The Kalman condition for the boundary controllability of coupled parabolic systems. Bounds on biorthogonal families to complex matrix exponentials

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Abstract

This paper is devoted to prove the controllability to trajectories of a system of n one-dimensional parabolic equations when the control is exerted on a part of the boundary by means of m controls. We give a general *Kalman condition* (necessary and sufficient) and also present a construction and sharp estimates of a biothorgonal family in $L^2(0,T;\mathbb{C})$ to $\{t^j e^{-\Lambda_k t}\}$.

Résumé

Cet article a pour but de prouver la contrôlabilité aux trajectoires d'un système de n équations paraboliques en une dimension d'espace par m contrôles exercés sur une une partie du bord. Nous obtenons une condition de *Kalman* (nécessaire et suffisante). La preuve passe par la construction dans $L^2\left(0,T;C\right)$ d'une famille biorthogonale à la suite $\{t^j e^{-\Lambda_k t}\}$ et par une estimation de la norme de ses éléments.

Keywords: Parabolic systems, Boundary Controllability, Biorthogonal families, Kalman Rank condition.

1. Statement of the problem. Main results

This work is devoted to the study of the controllability properties of the following parabolic system

$$\begin{cases} y_t = y_{xx} + Ay & \text{in } Q = (0,\pi)\times(0,T), \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases} \tag{1}$$

where $T > 0$ is given, $A \in \mathcal{L}(\mathbb{C}^n)$ and $B \in \mathcal{L}(\mathbb{C}^m;\mathbb{C}^n)$ are two given matrices and $y_0 \in H^{-1}(0,\pi;\mathbb{C}^n)$ is the initial datum. In system (1), $v \in L^2(0,T;\mathbb{C}^m)$ is a control function (to be determined) which acts on the system by means of the Dirichlet boundary condition at point $x = 0$.

The aim of this work is to give a necessary and sufficient condition for the exact controllability to trajectories of system (1). Let us remark that, for every $v \in L^2(0,T;\mathbb{C}^m)$ and $y_0 \in H^{-1}(0,\pi;\mathbb{C}^n)$, system (1) possesses a unique solution (defined by transposition; see Section 2) which satisfies

$$y \in L^2(Q;\mathbb{C}^n) \cap C^0([0,T];H^{-1}(0,\pi;\mathbb{C}^n))$$

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[1]Supported by l'Agence Nationale de la Recherche under grant ANR-07-JCJC-0139-01

[2]Supported by grant MTM2006-07932 of the D.G.E.S. (Spain).

[3]Supported by CONACyT (Mexico).

and depends continuously on the data v and y_0, i.e., there exists a constant $C = C(T) > 0$ such that

$$\|y\|_{L^2(Q;\mathbb{C}^n)} + \|y\|_{C^0([0,T];H^{-1}(0,\pi;\mathbb{C}^n))} \leq C \left(\|y_0\|_{H^{-1}(0,\pi;\mathbb{C}^n)} + \|v\|_{L^2(0,T;\mathbb{C}^m)} \right).$$

It will be said that system (1) is approximately controllable in $H^{-1}(0,\pi;\mathbb{C}^n)$ at time T if for every $y_0, y_d \in H^{-1}(0,\pi;\mathbb{C}^n)$ and for every $\varepsilon > 0$, there exists a control $v \in L^2(0,T;\mathbb{C}^m)$ such that the solution y to (1) satisfies

$$\|y(\cdot,T) - y_d\|_{H^{-1}(0,\pi;\mathbb{C}^n)} \leq \varepsilon.$$

Also, it will be said that system (1) is exactly controllable to trajectories at time T if, for every initial datum $y_0 \in H^{-1}(0,\pi;\mathbb{C}^n)$ and every trajectory $\widehat{y} \in L^2(Q;\mathbb{C}^n) \cap C^0([0,T];H^{-1}(0,\pi;\mathbb{C}^n))$ of system (1) (i.e., a solution to (1) associated to fixed $\widehat{v} \in L^2(0,T;\mathbb{C}^m)$ and $\widehat{y}_0 \in H^{-1}(0,\pi;\mathbb{C}^n)$), there exists a control $v \in L^2(0,T;\mathbb{C}^m)$ such that the corresponding solution y of (1) satisfies

$$y(\cdot,T) = \widehat{y}(\cdot,T) \text{ in } H^{-1}(0,\pi;\mathbb{C}^n).$$

Thanks to the linear character of system (1), this last property is equivalent to the null controllability at time T. That is, for every $y_0 \in H^{-1}(0,\pi;\mathbb{C}^n)$ there exists a control $v \in L^2(0,T;\mathbb{C}^m)$ such that the solution y to (1) satisfies

$$y(\cdot,T) = 0 \text{ in } H^{-1}(0,\pi;\mathbb{C}^n).$$

It is interesting to point out that we want to control the system (1), which has n equations, by means of the control v, which has m components. Of course, the most interesting case is the case in which the number of controls is less than the number of equations: $m < n$.

Nowadays, the controllability properties of system (1) are well known in the scalar case, i.e., in the case $n = 1$ (see for instance [12, 13, 29, 11, 25, 17, 16]). Thus, when $n = 1$ and $B \not\equiv 0$, system (1) is exactly controllable to trajectories, null controllable and approximately controllable in $H^{-1}(0,\pi;\mathbb{C}^n)$ at time T (see for instance [25, 17]). In fact, the boundary controllability results for system (1) can be easily obtained from the corresponding distributed controllability results and vice versa. As it is proved in [14, 15], the situation is quite different when $n \geq 2$. More details will be given below.

On the other hand, controllability of linear ordinary differential systems is well-known. In particular we have at our disposal the famous Kalman rank condition (see for example [23, Chapter 2, p. 35]), that is to say, if $n, m \in \mathbb{N}$ with $n, m \geq 1$ and $A \in \mathcal{L}(\mathbb{C}^n)$ and $B \in \mathcal{L}(\mathbb{C}^m;\mathbb{C}^n)$, then the linear ordinary differential system $Y' = AY + Bu$ is controllable at time $T > 0$ *if and only if*

$$\operatorname{rank}[A \,|\, B] = \operatorname{rank}[A^{n-1}B\,,\, A^{n-2}B\,,\, \cdots\,,\, B] = n, \tag{2}$$

where $[A^{n-1}B\,,\, A^{n-2}B\,,\, \cdots\,,\, B] \in \mathcal{L}(\mathbb{C}^{mn};\mathbb{C}^n)$.

In the framework of distributed controllability, an extension of this algebraic condition to a class of coupled second order parabolic equations has been obtained in [6] and [5]. Let us describe the Kalman condition and the result of controllability proved in [6]. Let R be a scalar second order elliptic selfadjoint operator. Let us also consider the matrices $D = \operatorname{diag}(d_1, d_2, ..., d_n) \in \mathcal{L}(\mathbb{R}^n)$ (where $d_i > 0$ for every $i : 1 \leq i \leq n$), $A \in \mathcal{L}(\mathbb{R}^n)$ and $B \in \mathcal{L}(\mathbb{R}^m;\mathbb{R}^n)$. Let Ω be a bounded and regular domain of $\mathbb{R}^d$ and $\omega \subset\subset \Omega$ a nonempty open subset. Denoting by L the operator given by $L := DR + A$, the authors define the Kalman operator associated with (L, B) by the matrix operator

$$\left\{ \begin{array}{l} \mathcal{K} := [L \,|\, B] : D(\mathcal{K}) \subset L^2(\Omega)^{nm} \to L^2(\Omega)^n, \text{ with} \\ D(\mathcal{K}) := \{u \in L^2(\Omega)^{nm} : \mathcal{K}u \in L^2(\Omega)^n\}, \end{array} \right.$$

where

$$[L \,|\, B] = \left[L^{n-1}B\,,\, L^{n-2}B\,,\, ...\,,\, LB\,,\, B\right].$$

They prove that the following system

$$\left\{ \begin{array}{l} \partial_t y = (DR + A)y + Bv1_\omega \text{ in } \Omega_T = \Omega \times (0,T), \\ y = 0 \text{ on } \Sigma_T = \partial\Omega \times (0,T), \quad y(\cdot,0) = y_0(\cdot) \text{ in } \Omega, \end{array} \right. \tag{3}$$

is null controllable if and only if

$$\operatorname{Ker}(\mathcal{K}^*) = \{0\}. \tag{4}$$

Let us point out that when $D = I_d$, this last condition is equivalent to the algebraic Kalman condition (2). In fact, in [5] the authors study this case and consider an operator L and coupling matrices A and B which depend on t.

In [14, 15], the authors study the controllability properties of system (1) when $n = 2$ and $m = 1$. They prove that, unlike of system (3) with $D = I_d$, the algebraic Kalman rank condition (2) is not a sufficient condition for the null controllability of system (1). They also exhibit an additional condition which is equivalent to the exact controllability to trajectories of system (1) ($n = 2$ and $m = 1$): Denote by μ_1 and μ_2 the eigenvalues of A^*. Then, (1) is exactly controllable to trajectories at any time T if and only if $\operatorname{rank}[A \,|\, B] = 2$ and

$$\mu_1 - \mu_2 \neq k^2 - l^2, \quad \forall k, l \in \mathbb{N} \text{ with } k \neq l.$$

This work is an extension of both [6] and [15]. For $n, m \in \mathbb{N}^*$, we give a suitable extension of the finite-dimensional Kalman rank condition. We show that the exact controllability to trajectories for system (1) is equivalent to this *Kalman condition* (see Theorem 1.1).

In the last ten years, the study of the controllability properties of coupled parabolic systems has had an increasing interest (see for instance [31], [3], [9], [7], [4], [19], [21], [5], [6], [20], [24] and [10]). In these papers, almost all the results have been established for 2×2 systems where the distributed control is exerted on one equation ($n = 2$ and $m = 1$). The most general results in this context seem to be those in [20], [5] and [6]. In [20], the authors study a *cascade* parabolic system of n equations ($n \geq 2$) controlled with one single distributed control.

