The k-Fibonacci group and periods of the k-step Fibonacci sequences

MUNESH KUMARI[1], KALIKA PRASAD[1], BAHAR KULOĞLU[2], ENGIN ÖZKAN[3]
[1]Department of Mathematics,Central University of Jharkhand, 835205, INDIA
[2] Graduate School of Natural and Applied Sciences, Erzincan Binali Yıldırım University, TURKEY
[3] Department of Mathematics, Erzincan Binali Yıldırım University, TURKEY

Abstract: - In this paper, we have introduced a new infinite cyclic group called the k-Fibonacci group and studied its algebraic properties. Further, we have obtained periods for k-step Fibonacci sequences in the 2-generator groups such as S_3, D_3, A_3, Q_8 and in the 3-generator group, $Q_8 \times \mathbb{Z}_{2m}$.

Key-Words: - k-step Fibonacci sequence; Fibonacci Group; Period and Basic Period.

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1 Introduction

Wall [1] studied Fibonacci sequences in groups first. Wall showed that the Fibonacci sequence formed in the cyclic group is periodic according to a prime number and gave the relationship between the period and p. Wilcox [2] carried this problem to abelian groups. Campell et al [3] introduced the Fibonacci length and Fibonacci orbit concepts and moved this problem to some finite simple groups. Özkan et al [11] reported that 3-step Fibonacci sequences are periodic to an m number and the relationship between the period and m. Lu and Wang generalized similar work to k-step Fibonacci sequences[4]. Some recent work in this direction can be seen in[5, 6, 7].

Similar studies have been done on some finite groups, nilpotent groups, and binary polyhedral groups [8, 9, 10, 12, 13, 14].

Gwang-Yeon Lee, et. al.[15] defined the k-generalized Fibonacci matrix Q_k and gave some of its properties. Later, Gwang-Yeon Lee and Jin-Soo Kim [16] defined the k-Fibonacci and the symmetric k-Fibonacci matrix from the k-Fibonacci sequences and discussed its algebra.

Let $\{f_{k,n}\}_{n\geq 0}$ denote the generalized Fibonacci sequence of order $k(\geq 2) \in \mathbb{N}$ given by

$$f_{k,k+n} = f_{k,k+n-1} + f_{k,k+n-2} + f_{k,k+n-3} + \dots + f_{k,n+1} + f_{k,n}, \quad n \geq 0,$$

with $f_{k,0} = f_{k,1} = \dots = f_{k,k-2} = 0$ and $f_{k,k-1} = 1$. The generalized Fibonacci sequence $\{f_{k,n}\}_{n\geq 0}$ is also known as the k-step Fibonacci sequence.

Let the matrix Q_k of order k be given as

$$Q_k = Q_k^1 = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}.$$

On the usual multiplication of Q_k-matrix to n times, we get Q_k^n, where Q_k^n is the generalized Fibonacci matrix[17]. The generalized Fibonacci matrix Q_k^n is defined as

$$Q_k^n = \begin{bmatrix} f_{k,n+k-1} & f_{k,n+k-2} + f_{k,n+k-3} + \dots + f_{k,n} & \dots & f_{k,n+k-2} \\ f_{k,n+k-2} & f_{k,n+k-3} + f_{k,n+k-4} + \dots + f_{k,n-1} & \dots & f_{k,n+k-3} \\ \vdots & \vdots & \vdots & \vdots \\ f_{k,n} & f_{k,n-1} + f_{k,n-2} + \dots + f_{k,-k+n+1} & \dots & f_{k,n-1} \end{bmatrix},$$

where

$$Q_k^0 = I_k, \ (Q_k^1)^n = Q_k^n, \ Q_k^m Q_k^n = Q_k^{m+n} \text{ and } det(Q_k^n) = (-1)^{(k-1)n} \text{ for } m, n \in \mathbb{Z}. \quad (1)$$

2 The k-Fibonacci Group

We will now show that the set of Q_k^n, $n \in \mathbb{Z}$ forms a commutative group according to the matrix product. We will also show that this group is isomorphic to the group of integers $\mathbb{Z}$ and hence is cyclic.

