed points of  $C_p$ , counting multiplicity is  $\deg(C_p) + 1$ . Each root of  $p$  is an attracting or a superattracting fixed point of  $C_p$ , and these are simple roots of  $F(z) - zG(z) = 0$ . Every other fixed point of  $C_p$  are extraneous and is a root of  $L_p(z) + 2 = 0$ . As is evident, a multiple fixed point of  $C_p$  with multiplicity  $k$  is a multiple root of  $L_p(z) + 2 = 0$  with the same multiplicity and vice-versa. Thus,

$$\deg(C_p) = m + n + r + \deg(L_p). \quad (4)$$

Now we need to find  $\deg(L_p)$  in order to determine  $\deg(C_p)$ .

If  $\alpha$  be a root of  $p$  with multiplicity  $k$ , then it is a root of  $p'$  with multiplicity  $k - 1$ . Therefore

$$p'(z) = \prod_{j=1}^n (z - \beta_j) \prod_{k=1}^r (z - \gamma_k)^{a_k - 1} g(z)$$

and

$$p''(z) = \prod_{k=1}^r (z - \gamma_k)^{a_k - 2} h(z), \quad (5)$$

where  $g$  and  $h$  are some polynomials such that  $g$  is non-zero at each  $\alpha_i, \beta_j$  and  $\gamma_k$  and  $h$  is non-zero at each  $\beta_j$  and  $\gamma_k$ , for  $i = 1, 2, \dots, m$ ,  $j = 1, 2, \dots, n$ , and  $k = 1, 2, \dots, r$ . Here we donot rule out  $h(\alpha_i) = 0$  and that is possible, but not relevant here. Note that

$$\deg(g) = \deg(p') - n - \sum_{k=1}^r (a_k - 1) = (d - 1) - n - M + r = m + n + r - 1$$

and

$$\deg(h) = \deg(p'') - \sum_{k=1}^r (a_k - 2) = (d - 2) - M + 2r = m + 2n + 2r - 2.$$

Now

$$\begin{aligned} L_p(z) &= \frac{p(z)p''(z)}{[p'(z)]^2} \\ &= \frac{\prod_{i=1}^m (z - \alpha_i) \prod_{j=1}^n (z - \beta_j)^2 \prod_{k=1}^r (z - \gamma_k)^{2a_k - 2} h(z)}{\prod_{j=1}^n (z - \beta_j)^2 \prod_{k=1}^r (z - \gamma_k)^{2a_k - 2} [g(z)]^2} = \frac{\prod_{i=1}^m (z - \alpha_i) h(z)}{[g(z)]^2} \end{aligned}$$

Letting  $L_p(z) = \frac{P(z)}{Q(z)}$ , we note that every common root of  $P$  and  $Q$  is a root of  $g$  and hence is different from each  $\alpha_i, \beta_j$  and  $\gamma_k$ . Thus any such common root is not a root of  $p$ . Further, it is a common root of  $g$  and  $h$ , i.e., it is a critical point as well as an inflection point of  $p$  ( $p''$  vanishes at this point). In other words, every common root of  $P$  and  $Q$ , if exists, is a special critical point of  $p$ . Conversely, every special critical point is a common root of  $P$  and  $Q$  (in fact of  $g$  and  $h$ ).

Let  $p$  has  $s$  number of distinct special critical points, say  $c_j$ , with multiplicity  $b_j$  for  $j = 1, 2, \dots, s$ . Then  $g(z) = \prod_{j=1}^s (z - c_j)^{b_j} \tilde{g}(z)$  and  $h(z) = \prod_{j=1}^s (z - c_j)^{b_j - 1} \tilde{h}(z)$  where  $\tilde{g}$  and  $\tilde{h}$  are polynomials without any common root. In this case,

$$L_p(z) = \frac{\prod_{i=1}^m (z - \alpha_i) \prod_{j=1}^s (z - c_j)^{b_j - 1} \tilde{h}(z)}{\prod_{j=1}^s (z - c_j)^{2b_j} [\tilde{g}(z)]^2} = \frac{\prod_{i=1}^m (z - \alpha_i) \tilde{h}(z)}{\prod_{j=1}^s (z - c_j)^{b_j + 1} [\tilde{g}(z)]^2}.$$

Now  $\deg(\tilde{g}) = \deg(g) - \sum_{j=1}^s b_j = m + n + r - 1 - B$  and  $\deg(\tilde{h}) = \deg(h) - \sum_{j=1}^s (b_j - 1) = m + 2n + 2r - 2 - B + s$ . Therefore,  $\deg(\prod_{i=1}^m (z - \alpha_i) \tilde{h}(z)) = m + \deg(\tilde{h}) = 2m + 2n + 2r - 2 - B + s$  and  $\deg(\prod_{j=1}^s (z - c_j)^{b_j + 1} \tilde{g}(z)^2) = \sum_{j=1}^s (b_j + 1) + 2 \deg(\tilde{g}) = B + s + 2m + 2n + 2r - 2 - 2B = 2m + 2n + 2r - 2 - B + s$ . This implies that  $\deg(L_p) = 2m + 2n + 2r - 2 - B + s$ . Hence by Equation (4),

$$\deg(C_p) = 3(m + n + r) - 2 - B + s.$$

If  $p$  has no special critical point then  $s = B = 0$  and we get,

$$\deg(C_p) = 3(m + n + r) - 2.$$

□

*Proof of Corollary 1.2.* 1. If  $p$  has two distinct roots and  $p(z) = (z - a)^m(z - b)^n$  for some  $m, n \in \mathbb{N}, a, b \in \mathbb{C}$  then it has only one critical point different from the roots, namely  $\frac{mb+na}{m+n}$ . Further, it is a simple critical point. Hence  $p$  has no special critical point and  $\deg(C_p) = 4$ .

We assert that four is the minimum possible degree of  $C_p$  for every polynomial  $p$ . To see it, let  $p$  have at least three distinct roots. Then  $\deg(p) \geq 3$ . If  $p$  has no special critical point then it follows from Theorem 1.1 that  $\deg(C_p) \geq 7$ . Now assume that  $p$  has at least one special critical point. Since each special critical point of  $p$  is a root of  $g$  and  $\deg(g) = (m + n + r) - 1$  where  $g, m, n, r$  are as given in the proof of Theorem 1.1,  $s \geq 1$  and  $B \leq m + n + r - 1$ . Then  $\deg(C_p) \geq 3(m + n + r) - 2 - (m + n + r - 1) + 1 = 2(m + n + r)$ . Since  $m + n + r \geq 3$ , we have  $\deg(C_p) \geq 6$ . It is clear that  $\deg(C_p) = 5$  is never possible for any polynomial  $p$ .

