THE KAKEYA MAXIMAL FUNCTION AND THE SPHERICAL SUMMATION MULTIPLIERS.

By Antonio Cordoba.

I. Introduction. The purpose of this paper is to start the program of getting a real variable understanding of the Bochner-Riesz spherical summation operators. These operators are defined on functions on $\mathbf{R}^n$ by the formula

$$\widehat{T_\lambda f}(\xi) = m_\lambda(\xi)\hat{f}\,(\xi)$$

where $m_\lambda(\xi) = (1-|\xi|^2)^\lambda$ if $|\xi| \leqslant 1$ and $m_\lambda(\xi) = 0$ otherwise.

They were first studied by Bochner [1] and Stein [11], [12], [13] in connection with summation of multiple Fourier Series. If λ is bigger than a critical exponent depending of the dimension $(\lambda > (n-1)/2)$, then the kernel of T_λ is integrable and therefore T_λ is bounded on every $L^p(\mathbf{R}^n)$. Stein [11] and Calderon and Zygmund [2], showed that T_λ is bounded on L^p, $1 < p < \infty$ when $\lambda = \text{critical index} = (n-1)/2$. The problem then, arises when we consider λ smaller than that critical index. Herz [9] (See also Fefferman [4]) pointed out that T_λ is unbounded outside the range

$$p(\lambda) = \frac{2n}{n+1+2\lambda} < p < \frac{2n}{n-1-2\lambda} = p'(\lambda).$$

Fefferman [5] showed that T is never bounded on L^p except for the obvious cases: $n = 1$ or $p = 2$; and also Fefferman [4] proved that T_λ is bounded on $L^p(\mathbf{R}^n)$ provided that $p(\lambda) < p < p'(\lambda)$ and $\lambda > (n-1)/4$.

This result has been sharpened by Tomas [15] to $\lambda > (n-1)/2(n+1)$.

Finally Carleson and Sjölin [3], Fefferman [6] and Hörmander [10] proved that, in $\mathbf{R}^2$, T_λ is bounded on L^p whenever $\lambda > 0$ and $p(\lambda) < p < p'(\lambda)$.

So for $n > 2$ we have the natural question: is $T_\lambda(\lambda > 0)$ bounded on $L^p(\mathbf{R}^n)$, $p(\lambda) < p < p'(\lambda)$?

Our approach to the problem is inspired by the work of Fefferman and it is as follows: The multiplier theorem for T_λ can be easily reduced to this problem:

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Suppose that $\varphi: R \to R$ is a smooth function supported on $[-1,1]$ and let $\varphi(r) = \bar{\varphi}(\delta^{-1}(r-1))$, where δ is a small number. Consider the Fourier multiplier defined by $\widehat{Tf}(\xi) = \varphi(|\xi|)\hat{f}(\xi)$.

Is it true that

$$\|Tf\|_{\frac{2n}{n-1}} \leq C|\mathrm{Log}\,\delta|^p \|f\|_{\frac{2n}{n-1}}$$

for some constants C and p independent of δ?

Now, Fefferman's approach to the problem is in the spirit of Cotlar's lemma and we can interpret it, in relation to the multiplier φ, as follows: The support of the kernel for T can be decomposed into a family of rectangles of eccentricity $\delta^{-1/2}$ and the convolution operators, obtained by restricting the kernel to these rectangles, are "almost orthogonal". It happens that, in dimension two, the key estimate is on $L^4(\mathbf{R}^2)$ and this fact is decisive in applying orthogonality methods that cannot be used in higher dimensions, when the important estimate is on

$$L^{\frac{2n}{n-1}}(\mathbf{R}^n).$$

However that proof suggests the idea that, if we consider the maximal function:

$$M_{\delta^{-1/2}}f(x) = \operatorname*{Sup}_{x \in R} \frac{1}{|R|} \int_R |f(y)|\,dy$$

(where the "Sup" is taken over rectangles of eccentricity $\delta^{-1/2}$ and arbitrary direction), then this maximal function controls the multiplier φ.

Part II is devoted to the maximal function and our main result is:

THEOREM 1.1. *M_N is bounded on $L^2(\mathbf{R}^2)$ and there exists a constant C, independent of N, such that:*

$$\|M_N f\|_2 \leq C[\mathrm{Log}\,3N]^2 \|f\|_2 \qquad \forall f \in L^2(\mathbf{R}^2).$$

In Part III we prove that:

$$\|Tf\|_4 \leq C|\mathrm{Log}\,\delta|^{5/4} \|f\|_4, \qquad \forall f \in L^4(\mathbf{R}^2).$$

It would be interesting to answer the following questions for $n > 2$:
(1) Are there constants C, p, independent of N, such that

$$\|M_N f\|_n \leq C[\mathrm{Log}\,N]^p \|f\|_n \qquad \forall f \in L^n(\mathbf{R}^n)?$$

(2) Is there a constant C such that

$$\int |Tf(x)|^2 |g(x)|\, dx \leqslant C \int |f(x)|^2 \left[M_{\delta^{-1/2}} g^2(x) \right]^{1/2} dx$$

Finally it is a pleasure to express my gratitude to my teacher and friend, Charles Fefferman, who introduced me to these problems and guided and helped me in this work. I would like to thank Karen McKeown for her excellent typing of my manuscript.

II. The Maximal Function. Let $N \geqslant 1$ be a real number. By a rectangle of eccentricity N we mean a rectangle R such that:

$$\frac{\text{Length of the bigger side of } R}{\text{Length of the smaller side of } R} = N$$

Consider $\mathfrak{R}_N = \{\text{rectangles of eccentricity } N\}$; given a locally integrable function f we consider the maximal function

$$Mf(x) = \operatorname*{Sup}_{x \in R \in \mathfrak{R}_N} \frac{1}{|R|} \int_R |f(y)|\, dy.$$

The purpose of this chapter is to prove the following theorem:

THEOREM 1.1. *The operator M is bounded in $L^2(\mathbf{R}^2)$ and there exists a constant C (independent of N) such that*:

$$\|Mf\|_2 \leqslant C(\operatorname{Log} 3N)^2 \|f\|_2 \tag{1}$$

$$|\{x : Mf(x) > \alpha > 0\}| \leqslant C(\operatorname{Log} 3N)^3 \frac{\|f\|_2^2}{\alpha^2} \quad f \in L^2(\mathbf{R}^2). \tag{1'}$$

In order to prove Theorem 1.1 we fix a number $\delta > 0$ and consider the maximal function

$$M_\delta f(x) = \operatorname*{Sup}_{x \in R} \frac{1}{|R|} \int_R |f(y)|\, dy$$

where the "Sup" is taken over all the rectangles R such that:

$$\begin{cases} \text{Length of the smaller side of } R = \delta \\ \text{Length of the bigger side of } R = \delta N. \end{cases}$$

(In the following we shall describe this situation by saying that R has dimension $\delta\times\delta N$; we shall define the direction of R as the direction of its bigger side.)

PROPOSITION 1.2. *There exists a constant* C (*independent of* δ *and* N) *such that*

$$\|M_\delta f\|_2 \leqslant C(\operatorname{Log}3N)^{1/2}\|f\|_2 \qquad \forall f\in L^2(\mathbf{R}^2). \tag{2}$$

(When we know that Proposition 1.2 is true, then we shall put together the operators M_δ to obtain the estimates [1], [1′]).