Theorem 2.1. *The collection of all generalized Fibonacci matrices forms a group with respect to matrix multiplication.*
That is, given the following set G,

$$G = \{Q_k^n : k(\geq 2) \in \mathbb{N}, n \in \mathbb{Z}\}. \quad (2)$$

Then G forms an abelian group with respect to the matrix multiplication.

Proof. For $n = 0$, since $I_k \in G$, so G is non-empty. It can easily be seen that the set G satisfies the grouping conditions, so we omit it. □

We refer to the group G defined in Theorem 2.1 as the k-Fibonacci group.

Example 1. *The following is a 3-Fibonacci group with entries from a combination of tribonacci numbers:*

$$G = \left\{ Q_3^n = \begin{bmatrix} f_{3,n+2} & f_{3,n+1} + f_{3,n} & f_{3,n+1} \\ f_{3,n+1} & f_{3,n} + f_{3,n-1} & f_{3,n} \\ f_{3,n} & f_{3,n-1} + f_{3,n-2} & f_{3,n-1} \end{bmatrix} : n \in \mathbb{Z} \right\}.$$

Clearly, G satisfies all the axioms of the multiplicative abelian group.

Also, G is an infinite group.

Theorem 2.2. *The k-Fibonacci group G is isomorphic to $\mathbb{Z}$.*

Proof. Let us consider the set $G = \{Q_k^n : k(\geq 2) \in \mathbb{N}, n \in \mathbb{Z}\}$.
Define a mapping $\phi : \mathbb{Z} \to G$ such that $\phi(n) = Q_k^n$, clearly ϕ is well defined.
Let $n_1, n_2 \in \mathbb{Z}$, then we have

$$\phi(n_1) = \phi(n_2) \implies Q_k^{n_1} = Q_k^{n_2} \implies Q_k^{n_1 - n_2} = Q_k^0 \implies n_1 - n_2 = 0.$$

Thus the mapping $\phi(n)$ is one-one.
Now, let $Q_k^m \in G$, then there exists $m \in \mathbb{Z}$ such that $\phi(m) = Q_k^m$ implies ϕ is onto.
Since, for $n_1, n_2 \in \mathbb{Z}$, we have

$$\phi(n_1 + n_2) = Q_k^{n_1+n_2} = Q_k^{n_1} * Q_k^{n_2} = \phi(n_1) * \phi(n_2) \implies \phi \text{ is homomorphism.}$$

Thus ϕ is one-one, onto and homomorphism implies $G \cong \mathbb{Z}$. □

Hence, the following corollaries are consequences of the above theorem.

Corollary 2.3. *The k-Fibonacci group is an infinite cyclic group and its generators are Q_k^1 and Q_k^{-1}.*

Corollary 2.4. *The center of the k-Fibonacci group G is the group itself i.e $Z(G) = G$.*

Theorem 2.5. *For a positive integer $k \geq 2$, let $G = \{Q_k^n : n \in \mathbb{Z}\}$ and $H = \{(Q_k^n)^m : m, n \in \mathbb{Z}\}$. Then H forms a subgroup of G. Moreover, only possible subgroups of G are of the form $< Q_k^{n\mathbb{Z}} >$.*

Proof. Since $(Q_k^n)^0 = I_k \in H$, thus H is non-empty. Let two elements of H are $(Q_k^n)^{m_1}$ and $(Q_k^n)^{m_2}$ where $m_1, m_2 \in \mathbb{Z}$, then from equation (1), we have

$$\begin{aligned} (Q_k^n)^{m_1}[(Q_k^n)^{m_2}]^{-1} &= (Q_k^n)^{m_1}(Q_k^n)^{-m_2} \\ &= (Q_k^n)^{m_1 - m_2} \in H \\ &\text{as } m_1 - m_2 \in \mathbb{Z}. \end{aligned}$$