To our knowledge, the unique works that study the boundary controllability problem for general coupled parabolic systems are [14] and [15]. It is also worth mentioning the paper [1] where a boundary controllability result for a particular hyperbolic coupled system is proved.

In this work we will use the following notation: Given $z \in \mathcal{L}(\mathbb{C}^N; \mathbb{C}^M)$, $N, M \geq 1$, $z^* \in \mathcal{L}(\mathbb{C}^M; \mathbb{C}^N)$ stands for the conjugate transpose of z. If $N = M = 1$, i.e., if $z = a + bi \in \mathbb{C}$, then $z^* = \overline{z} = a - bi$ is the complex conjugate of z.

Let us now precise our controllability result.

It is well known that the operator $-\partial_{xx}$ on $(0, \pi)$ with homogenous Dirichlet boundary conditions admits a sequence of eigenvalues and normalized eigenfunctions given by

$$\lambda_k = k^2, \quad \phi_k(x) = \sqrt{\frac{2}{\pi}} \sin kx, \quad k \geq 1, \quad x \in (0, \pi), \tag{5}$$

which is a Hilbert basis of $L^2(0, \pi)$. Thus, if $y \in L^2(0, \pi; \mathbb{C}^n)$ there exists a unique sequence $\{y_k\}_{k \geq 1} \subset \mathbb{C}^n$ such that

$$y = \sum_{k \geq 1} y_k \phi_k.$$

Let $L : D(L) \subset L^2(0, \pi; \mathbb{C}^n) \to L^2(0, \pi; \mathbb{C}^n)$ the unbounded linear operator defined by

$$L = I_d\, \partial_{xx} + A, \quad D(L) = H^2(0, \pi; \mathbb{C}^n) \cap H_0^1(0, \pi; \mathbb{C}^n).$$

Its adjoint operator is given by

$$L^* = I_d\, \partial_{xx} + A^*, \quad D(L^*) = H^2(0, \pi; \mathbb{C}^n) \cap H_0^1(0, \pi; \mathbb{C}^n).$$

Then for any $y = \sum_{k \geq 1} y_k \phi_k \in D(L) = D(L^*)$, we have

$$Ly = \sum_{k \geq 1} \left[(-\lambda_k I_d + A)\, y_k \right] \phi_k, \quad L^* y = \sum_{k \geq 1} \left[(-\lambda_p I_d + A^*)\, y_k \right] \phi_k.$$

In what follows, we set:

$$L_k = -\lambda_k I_d + A \in \mathcal{L}(\mathbb{C}^n) \text{ and } L_k^* = -\lambda_k I_d + A^* \in \mathcal{L}(\mathbb{C}^n), \quad \forall k \geq 1. \tag{6}$$

For $k \geq 1$, let us introduce the matrices

$$B_k = \begin{pmatrix} B \\ \vdots \\ B \end{pmatrix} \in \mathcal{L}(\mathbb{C}^m; \mathbb{C}^{nk}), \quad \mathcal{L}_k = \begin{pmatrix} L_1 & 0 & \cdots & 0 \\ 0 & L_2 & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & \cdots & 0 & L_k \end{pmatrix} \in \mathcal{L}(\mathbb{C}^{nk}), \tag{7}$$

and let us write the Kalman matrix associated with the pair $(\mathcal{L}_k, B_k)$:

$$\mathcal{K}_k = [\mathcal{L}_k \mid B_k] = [B_k \,,\, \mathcal{L}_k B_k \,,\, \mathcal{L}_k^2 B_k \,,\, \cdots \,,\, \mathcal{L}_k^{nk-2} B_k \,,\, \mathcal{L}_k^{nk-1} B_k] \in \mathcal{L}(\mathbb{C}^{mnk}; \mathbb{C}^{nk}). \tag{8}$$

The main result of this work is the following characterization of the exact controllability to trajectories at time T of system (1):

Theorem 1.1. *Let us fix $A \in \mathcal{L}(\mathbb{C}^n)$ and $B \in \mathcal{L}(\mathbb{C}^m; \mathbb{C}^n)$. Then, system (1) is exactly controllable to trajectories at any time T if and only if the pair $(\mathcal{L}_k, B_k)$ is controllable for all $k \geq 1$, i.e., if and only if*

$$\operatorname{rank} \mathcal{K}_k = nk, \quad \forall k \geq 1. \tag{9}$$

Remark 1.1.

1. *Actually, condition (9) only has to be checked for a frequency. In Corollary 3.3, we will show that there exists a positive integer k_0, only depending on A, such that* $\operatorname{rank} \mathcal{K}_{k_0} = nk_0$ *if and only if* $\operatorname{rank} \mathcal{K}_k = nk$ *for every $k \geq 1$.*
2. *Note that the algebraic Kalman condition, rank $[A \mid B] = n$, corresponds to $k = 1$ and then, it is a necessary condition for the exact controllability to trajectories of system (1).*
3. *We will see that when $B \in \mathbb{C}^n$, i.e., when $m = 1$ (one control force), condition (9) is equivalent to the algebraic Kalman condition,* $\operatorname{rank}[A \mid B] = n$ *and*
$$\mu_i - \mu_j \neq \lambda_k - \lambda_l, \quad \forall (k, i), (l, j) \in \mathbb{N} \times \{1, 2, ..., p\} \text{ with } (k, i) \neq (l, j),$$
where $\{\mu_l\}_{1 \leq l \leq p} \subset \mathbb{C}$ is the set of distinct eigenvalues of A^*. In this sense, Theorem 1.1 generalizes the results obtained in [14] and [15].
4. *We will also see that if* $\operatorname{rank} B = n$ *(and therefore $m \geq n$), then the pair (A, B) fulfills condition (9) and system (1) is exactly controllable to trajectories at time T. This boundary controllability result has been obtained in [15] in the N-dimensional case.*
5. *From Theorem 1.1 we can conclude that, unlike the scalar case, $n = 1$, the distributed controllability property of parabolic systems in not equivalent to the boundary control property: the Kalman rank condition is a necessary condition for the controllability of both systems but is not a sufficient condition for the boundary controllability of system (1). This shows that there is an important difference between the controllability properties for scalar parabolic problems and coupled parabolic systems.* ■

The sufficient part of Theorem 1.1 is proved through a moment problem. This method has been successfully used to prove the boundary controllability problem for the scalar heat equation (see [12]). Let us briefly remember this method in the case of the scalar heat equation.

Let us fix $y_0 \in H^{-1}(0, \pi)$. Then, there exists $v \in L^2(0, T)$ such that the solution to

$$\begin{cases} y_t - y_{xx} = 0 & \text{in } Q, \\ y(0, \cdot) = v, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0 & \text{in } (0, \pi), \end{cases}$$

satisfies $y(\cdot, T) = 0$ in $(0, \pi)$ if and only if $v \in L^2(0, T)$ satisfies

$$-\langle y_0, e^{-\lambda_k T} \phi_k \rangle_{H^{-1}(0,\pi), H_0^1(0,\pi)} = \int_0^T v(t) e^{-\lambda_k (T-t)} \partial_x \phi_k(0) \, dt, \quad \forall k \geq 1.$$

In the previous equality λ_k and ϕ_k are given in (5).

Using the Fourier decomposition of y_0, $y_0 = \sum_{k\geq 1} y_{0,k}\phi_k$, it will be sufficient to find a control $v \in L^2(0,T)$ such that

$$k\sqrt{\frac{2}{\pi}}\int_0^T e^{-\lambda_k(T-t)}v(t)\,dt = -e^{-\lambda_k T}y_{0,k}\,,\quad \forall k\geq 1.$$

This problem is called a *moment problem.*

Let us recall that a family $\{p_k\}_{k\geq 1}\subset L^2(0,T)$ is biorthogonal to $\{e^{-\lambda_k t}\}_{k\geq 1}$ if it satisfies

$$\int_0^T e^{-\lambda_k t}p_l(t) = \delta_{kl},\quad \forall (k,l): k,l\geq 1.$$

In [12] and [13], the authors solve the previous moment problem by proving the existence of a biorthogonal family $\{p_k\}_{k\geq 1}$ to $\{e^{-\lambda_k t}\}_{k\geq 1}$ which satisfies the additional property: for every $\epsilon > 0$, there exists a constant $C(\epsilon,T)>0$ such that

$$\|p_k\|_{L^2(0,T)} \leq C(\epsilon,T)e^{\epsilon\lambda_k}.$$

In fact, the control v is obtained as a linear combination of $\{p_k\}_{k\geq 1}$ and the previous bounds are used to prove that this combination converges in $L^2(0,T)$.

In this paper we will follow the previous technique for proving the sufficient part of Theorem 1.1. In the case considered here, we have two difficulties: first, the control v may act only on some of the n equations of the system (in general $m<n$); second, the spectrum of the operator $L^* = \partial_{xx}I_d + A^*$ may be complex with eigenvalues of multiplicity greater than 1. This leads to a moment problem associated to families as $\{t^j e^{-\Lambda_k t}\}_{k\geq 1, 0\leq j\leq \eta-1}$ with $\eta\geq 1$ a positive integer.

In [15], the authors considered the case $\eta = 2$ and proved the existence of a suitable family $\{q_k,\ \widetilde{q}_k : k\geq 1\}$ biorthogonal to $\{e^{-\Lambda_k t},\ te^{-\Lambda_k t} : k\geq 1\}$. We extend this result to any $\eta\geq 1$ and to a large class of sequences $\{\Lambda_k\}_{k\geq 1}\subset\mathbb{C}$. To our knowledge, this construction produces a new and interesting result by itself and it is our second main result.

Let us fix $\eta > 1$, a positive integer, and let us consider a sequence of complex numbers $\Lambda = \{\Lambda_k\}_{k\geq 1}\subset \mathbb{C}_+ = \{\lambda\in\mathbb{C} : \Re\lambda > 0\}$. Throughout this work we will use the notation:

$$e_{k,j}(t) = t^j e^{-\Lambda_k t},\quad \forall t>0,$$

with $k\geq 1$ and $j: 0\leq j\leq \eta-1$.