2. Recall that a special critical point of a rational map is a critical point with multiplicity at least two which is not a root. If for a polynomial  $p$  with degree  $d$ , all the critical points are special, then  $p$  has no multiple roots, i.e.,  $p$  is generic. Clearly the number of roots of  $p$  is  $d$ . Let  $p'(z) = \prod_{j=1}^s (z - c_j)^{b_j}$  where  $b_j \geq 2$  and  $p(c_j) \neq 0$  for any  $j = 1, 2, \dots, s$ . Then  $\deg(p') = d - 1 = \sum_{j=1}^s b_j$  and  $p''(z) = \prod_{j=1}^s (z - c_j)^{b_j-1} q(z)$  where  $q(c_j) \neq 0$  for any  $j = 1, 2, \dots, s$ . So  $\deg(p'') = d - 2 = \sum_{j=1}^s (b_j - 1) + \deg(q)$ . This implies that  $\deg(q) = d - 2 - \sum_{j=1}^s (b_j - 1) = s - 1$ . Thus  $\deg(L_p(z)) = \deg\left(\frac{p'(z)q(z)}{\prod_{j=1}^s (z - c_j)^{b_j+1}}\right) = d + s - 1$ . Now it follows from Equation (4) that  $\deg(C_p) = 2d + s - 1$ . Further, if  $p$  is unicritical then  $p(z) = (z - a)^d + b$ , for some  $a, b \in \mathbb{C}, b \neq 0$ . If  $d = 2$  then by the previous part of this corollary,  $\deg(C_p) = 4$ . If  $d \geq 3$  then its only critical point is  $a$  and that is special. Now, it follows from the preceding lines that  $\deg(C_p) = 2d$ .

□

## 4 Dynamics of $C_\lambda$

We need the following well-known results for the proofs. For a rational map  $R$ , let  $C_R$  denote the set of all critical points of  $R$ .

**Lemma 4.1.** *Let  $U$  be a periodic Fatou component of a rational map  $R$ .*

1. *If  $U$  is an immediate attracting basin or an immediate parabolic basin then  $U \cap C_R \neq \emptyset$ .*

2. If  $U$  is a Siegel disk or a Herman ring then its boundary is contained in the closure of  $\{R^n(c) : n \geq 0 \text{ and } c \in C_R\}$ .

**Lemma 4.2** (Riemann-Hurwitz formula). *If  $R : U \rightarrow V$  is a rational map between two of its Fatou components  $U$  and  $V$  then it is a proper map of some degree  $d$  and  $c(U) - 2 = d(c(V) - 2) + C$  where  $c(\cdot)$  denotes the connectivity of a domain and  $C$  is the number of critical points of  $R$  in  $U$  counting multiplicity. Further, if  $c(V) = 1$  and there is no critical point of  $R$  in  $U$  then  $c(U) = 1$ .*

The following lemma is crucial to prove the simple connectivity of Chebyshev's method applied to polynomials.

**Lemma 4.3.** *Let  $R$  be a rational map for which  $\infty$  is a repelling fixed point. If  $\mathcal{A}$  is an unbounded invariant immediate basin of attraction then its boundary contains at least one pole of  $R$ . Further, if all the poles of  $R$  are on the boundary of  $\mathcal{A}$  and  $\mathcal{A}$  is simply connected then the Julia set of  $R$  is connected.*

*Proof.* Let  $s > 0$  and  $B_s = \{z : \sigma(z, \infty) < s\}$  where  $\sigma$  denotes the spherical metric in  $\widehat{\mathbb{C}}$ . Choose a sufficiently small  $s$  such that  $B_s$  does not contain any critical value of  $R$ . This is possible as  $\infty$  is a repelling fixed point of  $R$ . Then the set  $R^{-1}(B_s)$  has  $d = \deg(R)$  components one of which, say  $N_0$  contains  $\infty$ . Further,  $R$  is one-one on  $N_0$ . Let all other components of  $R^{-1}(B_s)$  be denoted by  $N_i, 1 \leq i \leq d-1$ . Let  $w \in (B_s \cap \mathcal{A}) \setminus \{\infty\}$ . As the degree of  $R : \mathcal{A} \rightarrow \mathcal{A}$  is at least two (as  $\mathcal{A}$  contains at least one critical point by Lemma 4.1), at least two pre-images of  $w$  are in  $\mathcal{A}$ . Since  $R$  is one-one in  $N_0$ , there is a pre-image of  $w$  in  $\mathcal{A} \cap N_j$  for some  $j, 1 \leq j \leq d-1$ . This is true for all  $s' < s$  and for each  $w \in (B_{s'} \cap \mathcal{A}) \setminus \{\infty\}$ . By considering a sequence  $s_n \rightarrow 0$  and  $w_n \in (B_{s_n} \cap \mathcal{A}) \setminus \{\infty\}$  so that  $w_n$ s are distinct and  $w_n \rightarrow \infty$ , we get a sequence  $z_n$  in  $\cup_{1 \leq i \leq d-1} \mathcal{A} \cap N_i$ . Since  $z_m \neq z_n$  for all  $m \neq n$ , there is a subsequence  $z_{n_k}$  and  $j^* \in \{1, 2, \dots, d-1\}$  such that  $z_{n_k} \in \mathcal{A} \cap N_{j^*}$  for all  $k$ . This subsequence has a limit point and that cannot be anything but a pole of  $R$ . This pole is clearly in the Julia set of  $R$  and thus on the boundary of  $\mathcal{A}$ .

If all the poles of  $R$  are on the boundary of  $\mathcal{A}$  and  $\mathcal{A}$  is simply connected then the unbounded component of the Julia set contains all the poles of  $R$ . Let  $U$  be a multiply connected Fatou component of  $R$ . Consider a Jordan curve  $\gamma$  in  $U$  that surrounds a point of the Julia set i.e., the bounded component of  $\widehat{\mathbb{C}} \setminus \gamma$  intersects the Julia set. As  $\infty \in \mathcal{J}(R)$  and the backward orbit of  $\infty$  is dense in the Julia set, there is a point  $z$  surrounded by  $\gamma$  such that  $R^k(z)$  is a pole of  $R$ . Without loss of generality, assume that  $k$  is the smallest natural number such that  $R^k(z)$  is a pole. Then the curve  $R^k(\gamma)$  surrounds a pole of  $R$  by the Open mapping theorem. The set  $R^k(\gamma)$  is completely contained in the Fatou set whereas there is a Julia

component containing  $\infty$  and all the poles of  $R$ . This is not possible proving that all the Fatou components are simply connected. In other words, the Julia set of  $R$  is connected.  $\square$

**Remark 4.1.** *It can follow from the arguments used in the above proof that even if  $\mathcal{A}$  is not simply connected, all the Fatou components other than itself are simply connected whenever the boundary of  $\mathcal{A}$  contains all the poles of  $R$ .*

**Proposition 4.4.** *Let  $p$  be a cubic polynomial.*

1. *If  $p$  is unicritical then its Chebyshev's method is conjugate to  $\frac{5z^6+5z^3-1}{9z^5}$  and its Julia set is connected.*
2. *If  $p$  is not generic then its Chebyshev's method is conjugate to  $\frac{5z^4+15z^3+24z^2+22z+6}{9(z+1)^3}$  and its Julia set is connected.*

*Proof.* 1. Let  $p(z) = (z - \alpha)^3 + \beta$  for some  $\alpha, \beta \in \mathbb{C}, \beta \neq 0$ . If  $\beta = re^{i\theta}$  for  $r > 0$  and  $\theta \in [0, 2\pi)$  then  $(p \circ T)(z) = -re^{i\theta}(z^3 - 1)$  where  $T(z) = -r^{\frac{1}{3}}e^{i\frac{\theta}{3}}z + \alpha$ . In view of the Scaling theorem, we assume without loss of any generality that  $p(z) = z^3 - 1$ . Its Chebyshev's method is

$$C_p(z) = \frac{5z^6 + 5z^3 - 1}{9z^5} \quad \text{and} \quad C'_p(z) = \frac{5(z^3 - 1)^2}{9z^6}.$$

Note that a rational map with degree  $d$  has  $2d - 2$  critical points counting multiplicity. As  $\deg(C_p) = 6$ , there are ten critical points counted with multiplicities and those are the three roots of  $p$  each with multiplicity two and the pole 0 with multiplicity four.