So, H is a subgroup of G .
Now, to prove other parts of the theorem, let Q_k^n be the element of H such that n is the smallest positive integer. Let $Q_k^m, m \in \mathbb{Z}$ be any arbitrary element of H then by division algorithm, we have $m = nq + r$ where $q, r \in \mathbb{Z}$ and $0 \leq r < n$.
Since $r = m - nq \in \mathbb{Z}$, so $Q_k^n = Q_k^{m-nq} \in H$ and $m, n \in \mathbb{Z}$ implies $Q_k^m, Q_k^n \in H$ hence $Q_k^r = Q_k^{m-nq} = Q_k^0 = I$.
This implies $r = 0$, which gives

$$\begin{aligned} m &= nq \\ \implies Q_k^m &= Q_k^{nq} \in H, q \in \mathbb{Z} \end{aligned}$$

and Q_k^m is an arbitrary element. □

Corollary 2.6. *All the subgroups of the k-Fibonacci group are normal.*

Proof. Since the k-Fibonacci group G is abelian it implies that all the subgroups of G are normal. □

3 Periods for the k-step Fibonacci Sequence

Throughout, we use $BPer_k(G; x_0, x_1, ..., x_{j-1})$ and $Per_k(G; x_0, x_1, ..., x_{j-1})$ notation to denote the basic period and period of the k-step Fibonacci sequence $F_{k,k+n}(G; x_0, x_1, ..., x_{j-1})$, respectively.

From [18], we note the following definition.

Definition 3.1. *In a finite group, a k-nacci sequence is a sequence of group elements $\{x_0, x_1, ..., x_n, x_{n+1}, ...\}$ where initial set $\{x_0, x_1, ..., x_{j-1}\}$ is provided and*

$$x_n = \begin{cases} x_0 x_1 ... x_n ..., & : j \leq n < k \\ x_{n-k} x_{n-k+1} ... x_{n-1} ..., & : n \geq k. \end{cases}$$

It is denoted by $F_k(G; x_0, x_1, ..., x_{j-1})$ where G is a group generated by the initial set.

A sequence is simply periodic with period k if the first k elements in the sequence form a repeating subsequence. Let $k(p)$ denote the fundamental period of the sequence and call it the Wall number.

A sequence of group elements is said to be periodic if, after a particular point, it contains only a fixed subsequence repeatedly.

Definition 3.2. *The action of automorphism group AutG of G on X and on the k-nacci sequences $F_k(G; x_0, x_1, ..., x_{j-1})$, $(x_0, x_1, ..., x_{j-1}) \in X$, AutG consists of all isomorphism $\theta : G \to G$ and if $\theta \in AutG$ and $(x_0, x_1, ..., x_{j-1}) \in X$ then $(x_0\theta, x_1\theta, ..., x_{j-1}\theta) \in X$.*

Let A be a subset of G and $\theta \in$ Aut$G$, then the image of A is $A\theta = \{a\theta : a \in A\}$ under θ.

Theorem 3.1. *Let $S_3 = \langle x, y : x^3 = y^2 = (xy)^2 = e\rangle$ be the presentation for the group S_3. Then in S_3, the periods of the k-step Fibonacci sequences and the basic periods of the basic k-step Fibonacci sequences are given as:*

(i) For $k = 2$, $Per_2(S_3; x, y)$=6 and $BPer_2(S_3; x, y)$=6.

(ii) For $k \geq 3$, $Per_k(S_3; x, y) = 2k + 2$ and $BPer_k(S_3; x, y) = 2k + 2$.

Proof. (i). For $k = 2$, we have the following sequence,

$$x,\ y,\ xy,\ x^2,\ yx,\ xy,\ x,\ y, ...,$$

which repeats after 6 terms hence its period is 6. Since, we have $x\theta = x$, $y\theta = yx$, $xy\theta = y$, for the inner automorphism θ induced by conjugation by x, so the basic period is 6. □

(ii). For $k \geq 3$, the first k terms of sequence are

$$x,\ y,\ x_2 = (xy)^{2^0},\ x_3 = (xy)^{2^1},\ x_4 = (xy)^{2^2},\ ...,\ x_{k-1} = (xy)^{2^{k-3}}.$$

So, from the above, the following sequence is obtained:

$$x_0 = x, x_1 = y, x_2 = xy \text{ and } x_j = e \text{ for } 3 \leq j \leq k-1.$$