Given $T\in(0,\infty]$, we define

$$A(\Lambda,\eta,T) = \overline{\text{span}\ \{e_{k,j} : k\geq 1,\ 0\leq j\leq \eta-1\}}^{L^2(0,T;\mathbb{C})}.$$

Let us recall that the family $\{\varphi_{k,j}\}_{k\geq 1, 0\leq j\leq \eta-1}\subset A(\Lambda,\eta,T)$ is biorthogonal to $\{e_{k,j}\}_{k\geq 1,0\leq j\leq \eta-1}$ if the equalities

$$(e_{k,j},\varphi_{l,i})_{L^2(0,T;\mathbb{C})} := \int_0^T t^j e^{-\Lambda_k t}\varphi^*_{l,i}(t)\,dt = \delta_{kl}\delta_{ij},\quad \forall (k,j),(l,i): k,l\geq 1,\ 0\leq i,j\leq \eta-1, \tag{10}$$

holds.

Our second main result is the following one:

Theorem 1.2. *Let us fix $\eta\geq 1$, a positive integer, and $T\in(0,\infty]$. Assume that $\{\Lambda_k\}_{k\geq 1}$ is a sequence of complex numbers such that,*

$$\begin{cases} \Re\Lambda_k \geq \delta|\Lambda_k|, \quad |\Lambda_k - \Lambda_l| \geq \rho|k-l|, \quad \forall k,l\geq 1, \\ \displaystyle\sum_{k\geq 1}\frac{1}{|\Lambda_k|} < \infty, \end{cases} \tag{11}$$

for two positive constants δ and ρ. Then, there exists a family $\{\varphi_{k,j}\}_{k\geq 1, 0\leq j\leq \eta-1} \subset A(\Lambda,\eta,T)$ biorthogonal to $\{e_{k,j}\}_{k\geq 1, 0\leq j\leq \eta-1}$ such that, for every $\varepsilon>0$, there exists $C(\varepsilon,T)>0$ for which

$$\|\varphi_{k,j}\|_{L^2(0,T;\mathbb{C})} \leq C(\varepsilon,T)e^{\varepsilon\Re\Lambda_k}, \quad \forall (k,j): k\geq 1,\ 0\leq j\leq \eta-1. \tag{12}$$

■

The plan of the paper is the following: In Section 2, we will address some preliminary results. In Section 3 we will study the Kalman condition (9) in some interesting cases. Section 4 concerns one of the main results, the construction and estimates of a biorthogonal family (proof of Theorem 1.2). Section 5 is devoted to the proof of Theorem 1.1. Finally, in Section 6 we give some comments and open problems.

2. Preliminary results

We devote this section to recalling some known results that will be used below.

We begin by recalling some results for system (1). First, we introduce the concept of *solution by transposition* to system (1). To this end, let us consider the linear backward in time problem

$$\begin{cases} -\varphi_t - \varphi_{xx} = A^*\varphi + g & \text{in } \ Q, \\ \varphi(0,\cdot)=0, \quad \varphi(\pi,\cdot)=0 & \text{in } (0,T), \\ \varphi(\cdot\,,T)=0 & \text{in } \ (0,1), \end{cases} \tag{13}$$

where $g\in L^2(Q;\mathbb{C}^n)$ is given. It is well known that, for every $g\in L^2(Q;\mathbb{C}^n)$, this problem has a unique strong solution

$$\varphi\in L^2(0,T;H^2(0,\pi;\mathbb{C}^n))\cap C^0([0,T];H_0^1(0,\pi;\mathbb{C}^n)),$$

which depends continuously on g, i.e., there exists a constant $C=C(T)>0$ such that

$$\|\varphi\|_{L^2(0,T;H^2(0,\pi;\mathbb{C}^n))} + \|\varphi\|_{C^0([0,T];H_0^1(0,\pi;\mathbb{C}^n))} \leq C\|g\|_{L^2(Q;\mathbb{C}^n)}.$$

Thanks to the previous properties, we can introduce the following definition:

Definition 2.1. *Let $y_0\in H^{-1}(0,\pi;\mathbb{C}^n)$ and $v\in L^2(0,T;\mathbb{C}^m)$ be given. It will be said that $y\in L^2(Q;\mathbb{C}^n)$ is a solution by transposition to (1) if, for each $g\in L^2(Q;\mathbb{C}^n)$, one has*

$$\iint_Q (y\,,\,g)_{\mathbb{C}^n}\,dx\,dt = \langle y_0,\varphi(\cdot,0)\rangle + \int_0^T (v(t)\,,\,B^*\varphi_x(0,t))_{\mathbb{C}^m}\,dt,$$

where φ is the solution to (13) associated to g and $\langle\cdot\,,\cdot\rangle$, $(\cdot,\cdot)_{\mathbb{C}^n}$ and $(\cdot,\cdot)_{\mathbb{C}^m}$ stands for, resp., the usual duality pairing between $H^{-1}(0,\pi;\mathbb{C}^n)$ and $H_0^1(0,\pi;\mathbb{C}^n)$ and the scalar products in $\mathbb{C}^n$ and $\mathbb{C}^m$. ■

We can now state the existence and uniqueness of solution to system (1). One has:

Proposition 2.2. *Assume that $y_0\in H^{-1}(0,\pi;\mathbb{C}^n)$ and $v\in L^2(0,T;\mathbb{C}^m)$ are given. Then problem (1) admits a unique solution by transposition y that satisfies:*

$$\begin{cases} y\in L^2(Q;\mathbb{C}^n)\cap C^0([0,T];H^{-1}(0,\pi;\mathbb{C}^n)), \quad y_t\in L^2(0,T;D(-\Delta;\mathbb{C}^n)'), \\ y_t - y_{xx} = Ay \ \text{ in } \ L^2(0,T;D(-\Delta;\mathbb{C}^n)'), \\ y(\cdot\,,0)=y_0 \ \text{ in } \ H^{-1}(0,\pi;\mathbb{C}^n) \ \text{ and} \\ \|y\|_{L^2(Q;\mathbb{C}^n)} + \|y_t\|_{L^2(0,T;D(-\Delta;\mathbb{C}^n)')} \leq C\left(\|y_0\|_{H^{-1}(0,\pi;\mathbb{C}^n)} + \|v\|_{L^2(0,T;\mathbb{C}^m)}\right), \end{cases}$$

for a positive constant $C=C(T)$. ■

This result can be proved using standard arguments. Anyway, for a detailed proof of the result, see for instance [15].

As it is well-known, the controllability properties of system (1) are equivalent to appropriate properties of the following adjoint system

$$\begin{cases} -\varphi_t = \varphi_{xx} + A^*\varphi & \text{in } Q, \\ \varphi(0,\cdot) = 0, \quad \varphi(\pi,\cdot) = 0 & \text{on } (0,T), \\ \varphi(\cdot,T) = \varphi_0 & \text{in } (0,\pi), \end{cases} \tag{14}$$

where $\varphi_0 \in H_0^1(0,\pi;\mathbb{C}^n)$. Let us observe that, for every $\varphi_0 \in H_0^1(0,\pi;\mathbb{C}^n)$, system (14) admits a unique solution $\varphi \in L^2(0,T;H^2(0,\pi;\mathbb{C}^n)) \cap C^0([0,T];H_0^1(0,\pi;\mathbb{C}^n))$ and, for a positive constant $C = C(T)$, one has

$$\|\varphi\|_{L^2(0,T;H^2(0,\pi;\mathbb{C}^n))} + \|\varphi\|_{C^0([0,T];H_0^1(0,\pi;\mathbb{C}^n))} \le C\|\varphi_0\|_{H_0^1(0,\pi;\mathbb{C}^n)}.$$

The solutions y to problem (1) and φ to the adjoint system (14) are related by means of the following result:

Proposition 2.3. *Let $y_0 \in H^{-1}(0,\pi;\mathbb{C}^n)$, $\varphi_0 \in H_0^1(0,\pi;\mathbb{C}^n)$ and $v \in L^2(0,T;\mathbb{C}^m)$ be given. Let y and φ be, resp., the solution to (1) associated to y_0 and v and the solution to the adjoint system (14) associated to φ_0. Then:*

$$\langle y(\cdot,t),\varphi(\cdot,t)\rangle - \langle y_0,\varphi(\cdot,0)\rangle = \int_0^t (v(s)\,,\,B^*\varphi_x(0,s))_{\mathbb{C}^m}\,ds, \quad \forall t \in [0,T]. \tag{15}$$

This proposition is an easy consequence of Proposition 2.2 and the details are left to the reader.

As said above, the controllability properties of (1) can be characterized in terms of appropriate properties of the solutions to (14). More precisely, we have:

Proposition 2.4. *The following properties are equivalent:*

1. *There exists a positive constant C such that, for any $y_0 \in H^{-1}(0,\pi;\mathbb{C}^n)$, there exists a control $v \in L^2(0,T;\mathbb{C}^m)$ such that*

$$\|v\|^2_{L^2(0,T;\mathbb{C}^m)} \le C\|y_0\|^2_{H^{-1}(0,\pi;\mathbb{C}^n)}$$

   and the associated solution to (1) satisfies $y(\cdot,T) = 0$ in $H^{-1}(0,\pi;\mathbb{C}^n)$.
2. *There exists a positive constant C such that, for any trajectory $\widehat{y} \in C^0([0,T];H^{-1}(0,\pi;\mathbb{C}^n))$ of (1) and any $y_0 \in H^{-1}(0,\pi;\mathbb{C}^n)$, there exists a control $v \in L^2(0,T;\mathbb{C}^m)$ such that*

$$\|v - \widehat{v}\|^2_{L^2(0,T;\mathbb{C}^m)} \le C\|y_0 - \widehat{y}(\cdot\,,0)\|^2_{H^{-1}(0,\pi;\mathbb{C}^n)}$$

   and the associated solution y to (1) satisfies $y(\cdot\,,T) = \widehat{y}(\cdot\,,T)$ in $H^{-1}(0,\pi;\mathbb{C}^n)$.
3. *There exists a positive constant C such that the observability inequality*

$$\|\varphi(\cdot,0)\|^2_{H_0^1(0,\pi;\mathbb{C}^n)} \le C\int_0^T |B^*\varphi_x(0,t)|^2\,dt \tag{16}$$

   holds for every $\varphi_0 \in H_0^1(0,\pi;\mathbb{C}^n)$. In (16), φ is the solution to the adjoint system (14) associated to φ_0.