Proof of 1.2.

(a) First of all we decompose the interval of directions $[0,2\pi]$ into eight subintervals of the same length:

$$[0,2\pi]=\left[0,\frac{\pi}{4}\right]\cup\left[\frac{\pi}{4},\frac{\pi}{2}\right]\cup\cdots\cup\left[\frac{7\pi}{4},2\pi\right]=\bigcup_{i=1}^{8}I_i.$$

And consider, for every piece I_i, the maximal function M_δ^i defined analogously as M_δ but with rectangles of directions in the interval I_i. Obviously we have:

$$M_\delta f(x)\leqslant\sum_{i=1}^{8}M_\delta^i f(x)$$

So in order to prove the estimate [2], it is enough to prove it for each M_δ^i. By the symmetry of the situation it is sufficient to show:

$$\|M_\delta^1 f\|_2\leqslant C(\operatorname{Log}3N)^{1/2}\|f\|_2, \qquad \text{with } \mathbf{C} \text{ independent of d and } N.$$

In the following we shall drop the index and we shall consider our maximal function $M_\delta=M_\delta^1$.

(b) We divide the plane, by vertical and horizontal lines, into a grid of squares of side δN. The operator M_δ acts "independently" on the squares of the grid and so we can simplify the problem by considering only functions f supported on one of the squares of the grid. More precisely:

Let $\mathbf{R}^2=\cup Q_\alpha$ (where $\mathring{Q}_\alpha\cap\mathring{Q}_\beta=\phi$ if $\alpha\neq\beta$ and side $(Q_\alpha)=\delta N$) and let $f=\sum f^\alpha$ where $f^\alpha=f/Q_\alpha$. Then: $M_\delta f^\alpha(x)M_\delta f^\beta(x)=0$ if $Q_\alpha^*\cap Q_\beta^*=\Phi$†. Therefore

$$|M_\delta f(x)|^2\leqslant\left|\sum_\alpha M_\delta f^\alpha(x)\right|^2\leqslant 9\sum_\alpha|M_\delta f^\alpha(x)|^2.$$

†Where Q^* is the square with the same center than Q but expanded by the factor 2.

Suppose that we have proved [2] for functions g with support on a square of side δN, then, given any $f \in L^2(\mathbf{R}^2)$ we have:

$$\int |M_\delta f(x)|^2 dx \leqslant 9 \sum_\alpha \int |M_\delta f^\alpha(x)|^2 dx \leqslant 9 \sum_\alpha C \operatorname{Log} N \int |f^\alpha(x)|^2 dx$$

$$= 9C \operatorname{Log} N \int \left| \sum_\alpha f^\alpha(x) \right|^2 dx = C' \operatorname{Log} N \|f\|_2^2$$

and we are done.

So let Q be a square with sides parallel to the coordinates axes and sides δN; suppose that $f \in L^2(Q)$.

Then $M_\delta f(x) = 0$ if $x \notin Q^*$.

We decompose the square Q^* into $9N^2$ small squares $\{Q_{ip}\}$ of side δ, by vertical and horizontal lines. The point is that for every square Q_{ip} we can find a rectangle R_{ip} (of direction in the interval $[0, \pi/4]$ and dimensions $\delta \times \delta N$) such that:

(i) $Q_{ip} \cap R_{ip} \neq \phi$

(ii) $M_\delta f(x) \leqslant 2 \dfrac{1}{|R_{ip}|} \displaystyle\int_{R_{ip}} |f(y)| \, dy \cdot \chi_{Q_{ip}}(x)$

So, if we define the linear operator: (f is fixed)

$$T_f(g)(x) = \sum_{i,p} \frac{1}{|R_{ip}|} \int_{R_{ip}} g(y) \, dy \cdot \chi_{Q_{ip}}(x)$$

we have that $M_\delta f(x) \leqslant 2T_f(|f|)(x)$. Then, in order to show the inequality [2], it is enough to prove that $\|T_f(g)\|_2 \leqslant C(\operatorname{Log} 3N)^{1/2} \|g\|_2$ $\forall g \in L^2(Q^*)$, with C independent of f, δ and N.

(c) Thus we have linearized the problem and we can consider the adjoint of T_f, T_f^*. Given h and g in $L^2(Q^*)$ we have:

$$\int_{Q^*} g(y) T_f^*(h)(y) \, dy = \int_{Q^*} T_f(g)(x) h(x) \, dx = \sum_{i,p} \int_{Q_{ip}} T_f(g)(x) h(x) \, dx$$

$$= \sum_{i,p} \int_{Q_{ip}} h(x) \left[\frac{1}{|R_{ip}|} \int_{R_{ip}} g(y) \, dy \right] dx$$

$$= \int_{Q^*} g(y) \left[\sum_{i,p} \frac{1}{|R_{ip}|} \int_{Q_{ip}} h(x) \, dx \cdot \chi_{R_{ip}}(y) \right] dy.$$

So we have the formula:

$$T_f^*(h)(y) = \sum_{i,p} \frac{1}{|R_{ip}|} \left(\int_{Q_{ip}} h(x) \, dx \right) \chi_{R_{ip}}(y).$$

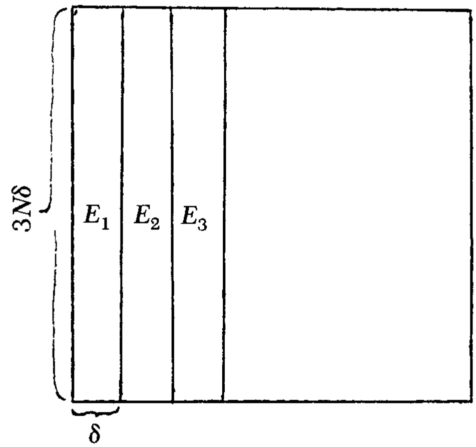

Fig. 1

Now given $h \in L^2(Q^*)$ we have the decomposition $h = h_1 + \cdots + h_{3N}$ (Fig. 1) where $h_i = h/E_i$ is the restriction of h to the vertical strip E_i of width δ. Then, in order to prove that $\|T_f^*(h)\|_2 \leqslant C(\mathrm{Log}\,3N)^{1/2}\|h\|_2$ it is enough to show that:

$$\|T_f^*(h_i)\|_2 \leqslant CN^{-1/2}(\mathrm{Log}\,3N)^{1/2}\|h_i\|_2 \qquad i = 1,\dots,3N$$

because then

$$\|T_f^*(h)\|_2 = \left\|\sum_{i=1}^{3N} T_f^*(h_i)\right\|_2 \leqslant \sum_i \|T_f^*(h_i)\|_2 \leqslant CN^{-1/2}(\mathrm{Log}\,3N)^{1/2}\sum_i \|h_i\|_2$$
$$\leqslant C(\mathrm{Log}\,3N)^{1/2}\|h\|_2.$$

(d) So, suppose that the function h lies on the strip E_i. We decompose E_i into $3N$ squares $\{Q_{ip}\}_{p=1\cdots 3N}$ of side δ and also we decompose the function $h = h_1 + \cdots + h_{3N}$ where $h_p = h/Q_{ip}$.
Then we have

$$T_f^*(h)(x) = \sum_p T_f^*(h_p)(x) = \sum_p \frac{1}{|R_{ip}|}\int_{Q_{ip}} h_p(y)\,dy\,\chi_{R_{ip}}(x)$$

which implies

$$|T_f^*(h)(x)| \leqslant \sum_p \frac{1}{\delta^2 N}\|h_p\|_2 \delta \chi_{R_{ip}}(x) = \frac{1}{\delta N}\sum_{p=1}^{3N}\|h_p\|_2 \chi_{R_{ip}}(x).$$