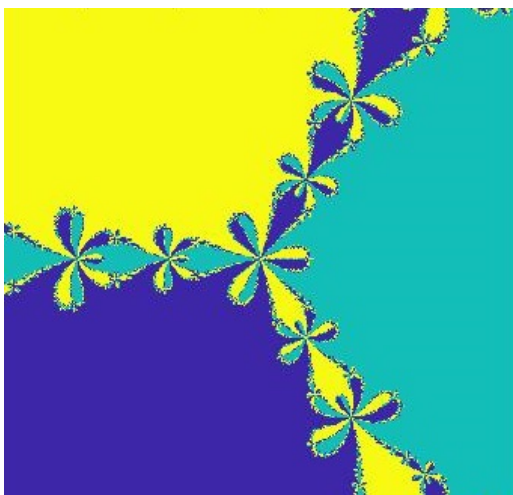
As  $\infty$  is a repelling fixed point, it is in the Julia set of  $C_p$  giving that  $0 \in \mathcal{J}(C_p)$ . Hence none of the superattracting immediate basins contains any critical point other than the superattracting fixed point. Hence each immediate basin is simply connected by Theorem 3.9 [8]. These are the only periodic Fatou components by Lemma 4.1. Every Fatou component different from these are simply connected by the Riemann-Hurwitz formula (Lemma 4.2).

2. Let  $p(z) = (z - a)^2(z - b)$  for some  $a, b \in \mathbb{C}$  and  $a \neq b$ . Then for the affine map  $T(z) = \frac{a-b}{3}z + \frac{2a+b}{3}$ ,  $(\frac{3}{a-b})^3 p(T(z)) = (z - 1)^2(z + 2)$ . In view of the Scaling theorem, we assume without loss of generality that  $p(z) = (z - 1)^2(z + 2)$ . Then

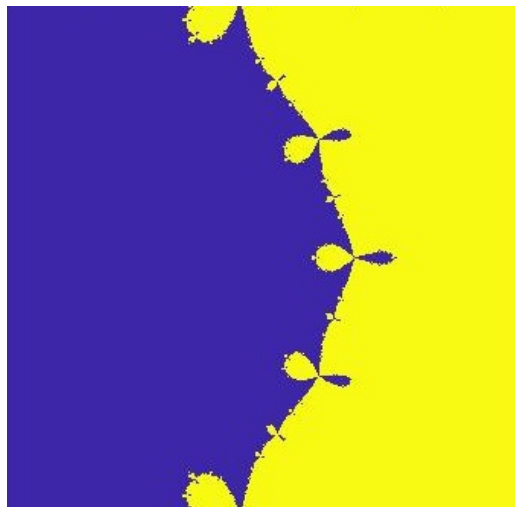
$$C_p(z) = \frac{5z^4 + 15z^3 + 24z^2 + 22z + 6}{9(z + 1)^3}$$

and  $C_p'(z) = \frac{(z+2)^2(5z^2+1)}{9(z+1)^4}$ . The critical points are  $-1, -2$  and  $\pm \frac{i}{\sqrt{5}}$ . Also  $-2$  is a superattracting fixed point of  $C_p$  whereas  $1$  is an attracting fixed point. Let  $\mathcal{A}_{-2}$  and  $\mathcal{A}_1$  be the immediate basins of attraction of  $-2$  and  $1$  respectively. Note that  $-1 \in \mathcal{J}(C_p)$ . Since  $\mathcal{A}_1$  must contain a critical point of  $C_p$ , it is either  $\frac{i}{\sqrt{5}}$  or  $-\frac{i}{\sqrt{5}}$ . Since all the coefficients of  $C_p$  are real,  $C_p^n(\bar{z}) = \overline{C_p^n(z)}$  for all  $n$  and  $z \in \mathbb{C}$ . This gives that each Fatou component intersecting the real line is symmetric with respect to  $\mathbb{R}$ . In particular,  $\mathcal{A}_1$  is symmetric with respect to  $\mathbb{R}$ . This gives that  $\mathcal{A}_1$  contains both these critical points  $\pm \frac{i}{\sqrt{5}}$ . Thus the only periodic Fatou components of  $C_p$  are  $\mathcal{A}_1$  and  $\mathcal{A}_{-2}$ , by Lemma 4.1.

If  $C_p(x) < x$  for any  $x < -2$ , then strictly increasingness of  $C_p$  in  $(-\infty, -2)$  will give that  $\lim_{n \rightarrow \infty} C_p^n(x) = \infty$ , which is not possible as  $\infty$  is a repelling fixed point. Therefore,  $C_p(x) > x$  for all  $x < -2$  and  $\lim_{n \rightarrow \infty} C_p^n(x) = -2$ . In other words,  $(-\infty, -2) \subset \mathcal{A}_{-2}$  showing that  $\mathcal{A}_{-2}$  is unbounded. By Lemma 4.3, there is a pole of  $C_p$  on the boundary of  $\mathcal{A}_{-2}$ . But  $C_p$  has only one pole. As  $\mathcal{A}_{-2}$  does not contain any critical point other than  $-2$ , it is simply connected (Theorem 3.9, [8]). Now it follows from Lemma 4.3 that the Julia set of  $C_p$  is connected.



(a) Unicritical:  $p(z) = z^3 - 1$



(b) Non-generic:  $p(z) = (z - 1)^2(z + 2)$

Figure 4: The Julia set of  $C_p$  for unicritical and non-generic  $p$

□

The Fatou set of the unicritical polynomial  $z^3 - 1$  is given in Figure 4(a). The three superattracting basins are shown in blue, yellow and green. The two basins of the roots of the non-generic polynomial  $(z - 1)^2(z + 2)$  are shown in yellow and blue in Figure 4(b).



**Remark 4.2.** *The Chebyshev's method of each unicritical cubic polynomial has three finite extraneous fixed points, each with multiplier 5. For each non-generic cubic polynomial  $p$ , the finite extraneous fixed points of  $C_p$  are with multipliers 9 and  $\frac{49}{9}$ . This is because the multipliers of fixed points remain unchanged under conformal conjugacy.*

Note that  $\infty$  is always an extraneous fixed point of  $C_p$  and its multiplier is  $\frac{2d^2}{2d^2-3d+1}$  where  $d = \deg(C_p)$ . In order to deal with all non-unicritical and generic cubic polynomials, we use a parameterization in terms of the multiplier of a finite extraneous fixed point of the Chebyshev's method. The forbidden value 5 in the following lemma corresponds precisely to unicritical polynomials whereas there is no cubic polynomial whose Chebyshev's method has a finite extraneous fixed point with multiplier 6. In fact, the multiplier of  $\infty$  can also never be 6.