Again, this result is well known and is a consequence of formula (15). For its proof, see for instance [15].

Remark 2.1. *It is also well known that the approximate controllability of (1) can be characterized in terms of a property of the solutions to (14). More precisely, (1) is approximately controllable if and only if the following unique continuation property holds:*

"Let $\varphi_0 \in H_0^1(0,\pi;\mathbb{C}^n)$ be given and let φ be the associated adjoint state. Then, if $B^*\varphi_x(0,t) = 0$ on $(0,T)$, one has $\varphi \equiv 0$ on Q." ■

3. The Kalman condition

We will devote this section to showing some properties related to the Kalman condition (9). To be precise, we will give equivalent conditions to (9) in two important cases, $m = 1$ and $m \geq n$, and will clarify Remark 1.1.

Throughout this work we will use the following notation:

Notation: Let us denote by $\{\mu_l\}_{1\leq l\leq p} \subset \mathbb{C}$ the set of distinct eigenvalues of A^*. For $l : 1 \leq l \leq p$, we denote by n_l the geometric multiplicity of μ_l and assume that we have

$$n_1 \geq n_l, \quad 2 \leq l \leq p.$$

The sequence $\{v_{l,j}\}_{1\leq j\leq n_l}$ will denote a basis of eigenvectors of A^* associated to μ_l, i.e., a basis of the eigenspace associated to μ_l. To each eigenvector $v_{l,j}$ we associate its Jordan chain (of dimension $\tau_{l,j}$) and the corresponding set of generalized eigenvectors $\{v_{l,j}^i\}_{1\leq i\leq \tau_{l,j}}$ defined by:

$$\begin{cases} A^* v_{l,j}^i = \mu_l v_l^i + v_l^{i+1}, & 1 \leq i < \tau_{l,j}\,, \\ A^* v_{l,j}^{\tau_{l,j}} = \mu_l v_{l,j}^{\tau_{l,j}} \end{cases}$$

(so that $v_{l,j}^{\tau_{l,j}} = v_{l,j}$).

We will first present an equivalent condition to the Kalman rank condition that will be used later. To this end, let us consider $\mathcal{A} \in \mathcal{L}(\mathbb{C}^N)$ and $\mathcal{B} \in \mathcal{L}(\mathbb{C}^M; \mathbb{C}^N)$ two matrices, (N and M are positive integers). Let $\{\theta_l\}_{1\leq l\leq \hat{p}} \subset \mathbb{C}$ be the set of distinct eigenvalues of $\mathcal{A}^*$. For $l : 1 \leq l \leq \hat{p}$, we denote by m_l the geometric multiplicity of θ_l. The sequence $\{w_{l,j}\}_{1\leq j\leq m_l}$ will denote a basis of eigenvectors of A^* associated to θ_l, i.e., a basis of the eigenspace associated to θ_l. With this notation, one has:

Proposition 3.1. *Under the previous notations for the pair $(\mathcal{A}, \mathcal{B})$, the following conditions are equivalent:*

1. $\operatorname{rank}[\mathcal{A} \,|\, \mathcal{B}] = \operatorname{rank}[\mathcal{B}\,, \mathcal{A}\mathcal{B}\,, \mathcal{A}^2\mathcal{B}\,, \cdots, \mathcal{A}^{N-1}\mathcal{B}] = N$.
2. *The set $\{\mathcal{B}^* w_{l,1}, \mathcal{B}^* w_{l,2}, \cdots, \mathcal{B}^* w_{l,m_l}\} \subset \mathbb{C}^M$ is linearly independent (i.e.*

$$\operatorname{rank}[\mathcal{B}^* w_{l,1}\,, \mathcal{B}^* w_{l,2}\,, \cdots, \mathcal{B}^* w_{l,m_l}] = m_l)$$

for every l, with $1 \leq l \leq \hat{p}$.

Proof: We will deduce the proof from the Hautus test which is an equivalent condition to the Kalman rank condition. Indeed, it is well known (for instance, see [32], page 15) that $\operatorname{rank}[\mathcal{A} \,|\, \mathcal{B}] = N$ if and only if

$$\operatorname{rank} \begin{pmatrix} \mathcal{A}^* - \theta_l I_d \\ \mathcal{B}^* \end{pmatrix} = N, \quad \forall l : 1 \leq l \leq \hat{p}. \tag{17}$$

Let us assume that the Kalman rank condition holds and let us proof that the set

$$\{B^* w_{l,j}\}_{1\leq j\leq m_l} \subset \mathbb{C}^M$$

is linearly independent for every $l : 1 \leq l \leq \hat{p}$. To this end, let us suppose that for $\{\alpha_j\}_{1\leq j\leq m_l} \subset \mathbb{C}$ one has

$$\sum_{j=1}^{m_l} \alpha_j B^* w_{l,j} = 0.$$

In particular, $w = \sum_{j=1}^{m_l} \alpha_j w_{l,j} \in \mathbb{C}^n$ is an eigenvector of $\mathcal{A}^*$ associated to θ_l and is a solution to the linear system

$$\begin{pmatrix} \mathcal{A}^* - \theta_l I_d \\ B^* \end{pmatrix} w = 0.$$

Using (17), we conclude $w = \sum_{j=1}^{m_l} \alpha_j w_{l,j} \equiv 0$, i.e., $\alpha_j \equiv 0$, for every $j : 1 \le j \le m_l$.

Let us now assume that the set $\{\mathcal{B}^* w_{l,j}\}_{1\le j\le m_l} \subset \mathbb{C}^M$ is linearly independent for every $l : 1 \le l \le \hat{p}$ and let us proof that $(\mathcal{A}, \mathcal{B})$ fulfills condition (17). Thus, we consider $w \in \mathbb{C}^N$ a solution to the previous linear system. In particular, $\mathcal{B}^* w = 0$ and w is an eigenvector of $\mathcal{A}^*$ associated to θ_l. As a consequence, w can be written as $w = \sum_{j=1}^{m_l} \alpha_j w_{l,j}$, with $\alpha_j \in \mathbb{C}$. Evidently, the equality $\mathcal{B}^* w = 0$ implies $\alpha_j = 0$ for every j whence $w \equiv 0$. This finalizes the proof. ■

Our next task will be to clarify the first point in Remark 1.1. Before let us prove the following result:

Proposition 3.2. *Let $A \in \mathcal{L}(\mathbb{C}^n)$ be given and let us denote by $\{\mu_l\}_{1\le l\le p} \subset \mathbb{C}$ the set of distinct eigenvalues of A^*. Then, there exists an integer $k_0 = k_0(A) \in \mathbb{N}$, only depending on A, such that,*

$$\mu_i - \mu_j \neq \lambda_k - \lambda_l, \tag{18}$$

for every $k > k_0$, $l \ge 1$, $k \neq l$, and $i, j : 1 \le i, j \le p$.

Proof: First, let us observe that λ_k is given by (5) and then, for $\rho_0 = 1$ and $K_0 = 1$, one has

$$|\lambda_k - \lambda_l| \ge \rho_0 |k^2 - l^2|, \quad \forall k, l \ge K_0.$$

Let us consider

$$k_0 = \max\left\{K_0, \left[\frac{1}{\rho_0} \max_{1\le i,j\le p} |\mu_i - \mu_j|\right] + 1\right\},$$

and let us take $k > k_0$, $l \ge 1$, with $k \neq l$, and i, j, with $1 \le i, j \le p$. Then, if $\mu_i - \mu_j \notin \mathbb{R}$, we can conclude the result. If $\mu_i - \mu_j \in \mathbb{R}$ and, for instance, $k > l$,

$$\mu_i - \mu_j \le |\mu_i - \mu_j| \le \rho_0 k_0 < \rho_0 (k + l) \le \rho_0 (k + l)(k - l) = \rho_0 (k^2 - l^2) \le \lambda_k - \lambda_l.$$

Finally, if $\mu_i - \mu_j \in \mathbb{R}$ and $k < l$, one has

$$\mu_j - \mu_i \le |\mu_i - \mu_j| \le \rho_0 k_0 < \rho_0 (k + l) \le \rho_0 (k + l)(l - k) \le \lambda_l - \lambda_k.$$

We have thus the proof. ■

By means of the previous result we can establish an equivalent condition to (9). Thus, one has:

Corollary 3.3. *Let $A \in \mathcal{L}(\mathbb{C}^n)$ and $B \in \mathcal{L}(\mathbb{C}^m; \mathbb{C}^n)$ be given and let us consider $k_0 \ge 1$ provided by Proposition 3.2. Then, the three following conditions are equivalent:*

1. $\operatorname{rank} \mathcal{K}_k = nk$ *for every* $k \ge 1$.
2. $\operatorname{rank} \mathcal{K}_k = nk$ *for every* $k : 1 \le k \le k_0$.
3. $\operatorname{rank} \mathcal{K}_{k_0} = nk_0$.

Proof: Of course, condition 1 implies condition 2 and this one implies condition 3.

Let us now prove that condition 3 implies condition 1 and, to this end, let us take k_0 such that (18) holds. In order to prove the result, let us denote by $\sigma(\mathcal{L}_k^*)$ the set of eigenvalues of the matrix $\mathcal{L}_k^* \in \mathcal{L}(\mathbb{C}^{nk})$. From the definition of $\mathcal{L}_k$ (see (6) and (7)) we get

$$\sigma(\mathcal{L}_k) = \{-\lambda_l + \mu_i : 1 \le l \le k, \ 1 \le i \le p\},$$

(remember that $\{\mu_i\}_{1\le i\le p} \subset \mathbb{C}$ is the set of distinct eigenvalues of A^*).