Therefore

$$\int |T_f^*\,(h)(x)|^2 dx \leqslant \frac{1}{\delta^2 N^2} \int \Big(\sum_p \|h_p\|_2 \chi_{R_{ip}}(x) \Big)^2 dx$$
$$= \frac{1}{\delta^2 N^2} \sum_{p,q} \|h_p\|_2 \|h_q\|_2 |R_{ip} \cap R_{iq}|$$

Now it is an easy geometrical fact that $|R_{ip} \cap R_{iq}| \leqslant 16\,(N\delta^2)/(|p-q|+1)$. Thus

$$\int |T_f^*\,(h)(x)|^2 dx \leqslant C \frac{1}{N} \sum_{p,q} \frac{\|h_p\|_2 \|h_q\|_2}{1+|p-q|} \leqslant CN^{-1} \operatorname{Log} 3N \|h\|_2^2.$$

Remark 1. Part (d) of the proof of Proposition 1.2 admits the following description: Suppose that we have a square room Q of side 1, and we want to illuminate the side AB with beams of light placed on the opposite side CD.

Suppose that our beams have width N^{-1} (i.e., each one illuminates only an interval of length N^{-1} on AB) but we can place them arbitrarily on CD and also we have freedom to choose the direction of the light for each beam (Fig. 2). Then, if the whole wall AB is illuminated and if P is the portion of room illuminated, we have the estimate:

$$|P| \geqslant \frac{1}{\operatorname{Log} N}$$

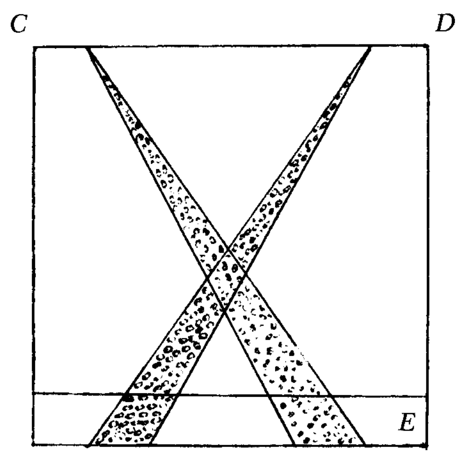

Fig. 2

(This result has been discovered independently by Rolf Anderson in relation with the following problem: Suppose that each interval $[((i-1)/n,0),\ (i/n,0)]$, $i=1,\ldots,n$, is the base of a strip not parallel with the x axis. Let $E(n,k)$ denote the linear measure of the intersection of the union of these strips with the lines $y=1,\cdots k$. Is it true that $E(n,k) \geqslant Cn^{-1/k}$?)

To see that, we can consider a strip E of width N^{-1} over AB. We divide E into N small squares $\{Q_j\}$ and we can suppose that for every square Q_j we have a triangle of light $R_j (R_j \cap Q_j \neq \phi)$.

Then we have the operator $T^*: L^2(E) \to L^2(Q)$ defined as follows: if $f \in L^2(E)$ then

$$T^* f(x) = \sum_{i=1}^{N} \frac{1}{|R_i|} \left(\int_{Q_i} f(y)\, dy \right) \chi_{R_i}(x)$$

By (d) we know that $\|T^*\| \leqslant N^{-1/2} (\operatorname{Log} N)^{1/2}$.

The adjoint of T^*, T is an "average" defined on $g \in L^2(Q)$ by

$$Tg(x) = \sum_{i=1}^{N} \frac{1}{|R_i|} \left(\int_{R_i} g(y)\, dy \right) \chi_{Q_i}(x)$$

Consider $g = \chi_p$ (P is the illuminated set), then obviously $T\chi_p(x) = 1 \forall x \in E$, so we have $\|T\chi_p\|_2 = \|\chi_E\|_2 = N^{-1/2}$.

On the other hand, $\|T\chi_p\|_2 \leqslant N^{-1/2} (\operatorname{Log} N)^{1/2} |P|^{1/2}$. So $|P| \geqslant (\operatorname{Log} N)^{-1}$.

Q.E.D.

The following proposition tells us that the estimate of proposition 1.2 is rather sharp.

PROPOSITION 1.3. *For every N and for every $\delta > 0$ we can find a function $f \in L^2(\mathbf{R}^2)$ such that if M is the maximal function of proposition* 1.2 *corresponding to such N and δ, then we have:*

$$\|Mf\|_2 \geqslant \left[(\operatorname{Log} 3N)^{1/2} \big/ (\operatorname{Log}\operatorname{Log} 3N)^{1/2} \right] \cdot \|f\|_2$$

Proof of 1.3. (a) We start with a triangle $\triangle_0$ of base with length 1 and height $h_0 = 1$. We "sprout" the triangle $\triangle_0$ to the height $h_1 = 2$ to get the tree P_1 composed of two triangles: $\triangle_1^1$, $\triangle_1^2$ (as in Fig. 3). We have the estimate

$$|P_1| \leqslant |\triangle_0| + 4 \cdot 1/4 |\triangle_0| = 2|\triangle_0|$$

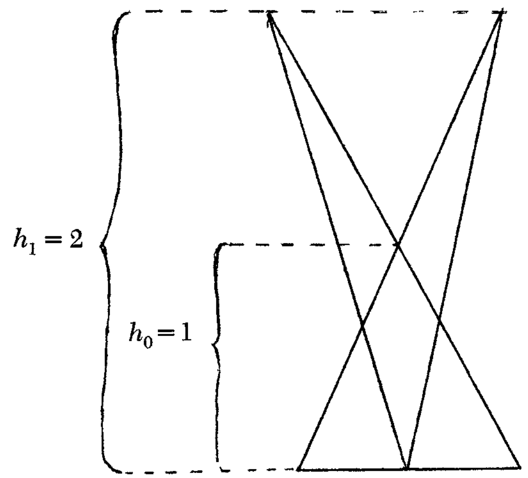

Fig. 3

(b) We repeat the preceding process with each one of the triangles $\triangle_1^1$, $\triangle_1^2$ to get the tree P_2 composed of four triangles: $\triangle_2^1$, $\triangle_2^2$, $\triangle_2^3$, $\triangle_2^4$. We have

$$|P_2| \leqslant |P_1| + 2/3|\triangle_0| \leqslant |\triangle_0| + 2(1/2 + 1/3)|\triangle_0|.$$

Suppose now that we iterate the process until the stage k. We get a tree P_k composed of 2^k triangles of height $h_k = k$ and base 2^{-k}. Furthermore,

$$|P_k| \leqslant |\triangle_0|\left[1 + 2(1/2 + 1/3 + \cdots + 1/k)\right] \cong \operatorname{Log} k.$$

(c) Now for every triangle T on the tree P_k we consider the region $\tilde{T}$ (Fig. 4).

And the point is that the regions $\tilde{T}$ corresponding to the different triangles of the tree P_k are pairwise disjoint.

So if $E_k = U_{T_i \in P_k} \tilde{T}_i$ we have $|E_k| \sim 2^k \cdot k \cdot 2^{-k} = k$.