**Lemma 4.5.** *For  $\lambda \in \mathbb{C} - \{5, 6\}$ , if  $p$  is a non-unicritical and generic cubic polynomial whose Chebyshev's method  $C_p$  has a finite extraneous fixed point with multiplier  $\lambda$  then  $C_p$  is conjugate to the Chebyshev's method of  $p_\lambda(z) = z^3 + 3z + \psi(\lambda)$ , where  $\psi(\lambda) = \frac{3\lambda^2 - 39\lambda + 124}{(5-\lambda)\sqrt{5-\lambda}}$  and  $\sqrt{5-\lambda}$  denotes the principal branch.*

*Proof.* Let  $p$  be a non-unicritical and generic cubic polynomial. Then  $p(z) = z^3 + az + b$  where  $a \neq 0, b \in \mathbb{C}$  and all the roots of  $p$  are simple. Further,  $L_p(z) = \frac{6z(z^3+az+b)}{(3z^2+a)^2}$  and  $L_{p'}(z) = \frac{3z^2+a}{6z^2}$ . If  $z$  is a finite extraneous fixed point of  $C_p$  with multiplier  $\lambda$  then, in view of Equation (2),

$$5 - \frac{a}{3z^2} = \lambda. \quad (6)$$

Note that the above equation has no finite solution for  $\lambda = 5$ . Now either  $\sqrt{\frac{a}{3(5-\lambda)}}$  or  $-\sqrt{\frac{a}{3(5-\lambda)}}$  is the extraneous fixed point of  $C_p$ . Recall that, a fixed point  $z$  of  $C_p$  is extraneous if and only if  $p(z) \neq 0$  and  $L_p(z) = -2$ . Since  $p$  is cubic, generic and non-unicritical, neither  $p$  and  $p'$  nor  $p'$  and  $p''$  have any common root. A finite extraneous fixed point of  $C_p$  is a solution of

$$\frac{p(z)p''(z)}{[p'(z)]^2} = -2, \quad (7)$$

and any such solution is neither a root of  $p$  nor a root of  $p'$ . Equation (7) becomes

$$12z^4 + 9az^2 + 3bz + a^2 = 0. \quad (8)$$

For  $\lambda = 6$ , the point  $\pm\sqrt{\frac{a}{3(5-\lambda)}} = \pm i\sqrt{\frac{a}{3}}$  becomes a root of  $p'$  and hence has been avoided.

Considering  $-\sqrt{\frac{a}{3(5-\lambda)}}$  to be an extraneous fixed point, from Equation (8), we have  $b = \frac{a\sqrt{a}}{3\sqrt{3}}\psi(\lambda)$ . Similarly assuming that  $\sqrt{\frac{a}{3(5-\lambda)}}$  is the extraneous fixed point we have  $b = -\frac{a\sqrt{a}}{3\sqrt{3}}\psi(\lambda)$ .

Let  $p_1(z) = z^3 + az + \frac{a\sqrt{a}}{3\sqrt{3}}\psi(\lambda)$  and  $p_2(z) = z^3 + az - \frac{a\sqrt{a}}{3\sqrt{3}}\psi(\lambda)$ . Then  $p_1(z) = -p_2(-z)$  and by the Scaling Theorem,  $C_{p_1}$  and  $C_{p_2}$  are conformally (in fact, affine) conjugate. Now, for  $\phi(z) = \frac{\sqrt{a}}{\sqrt{3}}z$ ,  $p_1(\phi(z)) = \frac{a\sqrt{a}}{3\sqrt{3}}(z^3 + 3z + \psi(\lambda))$ . Again applying the Scaling Theorem, we conclude that the Chebyshev's methods applied to  $p_1$  and  $z^3 + 3z + \psi(\lambda)$  are conjugate.  $\square$

**Remark 4.3.** 1. *The extraneous fixed point of  $C_\lambda$ ,  $\lambda \neq 5, 6$  having its multiplier equal to  $\lambda$  is  $-\frac{1}{\sqrt{5-\lambda}}$ .*

2. *If  $\lambda = 5$  then Equation (6) gives that  $a = 0$  and  $p(z) = z^3 + b$  becomes an unicritical polynomial.*

3. *For  $\lambda = 6$ , the point that qualifies to be a finite extraneous fixed point of  $C_p$  is  $\pm i\sqrt{\frac{a}{3}}$ . But  $p'$  is zero whereas  $p''$  is not zero at this point. It cannot be a solution of Equation (7). This gives that there is no extraneous fixed point of  $C_p$  with multiplier equal to 6 for any non-unicritical and generic polynomial  $p$ . As seen in Proposition 4.4 and the remark following it, there is also no unicritical or non-generic cubic polynomial with an extraneous fixed point with multiplier equal to 6.*

4. *Since  $\psi'(\lambda) = -\frac{3}{2} \frac{(6-\lambda)(1-\lambda)}{(5-\lambda)^2\sqrt{5-\lambda}}$ ,  $\psi'(\lambda) < 0$  for all  $\lambda < 1$  and is  $> 0$  for all  $1 < \lambda < 5$ . The function  $\psi : (-\infty, 5) \rightarrow [11, \infty)$  is strictly decreasing in  $(-\infty, 1)$ , attains its minimum at 1 and then it strictly increases. The minimum value is  $\psi(1) = 11$ . See Figure 5 for the graph of  $\psi$ .*

5. *For all  $\lambda < 5$ ,  $\psi(\lambda)$  is a real number and  $p_\lambda(z) = z^3 + 3z + \psi(\lambda)$  preserves the real axis. In fact, all the coefficients in the numerator and the denominator of  $C_\lambda$  are real and therefore  $C_\lambda(\bar{z}) = \overline{C_\lambda(z)}$  for all  $z$ . This gives that  $C_\lambda^n(\bar{z}) = \overline{C_\lambda^n(z)}$  for all  $n$ . In other words, the Fatou set of  $C_\lambda$  is symmetric about the real line. If a Fatou component of  $C_\lambda$  intersects the real line then it is also symmetric about the real line.*

Now onwards, we consider  $p_\lambda(z) = z^3 + 3z + \psi(\lambda)$  and let  $C_{p_\lambda}$  be denoted by  $C_\lambda$  for  $\lambda \in \mathbb{C} \setminus \{5, 6\}$ . Then  $L_{p_\lambda}(z) = \frac{p_\lambda(z)2z}{3(z^2+1)^2}$ ,

$$C_\lambda(z) = \frac{5z^7 + 6z^5 - 5\psi(\lambda)z^4 - 3z^3 - 12\psi(\lambda)z^2 - \psi(\lambda)^2z - 3\psi(\lambda)}{9(z^2 + 1)^3}, \quad (9)$$

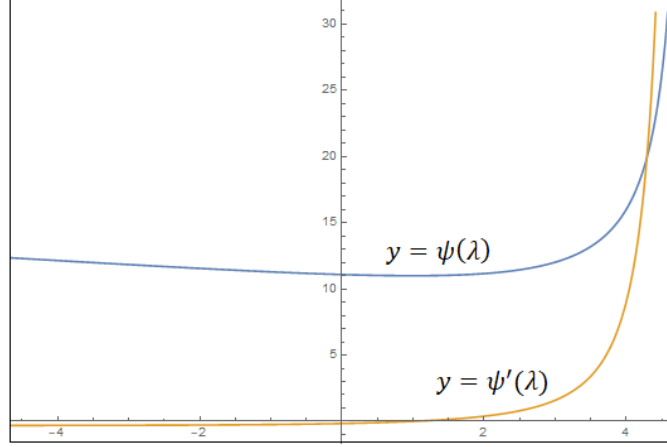


Figure 5: Graph of  $\psi$

and

$$C'_\lambda(z) = \frac{[p_\lambda(z)]^2(5z^2 - 1)}{9(z^2 + 1)^4}. \quad (10)$$

The following are some consequences of the above expressions.