Let us start with the case $k < k_0$. Actually, we can prove that if $\operatorname{rank} \mathcal{K}_k = nk$, then $\operatorname{rank} \mathcal{K}_{k-1} = n(k-1)$. Indeed, by contradiction, if $\operatorname{rank} \mathcal{K}_{k-1} < n(k-1)$, using Proposition 3.1, there exists an eigenvector $V \in \mathbb{C}^{n(k-1)}$ of the matrix $\mathcal{L}_{k-1}^*$ associated to $\theta \in \mathbb{C}$ such that

$B^*_{k-1}V = 0$. It is easy to check that θ is also an eigenvalue of $\mathcal{L}^*_k$ which has as associated eigenvector

$$\widetilde{V} = \begin{pmatrix} V \\ 0 \end{pmatrix} \in \mathbb{C}^{nk}$$

and $B^*_k \widetilde{V} = 0$. This contradicts the previous assumption. In particular $\operatorname{rank}[A \,|\, B] = \operatorname{rank} \mathcal{K}_1 = n$.

Let us now prove the case $k = k_0 + 1$. By contradiction, let us suppose that $\operatorname{rank} \mathcal{K}_{k_0+1} < n(k_0+1)$. It is clear that

$$\sigma(\mathcal{L}^*_{k_0+1}) = \sigma(\mathcal{L}^*_{k_0}) \cup \{-\lambda_{k_0+1} + \mu_i : 1 \le i \le p\}.$$

Using the Hautus criterium (17) for the couple $(\mathcal{L}^*_{k_0+1}, B_{k_0+1})$ and taking into account that $\operatorname{rank} \mathcal{K}_{k_0} = nk_0$ we deduce that there exists an eigenvector $V \in \mathbb{C}^{n(k_0+1)}$ of $\mathcal{L}^*_{k_0+1}$ associated to $-\lambda_{k_0+1} + \mu_i$, with $i : 1 \le i \le p$, such that

$$B^*_{k_0+1} V = 0. \tag{19}$$

From Proposition 3.2, we deduce that $\sigma(\mathcal{L}^*_{k_0}) \cap \{-\lambda_{k_0+1} + \mu_i : 1 \le i \le p\} = \emptyset$ and therefore, the vector $V = (V_j)_{1 \le j \le k_0+1} \in \mathbb{C}^{n(k_0+1)}$ is an eigenvector of $\mathcal{L}^*_{k_0+1}$ associated to $-\lambda_{k_0+1} + \mu_i$ if and only if

$$V_j = 0, \quad \forall j \ne k_0 + 1, \text{ and } V_{k_0+1} = v,$$

where $v \in \mathbb{C}^n$ is an eigenvector of A^* associated to μ_i. Thus, condition (19) implies that v belongs to the kernel of B^*. Using again the Hautus test (17), this time applied to (A, B), we infer that $\operatorname{rank}[A \,|\, B] = \operatorname{rank} \mathcal{K}_1 < n$. But this last inequality contradicts condition 2.

The general case $k \ge k_0$ can be obtained combining an induction argument and the previous reasoning. This ends the proof. ∎

In the next result we will study the Kalman condition (9) in the particular case $m = 1$. Thus, one has

Proposition 3.4. *Let us fix $A \in \mathcal{L}(\mathbb{C}^n)$ and $B \in \mathbb{C}^n$ $(m = 1)$. Then, the following conditions are equivalent:*

1. $\operatorname{rank} \mathcal{K}_k = nk$ *for every* $k \ge 1$.
2. $\operatorname{rank}[A \,|\, B] = n$ *and*

$$\mu_i - \mu_j \ne \lambda_k - \lambda_l, \quad \forall (k,i), (l,j) \in \mathbb{N} \times \{1, 2, ..., p\} \textit{ with } (k,i) \ne (l,j).$$

Proof: Let us assume that $\operatorname{rank} \mathcal{K}_k = nk$ for every $k \ge 1$. It is then clear that $\operatorname{rank}[A \,|\, B] = n$. By contradiction, assume that there exist $k, l \ge 1$, with $k > l$, and $i, j : 1 \le i, j \le p$ such that

$$-\lambda_k + \mu_i = -\lambda_l + \mu_j := \theta.$$

Then, $\theta \in \sigma(\mathcal{K}^*_k)$. Let us take $w_1 \in \mathbb{C}^n$ and $w_2 \in \mathbb{C}^n$ eigenvectors of A^* associated, resp., to μ_i and μ_j. Then, $V_1 = (V_{1,\ell})_{1 \le \ell \le k}, V_2 = (V_{2,\ell})_{1 \le \ell \le k} \in \mathbb{C}^{nk}$, with

$$\begin{cases} V_{1,k} = w_1 \text{ and } V_{1,\ell} = 0, & \forall \ell \ne k, \\ V_{2,l} = w_2 \text{ and } V_{2,\ell} = 0, & \forall \ell \ne l, \end{cases}$$

are two independent eigenvectors of $\mathcal{K}^*_k$ associated to θ. Using Proposition 3.1 we deduce that the set $\{B^*_k V_1, B^*_k V_2\} \subset \mathbb{C}$ is linearly independent $(m = 1)$. This is evidently absurd. So, we have condition 2.

On the other hand, let us assume that (A, B) satisfies condition 2. We deduce that (18) holds with $k_0 \equiv 1$. Applying directly Corollary 3.3 with $k_0 = 1$ we obtain $\operatorname{rank} \mathcal{K}_k = nk$ for every $k \ge 1$. This ends the proof. ∎

Remark 3.1. *As said in Remark 1.1 and in view of this last result, we deduce that Theorem 1.1 generalizes the controllability result for system* (1) *stated in [14] and [15] for* $n = 2$ *and* $m = 1$. ■

Let us now analyze condition (9) when $m \geq n$ (at least the same number of controls than equations) and the matrix $B \in \mathcal{L}(\mathbb{C}^m; \mathbb{C}^n)$ satisfies $\operatorname{rank} B = n$. One has:

Proposition 3.5. *Let us fix* $A \in \mathcal{L}(\mathbb{C}^n)$ *and* $B \in \mathcal{L}(\mathbb{C}^m; \mathbb{C}^n)$ *such that* $\operatorname{rank} B = n$. *Then, the pair* (A, B) *satisfies condition* (9) *for any* $k \geq 1$.

Proof: Let us fix $k \geq 1$. In order to prove (9), we will use Proposition 3.1 for the pair $(\mathcal{L}_k, B_k)$ (see (7) for the definition of these matrices). Therefore, let us consider $\theta \in \sigma(\mathcal{L}_k^*)$, an eigenvalue of $\mathcal{L}_k^*$, and the set

$$I(\theta) = \{(l, i) : 1 \leq l \leq k,\ 1 \leq i \leq p,\ \theta = -\lambda_l + \mu_i\}.$$

Aided by this set we can obtain a basis of eigenvectors of $\mathcal{L}_k^*$ associated to θ:

$$EV(\theta) = \bigcup_{(l,i) \in I(\theta)} \{V_{i,j}^l \in \mathbb{C}^{kn} : 1 \leq j \leq n_i\},$$

with $V_{i,j}^l = (V_{i,j}^{l,\ell})_{1 \leq \ell \leq k}$ and $V_{i,j}^{l,\ell} \in \mathbb{C}^n$ given by $V_{i,j}^{l,\ell} = \delta_{l,\ell} v_{i,j}$.

The set $\{B_k^* V : V \in EV(\theta)\}$ can be written as

$$\begin{aligned}\{B_k^* V : V \in EV(\theta)\} &= \{B^* v_{i,j} : i \text{ such that, for } l \geq 1,\ (l, i) \in I(\theta) \text{ and } 1 \leq j \leq n_i\} \\ &\subset \{B^* v_{i,j} : 1 \leq i \leq p,\ 1 \leq j \leq n_i\}.\end{aligned}$$

Taking into account that $\operatorname{rank} B^* = n$ and the set $\{v_{i,j} : 1 \leq i \leq p,\ 1 \leq j \leq n_i\} \subset \mathbb{C}^n$ is linearly independent, we deduce that $\{B_k^* V : V \in EV(\theta)\}$ is also linearly independent. This finalizes the proof of the result. ■

4. Biorthogonal families: construction and estimates

This section will be devoted to proving Theorem 1.2. To this end, we will follow here the strategy based on the Laplace transform which explicitly constructs the biorthogonal family from the fixed complex sequence. This strategy has been used for instance in [30] in order to construct a biorthogonal family to the set $\left\{e^{-\Lambda_k t}\right\}_{k \geq 1}$, where $\{\Lambda_k\}_{k \geq 1}$ is a real positive sequence satisfying suitable properties.

Throughout this section $\eta \geq 1$ will denote a positive integer and $\Lambda = \{\Lambda_k\}_{k \geq 1} \subset \mathbb{C}_+ = \{\lambda \in \mathbb{C} : \Re\lambda > 0\}$ is a complex sequence satisfying

$$\Lambda_k \neq \Lambda_j, \quad \forall k, j \in \mathbb{N} \text{ with } k \neq j. \tag{20}$$

We will obtain the proof of the Theorem 1.2 reasoning as follows:

1. First, we will prove the existence of a biorthogonal family $\{\varphi_{k,j}\}_{k \geq 1, 0 \leq j \leq \eta - 1} \subset A(\Lambda, \eta, \infty)$ to $\left\{t^j e^{-\Lambda_k t}\right\}_{k \geq 1, 0 \leq j \leq \eta - 1}$ (see Proposition 4.1). In view of other applications, we will consider more general Λ_k than those satisfying condition (11) in Theorem 1.2. In this proposition we will also give an estimate of the norm in $L^2(0, \infty; \mathbb{C})$ of $\varphi_{k,j}$ in terms of a Blaschke product associated to the sequence Λ (see (22)).
2. Secondly, we will use assumptions (11) in order to get an estimate of the Blaschke product $\mathcal{P}_k$ in (22) (see Proposition 4.5).
3. Finally, we will directly prove Theorem 1.2 when $T = \infty$ and will deduce the general case $T \in (0, \infty)$ using a well known argument (see Corollary 4.6).