Taking $\delta = 2^{-k}$, $N = k \cdot 2^k$ and M_δ the corresponding maximal function of proposition 1.2, we have

$$\|M_\delta \chi_{P_k}\|_2 \geqslant \frac{1}{4}|E_k|^{1/2} \sim k^{1/2}$$

and $|P_k| \leqslant 2$ Log k so that

$$\|M_\delta \chi_{P_k}\|_2 \geqslant (\operatorname{Log} N)^{1/2} \Big/ (\operatorname{Log}\operatorname{Log} N)^{1/2} \|\chi_{P_k}\|_2$$

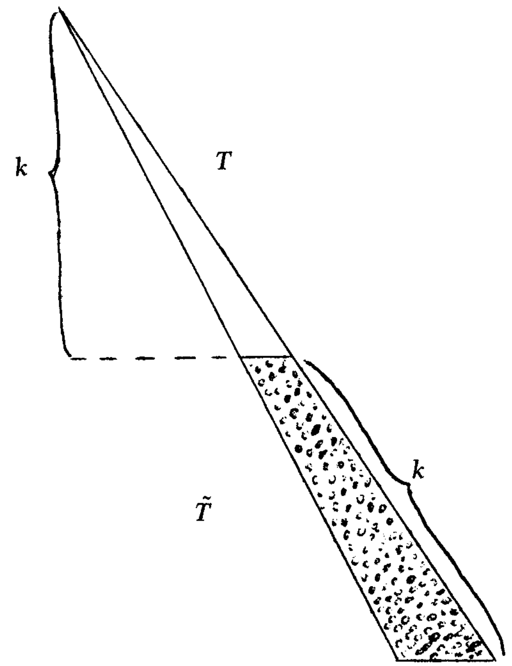

Fig. 4

This establishes proposition 1.3 for this particular δ; in order to do it with a general δ we can work with the same construction but expanded by a convenient factor.

Proof of Theorem 1.1. We divide the interval $[0,2\pi]$ into N pieces and we shall consider only the directions given by the angles: $0, 2\pi/N,\ldots,2\pi$.

Now given any rectangle R of eccentricity N, we can find a rectangle R_1 with the same dimensions as R and direction in the set $\{2k\pi N^{-1}\}_{k=1\cdots N}$ such that $R \subset \widetilde{R_1}$, (where $\widetilde{R_1}$ is the double of R_1).

From this fact it is clear that, in order to estimate the norm of the maximal function, we can consider only rectangles with direction in the set $\{2k\pi N^{-1}\}_{k=1\cdots N}$.

By a similar argument we can consider δ only of the form 2^n, $n \in Z$.

Some Notation:

$$\text{(i)}\quad T^{j}_{2^n}f(x) = \operatorname*{Sup}_{x\in R} \frac{1}{|R|}\int_R |f(y)|\,dy$$

where the "Sup" is taken over all the rectangles of dimensions $2^n \times 2^n N$ and direction $\pi j N^{-1}$.

$$\text{(ii)}\quad T_{2^n} = \operatorname*{Sup}_j T^j_{2^n}, \quad T^j = \operatorname*{Sup}_n T^j_{2^n}, \quad T = \operatorname*{Sup}_j T^j = \operatorname*{Sup}_n T_{2^n}.$$

Now, given $\alpha > 0$, we can apply the standard covering lemma to get, for every i, a sequence of rectangles $\{R^i_n\}$ with direction $\pi N^{-1} i$, pairwise disjoint and such that:

$$E^i_\alpha = \{x : T^i f(x) \geqslant \alpha\} \subset \widetilde{UR^i_n}$$

$$\frac{1}{|R^i_n|} \int_{R^i_n} |f(y)|\, dy \geqslant \alpha$$

By the preceding remarks we know that

$$E_\alpha = \{x : Mf(x) \geqslant 4\alpha\} \subset \bigcup_{i=1}^{N} E^i_\alpha$$

The Sieve. Thus we get N sequences of rectangles and we know that the sides of these rectangles are bounded. Let n_0 be the biggest integer such that there exists in our N collections a rectangle of dimensions $2^{n_0} \times 2^{n_0} N$.

Consider the family of rectangles in the N-collections that have dimensions $2^{n_0} \times 2^{n_0} N$, then we can get a subfamily B_0 with the following properties:

(1°) No rectangle in B_0 is contained in the double of another rectangle in B_0.

(2°) If a rectangle has been eliminated then it is contained in the double of a rectangle of B_0.

Now let n_1 be the biggest integer such that $n_1 < n_0$ and there are rectangles in our primitive N-collections with dimensions $2^{n_1} \times 2^{n_1} N$. Consider the set of such rectangles and eliminate all of them that are contained in the double of a rectangle in $B_0 \ldots$

By induction we get a family of rectangles B_k of side $2^{n_k} \times 2^{n_k} N$ $(n_0 > n_1 > \cdots > n_k > \cdots)$ in such a way that:

(1°) No rectangle of B_k is contained in the double of another rectangle in B_j, $j \leqslant k$.

(2°) If R is a rectangle in our primitive N-collection with dimensions $2^{n_k} \times 2^{n_k} N$ then, either R is in B_k or R is contained in the double of a rectangle in $\bigcup_{j=0}^{k} B_j$.

Obviously $E_\alpha \subset_{R \in UB_k} U\tilde{\tilde{R}}$.

More Notation.

With $k=0,1,2,\dots,$ let us define

$$\triangle_k=\{\text{rectangles in } UB_j \text{ of side } 2^n\times 2^nN$$
$$\text{with: } n_0-k\operatorname{Log}N\geqslant n>n_0-(k+1)\operatorname{Log}N\}.$$

Let

$$E_i=\underset{R\in\triangle_i}{U}R,\qquad \tilde{\tilde{E}}_i=\underset{R\in\triangle_i}{U}\tilde{\tilde{R}}$$

then we know that $E_\alpha\subset U\tilde{\tilde{E}}_i$.

Observe that the family of sets $\{E_i\}$ is "almost disjoint" i.e. $E_i\cap E_j=\phi$ if $|i-j|\geqslant 2$. This is because if $R_i\in\triangle_i$, $R_j\in\triangle_j$ and $i-j\geqslant 2$ then the big side of R_j is smaller than the small side of R_i and so, if $R_i\cap R_j\neq\phi$ we have that $R_j\subset\tilde{\tilde{R}}_i$ and this is impossible.

Let $f_i=f|E_i$ $i=0,1,\dots$ and let S_i be the maximal function defined as follows:

$$S_ig(x)=\operatorname*{Sup}_{x\in R}\frac{1}{|R|}\int_R|g(y)|\,dy$$

where the "Sup" is taken over rectangles of dimensions $2^n\times 2^nN$ where

$$n_0+2-i\operatorname{Log}N\geqslant n>n_0+2-(i+1)\operatorname{Log}N$$

By proposition 1.2 we know that S_i is bounded in $L^2(\mathbf{R}^2)$ with norm $\leqslant C[\operatorname{Log}N]^{3/2}$ (C independent of N and i).