- Lemma 4.6.**
1. For  $\lambda < 5$ , the polynomial  $p_\lambda$  has a unique real root  $r_\lambda$  and the other two roots are complex conjugates of each other.
  2. For  $\lambda \in [-1, 1)$ , in addition to  $-\frac{1}{\sqrt{5-\lambda}}$  there are three extraneous fixed points, one is real, we denote it by  $\alpha_\lambda$  and the other two are complex conjugates of each other.
  3. For  $\lambda = 1$ , the extraneous fixed point  $-\frac{1}{2}$  is multiple with multiplicity two and the other two are complex conjugates of each other.
  4. For  $\lambda \in [-1, 1]$ , all the three roots of  $p_\lambda$  and the poles  $\pm i$  are critical points of  $C_\lambda$  each with multiplicity two. The other two simple critical points of  $C_\lambda$  are  $\frac{1}{\sqrt{5}}$  and  $-\frac{1}{\sqrt{5}}$ .

*Proof.*

1. Since  $p_\lambda$  is monic and is of odd degree and also preserves the real line,  $\lim_{x \rightarrow \infty} p_\lambda(x) = \infty$  and  $\lim_{x \rightarrow -\infty} p_\lambda(x) = -\infty$ . Since  $p'_\lambda(x) > 0$  for all  $x \in \mathbb{R}$ , it is strictly increasing and hence it has a unique real root. Clearly the other two roots are complex conjugates of each other as all the coefficients of  $p_\lambda$  are real for all  $\lambda < 5$ .

2. Putting  $a = 3$  and  $b = \psi(\lambda)$  in Equation (8) we get that the extraneous fixed points of  $C_\lambda$  are the solutions of

$$4z^4 + 9z^2 + \psi(\lambda)z + 3 = 0. \quad (11)$$

These are nothing but the solutions of  $(z + \frac{1}{\sqrt{5-\lambda}})q(z) = 0$  where  $q(z) = 4z^3 - \frac{4z^2}{\sqrt{5-\lambda}} + \frac{(49-9\lambda)z}{5-\lambda} + 3\sqrt{5-\lambda}$ . Note that  $q'(z) = 12z^2 - \frac{8z}{\sqrt{5-\lambda}} + \frac{49-9\lambda}{5-\lambda}$  and its discriminant is  $-16(27 + \frac{8}{5-\lambda})$ , and that is negative for  $-1 \leq \lambda < 1$ . The two roots of  $q'$  are non-real. This gives that  $q'(z) \neq 0$  and is either positive or negative for all  $z \in \mathbb{R}$ , i.e.,  $q$  is either strictly increasing or strictly decreasing on the real line. Since  $q : \mathbb{R} \rightarrow \mathbb{R}$ ,  $\lim_{z \rightarrow -\infty} q(z) = -\infty$  and  $\lim_{z \rightarrow \infty} q(z) = \infty$ ,  $q$  is strictly increasing on  $\mathbb{R}$  and hence has a unique real root. This is the other real extraneous fixed point  $\alpha_\lambda$  of  $C_\lambda$ . As all the coefficients of  $q$  are real, the other two roots are complex conjugates of each other which are nothing but the non-real extraneous fixed points of  $C_\lambda$  for  $\lambda \in [-1, 1)$ .

3. For  $\lambda = 1$ , it follows from Equation (11) that  $-\frac{1}{2}$  is a double root of  $4z^4 + 9z^2 + 11z + 3 = 0$ . The other two roots are found to be  $\beta_1 = \frac{1}{2}(1 + i\sqrt{11})$  and  $\beta_2 = \frac{1}{2}(1 - i\sqrt{11})$ . These are the extraneous fixed points of  $C_1$ .
4. Since  $p_\lambda$  is generic, its three roots are simple and are superattracting fixed points of  $C_\lambda$ , each of which is a critical point with multiplicity two. The rest follows from Equation (9) and Equation (10).

□

Some estimates are going to be useful.

**Lemma 4.7.** *Let  $-1 \leq \lambda \leq 1$ .*

1. *If  $r_\lambda$  is the real root of  $p_\lambda$  then  $-2 < r_\lambda < -1$  and  $C_\lambda(\frac{1}{\sqrt{5}}) < r_\lambda$ . Further, there is  $x_0 \in (-\frac{1}{\sqrt{5}}, 0)$  such that  $C_\lambda(x_0) = r_\lambda$ .*
2. *If  $-1 \leq \lambda < 1$  and  $\alpha_\lambda$  is the real extraneous fixed point of  $C_\lambda$  different from  $-\frac{1}{\sqrt{5-\lambda}}$  then  $-\frac{2}{\sqrt{5-\lambda}} < \alpha_\lambda < -\frac{1}{\sqrt{5-\lambda}}$ . For  $\lambda = 1$ ,  $\alpha_\lambda = -\frac{1}{\sqrt{5-\lambda}}$ .*

*Proof.* For  $-1 \leq \lambda \leq 1$ ,  $11 \leq \psi(\lambda) \leq \psi(-1) \approx 11.294$ .

1. Note that  $p_\lambda(-1) = -4 + \psi(\lambda) > 0$  and  $p_\lambda(-2) = -14 + \psi(\lambda) < 0$  for  $-1 \leq \lambda \leq 1$ . This gives that  $-2 < r_\lambda < -1$ . For  $-1 \leq \lambda \leq 1$ ,  $\psi(\lambda) \geq 11$  and we have  $C_\lambda(\frac{1}{\sqrt{5}}) = -\frac{\sqrt{5}[25\psi(\lambda)^2 + 140\sqrt{5}\psi(\lambda) + 8]}{1944} < -7.449$ . Consequently, we have  $C_\lambda(\frac{1}{\sqrt{5}}) < r_\lambda$ .

It follows from Equation (9) that  $C_\lambda(z) = r_\lambda$  if and only if  $S(z) = 5z^7 + 6z^5 - 5\psi(\lambda)z^4 - 3z^3 - 12\psi(\lambda)z^2 - \psi(\lambda)^2z - 3\psi(\lambda) - 9r_\lambda(z^2 + 1)^3 = 0$ . Note that  $S(0) = -3(\psi(\lambda) + 3r_\lambda) < 0$  (because  $\psi(\lambda) \geq 11$ ). Further,  $S(-\frac{1}{\sqrt{5}}) = \frac{8}{25\sqrt{5}} - \frac{1944r_\lambda}{125} + \frac{\psi(\lambda)}{\sqrt{5}}(\psi(\lambda) - \frac{28}{\sqrt{5}})$ . Note that  $15 < \frac{8}{25\sqrt{5}} - \frac{1944r_\lambda}{125} < 31$  and the

last term  $\frac{\psi(\lambda)}{\sqrt{5}}(\psi(\lambda) - \frac{28}{\sqrt{5}})$  is increasing as a function of  $\psi(\lambda)$  (not of  $\lambda$ ) with its minimum value  $\frac{11}{\sqrt{5}}(11 - \frac{28}{\sqrt{5}}) \approx -7.487$ . Thus  $S(-\frac{1}{\sqrt{5}}) > 7$  and we are done by the Intermediate value Theorem.