Hence, we have

$$\begin{aligned} &x_{k-1} = e, x_k = e, x_{k+1} = x^2, x_{k+2} = yx, \\ &\quad x_{k+3} = xy, x_{k+4} = e, x_{k+5} = e, ... \\ &x_{2k+2} = x, x_{2k+3} = y, x_{2k+4} = xy, x_{2k+5} = e, \\ &\quad x_{2k+6} = e, ... \\ &x_{3k+3} = x^2, x_{3k+4} = yx, x_{3k+5} = xy, x_{3k+6} = e, \\ &\quad x_{3k+7} = e, ..., x_{4k+4} = x, ..., \end{aligned}$$

and whenever $nk + (n + 3) \leq j \leq (n + 1)k + n$, $n = 1, 2, 3, ...$ then $x_j = e$.
Also, the following hold:

$$x_{2k+2} = x = \prod_{j=k+2}^{2k+1} x_j, \quad x_{2k+3} = y = \prod_{j=k+3}^{2k+2} x_j,$$

$$x_{2k+4} = xy = \prod_{j=k+4}^{2k+3} x_j.$$

Observe that the values of consecutive terms x_{2k+2}, x_{2k+3}, and x_{2k+4} rely on x, y and the cycle starts again with the $(2k + 2)^{th}$ term, which is, $x_0 = x_{2k+2}$, $x_1 = x_{2k+3}$, $x_2 = x_{2k+4}$, ...
Therefore, $Per_k(S_3; x, y) = 2k + 2$.

From the above sequence, we can see that $BPer_k(S_3; x, y) = 2k + 2$, since $x\theta = x$, $y\theta = x^2y$, $xy\theta = y$, where θ is an outer automorphism. □

Also, this theorem is valid in the group D_3.

Theorem 3.2. *Let $A_3 = \langle x, y : x^3 = y^2 = (xy)^3 = e\rangle$ be the presentation for the group A_3. Then the periods and the basic periods of the basic k-step Fibonacci sequences are given as:*

(i) For $k = 2$, $Per_2(A_3; x, y) = 16$ and $BPer_2(A_3; x, y) = 16$.

(ii) For $k = 3$, $Per_3(A_3; x, y) = 13$ and $BPer_3(A_3; x, y) = 13$.

Proof. (i). For $k = 2$, we have the sequence,

$$x, y, xy, yxy, x^2, xyx^2, xyx, x^2, xy, y, x, yx, xyx, xyx^2, x^2y, yx^2, x, y, ...,$$

which repeats after 16 terms therefore the period is 16. Similarly, for the basic sequence,

$$x, xyx^2, x^2yx^2, yx, yx^2, x^2yx, x^2y, x^2, x^2yx^2, xyx^2, x, xy, x^2y, x^2yx, yx^2, xyx, x, xyx^2, ...,$$

the basic period is 16, since $x\theta = x$, $y\theta = xyx^2$, where θ is the inner automorphism induced by conjugation by x. So, $Per_2(A_3; x, y) = 16$ and $BPer_2(A_3; x, y) = 16$. □

(ii). For $k = 3$, the first few terms of the sequence are

$$\begin{aligned} &x, y, xy, (xy)^2, y, y, yx^2, yx^2, x^2yx^2, x^2y, x^2, xyx, \\ &e, yx, y, yxy, xyx, y, y, xyx, xyx, e, x^2yx^2, \\ &e, x^2yx^2, xyx, e, e, xyx, xyx, yxy, yx^2y, xyx, xyx, \\ &e, yxy, e, yxy, xyx, e, e, xyx, xyx, yxy, ... \end{aligned}$$

Here,

$$\begin{aligned} &x_{26} = e, x_{27} = e, x_{28} = xyx, x_{29} = yxy, ..., \\ &x_{39} = e, x_{40} = e, x_{41} = xyx, x_{42} = yxy, ..., \\ &x_{52} = e, x_{53} = e, x_{54} = xyx, x_{55} = yxy, ... \end{aligned}$$

Also, for $k = 3$,

$$\begin{aligned} &x_{uh_k(3)-(k-4)} = e,\ x_{uh_k(3)-(k-3)} = e, \\ &x_{uh_k(3)-(k-2)} = xyx, x_{uh_k(3)-(k-1)} = xyx, \\ &x_{uh_k(3)-k} = yxy, \end{aligned}$$

where $u \in \mathbb{Z}^+$ and $h_k(3)$ refer to the Wall number for the k-step Fibonacci sequence modulo 3.
So that, $Per_2(A_3; x, y) = 13$.
Similarly, $BPer_2(A_3; x, y) = 13$ because $x\theta = x$, $y\theta = xyx^2$. □

Also, this theorem is valid for $\langle 2, 3, 3\rangle$ polyhedral group.