Now if $x\in\tilde{\tilde{E}}_i=U_{R\in\triangle_i}\tilde{\tilde{R}}$ we know that there exists $R\in\triangle_i$ such that $x\in\tilde{\tilde{R}}$, and then

$$S_if_i(x)\geqslant\frac{1}{|\tilde{\tilde{R}}|}\int_R|f_i(y)|\,dy\geqslant\frac{1}{16}\frac{1}{|R|}\int_R|f_i(y)|\,dy\geqslant\frac{1}{16}\alpha$$

which yields $\tilde{\tilde{E}}_i\subset\{x:S_i\,f_i(x)\geqslant(1/16)\alpha\}$, so that

$$|\tilde{\tilde{E}}_i|\leqslant C(\operatorname{Log}3N)^3\frac{\|f_i\|_2^2}{\alpha^2}.$$

Then

$$|E_\alpha|\leqslant\sum_i|\tilde{\tilde{E}}_i|\leqslant C(\operatorname{Log}3N)^3\frac{1}{\alpha^2}\sum_i\|f_i\|_2^2=C(\operatorname{Log}3N)^3\frac{1}{\alpha^2}\sum_i\int f(x)^2\chi_{E_i}(x)\,dx$$
$$\leqslant C(\operatorname{Log}3N)^3\frac{\|f\|_2^2}{\alpha^2}$$

and this proves inequality [1′].

We can get the strong type inequality [1] from [1′] by using the interpolation theorems of Riesz-Thorin and Marcinkiewicz. Q.E.D.

III. The Carleson-Sjolin-Fefferman-Hormander Multiplier Theorem. Suppose that $\varphi_0 : R \to R$ is a smooth function with support on $(-1,1)$ and let $\varphi(r) = \varphi_0((\mathrm{r}-1)/\delta)$ where $\delta > 0$ is a small number.

Consider the Fourier multiplier defined by

$$\widehat{Tf}(\xi) = \varphi(|\xi|)\hat{f}\,(\xi), \qquad f \in C_0^\infty(R^2).$$

THEOREM 2.1. *There exists a constant C independent of δ such that*

$$\|Tf\|_4 \leqslant C|\log\delta|^{5/4}\|f\|_4, \qquad \forall f \in C_0^\infty(R^2). \tag{1}$$

Proof. (a) First of all let us compute the kernel

$$K(x) = \int \varphi(r) r J_0(2\pi|x|r)\,dr$$

where J_0 is the Bessel function of order zero.

Considering the asymptotic expansion of J_0 it follows that, modulo an L^1-kernel with norm independent of δ, $K(x)$ looks like:

$$K(x) = \frac{1}{2\pi i}\,\frac{\exp(-2\pi i|x|)}{|x|^{3/2}} \int \bar{\varphi}_0'(r)\exp(-2\pi i|x|r)\,dr$$

$$= \frac{1}{4\pi^2}\,\frac{\exp(-2\pi i|x|)}{|x|^{5/2}} \int \bar{\varphi}_0''(r)\exp(-2\pi i|x|r)\,dr.$$

where $\bar{\varphi}_0(r) = \varphi_0(r)(1+\delta r)^{1/2}$ has approximately the same bounds that φ_0 as a function in the Schwartz class S.

This estimate tell us that, in order to get (1), it is enough to consider functions f supported on a square of side δ^{-2}.

We need a decomposition of the kernel and of the multiplier: Let $\{\psi_j\}_{j=1\ldots 2\pi\delta^{-1/2}}$ be a smooth partition of unity on the circle such that

(i) $\psi_1(w) = \psi(\delta^{-1/2}w)$

(ii) $\psi_j(w) = \psi_1(w - j\delta^{1/2})$

Where ψ is a smooth function with support on $(-1,1)$.

As in I we will consider only the part of the operator given by

$$Tf(x)=\sum_{j=1}^{\frac{1}{2}\delta^{-1/2}} T_jf(x)$$

where $\widehat{T_jf}(\xi)=m_j(\xi)\hat{f}(\xi)$, and $m_j(\xi)=\varphi(|\xi|)\psi_j(\theta)$ with $\xi=(|\xi|,\theta)$ the polar coordinates in the plane.

We have

$$\int|Tf(x)|^4dx=\int\left|\sum_{ij}T_if(x)T_jf(x)\right|^2dx=\int\left|\sum_{ij}\widehat{T_if}*\widehat{T_jf}(\xi)\right|^2d\xi$$

$$\leqslant C\sum_{ij}\int|\widehat{T_if}*\widehat{T_jf}(\xi)|^2d\xi=C\sum_{ij}\int|T_if(x)T_jf(x)|^2dx,$$

C independent of δ.

This is because no point belongs to more than 4 of the sets $A_{ij}=\operatorname{Supp}m_i+\operatorname{Supp}m_j$.

We will decompose K_1 the kernel of T_1, and then we will get the decomposition of K_i by rotation.

Now by integration by parts we can observe that:

(i) If, in polar coordinates $x=(R,\theta)$, we are in the "rectangle"

$$R\sim 2^m\delta^{-1},\ |\sin(\theta)|\sim 2^n\delta^{1/2},\qquad m,n\geqslant 0.$$

Then we have the estimate $|K_1(R,\theta)|\leqslant A_p\delta^{3/2}(2^m2^n)^{-p}$ with A_p independent of δ (we use a $p\geqslant 1$ to be fixed later).

(ii) In the region $R\sim 2^{-m}\delta^{-1}$ ($m\geqslant 0$), and $|\sin(\theta)|\leqslant 2^m\delta^{1/2}$ we can use the obvious estimate $|K_1(R,\theta)|\leqslant\delta^{3/2}\|\varphi\|_\infty$.

(iii) Finally if $R\sim 2^{-m}\delta^{-1}$ and $|\sin\theta|\sim 2^n2^m\delta^{1/2}$ then as before, integration by parts shows that for each $p\gg 1$ there exists a constant A_p such that

$$|K_1(R,\theta)|\leqslant A_p2^{-np}\delta^{3/2}$$

if (R,θ) lies in the "rectangle" $R\sim 2^{-m}\delta^{-1}$, $|\sin\theta|\sim 2^n2^m\delta^{1/2}$.

We can observe also that K_1 is negligible outside the region $|\theta|\leqslant\pi/8$.

These estimates suggest the following decomposition of K_1:

$$K_1=\sum_{k=0}^{|\operatorname{Log}\delta|}G_k^1+\text{negligible}.$$

where G_k^1 lies on the "rectangle" R_k^1 defined as follows:

$$R_0^1 = \{(r,\theta) | \delta^{-1} \leqslant r \leqslant 2\delta^{-1} \quad \text{and} \quad |\sin\theta| \leqslant \delta^{1/2}$$
$$\text{or} \quad r \leqslant \delta^{-1} \quad \text{and} \quad |r\sin\theta| \leqslant \delta^{-1/2}\}.$$
$$R_k^1 = \{(r,\theta) | 2^k\delta^{-1} \leqslant r \leqslant 2^{k+1}\delta^{-1} \quad \text{and} \quad |\sin\theta| \leqslant 2^k\delta^{1/2} \text{ or}$$
$$\delta^{-1} \leqslant r \leqslant 2^k\delta^{-1} \quad \text{and} \quad 2^{k-1}\delta^{1/2} \leqslant |\sin\theta| \leqslant 2^k\delta^{1/2} \text{ or}$$
$$0 \leqslant r \leqslant \delta^{-1} \quad \text{and} \quad 2^{k-1}\delta^{-1/2} \leqslant |r\sin\theta| \leqslant 2^k\delta^{-1/2}\}.$$