For $T \in (0, \infty]$, let us recall that $A(\Lambda, \eta, T)$ is the space given by

$$A(\Lambda, \eta, T) = \overline{\text{span } \{t^j e^{-\Lambda_k t} : k \geq 1,\ 0 \leq j \leq \eta - 1\}}^{L^2(0,T;\mathbb{C})}$$

and is a closed subspace of $L^2(0, T; \mathbb{C})$.

Let us also recall that the function $e_{k,j}$ is given by

$$e_{k,j}(t) = t^j e^{-\Lambda_k t}, \quad t \geq 0,$$

with (k, j) such that $k \geq 1$ and $0 \leq j \leq \eta - 1$.

We will obtain the proof of Theorem 1.2 from several previous results. Let us start with the following one:

Proposition 4.1. *Assume that* $\Lambda = \{\Lambda_k\}_{k\geq 1} \subset \mathbb{C}_+$ *satisfies* (20) *and the assumption*

$$\sum_{k\geq 1} \frac{\Re\Lambda_k}{(1 + \Re\Lambda_k)^2 + (\Im\Lambda_k)^2} < \infty. \tag{21}$$

Then, there exists a biorthogonal family $\{\varphi_{k,j}\}_{k\geq 1, 0\leq j\leq \eta-1} \subset A(\Lambda, \eta, \infty)$ *to* $\{e_{k,j}\}_{k\geq 1, 0\leq j\leq \eta-1}$ *such that*

$$\|\varphi_{k,j}\|_{L^2(0,\infty;\mathbb{C})} \leq C \left[1 + \left(\frac{1}{\Re\Lambda_k}\right)^{(2\eta-j)(\eta-j-1)+1}\right] (\Re\Lambda_k)^{\eta(\eta-j)} |1 + \Lambda_k|^{2\eta(\eta-j)} \mathcal{P}_k^{\eta(\eta-j)}, \tag{22}$$

where C *is a positive constant, only depending on* η, *and* $\mathcal{P}_k$ *is given by*

$$\mathcal{P}_k := \prod_{\substack{\ell\geq 1 \\ \ell\neq k}} \left| \frac{1 + \Lambda_k/\Lambda_\ell^*}{1 - \Lambda_k/\Lambda_\ell} \right|. \qquad \blacksquare$$

To prove this result, we need some preliminary lemmata.

Lemma 4.2. *Under the assumptions of Proposition 4.1, let us consider the Blaschke product associated to* Λ, $W : \mathbb{C}_+ \to \mathbb{C}$, *defined by:*

$$\begin{cases} W(\lambda) = W(\lambda, \Lambda) = \displaystyle\prod_{k\geq 1} \delta_k \frac{1 - \lambda/\Lambda_k}{1 + \lambda/\Lambda_k^*}, & \lambda \in \mathbb{C}_+ \\ \delta_k = \dfrac{\Lambda_k}{\Lambda_k^*} \dfrac{|\Lambda_k - 1|}{|\Lambda_k + 1|} \dfrac{\Lambda_k^* + 1}{\Lambda_k^* - 1} & (\delta_k = 1 \text{ if } \Lambda_k = 1). \end{cases}$$

Then, $W \in H^\infty(\mathbb{C}_+)$, *the space of bounded and holomorphic functions defined on* $\mathbb{C}_+$, *is defined almost everywhere on* $i\mathbb{R}$ *and satisfies* $|W(\lambda)| < 1$ *for* $\Re\lambda > 0$, $|W(i\tau)| = 1$ *for almost every* $\tau \in \mathbb{R}$ *and*

$$W(\lambda_0) = 0 \iff \lambda_0 = \Lambda_k \text{ with } k \geq 1.$$

Moreover, Λ_k *is a simple root of* W, *for any* $k \geq 1$.

Proof: Let U be the unit ball of $\mathbb{C}$ and let us consider a sequence $\{\alpha_k\}_{k\geq 1} \subset U$ such that

$$\sum_{k\geq 1} (1 - |\alpha_k|) < \infty.$$

Then, it is well known (see for instance [28]) that the following function

$$C(z) = \prod_{k\geq 1} \frac{|\alpha_k|}{\alpha_k} \frac{\alpha_k - z}{1 - \alpha_k^* z}, \quad \text{with } z \in U,$$

is well defined in U and everywhere on ∂U and satisfies $C \in H^\infty(U)$ and $|C(e^{i\theta})| = 1$ for almost all $\theta \in (-\pi, \pi)$.

We will obtain the proof of the lemma from the previous properties of the function C. Indeed, it is not difficult to check that

$$h : z \in U \mapsto h(z) = \frac{1+z}{1-z} \in \mathbb{C}_+$$

is a bijective map. In addition, h is holomorphic in U and $W(\lambda) = C(h^{-1}(\lambda))$ with $\alpha_k = h^{-1}(\Lambda_k)$. Observe that

$$(1 - |\alpha_k|) = 1 - \left(1 - \frac{4\Re\Lambda_K}{(1+\Re\Lambda_K)^2 + (\Im\Lambda_k)^2}\right)^{1/2}$$

and $\sum_{k\geq 1}(1 - |\alpha_k|) < \infty$ if and only if (21) holds. Combining the previous properties we conclude that $W \in H^\infty(\mathbb{C}_+)$ if (21) is fulfilled.

Finally, it can be easily checked that $|W(\lambda)| < 1$ for $\Re\lambda > 0$, $|W(i\tau)| = 1$ for almost all $\tau \in \mathbb{R}$ and the two last properties. ■

As a direct consequence of the previous result we deduce:

Corollary 4.3. *Under the assumptions of Proposition 4.1, one has that $A(\Lambda, \eta, \infty)$ is a proper closed subspace of $L^2(0, \infty; \mathbb{C})$.*

Proof: Since conditions (20) and (21) are assumed, it follows that $W \in H^\infty(\mathbb{C}_+)$, given by Lemma 4.2, satisfies $W \not\equiv 0$. Let us set

$$\Phi(\lambda) = \left[\frac{W(\lambda)}{(1+\lambda)^2}\right]^\eta, \quad \text{for } \lambda \in \mathbb{C}_+\,. \tag{23}$$

Simple computations immediately show that $\Phi \in H^2(\mathbb{C}_+)$, the space of holomorphic functions on $\mathbb{C}_+$ such that

$$\int_{-\infty}^{+\infty} |\Phi(\sigma + i\tau)|^2 \, d\tau < \infty, \quad \forall \sigma > 0,$$

with norm

$$\|\Phi\|_{H^2(\mathbb{C}_+)} = \left(\int_{-\infty}^{+\infty} |\Phi(i\tau)|^2 \, d\tau\right)^{1/2}.$$

Using the properties of the function W, it is not difficult to check that, for a positive constant C_η (only depending on η), one has

$$\|\Phi\|_{H^2(\mathbb{C}_+)} \leq C_\eta.$$

It is well known that the Laplace transform is a homeomorphism[4] from $L^2(0, \infty; \mathbb{C})$ into $H^2(\mathbb{C}_+)$. Therefore, there exists a nontrivial function $\varphi \in L^2(0, \infty; \mathbb{C})$ such that

$$\Phi(\lambda) = \frac{1}{2\pi}\int_0^\infty e^{-\lambda t}\varphi^*(t)\, dt.$$

Observe that, thanks to Lemma 4.2, $\{\Lambda_k\}_{k\geq 1}$ are the zeros of Φ and have multiplicity η. In particular $\Phi^{(j)}(\Lambda_k) = 0$ for every $k \geq 1$ and $j : 0 \leq j \leq \eta - 1$. Thus

$$\int_0^\infty t^j e^{-\Lambda_k t}\varphi^*(t)\, dt = (e_{k,j}\,,\, \varphi)_{L^2(0,\infty;\mathbb{C})} = 0, \quad \forall (k,j) : k \geq 1,\ 0 \leq j \leq \eta - 1.$$

We have then proved that there exists $\varphi \in L^2(0, \infty; \mathbb{C})$, with $\varphi \not\equiv 0$, such that $\varphi \in A(\Lambda, \eta, \infty)^\perp$. This finalizes the proof. ■

Starting from the function Φ defined in (23), we would like to construct a family of functions $\{\Phi_{k,j}\} \subset H^2(\mathbb{C}_+)$ satisfying some additional conditions. These are given in the following result:

[4]For the space $H^2(\mathbb{C}_+)$ and the properties of the Laplace transform, see for instance [30, pp. 19–20]).

Lemma 4.4. *Assume that the sequence* $\{\Lambda_k\}_{k\geq 1}$ *satisfies* (20) *and* (21). *Then, there exists a family* $\{\Phi_{k,j}\}_{k\geq 1, 1\leq j\leq \eta} \subset H^2(\mathbb{C}_+)$ *such that*

$$\Phi_{k,j}^{(\nu)}(\Lambda_l) = (-1)^\nu \delta_{kl}\delta_{j\nu}, \quad \forall (k,j),(l,\nu) : k,l \geq 1,\ 0 \leq j,\nu \leq \eta - 1, \tag{24}$$

and

$$\|\Phi_{k,j}\|_{H^2(\mathbb{C}_+)} \leq C \frac{H_j(\Lambda_k)}{\left|\Phi^{(\eta)}(\Lambda_k)\right|^{\eta-j}}, \quad \forall (k,j) : k \geq 1,\ 0 \leq j \leq \eta - 1, \tag{25}$$

where C *is a positive constant, only depending on* η*, and* Φ *and* $H_j(\Lambda_k)$ *are respectively given by* (23) *and*

$$H_j(\Lambda_k) = 1 + \left(\frac{1}{\Re\Lambda_k}\right)^{(2\eta-j)(\eta-j-1)+1}. \qquad \blacksquare$$

Before presenting the proof of this lemma, let us complete the proof of Proposition 4.1.