Theorem 3.3. *Let $Q_8 = \langle x, y : x^4 = e, x^2 = y^2, y^{-1}xy = x^{-1}\rangle$ be the presentation for the group Q_8. Then the periods and the basic periods of the basic k-step Fibonacci sequences are given as:*

(i) For $k = 2$, $Per_2(Q_8; x, y) = 3$ and $BPer_2(Q_8; x, y) = 3$.

(ii) For $k \geq 3$, $Per_k(Q_8; x, y) = 2k + 2$ and $BPer_k(Q_8; x, y) = 2k + 2$.

Proof. (i). For $k = 2$, the sequence is as follows:

$$x, y, xy, x, y, xy, ...$$

and it has period $Per_2(Q_8; x, y) = 3$.
For the basic period, the sequence is

$$x, y^3, x^3y, x, y^3, x^3y, ...,$$

which has the period $BPer_2(Q_8; x, y) = 3$ since $x\theta = x, y\theta = y^3$ where θ is an inner automorphism induced by conjugation by x.
(ii). If $k \geq 3$;
For $k = 3$, the sequence is

$$x, y, xy, x^2, x^3, y, xy, e, x, y, xy, x^2, x^3, ...$$

For $k = 5$, the sequence is

$$x, y, xy, x^2, e, e, x^3, y^2, xy, e, , e, e, x, y, xy, x^2, ...$$

Here, it is seen that the period for each $k \geq 3$ is $2k+2$.
And similarly, $BPer_k(Q_8; x, y) = 2k + 2$. ☐

Theorem 3.4. *Let $Q_8 \times \mathbb{Z}_{2m} = \langle x, y, z : x^4 = e, x^2 = y^2, y^{-1}xyx = e, z^{2m} = e = [x, z] = [y, z]\rangle$ be the presentation for the group $Q_8 \times \mathbb{Z}_{2m}$. Then the periods are given as:*

(i) For $k = 2$, $Per_2(Q_8 \times \mathbb{Z}_{2m}; x, y, z)$= $lcm(h_2(2m), 3)$.

(ii) For $k = 3$, $Per_3(Q_8 \times \mathbb{Z}_{2m}; x, y, z) = 2lcm(2m, 2k + 2)$.

(iii) For $k \geq 4$, $Per_k(Q_8 \times \mathbb{Z}_{2m}; x, y, z) = \frac{lcm(2m,2k+2)(k+1)}{2}$.

Proof. (i). For $k = 2$, the sequence is as follows,

$$x, y, z, yz, z^2y, z^3y^2, z^5y^3, z^8y^5, z^{13}y^8, z^{21}y^{13}, z^{34}y^{21}, ...$$

From this sequence, we obtain a sub-sequence as follows:

$$y, z, yz, z^2y, z^3y^2, z^5y^3, ...$$

For $k = 2$, the sequence conforms to the following pattern,

$$x_{t+1} = z^{f_{2,t+1}}y^{f_{2,t}},\ x_{t+2} = z^{f_{2,t+2}}y^{f_{2,t+1}}.$$

We need the smallest t, satisfying $x_{t+1} = y, x_{t+2} = z$
Letting $lcm(h_2(2m), 3) = \lambda$, then we have $2m|f_{2,t+1},\ 3|f_{2,t+1}$ and hence $f_{2,\lambda} \equiv 1 \pmod{2m}$ and $f_{2,\lambda} \equiv 1 \pmod 3$. Also, $f_{2,\lambda+2} \equiv 1 \pmod{2m}$ and $f_{2,\lambda+2} \equiv 1 \pmod 3$.
If we choose $t = \lambda$, then we obtain $x_{\lambda+1} = y$ and $x_{\lambda+2} = z$.
So we get $Per_2(Q_8 \times \mathbb{Z}_{2m}; x, y, z)$=$lcm(h_2(2m), 3)$.
(ii). For $k = 3$, the first few terms of the sequence are:

$$x, y, z, xyz, z^2x, z^4y, z^7y^2, z^{13}xy^3, z^{24}x, z^{44}y, ...$$

Here,

$$x_8 = z^{24}x,\ x_9 = z^{44}y,\ x_{10} = z^{81}y^4, ...,$$
$$x_{16} = z^{3136}x,\ x_{17} = z^{5768}y,\ x_{18} = z^{10609}y^8, ...,$$
$$x_{24} = z^{410744}x,\ x_{25} = z^{755476}y,\ x_{26} = z^{1389537}y^{12}, ...$$

Using the above, we have

$$x_{8\beta} = z^{\frac{\beta}{j}4u_\beta}x,\ x_{8\beta+1} = z^{\frac{\beta}{j}4u_{\beta+1}}y,$$

$$x_{8\beta+2} = z^{\frac{\beta}{j}4u_{\beta+2}+1}y^{4\beta}, ...,$$

where j is odd and $\beta \in \mathbb{N}$, $\beta = 2^\sigma j$ and $u_\beta, u_{\beta+1}, u_{\beta+2} \in \mathbb{N}$, also $gcd(u_\beta, u_{\beta+1}, u_{\beta+2}) = 1$.
We need the smallest β, satisfying $x_{8\beta} = x, x_{8\beta+1} = y$ and $x_{8\beta+2} = z$.
If we choose $\beta = \frac{lcm(2m,2k+2)}{4}$, then we obtain

$$x_{2lcm(2m,2k+2)} = x, x_{2lcm(2m,2k+2)+1} = y,$$

$$x_{2lcm(2m,2k+2)+2} = z.$$

Thus, we get,

$$Per_3(Q_8 \times \mathbb{Z}_{2m}; x, y, z) = 2lcm(2m, 2k + 2).$$

(iii). For $k \geq 4$, the sequence is as follows,

$$x, y, z, xyz, z^2x^2, z^4x^3, z^8y, z^{15}x^2, z^{29}yx, z^{56}, z^{108}x, z^{208}y, z^{401}, z^{773}xy, z^{1490}x^2, z^{2872}x^3, z^{5536}y, z^{10671}x^2, z^{20569}yx, z^{39648}, ...$$

Using the above k-step Fibonacci sequence, we have

$$x_{\beta(2k+2)-k-1} = z^{\frac{\beta}{j}4\delta_{\beta-1}}x^3,\ x_{\beta(2k+2)-k} = z^{\frac{\beta}{j}4\delta_\beta}y,$$
$$x_{\beta(2k+2)-k+1} = z^{\frac{\beta}{j}4\delta_{\beta+1}+3}x^2,$$
$$x_{\beta(2k+2)-k+2} = z^{\frac{\beta}{j}4\delta_{\beta+2}+1}yx,$$
$$x_{\beta(2k+2)-k+3} = z^{\frac{\beta}{j}4\delta_{\beta+3}},$$
$$x_{\beta(2k+2)-k+4} = z^{\frac{\beta}{j}4\delta_{\beta+4}}x,$$
$$x_{\beta(2k+2)-k+5} = z^{\frac{\beta}{j}4\delta_{\beta+5}}y,$$
$$x_{\beta(2k+2)-k+6} = z^{\frac{\beta}{j}4\delta_{\beta+6}+1}x^2, ...,$$

where j is a positive odd integer and β is a positive integer, $\delta_{\beta-1}, \delta_\beta, \delta_{\beta+1}, ..., \delta_{\beta+4} \in \mathbb{N}$ and $gcd(\delta_{\beta-1}, \delta_\beta, \delta_{\beta+1}, ..., \delta_{\beta+4}) = 1$.