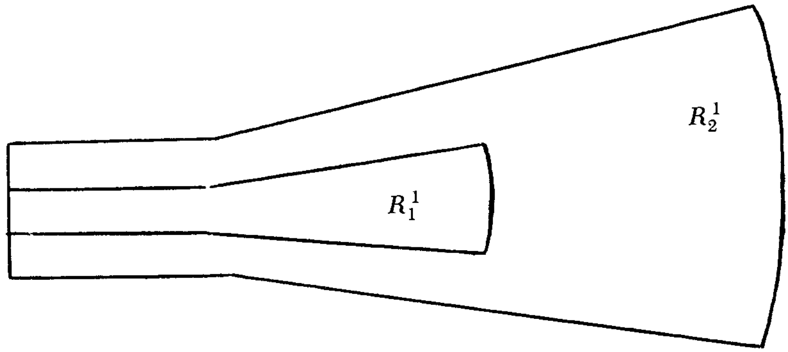

Fig. 5

In particular this decomposition shows that

$$\int |K_1(x)|\,dx \leqslant C, \qquad \text{independent of } \delta.$$

And we have the estimate $|G_k^1(x)| \leqslant A_p 2^{-kp} |R_k^1|^{-1}$.

By rotation we get for every $j = 1, \ldots 2\pi\delta^{-1/2}$ a family of "rectangles" $\{R_k^j\}_{k=1\cdots|\mathrm{Log}\,\delta|}$ and a decomposition of the kernel:

$$K_j = \sum_{k=0}^{|\mathrm{Log}\,\delta|} G_k^j + \text{negligible term}.$$

(b) Let us now introduce some more machinery. Let ϕ_j be a smooth function $j = 1, \cdots 2\pi\delta^{-1/2}$ such that:

(i) $\phi_j \equiv 1$ on $\|x - \omega_j\| \leqslant 2\delta^{1/2}$

(ii) $\phi_j \equiv 0$ on $\|x - \omega_j\| \geqslant 4\delta^{1/2}$

(iii) $|\phi_j(x)| \leqslant 1$ everywhere.

where $\omega_j = \cos(2\pi j\delta^{1/2}) + i\sin(2\pi j\delta^{1/2})$, $j = 1, 2, \ldots, [\delta^{-1/2}]$.

Also we can assume that $\phi_j(x) = \phi(\delta^{-1/2}(x - \omega_j))$ where ϕ is a smooth function supported on $\|x\| \leqslant 4$.

We need some information about the Fourier transform of ϕ_j.

$$\hat{\phi}_j(\xi) = \int \phi\left(\frac{x - \omega_j}{\delta^{1/2}}\right) e^{ix\cdot\xi}\, dx = \delta e^{-i\xi\cdot\omega_j}\hat{\phi}(\delta^{1/2}\xi)$$

Using the formula

$$e^{-i\delta^{1/2}\xi\cdot x} = \left(\frac{1}{-i\delta^{1/2}|\xi|}\right)^p \triangle_x^{p/2} e^{-i\delta^{1/2}\xi\cdot x}$$

we have that

$$|\hat{\phi}_j(\xi)| \leqslant A_p \delta\delta^{-p/2}|\xi|^{-p}, \quad A_p \quad \text{independent of } \delta.$$

This estimate implies the following: Suppose that f is supported in a square Q of side $\delta^{-1/2}$ and let $\hat{f}_j(\xi) = \phi_j(\xi)\hat{f}(\xi)$.

(i) Since

$$\int_{|\xi| \geqslant \delta^{-3/4}} |\hat{\phi}_j(\xi)|\, d\xi \leqslant A_p \delta\,\delta^{-p/2} \int_{\delta^{-3/4}}^{\infty} r^{-p+1}\, dr \leqslant A_p \delta^{p/4}$$

it follows that, with p large enough, the portion of f_j that lies outside the set $\{x : \operatorname{dist}(x, Q) \leqslant \delta^{-3/4}\}$ is negligible.

(ii) Let $\bar{\tau}_n(r) = \tau(2^{-n}\delta^{1/2}r)$ where τ is a smooth function supported on $(1,3)$ and such that: $\sum_{n=1}^{\infty}\bar{\tau}_n(r) = 1$ in the region $r \geqslant 2\delta^{-1/2}$ and let $\bar{\tau}_0$ be given by

$$\sum_{n=0}^{\infty} \bar{\tau}_n(r) = 1 \quad \text{on} \quad (o, \infty).$$

Define $\tau_n(x) = \bar{\tau}_n(|x|)$. Then we have

$$f_i = \hat{\phi}_i * f = \sum_{n=0}^{\frac{1}{4}|\mathrm{Log}\,\delta|} \tau_n \cdot \hat{\phi}_i * f + \text{negligible term}.$$

For each n we have

$$\sum_i \left(\int |\tau_n \cdot \hat{\phi}_i * f(x)|\, dx\right)^2 \leqslant 2^{2n}\delta^{-1} \sum_i \int |\tau_n \hat{\phi}_i * f(x)|^2\, dx$$

$$= 2^{2n}\delta^{-1} \int \sum_i |\hat{\tau}_n * \phi_i(\xi)|^2 |\hat{f}\ (\xi)|^2\, d\xi \leqslant A_p 2^{-np}\delta^{-1}\|f\|_2^2$$

This is because $\sum_i |\hat{\tau}_n * \phi_i(\xi)|^2 \leqslant A_p 2^{-np}$. To see this we start with the following estimate for $\hat{\tau}_n$:

$$|\hat{\tau}_n(\xi)| \leqslant A_p 2^{-n(p-2)} \delta^{-1} \delta^{p/2} |\xi|^{-p}$$

Now given ξ_0 and $k \geqslant 0$ there are, at most, 2^k indices i such that dist $(\xi_0, \operatorname{Supp} \phi_i) \sim 2^k \delta^{1/2}$ and for each one we have

$$|\hat{\tau}_n * \phi_i(\xi_0)| \leqslant A_p \delta \delta^{-1} 2^{-n(p-2)} \delta^{p/2} \left[2^k \delta^{1/2}\right]^{-p} = A_p 2^{-n(p-2)} 2^{-kp}.$$

Multiplying by 2^k and adding in k, we get

$$\sum_i |\hat{\tau}_n * \phi_i(\xi)|^2 \leqslant A_p 2^{-np}$$

(c) Given a square Q of side δ^{-2}, we decompose it into a family $\{Q_\alpha\}$ of squares of side δ^{-1} by horizontal and vertical lines. The index $\alpha = (\alpha_1, \alpha_2)$ for the square Q_α means that its center has coordinates $(\alpha_1 \delta^{-1}, \alpha_2 \delta^{-1})$. We shall prove:

(1) If f^α is supported in the square Q_α then

$$\| Tf^\alpha \|_4 \leqslant C |\operatorname{Log} \delta|^{5/4} \| f^\alpha \|_4$$

(2) If f^α lies on Q_α and f^β lies on Q_β then

$$\int |Tf^\alpha(x) Tf^\beta(x)|^2 dx \leqslant A_p |\operatorname{Log} \delta|^5 (\|\alpha - \beta\| + 1)^{-p} \| f^\alpha \|_4^2 \| f^\beta \|_4^2$$

Obviously estimates (1) and (2) imply theorem 2.1.