2. Recall from the previous lemma that  $q(z) = 4z^3 - \frac{4z^2}{\sqrt{5-\lambda}} + (9 + \frac{4}{5-\lambda})z + 3\sqrt{5-\lambda}$  and the real extraneous fixed point of  $C_\lambda$  different from  $-\frac{1}{\sqrt{5-\lambda}}$  is a root of  $q$ . As  $q(-\frac{2}{\sqrt{5-\lambda}}) = -\frac{1}{(5-\lambda)\sqrt{5-\lambda}}(71+12\lambda-3\lambda^2)$  and  $71+12\lambda-3\lambda^2 > 0$  for all  $\lambda \leq 1$ , we have  $q(-\frac{2}{\sqrt{5-\lambda}}) < 0$ . Similarly,  $q(-\frac{1}{\sqrt{5-\lambda}}) = \frac{1}{(5-\lambda)\sqrt{5-\lambda}}(3(6-\lambda)(1-\lambda)) > 0$  for  $-1 \leq \lambda < 1$ . It is already observed in Lemma 4.6(2) that  $q$  is strictly increasing on  $\mathbb{R}$ . Therefore  $-\frac{2}{\sqrt{5-\lambda}} < \alpha_\lambda < -\frac{1}{\sqrt{5-\lambda}}$ .

For  $\lambda = 1$ ,  $\alpha_\lambda = -\frac{1}{\sqrt{5-\lambda}}$  (See Lemma 4.6).

□

**Remark 4.4.** *Let  $-1 \leq \lambda \leq 1$ . Then*

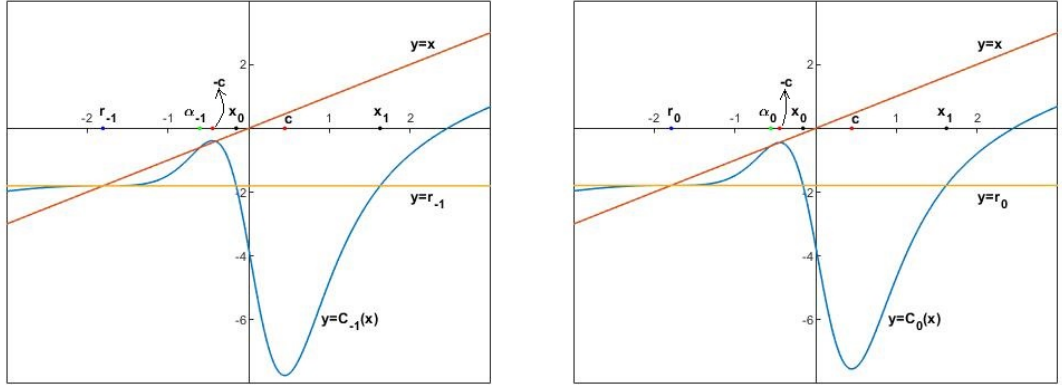
1.  $p_\lambda(\frac{-3}{\sqrt{5-\lambda}}) = \frac{3\lambda^2-30\lambda+52}{(5-\lambda)(\sqrt{5-\lambda})} > 0$ ,  $r_\lambda < \frac{-3}{\sqrt{5-\lambda}} < -\frac{1}{\sqrt{5}}$ , and
2.  $r_\lambda < -\frac{3}{\sqrt{5-\lambda}} < -\frac{2}{\sqrt{5-\lambda}} < \alpha_\lambda$ .

Note that the real root of  $p_\lambda$  is a superattracting fixed point of  $C_\lambda$ . The following lemma describes its immediate basin.

**Lemma 4.8.** *For  $-1 \leq \lambda \leq 1$ , let  $r_\lambda$  be the real root of  $p_\lambda$ . Then it is a superattracting fixed point of  $C_\lambda$  and its immediate basin  $\mathcal{A}_\lambda$  is unbounded. Further, it is simply connected and both the poles of  $C_\lambda$  are on its boundary.*

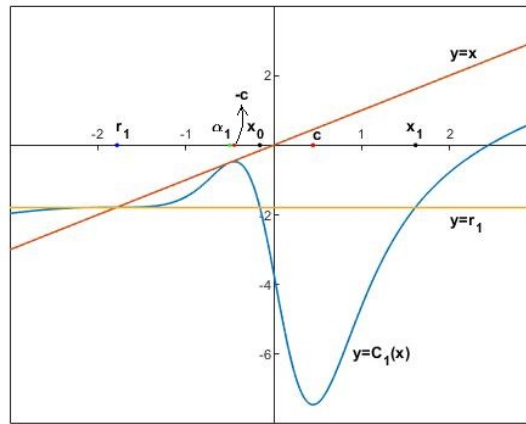
*Proof.* Clearly, the root  $r_\lambda$  of  $p_\lambda$  is simple and is a superattracting fixed point of  $C_\lambda$ . Note that  $C_\lambda$  is strictly increasing in  $(-\infty, -\frac{1}{\sqrt{5}})$ , strictly decreasing in  $(-\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}})$  and strictly increasing thereafter. Let  $c = \frac{1}{\sqrt{5}}$ . Since  $r_\lambda < -c$  (by Remark 4.4(1)),  $C_\lambda$  is strictly increasing in  $(-\infty, r_\lambda]$ . Further, by the preceding remark, the only other real extraneous fixed point of  $C_\lambda$  is greater than  $r_\lambda$ . Therefore, for all  $x \in (-\infty, r_\lambda)$ , either  $C_\lambda(x) < x$  or  $C_\lambda(x) > x$ . The first possibility leads to a strictly decreasing sequence  $\{C_\lambda^n(x)\}_{n>0}$  which must converge to  $-\infty$ . But this is not possible as  $\infty$  is a repelling fixed point of  $C_\lambda$ . Therefore  $C_\lambda(x) > x$  and  $C_\lambda^n(x) \rightarrow r_\lambda$  as  $n \rightarrow \infty$  for all  $x \in (-\infty, r_\lambda]$  giving that  $(-\infty, r_\lambda] \subset \mathcal{A}_\lambda$ . In other words,  $\mathcal{A}_\lambda$  is unbounded.

It follows from Lemma 4.7(1) that the critical value  $C_\lambda(c) \in \mathcal{A}_\lambda$ . As the extraneous fixed point  $-\frac{1}{\sqrt{5-\lambda}}$  is either attracting or parabolic, its basin (attracting or parabolic) contains a critical point. But, the only available critical point is  $-c$ . Therefore  $-c$  is in the immediate basin of attraction or immediate parabolic basin



(a)  $\lambda = -1$

(b)  $\lambda = 0$



(c)  $\lambda = 1$

Figure 6: The fixed point and the preimages of  $r_\lambda$  under  $C_\lambda$

of  $-\frac{1}{\sqrt{5-\lambda}}$ . So  $\mathcal{A}_\lambda$  contains at most one critical point other than  $r_\lambda$  and that can be  $c$  only. We are going to show that this is not the case, i.e.,  $c \notin \mathcal{A}_\lambda$ .