Proof of Proposition 4.1: From Lemma 4.4 we deduce that $\Phi_{k,j} \in H^2(\mathbb{C}_+)$ for every $(k,j) : k \geq 1$ and $0 \leq j \leq \eta - 1$. Thus, using again the Laplace transform, for any $k \geq 1$ and $j : 0 \leq j \leq \eta - 1$, there exists a nontrivial function $\widetilde{\varphi}_{k,j} \in L^2(0,\infty;\mathbb{C})$ such that

$$\Phi_{k,j}(\lambda) = \frac{1}{2\pi}\int_0^\infty e^{-\lambda t}\widetilde{\varphi}_{k,j}^*(t)\,dt, \quad \forall \lambda \in \mathbb{C}_+\,,$$

and $\|\widetilde{\varphi}_{k,j}\|_{L^2(0,\infty;\mathbb{C})} \leq C\|\Phi_{k,j}\|_{H^2(\mathbb{C}_+)}$ for a positive constant C. We also have

$$\Phi_{k,j}^{(\nu)}(\lambda) = (-1)^\nu \frac{1}{2\pi}\int_0^\infty t^\nu e^{-\lambda t}\widetilde{\varphi}_{k,j}^*(t)\,dt, \quad \forall \lambda \in \mathbb{C}_+\,, \quad \forall \nu \geq 0.$$

Let us consider the projection operator $\Pi_\Lambda : L^2(0,\infty;\mathbb{C}) \to A(\Lambda,\eta,\infty)$. One has

$$\int_0^\infty t^j e^{-\Lambda_k t}\Pi_\Lambda\varphi^*(t)\,dt = \int_0^\infty t^j e^{-\Lambda_k t}\varphi^*(t)\,dt, \quad \forall (k,j) : k \geq 1,\ 0 \leq j \leq \eta - 1, \quad \forall \varphi \in L^2(0,\infty;\mathbb{C}).$$

Taking into account (24) and the two previous equalities, we deduce that the set $\{\varphi_{k,j}\}_{k\geq 1, 0\leq j\leq \eta}$, with $\varphi_{k,j} = \Pi_\Lambda\widetilde{\varphi}_{k,j}/2\pi$, is a biorthogonal family associated to $\{t^j e^{-\Lambda_k}\}_{k\geq 1, 0\leq j\leq \eta}$ and

$$\|\varphi_{k,j}\|_{L^2(0,\infty;\mathbb{C})} \leq C\|\Phi_{k,j}\|_{H^2(\mathbb{C}_+)}$$

for a positive constant C.

From (25) and in order to prove (22), let us calculate $|\Phi^{(\eta)}(\Lambda_k)|$. First, the function Φ can be written as $\Phi(\lambda) = [f(\lambda)]^\eta$ with f a holomorphic function on $\mathbb{C}_+$. Since Λ_k is a simple zero of f, we get $\Phi^{(\eta)}(\Lambda_k) = \eta![f'(\Lambda_k)]^\eta$, i.e.,

$$\Phi^{(\eta)}(\Lambda_k) = \eta!\left[\frac{W'(\Lambda_k)}{(1+\Lambda_k)^2}\right]^\eta,$$

where W is given in Lemma 4.2. On the other hand, a simple calculation gives

$$W'(\Lambda_k) = -\delta_k \frac{-\Lambda_k^*}{2\Lambda_k\Re\Lambda_k}\prod_{\substack{\ell\geq 1\\ \ell\neq k}}\delta_\ell \frac{1-\Lambda_k/\Lambda_\ell}{1+\Lambda_k/\Lambda_\ell^*},$$

and therefore

$$|\Phi^{(\eta)}(\Lambda_k)| = \eta!\left[\frac{1}{2|1+\Lambda_k|^2\Re\Lambda_k}\prod_{\ell\neq k}\left|\frac{1-\Lambda_k/\Lambda_\ell}{1+\Lambda_k/\Lambda_\ell^*}\right|\right]^\eta.$$

Finally, from (25) we get (22). This ends the proof. $\blacksquare$

Proof of Lemma 4.4: Let us fix $k \geq 1$ and $j : 0 \leq j \leq \eta - 1$ and let us set

$$f_{k,j}(\lambda) := \frac{\Phi(\lambda)}{(\lambda - \Lambda_k)^{\eta - j}}, \quad \forall \lambda \in \mathbb{C}_+ , \tag{26}$$

where Φ is given by (23). From the properties of the function Φ, we get

$$\begin{cases} f_{k,j}^{(\nu)}(\Lambda_l) = 0, & \forall l \geq 1 \text{ with } l \neq k, \ \forall \nu : 0 \leq \nu \leq \eta - 1, \\ f_{k,j}^{(\nu)}(\Lambda_k) = 0, & \forall \nu : 0 \leq \nu \leq j - 1, \\ f_{k,j}^{(j+\nu)}(\Lambda_k) = \dfrac{(j+\nu)!}{(\eta+\nu)!} \Phi^{(\eta+\nu)}(\Lambda_k), & \forall \nu : \nu \geq 0. \end{cases} \tag{27}$$

In particular, $f_{k,j}^{(j)}(\Lambda_k) = \frac{j!}{\eta!} \Phi^{(\eta)}(\Lambda_k) \neq 0$ (remember that Λ_k is a zero of Φ of multiplicity η; see (23)).

We will obtain the proof after several steps.

Step 1: In this first step we will prove that there exists a polynomial function $p = p_{k,j}$ of degree $\eta - j - 1$ such that the function defined by

$$\Phi_{k,j}(\lambda) := p(\lambda) f_{k,j}(\lambda), \quad \forall \lambda \in \mathbb{C}_+ , \tag{28}$$

satisfies (24).

Clearly, for any polynomial p we have

$$\begin{cases} \Phi_{k,j}^{(\nu)}(\Lambda_l) = 0, & \forall l : l \neq k, \ \forall \nu : 0 \leq \nu \leq \eta - 1, \\ \Phi_{k,j}^{(\nu)}(\Lambda_k) = 0, & \forall \nu : 0 \leq \nu < j \leq \eta - 1. \end{cases}$$

Thus, in order to get (24), we have to show that there is a polynomial p such that $\Phi_{k,j}^{(j)}(\Lambda_k) = (-1)^j$ and $\Phi_{k,j}^{(j+\nu)}(\Lambda_k) = 0$ for $1 \leq \nu \leq \eta - j - 1$. In view of (27)), these relations lead to

$$\begin{cases} p(\Lambda_k) = \dfrac{(-1)^j}{f_{k,j}^{(j)}(\Lambda_k)} = \dfrac{\eta!}{j!} \dfrac{(-1)^j}{\Phi^{(\eta)}(\Lambda_k)}, \\ \displaystyle\sum_{\ell=0}^{\nu-1} a_{\nu\ell} p^{(\ell)}(\Lambda_k) + p^{(\nu)}(\Lambda_k) = 0, \quad \forall \nu : 1 \leq \nu \leq \eta - j - 1, \end{cases} \tag{29}$$

where

$$a_{\nu\ell} = \frac{\begin{pmatrix} j+\nu \\ \ell \end{pmatrix}}{\begin{pmatrix} j+\nu \\ \nu \end{pmatrix}} \frac{f_{k,j}^{(j+\nu-\ell)}(\Lambda_k)}{f_{k,j}^{(j)}(\Lambda_k)} = \frac{\nu! \eta!}{\ell! (\eta + \nu - \ell)!} \frac{\Phi^{(\eta+\nu-\ell)}(\Lambda_k)}{\Phi^{(\eta)}(\Lambda_k)}. \tag{30}$$

for every $\nu, \ell : 0 \leq \ell < \nu \leq \eta - j - 1$.

These relations allow us to compute $p^{(\nu)}(\Lambda_k)$ for $0 \leq \nu \leq \eta - j - 1$ and thus

$$p(\lambda) = \sum_{\nu=0}^{\eta-j-1} \frac{p^{(\nu)}(\Lambda_k)}{\nu!} (\lambda - \Lambda_k)^\nu .$$

Evidently $\Phi_{k,j}(\lambda) := p(\lambda) f_{k,j}(\lambda)$ satisfies (24).

Step 2: Let us now prove some estimates of the polynomial p constructed in the previous step.

We can rewrite the identities in (29) as a linear system of the form $\mathbf{A}P = \mathbf{B}$ with

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ a_{10} & 1 & \cdots & 0 & 0 \\ a_{20} & a_{21} & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ a_{\eta-j-1,0} & a_{\eta-j-1,1} & \cdots & a_{\eta-j-1,\eta-j-2} & 1 \end{pmatrix} \in \mathcal{L}(\mathbb{C}^{\eta-j}),\ \ \mathbf{B} = \begin{pmatrix} \frac{(-1)^j \eta!}{j!\Phi^{(\eta)}(\Lambda_k)} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{C}^{\eta-j}$$

and $P = \left(p^{(\nu)}(\Lambda_k)\right)_{0\le\nu\le\eta-j-1} \in \mathbb{C}^{\eta-j}$.

Thus $P = \mathbf{A}^{-1}\mathbf{B}$ and $|P|_{\mathbb{C}^{\eta-j}} \le \left\|\mathbf{A}^{-1}\right\| |\ \mathbf{B}|_{\mathbb{C}^{\eta-j}}$, where $\|\cdot\|$ stands for the Hilbert-Schmidt norm of a matrix, i.e., if $M \in \mathcal{L}(\mathbb{C}^{\eta-j})$, then

$$\|M\| = \left(\sum_{1\le r,s\le\eta-j} |m_{rs}|^2 \right)^{1/2}.$$

Let us write $\mathbf{A} = I_d - V$. Then (see [18, Theorem 1.4.1, p. 6]):

$$\|\ \mathbf{A}^{-1}\| \le \left(\frac{\|V\|^2 + \eta - j}{\eta - j - 1} \right)^{(\eta-j-1)/2}.$$

Recall that $V = I_d - \mathbf{A}$. Taking into account the expression of the elements $a_{\nu\ell}$ of V (see (30)) we have:

$$\begin{aligned}\eta - j + \|V\|^2 &= \eta - j + \sum_{0\le\ell<\nu\le\eta-j-1} |a_{\nu\ell}|^2 = \eta - j + \sum_{0\le\ell<\nu\le\eta-j-1} \frac{\nu!\eta!}{\ell!(\eta+\nu-\ell)!} \left| \frac{\Phi^{(\eta+\nu-\ell)}(\Lambda_k)}{\Phi^{(\eta)}(\Lambda_k)} \right|^2 \\ &= \sum_{0\le\ell\le\nu\le\eta-j-1} \frac{\nu!\eta!}{\ell!(\eta+\nu-\ell)!} \left| \frac{\Phi^{(\eta+\nu-\ell)}(\Lambda_k)}{\Phi^{(\eta)}(\Lambda_k)} \right|^2.\end{aligned}$$

Coming back to P, one has:

$$|P|_{\mathbb{C}^{\eta-j}} \le \left\|\mathbf{A}^{-1}\right\| |\ \mathbf{B}|_{\mathbb{C}^{\eta-j}} \le C \left(\sum_{0\le\ell\le\nu\le\eta-j-1} \left|\Phi^{(\eta+\nu-\ell)}(\Lambda_k)\right|^2 \right)^{\frac{\eta-j-1}{2}} \frac{1}{\left|\Phi^{(\eta)}(\Lambda_k)\right|^{\eta-j}}, \qquad (31)$$

for a positive constant C which only depends on η.