Proof of (1). Suppose that $f \in L^4(Q)$ (side of $Q = \delta^{-1}$). As usual we divide Q into $\delta^{-1/2}$ vertical strips $P_1, \ldots, P_{\delta^{-1/2}}$ of width $\delta^{-1/2}$ and we decompose $f = \sum f_k$, $f_k = f / P_k$.

It is enough to show that for each k

$$\| Tf_k \|_4 \leqslant C \delta^{3/8} |\operatorname{Log} \delta|^{5/4} \| f_k \|_4, \qquad C \text{ independent of } \delta.$$

Therefore we shall assume that f is supported in the strip P. Now we have

$$\begin{aligned}
\int |Tf(x)|^4 dx &\leqslant C \sum_{i,j} \int |T_i f_i(x) T_j f_j(x)|^2 dx \\
&\leqslant C \sum_{i,j} \int \left| \sum_k G_k^i * f_i(x) \sum_l G_l^j * f_j(x) \right|^2 dx + \text{negligible term} \\
&\leqslant C |\operatorname{Log} \delta|^2 \sum_{k,l=j}^{|\operatorname{Log} \delta|} \sum_{i,j} \int \left| G_k^i * f_i(x) G_l^j * f_j(x) \right|^2 dx \\
&\quad + \text{negligible term.}^{\dagger} \qquad [V]
\end{aligned}$$

†As always, a term is called "negligible" if its L^4 norm is dominated by $\delta \| f \|_4$.

Now we fix k, l (suppose $k \geqslant l$) and we consider

$$I_{k,l} = \sum_{i,j} \int \left| G_k^i * f_i(x) G_l^j * f_j(x) \right|^2 dx$$

We decompose the strip P into $\delta^{-1/2}$ squares $\{Q_u\}$ (enumerated from the top to the bottom) and set $f = \Sigma f_u$, $f_u = f/Q_u$.

Therefore $f_i = \Sigma f_{ui}$ where $f_{u,i} = \hat{\phi}_i * f_u$.

Then $f_{u,i} = \Sigma_{n=0}^{1/4|\mathrm{Log}\,\delta|} \tau_n \hat{\phi}_i * f_u$ + negligible term, and

$$\left| G_k^i * f_i(x) \right|^2 \leqslant C |\mathrm{Log}\,\delta| \sum_{n=0}^{1/4|\mathrm{Log}\,\delta|} \left| \sum_u G_k^i * \tau_n \hat{\phi}_i * f_u(x) \right|^2$$

$$+ \text{negligible term} \qquad \text{for every } k, i.$$

Thus

$$I_{k,l} \leqslant C |\mathrm{Log}\,\delta|^2 \sum_{m,n} \sum_{i,j} \int \left| \sum_u G_k^i * \tau_n \hat{\phi}_i * f_u(x) \sum_v G_l^j * \tau_m \hat{\phi}_j * f_v(x) \right|^2 dx$$

$$+ \text{negligible term.}$$

Now we fix m, n and consider

$$I_{k,l,m,n} = \sum_{i,j} \int \left| \sum_u G_k^i * \tau_n \hat{\phi}_i * f_u(x) \sum_v G_l^j * \tau_m \hat{\phi}_j * f_v(x) \right|^2 dx$$

If we fix i, j then for each u, v we have

$$A_{u,v}^{i,j} = \mathrm{Supp}\left(G_k^i * \tau_n \hat{\phi}_i * f_u \cdot G_l^j * \tau_m \hat{\phi}_j * f_v \right) \subset \{\mathrm{Supp}\, G_k^i + Q^{*n}_u\} \cap \{\mathrm{Supp}\, G_l^j + Q^{*m}_v\}$$

(where Q^{*s} is the square with the same center as Q but expanded by a factor of 2^s).

And, by the geometry of the situation, no point belongs to more than $2^{2(k+l+m+n)}$ of these sets A_{uv}^{ij}. Therefore

$$I_{k,l,m,n} \leqslant C 2^{4(k+l+m+n)} \sum_{u,v} \sum_{i,j} \int \left| G_k^i * \tau_n \hat{\phi}_i * f_u(x) \cdot G_l^j * \tau_m \hat{\phi}_j * f_v(x) \right|^2 dx$$

and

$$\left| G_k^i * \tau_n \hat{\phi}_i * f_u(x) \right| \leqslant A_p 2^{-kp} \delta^{3/2} \int |\tau_n \hat{\phi}_i * f_u(x)|\, dx$$

$$\left| G_l^j * \tau_m \hat{\phi}_j * f_v(x) \right| \leqslant A_p 2^{-lp} \delta^{3/2} \int |\tau_m \hat{\phi}_j * f_v(x)|\, dx$$

$$\left| \mathrm{Supp}\left(G_k^i * \tau_n \hat{\phi}_i * f_u \cdot G_l^j * \tau_m \hat{\phi}_j * f_v \right) \right| \leqslant C 2^{2(k+l+m+n)} \frac{\delta^{-3/2}}{|u-v|+1}$$

Thus

$$I_{k,l,m,n} \leqslant A_p 2^{6(k+l+m+n)} \cdot 2^{-2p(k+l)} \delta^{9/2}$$

$$\sum_{u,v} \frac{\sum_i \left(\int |\tau_n \hat{\phi}_i * f_u(y)| \, dy \right)^2 \sum_j \left(\int |\tau_m \hat{\phi}_j * f_v(y)| \, dy \right)^2}{|u-v|+1}$$

$$\leqslant A_p 2^{-p(k+l+m+n)} \delta^{5/2} \sum_{u,v} \frac{\| f_u \|_2^2 \| f_v \|_2^2}{|u-v|+1}$$

$$\leqslant A_p 2^{-p(k+l+m+n)} \delta^{3/2} \sum_{u,v} \frac{\| f_u \|_4^2 \| f_v \|_4^2}{|u-v|+1}$$

$$\leqslant A_p 2^{-p(k+l+m+n)} \delta^{3/2} |\operatorname{Log} \delta| \, \| f \|_4^4$$

So going back to [V] we get (if p is big enough)

$$\int |Tf(x)|^4 dx \leqslant C |\operatorname{Log} \delta|^5 \delta^{3/2} \sum_{k,l,m,n} 2^{-p(k+l+m+n)} \| f \|_4^4$$

$$\leqslant C \delta^{3/2} |\operatorname{Log} \delta|^5 \| f \|_4^4.$$

This completes the proof of (1).

Proof of (2). Suppose that f is supported in the square Q_1, g in Q_2 and $\operatorname{dist}(Q_1, Q_2) = d \cdot \delta^{-1}$. We have to show that

$$\int |Tf(x) Tg(x)|^4 \leqslant A_p (1+d)^{-p} |\operatorname{Log} \delta|^5 \| f \|_4^2 \| g \|_4^2$$

As before we decompose Q_1 and Q_2 into vertical strips and also we decompose the functions $f = \sum f_k$, $g = \sum g_l$.

Then it is enough to show that $\forall k, l$

$$\int |Tf_k(x) Tg_l(x)|^2 dx \leqslant A_p (1+d)^{-p} \delta^{3/2} |\operatorname{Log} \delta|^5 \| f_k \|_4^2 \| g_l \|_4^2$$

So in the following we shall assume that f lies on a vertical strip P and g on a vertical strip P' such that $\operatorname{dist}(p, p') \cong d\delta^{-1}$.