Suppose on the contrary that  $c \in \mathcal{A}_\lambda$ . Then  $C_\lambda([x_0, c]) = [C_\lambda(c), r_\lambda]$  and it gives that  $C_\lambda([x_0, c]) \subset \mathcal{A}_\lambda$ . Note that there is a real pre-image of  $r_\lambda$  in  $(c, \infty)$ . Let it be  $x_1$ . Then  $C_\lambda$  maps  $[c, x_1]$  onto the same interval  $[C_\lambda(c), r_\lambda]$  giving that  $x_1 \in \mathcal{A}_\lambda$  whenever  $c \in \mathcal{A}_\lambda$ . The locations of  $x_0, x_1, r_\lambda$  and the critical points are shown in Figure 6 for  $\lambda = -1, 0$  and  $1$ , where the red dots represent the critical points.

The Bottcher coordinate  $\phi$  is locally defined and univalent at  $r_\lambda$  (See Theorem 9.3, [8]), i.e., there is a simply connected domain  $\tilde{U} \subseteq \mathcal{A}_\lambda$  such that  $\phi : \tilde{U} \rightarrow \phi(\tilde{U}) \subseteq \{z : |z| < 1\}$  is conformal. Since the critical point  $c$  is assumed to be in

$\mathcal{A}_\lambda, \phi$  cannot be extended conformally to the whole of  $\mathcal{A}_\lambda$ . In other words, there is a maximal  $r \in (0, 1)$  such that  $\phi^{-1}$  is well-defined on  $D_r = \{z : |z| < r\}$ . Let  $U = \phi^{-1}(D_r)$ . Clearly  $c \in \partial U$ . Let  $V = C_\lambda(U)$ . Then  $\bar{V} \subset U$  because  $\phi(U) = D_r$  and  $\phi \circ C_\lambda \circ \phi^{-1}(z) = z^3$  on  $D_r$  (by Theorem 9.3, [8]), and  $\bar{D}_{r^3} \subset D_r$ . It also follows that  $C_\lambda : U \rightarrow V$  is a proper map of degree three. Since  $\phi^{-1}$  is well-defined and conformal on  $D_r \supsetneq D_{r^3}$ ,  $\phi^{-1}(D_{r^3}) = V$  is a Jordan domain. Let  $\gamma = \partial V \setminus \{C_\lambda(c)\}$ . As the local degree of  $C_\lambda$  at  $c$  is two, there are two branches of  $C_\lambda^{-1}$  and each is well-defined on  $\gamma$  by the Monodromy theorem. Since there is no critical value of  $C_\lambda$  on  $\gamma$ , the images of  $\gamma$  under each of these branches are Jordan arcs. Let these images be  $\sigma$  and  $\sigma'$ . Then  $\sigma \cap \sigma' = \emptyset$  and each of  $\bar{\sigma}$  and  $\bar{\sigma}'$  is a Jordan curve with  $\bar{\sigma} \cap \bar{\sigma}' = \{c\}$ . In fact, the bounded components of  $\widehat{\mathbb{C}} \setminus \bar{\sigma}$  and  $\widehat{\mathbb{C}} \setminus \bar{\sigma}'$  are the images of  $V$  under the two branches of  $C_\lambda^{-1}$ . This is because the unbounded components of  $\widehat{\mathbb{C}} \setminus \bar{\sigma}$  and  $\widehat{\mathbb{C}} \setminus \bar{\sigma}'$  contains a point of the Julia set of  $C_\lambda$ , namely  $\infty$  and therefore no such unbounded component can be mapped into  $V$ , which is in the Fatou set of  $C_\lambda$ . Clearly, these complementary bounded components are disjoint. One of these must be  $U$ . Assume without loss of generality that  $U$  is the bounded component of  $\widehat{\mathbb{C}} \setminus \bar{\sigma}$ . The possible figures of  $U$  and  $U'$  is given in the lefthand side image of Figure 7.

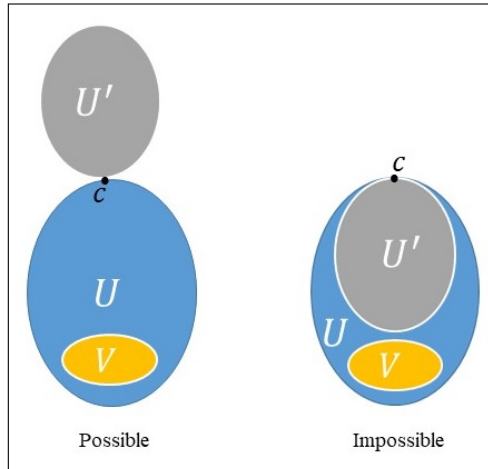


Figure 7: The possible position of  $U$  and  $U'$ .

Let  $U'$  be the bounded component of  $\widehat{\mathbb{C}} \setminus \bar{\sigma}'$ . Now  $r_\lambda \in U$  and  $U'$  contains a pre-image, say  $x^*$  of  $r_\lambda$  such that  $x^* \neq r_\lambda$ .

Now consider a simply connected open set  $W_0$  containing the closure of  $U \cup U'$  and let  $W_1$  be the component of  $C_\lambda^{-1}(W_0)$  containing  $r_\lambda$ . Then  $C_\lambda : W_1 \rightarrow W_0$  is a proper map with some degree  $d$ . Clearly  $d \geq 4$  as there are at least four pre-images of  $r_\lambda$  in  $W_1$  counting multiplicity, namely  $r_\lambda$  itself with multiplicity 3 and  $x^*$  with multiplicity 1. Since  $C_\lambda$  is a proper map (of degree 7) from  $\widehat{\mathbb{C}}$  onto

itself and  $W_1 \neq \widehat{\mathbb{C}}$ , the connectivity of  $W_1$  is non-zero and finite. In fact, each component of  $\widehat{\mathbb{C}} \setminus W_1$  is mapped onto  $\widehat{\mathbb{C}} \setminus W_0$  and there cannot be more than seven such components. Since  $W_1$  contains two critical points, namely  $r_\lambda$  with multiplicity 2 and  $c$  with multiplicity 1, it follows from the Riemann-Hurwitz formula (Lemma 4.2) that  $c(W_1) - 2 = d(c(W_0) - 2) + 3 = 3 - d$ . This gives that the connectivity of  $W_1$  is less than or equal to 1. Thus  $W_1$  is simply connected and  $d = 4$ . Note that  $W_1$  contains only one pre-image of  $r_\lambda$  different from itself and this must be  $x^*$ . Applying this argument again we get that  $C_\lambda : W_2 \rightarrow W_1$  is a proper map of degree 4,  $W_2$  is simply connected and  $x^*$  is the only pre-image of  $r_\lambda$  different from itself, belonging to  $W_2$  where  $W_2$  is the component of  $C_\lambda^{-1}(W_1)$  containing  $r_\lambda$ . It follows by induction that for each  $n \geq 1$ , if  $W_n$  is the component of  $C_\lambda^{-1}(W_{n-1})$  containing  $r_\lambda$  then  $C_\lambda : W_n \rightarrow W_{n-1}$  is a proper map of degree 4,  $W_n$  is simply connected and  $x^*$  is the only pre-image of  $r_\lambda$ , different from  $r_\lambda$  belonging to  $W_n$ .