Here, we need the smallest β satisfying

$$x_{\beta(2k+2)} = x, x_{\beta(2k+2)+1} = y, x_{\beta(2k+2)+2} = z.$$

Now, if we choose $\beta = \dfrac{lcm(2m, 2k+2)}{4}$, then we get

$$x_{\frac{lcm(2m,2k+2)(k+1)}{4}} = x,\ x_{\frac{lcm(2m,2k+2)(k+1)}{4}+1} = y,$$
$$x_{\frac{lcm(2m,2k+2)(k+1)}{4}+2} = z.$$

Thus, we obtain

$$Per_k(Q_8 \times \mathbb{Z}_{2m}; x, y, z) = \frac{lcm(2m, 2k+2)(k+1)}{2}.$$

□

4 Conclusion

In this study, the k-Fibonacci group was defined and its algebraic properties were examined. A Fibonacci sequence was created in groups, with some having two generators and others having three generators. They have been shown to be periodic, and their fundamental periods have been determined. This work can be applied to many different groups as well as to other sequences such as the Lucas and Pell sequences.

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References:

[1] D. D. Wall, "Fibonacci series modulo m", *Amer. Math. Monthly* 67 (1960) 525–532.

[2] H. J. Wilcox, "Fibonacci sequences of period n in groups", *Fibonacci Quart* 24 (1986) 356–361.

[3] C. Campbell, H. Doostie and E. Robertson, "Fibonacci length of generating pairs in groups", in *Applications of Fibonacci numbers*, eds. G. Bergum, A. Philippou and A. Horadam (Springer, 1990), volume 3, pp. 27–35.

[4] K. Lu and J. Wang, "k-step Fibonacci sequence modulo m", *Utilitas Mathematica* 71 (2006) 169–177.

[5] C. Campbell and P. Campbell, "The Fibonacci lengths of binary polyhedral groups and related groups", *Congr. Numer.* 194 (2009) 95–102.

[6] C. Campbell and P. Campbell, "The Fibonacci length of certain centro-polyhedral groups", *J. Appl. Math. Comput.* 19 (2005) 231–240.

[7] R. Dikici and E. Özkan, "An application of Fibonacci sequences in groups", *Appl Math Comput* 136 (2003) 323–331.

[8] E. Özkan, "3-step Fibonacci sequences in nilpotent groups", *Applied mathematics and computation* 144 (2003) 517–527.

[9] H. Aydin and R. Dikici, "General Fibonacci sequences in finite groups", *Fibonacci Quart* 36 (1998) 216–221.

[10] H. Aydin and G. C. Smith, "Finite p-quotients of some cyclically presented groups", *Journal of the London Mathematical Society* 49 (1994) 83–92.

[11] E. Özkan, H. Aydın and R. Dikici, "3-Step Fibonacci series modulo m", *Applied Mathematics and Computation* 143 (2003) 165–172.

[12] E. Özkan, H. Aydın and R. Dikici, "Applications of Fibonacci sequences in a finite nilpotent group", *Applied mathematics and computation* 141 (2003) 565–578.

[13] E. Özkan, "Fibonacci sequences in nilpotent groups with class n", *Chiang Mai J Sci* 31 (2004) 205–212.

[14] Ö. Deveci, M. Avcı and E. Karaduman, "The periods of the k-step Fibonacci and k-step Pell sequences in $D_{2n} \times z_{2^i}$ and $D_{2n} \times_\varphi Z_{2^i}$", *An. S tiint. Univ. Al. I. Cuza Ia si. Mat. (N.S.)* Tomul LXIV.

[15] G.-Y. Lee, S.-G. Lee and H.-G. Shin, "On the k-generalized Fibonacci matrix qk", *Linear Algebra and its applications* 251 (1997) 73–88.

[16] G.-Y. Lee and J.-S. Kim, "The linear algebra of the k-Fibonacci matrix", *Linear algebra and its applications* 373 (2003) 75–87.

[17] K. Prasad and H. Mahato, "Cryptography using generalized Fibonacci matrices with affine-hill cipher", *Journal of Discrete Mathematical Sciences and Cryptography* (2021) 1–12. DOI: 10.1080/09720529.2020.1838744

[18] S. W. Knox, "Fibonacci sequences in finite groups", *Fibonacci Quart.* 30 (1992) 116–120.