Then

$$\int |Tf(x) Tg(x)|^2 dx \leqslant C \sum_{i,j} \int |T_i f_i(x) T_j g_j(x)|^2 dx$$

We decompose the strips P and P' into a family of squares

$$P = UQ_u,\ P' = UQ'_v,\ |Q_u| = |Q'_v| = \delta^{-1}$$

and $f_u = f/Q_u$, $g_v = g/Q'_v$. Then with the same notation as in part (1) we have

$$\int |Tf(x)Tg(x)|^2\,dx$$
$$\leqslant C|\operatorname{Log}\delta|^4 \sum_{k,l=1}^{|\operatorname{Log}\delta|} \sum_{m,n=1}^{|\operatorname{Log}\delta|} \sum_{i,j} 2^{4(k+l+m+n)} \sum_{u,v} \int |G_k^i * \tau_n\hat{\phi}_i * f_u(x)$$
$$\cdot G_l^j * \tau_m\hat{\phi}_j * g_v(x)|^2\,dx + \text{negligible term.}$$

Now we decompose the sum into two parts; i.e. we consider the sets of indices:

$$J_1 = \{(k,l,m,n)|\sup\{2^k, 2^l, 2^n, 2^m\} \geqslant d/4\}$$
$$J_2 = \{(k,l,m,n)|\sup\{2^k, 2^l, 2^n, 2^m\} < d/4\}$$

And it is clear that if $(k,l,m,n) \in J_2$ then

$$|G_k^i * \tau_n\hat{\phi}_i * f_u(x) G_l^j * \tau_m\hat{\phi}_j * g_v(x)| \equiv 0$$

Now, suppose that $(k,l,m,n) \in J_1$. Then

$$\sum_{i,j} \sum_{u,v} \int \left|G_k^i * \tau_n\hat{\phi}_i * f_u(x)\right|^2 \left|G_l^j * \tau_m\hat{\phi}_j * g_v(x)\right|^2 dx$$
$$= \int \sum_i \sum_u \left|G_k^i * \tau_n\hat{\phi}_i * f_u(x)\right|^2 \sum_j \sum_v \left|G_l^j * \tau_m\hat{\phi}_j * g_v(x)\right|^2 dx$$
$$\leqslant \left(\int \sum_{i,i'} \sum_{u,u'} \left|G_k^i * \tau_n\hat{\phi}_i * f_u(x) G_k^{i'} * \tau_n\hat{\phi}_{i'} * f_{u'}(x)\right|^2 dx \right)^{1/2}$$
$$\cdot \left(\int \sum_{j,j'} \sum_{v,v'} \left|G_l^j * \tau_m\hat{\phi}_j * g_v(x) G_l^{j'} * \tau_m\hat{\phi}_{j'} * g_{v'}(x)\right|^2 dx \right)^{1/2}$$
$$\leqslant C|\operatorname{Log}\delta|^5 \delta^{3/2} 2^{-p(k+l+m+n)} \|f\|_4^2 \|g\|_4^2$$

And now we use the fact that in J_1 one of the four numbers 2^k, 2^l, 2^m, 2^n is bigger than $d/4$.

COROLLARY 2.3. (*Carleson-Sjolin-Fefferman-Hormander*). *The operator* T_λ *defined by* $\widehat{T_\lambda f}(\xi) = m_\lambda(\xi)\hat{f}(\xi)$, *where* $m_\lambda(\xi) = (1-|\xi|^2)^\lambda$ *if* $|\xi| \leqslant 1$ *and* $m_\lambda(\xi) = 0$ *otherwise, is bounded in* $L^p(\mathbf{R}^2)$ *if*

$$\frac{4}{3+2\lambda} < p < \frac{4}{1-2\lambda}, \qquad \left(\frac{1}{2} > \lambda > 0\right).$$

Proof. We define a partition of unity on $[0,1]$ as follows: For every n, χ_n is a smooth function with support on $[1-2^{-n+1}, 1-2^{-n-1}]$ such that $|D^p\chi_n(r)| \leqslant A_p 2^{np}$ (with A_p independent of n) and $\sum_{n=1}^\infty \chi_n(r) = 1$ on $[0,1]$. Then $m_\lambda(\xi) = \sum_{n=1}^\infty m_\lambda(\xi)\cdot\chi_n(|\xi|)$.

If we apply theorem 2.1 to the operator T_λ^n defined by the multiplier $m_\lambda(\xi)\chi_n(|\xi|)$ we get that

$$\|T_\lambda^n f\|_4 \leqslant C 2^{-n\lambda} n^{5/4} \|f\|_4$$

and then Corollary 2.3 can be deduced from this estimate by standard arguments of interpolation, duality and adding a geometric series. Q.E.D.

Remark. Theorem 2.1 can be used to prove a sharper version of corollary 2.3 i.e., suppose that m is a smooth function on $[0,1]$ such that behaves like

$$\left\{\mathrm{Log}\frac{1}{1-|x|}\right\}^{-\rho} \quad \text{near} \quad |x|=1.$$

Then m is a multiplier for $L^p(\mathbf{R}^2)$, $(4/3) \leqslant p \leqslant 4$ provided that $\rho > 9/4$.

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REFERENCES.

[1] Bochner, S., "Summation of multiple Fourier Series by spherical means," *Trans. Amer. Math. Soc.* **40** (1936), pp. 175–207.

[2] Calderon, A. P. and Zygmund, A. "On singular integrals," *Amer. J. Math.* **78** (1956), pp. 289–309.

[3] Carleson, L. and Sjolin, P., "Oscillatory integrals and a multiplier problem for the disc," *Studia Math.* **44** (1972), pp. 287–299.

[4] Fefferman, Ch., "Inequalities for strongly singular convolution operators," *Acta Math.* **124** (1970) pp. 9–36.
[5] ———, "The multiplier problem for the ball," *Annals of Math.* **94** (1972), pp. 330–336.
[6] ———, "A note on spherical summation multipliers," *Israeli J. Math.* **15** (1973), pp. 44–52.
[7] Gallego, A. and Cordoba, A., "Comunicación presentada en la XI R.A.M.E." Murcia, 1970.
[8] De Guzman, M., "Covering Lemma with applications to differentiability of measures and singular integrals operators," *Studia Math.* **34** (1970), pp. 299–317.
[9] Herz, S., "On the mean inversion of Fourier and Hankel transforms," *Proc. Nat. Acad., Sci. USA.* **40** (1954), pp. 996–999.
[10] Hormander, L., "Oscillatory integrals and multipliers on FL^p". *Arkiv. fur Mat.* **II** (1974), pp. 1–11.
[11] Stein, E., "Interpolation of linear operators," *Trans. Amer. Math. Soc.* **87** (1958), pp. 159–172.
[12] ———, "Localization and summability of multiple Fourier series," *Acta Math.* **100** (1958), pp. 93–147.
[13] ———, "On certain exponential sums arising in multiple Fourier series," *Annals of Math.* **73** (1961), pp. 87–109.
[14] ———, and Weiss, G., *Introduction to Fourier Analysis in Euclidean Spaces,* Princeton U. Press (1971).
[15] Tomas, P., "A restriction theorem for Fourier transforms," To appear.
[16] Watson, G., *Theory of Bessel Functions*, Cambridge U. Press (1962).
[17] Zygmund, A. "On Fourier coefficients and transforms of functions of two variables," To appear.
[18] ———, *Trigonometric Series.* Cambridge U. Press (1959).