Since  $x_0, x_1 \in \mathcal{A}_\lambda$ , one of them, say  $x_1$  is different from  $x^*$ . Thus  $x_1 \notin W_n$  for any  $n$ . Consider an arc in  $\mathcal{A}_\lambda$  joining  $r_\lambda$  with  $x_1$ . This arc cannot be contained in  $W_n$  and intersects its boundary for each  $n$ . Let  $w_n$  be a point of such intersection. Then  $w_n$  has an accumulation point, say  $w$  in  $\mathcal{A}_\lambda$ . Considering a sufficiently small neighborhood  $N_w$  of  $w$  contained in  $\mathcal{A}_\lambda$  we observe that  $C_\lambda^n(N_w)$  intersects the boundary of  $W_0$  for all sufficiently large  $n$ . However, there is an  $n_0$  such that  $C_\lambda^n(N_w) \subset V \subsetneq \bar{V} \subset W_0$  for all  $n > n_0$ . This is a contradiction and we prove that  $c \notin \mathcal{A}_\lambda$ .

It now follows from a well-known result (Theorem 9.3, [8]) that  $\mathcal{A}_\lambda$  is simply connected.

By Lemma 4.3, there is a pole of  $C_\lambda$  on the boundary of  $\mathcal{A}_\lambda$ . The Fatou component  $\mathcal{A}_\lambda$  is symmetric about the real line by Remark 4.3 (5). Since the poles are complex conjugates of each other, the other pole is also on the boundary of  $\mathcal{A}_\lambda$ . □

We now present the proof of Theorem 1.2.

*Proof of Theorem 1.2.* Note that  $C_\lambda$  has a fixed point at  $\infty$  and that is repelling. It follows from Lemma 4.8 that the immediate basin  $\mathcal{A}_\lambda$  of the real superattracting fixed point of  $C_\lambda$  corresponding to the real root of  $p_\lambda$  is unbounded, simply connected and contains both the poles of  $C_\lambda$  on its boundary. Now it follows from Lemma 4.3 that the Julia set of  $C_\lambda$  is connected. □



## 5 Concluding remarks

For  $C_\lambda, \lambda \in [-1, 1]$ , all the critical points except the poles are in the attracting or parabolic basins. In fact, the Fatou set of  $C_\lambda$  is the union of the basins of the three superattracting fixed points corresponding to the three roots of  $p_\lambda$  and the basin of the extraneous fixed point (which is parabolic for  $\lambda = \pm 1$  and attracting otherwise). In particular, the Fatou set of  $C_\lambda$  does not contain any Siegel disk or any Herman ring.

It is observed from the graph of  $\psi(\lambda)$  (Figure 5) that for each  $\lambda \in [-1, 1)$  there is a  $\lambda' \in (1, \delta]$  where  $\delta$  is the positive number satisfying  $\psi(\delta) = \psi(-1)$  such that  $\psi(\lambda) = \psi(\lambda')$ . This gives that  $p_\lambda = p_{\lambda'}$  and consequently,  $C_\lambda = C_{\lambda'}$ . Thus Theorem 1.2 is true for all  $\lambda \in [-1, \delta]$ . In terms of the real extraneous fixed points it means the following. For  $\lambda \in [-1, 1)$ , the multiplier of  $-\frac{1}{\sqrt{5-\lambda}}$  is  $\lambda$  whereas the multiplier of the second real extraneous fixed point  $\alpha_\lambda$  is in  $(1, \delta)$  and hence is repelling. For  $\lambda \in (1, \delta)$ , the extraneous fixed point  $-\frac{1}{\sqrt{5-\lambda}}$  becomes repelling making  $\alpha_\lambda$  attracting.

For  $\lambda \in [-1, 1]$ , the forward orbits of the critical points,  $\pm \frac{1}{\sqrt{5}}$  remains in the real line. This along with Lemma 4.1 gives that the two non-real extraneous fixed points cannot be attracting or parabolic. These are in fact, repelling. To see it, recall that each extraneous fixed point other than  $-\frac{1}{\sqrt{5-\lambda}}$  is a solution of  $q(z) = 4z^3 - \frac{4}{\sqrt{5-\lambda}}z^2 + \frac{49-9\lambda}{5-\lambda}z + 3\sqrt{5-\lambda} = 0$ . Among them one, namely  $\alpha_\lambda$  is already known to be real and other two say,  $\zeta$  and  $\bar{\zeta}$  are complex conjugates of each other. Since  $q(z) = 4(z-\alpha_\lambda)(z-\zeta)(z-\bar{\zeta})$ , comparing the constant terms we get,  $\alpha_\lambda\zeta\bar{\zeta} = -\frac{3\sqrt{5-\lambda}}{4}$ . In other word,  $|\zeta|^2 = -\frac{3\sqrt{5-\lambda}}{4\alpha_\lambda}$ . Since  $-\frac{2}{\sqrt{5-\lambda}} < \alpha_\lambda < -\frac{1}{\sqrt{5-\lambda}}$  (by Lemma 4.7 (2)), we have  $\frac{3(5-\lambda)}{8} < -\frac{3\sqrt{5-\lambda}}{4\alpha_\lambda} < \frac{3(5-\lambda)}{4}$ . Consequently,  $\frac{3}{2} < |\zeta|^2 < \frac{9}{2}$  and  $\frac{13}{3} < 5 - \frac{1}{|\zeta|^2} < \frac{43}{9}$ . Recall that  $|\lambda_{\bar{\zeta}}| = |\lambda_\zeta|$  and this is  $|5 - \frac{1}{\zeta^2}| \geq 5 - \frac{1}{|\zeta|^2} > \frac{13}{3}$ .

We conclude by presenting several problems for further investigation.

1. The Julia set of the Chebyshev's method applied to a cubic polynomial with an attracting extraneous fixed point (with non-real multiplier) may be connected. But the arguments used in this article seems to be insufficient to verify it.
2. The images in Figure 1, Figure 2 and Figure 3 suggest that the attracting/parabolic domain corresponding to the extraneous attracting/parabolic fixed point is bounded. However, this is yet to be proved.
3. We believe that all the immediate basins of attractions of the superattracting fixed points corresponding to the roots of  $p_\lambda$  are unbounded for  $\lambda \in [-1, 1]$ . This article proves it only for the real root of  $p_\lambda$ .

4. A rational map is called geometrically finite if the postcritical set  $\overline{\mathcal{P}_R} \cap \mathcal{J}(R)$  is finite where  $\mathcal{P}_R$  is the union of all the forward orbits of all the critical points of  $R$ . It follows from the proof of Theorem 1.2 that  $C_\lambda$  is geometrically finite for  $\lambda \in [-1, 1]$ . It is also clear from Proposition 4 that  $C_p$  is geometrically finite for all cubic unicritical and non-generic polynomials. Since the Julia set is connected in each of these cases, it is locally connected by [10]. The nature of the boundaries of the Fatou components can be explored.
5. For  $\lambda \in (-\infty, -1) \cup (\delta, 5)$ ,  $\psi(\lambda)$  is a real number and  $p_\lambda$  preserves the real line, and it has a unique real root. The dynamics of  $C_\lambda$  is symmetric about the real line by Remark 4.3(5). The forward orbits of the two real critical points of  $C_\lambda$  remain in the real line. It seems plausible to analyze these forward orbits and determine the dynamics of  $C_\lambda$ .

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