Finally, let us estimate $|\Phi^{(\eta+\nu-\ell)}(\Lambda_k)|$ for $(\nu,\ell) : 0 \le \ell \le \nu \le \eta-j-1$. Since Φ is a holomorphic function on $\mathbb{C}_+$, we can write:

$$\Phi^{(m)}(\Lambda_k) = \frac{m!}{2i\pi} \oint_{|z-\Lambda_k|=r} \frac{\Phi(z)}{(z-\Lambda_k)^{m+1}}\, dz, \quad \forall m \ge 0,$$

where $r > 0$ is such that $\{z \in \mathbb{C} : |z - \Lambda_k| = r\} \subset \mathbb{C}_+$. Observe that we can take $r = \Re\Lambda_k/2$. On the other hand, from the definition of Φ (see (23)) and the properties of W (Lemma 4.2), we deduce

$$|\Phi(z)| \le \frac{1}{|1+z|^{2\eta}} \le 1, \quad \forall z \in \mathbb{C}_+.$$

Thus:

$$\left|\Phi^{(m)}(\Lambda_k)\right| \le \frac{m!}{2\pi r^{m+1}} \oint_{|z-\lambda_k|=r} dz = \frac{m!}{r^m}, \quad \forall m \ge 0,$$

with $r = \Re\Lambda_k/2$.

Going back to (31), we get

$$|P|_{\mathbb{C}^{\eta-j}} \leq C \left(\sum_{0\leq \ell \leq \eta-j-1} \frac{1}{(\Re\Lambda_k)^{2(\eta+\ell)}} \right)^{(\eta-j-1)/2} \frac{1}{\left|\Phi^{(\eta)}(\Lambda_k)\right|^{\eta-j}} \leq C \frac{H_{0j}(\Lambda_k)}{\left|\Phi^{(\eta)}(\Lambda_k)\right|^{\eta-j}}\,, \tag{32}$$

for a new positive constant C only depending on η and where $H_{0,j}(\Lambda_k)$ given by

$$H_{0j}(\Lambda_k) = 1 + \left(\frac{1}{\Re\Lambda_k} \right)^{(2\eta-j-1)(\eta-j-1)}.$$

Indeed,

$$\sum_{0\leq \ell \leq \eta-j-1} \frac{1}{(\Re\Lambda_k)^{2(\eta+\ell)}} \leq \sum_{\ell=0}^{2(2\eta-j-1)} \frac{1}{(\Re\Lambda_k)^{\ell}} = P_{\eta,j}(1/\Re\Lambda_k)$$

with $P_{\eta,j}$ a polynomial of degree $2(2\eta-j-1)$. On the other hand, there exists a positive constant $C_{\eta,j}$ (only depending on η and j) such that

$$P_{\eta,j}(x) \leq C_{\eta,j} \left(1 + x^{2(2\eta-j-1)}\right), \quad \forall x \in (0,\infty).$$

From this last inequality it is not difficult to obtain (32).

Step 3: We finalize the proof of Lemma 4.4 showing that the function

$$\Phi_{k,j}(\lambda) = p(\lambda) f_{k,j}(\lambda) = \frac{\Phi(\lambda)}{(\lambda-\Lambda_k)^{\eta-j}} \sum_{\nu=0}^{\eta-j-1} \frac{p^{(\nu)}(\Lambda_k)}{\nu!}(\lambda-\Lambda_k), \quad \forall \lambda \in \mathbb{C}_+\,,$$

(the function Φ is given by (23)) satisfies $\Phi_{k,j} \in H^2(\mathbb{C}_+)$ and (25).

On the one hand, taking into account (32), we can infer

$$|p(i\tau)| \leq C \frac{H_{0j}(\Lambda_k)}{\left|\Phi^{(\eta)}(\Lambda_k)\right|^{\eta-j}} \sum_{\nu=0}^{\eta-j-1} |i\tau-\Lambda_k|^{\nu} \leq C \frac{H_{0j}(\Lambda_k)}{\left|\Phi^{(\eta)}(\Lambda_k)\right|^{\eta-j}} \left(1 + |i\tau-\Lambda_k|^{\eta-j-1}\right),$$

with C a positive constant only depending on η. On the other hand, we can estimate,

$$\left\{ \begin{aligned} \|\Phi_{k,j}\|^2_{H^2(\mathbb{C}_+)} &\leq C^2 \frac{H_{0j}(\Lambda_k)^2}{\left|\Phi^{(\eta)}(\Lambda_k)\right|^{2(\eta-j)}} \int_{-\infty}^{\infty} \frac{\left(1+|i\tau-\Lambda_k|^{\eta-j-1}\right)^2}{|i\tau-\Lambda_k|^{2(\eta-j)}} |\Phi(i\tau)|^2 \, d\tau \leq \\ &\leq C^2 \frac{H_{0j}(\Lambda_k)^2}{\left|\Phi^{(\eta)}(\Lambda_k)\right|^{2(\eta-j)}} \left(\frac{1}{(\Re\Lambda_k)^{\eta-j}} + \frac{1}{\Re\Lambda_k} \right)^2 \|\Phi\|^2_{H^2(\mathbb{C}_+)} \\ &\leq C^2 \frac{H_j(\Lambda_k)^2}{\left|\Phi^{(\eta)}(\Lambda_k)\right|^{2(\eta-j)}}\,, \end{aligned} \right.$$

where H_j is given in the statement of Lemma 4.4 and C is a new positive constant only depending on η. This last inequality shows that $\Phi_{k,j} \in H^2(\mathbb{C}_+)$, inequality (25) and finishes the proof of Lemma 4.4. ∎

In Proposition 4.1 we have proved that, under assumptions (20) and (21) on the sequence $\Lambda = \{\Lambda_k\}_{k\geq 1} \subset \mathbb{C}_+$, there exists a biorthogonal family $\{\varphi_{k,j}\}_{k\geq 1, 0\leq j\leq \eta-1} \subset A(\Lambda,\eta,\infty)$ to the set $\{t^j e^{-\Lambda_k t}\}_{k\geq 1, 0\leq j\leq \eta-1}$ ($\eta \geq 1$ is fixed) which satisfies (22). Now, we will see that if we impose slightly stronger assumptions upon the sequence Λ (see assumptions in Theorem 1.2) we can estimate the infinite product $\mathcal{P}_k$ given in the statement of Proposition 4.1. One has:

Proposition 4.5. *Let $\{\Lambda_k\}_{k\geq 1}$ be a sequence of complex numbers satisfying (11). Then, for every $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ such that*

$$\mathcal{P}_k := \prod_{\substack{\ell\geq 1\\ \ell\neq k}} \left| \frac{1+\Lambda_k/\Lambda_\ell^*}{1-\Lambda_k/\Lambda_\ell} \right| \leq C(\varepsilon) e^{\varepsilon \Re\Lambda_k}, \quad \forall k \geq 1. \qquad \blacksquare$$

The proof of this result can be found for instance in [27], [13] and [15] (see (25), p. 1730 in the last reference). See also [22] where a slightly stronger inequality is proved under assumptions on $\{\Lambda_k\}_{k\geq 1}$ which, in particular, imply (11).

Proof of Theorem 1.2: Let us start proving Theorem 1.2 in the case $T = \infty$.

First, if $\{\Lambda_k\}_{k\geq 1}$ satisfies (11), then one has (20) and there exists a positive constant C_δ such that

$$\frac{\Re\Lambda_k}{(1+\Re\Lambda_k)^2 + (\Im\Lambda_k)^2} \leq \frac{1}{\Re\Lambda_k} \leq \frac{1}{\delta}\frac{1}{|\Lambda_k|}.$$

Therefore (21) holds and we can apply Proposition 4.1 to the sequence $\{\Lambda_k\}_{k\geq 1}$ deducing the existence of a family $\{\varphi_{k,j}\}_{k\geq 1, 0\leq j\leq \eta-1} \subset A(\Lambda,\eta,\infty)$ biorthogonal to $\left\{t^j e^{-\Lambda_k t}\right\}_{k\geq 1, 0\leq j\leq \eta-1}$ which satisfies (22).

Secondly, taking $\varepsilon > 0$ and using that $\Re\Lambda_k \to \infty$ we infer that for a positive constant $C(\eta,\varepsilon)$ one has

$$\left[1 + \left(\frac{1}{\Re\Lambda_k}\right)^{(2\eta-j)(\eta-j-1)+1}\right] (\Re\Lambda_k)^{\eta(\eta-j)} |1+\Lambda_k|^{2\eta(\eta-j)} \leq C_1(\eta,\varepsilon) e^{\varepsilon\Re\Lambda_k/2},$$

for any (k,j) with $k\geq 1$ and $0 \leq j \leq \eta - 1$.

Finally, applying Proposition 4.5, with $\varepsilon/(2\eta(\eta-j))$, instead of ε and taking into account the previous inequality and (22) we obtain (12). This finalizes the proof in the case $T = \infty$.

Before continuing the proof of Theorem 1.2, let us present a consequence of the result proved for $T = \infty$:

Corollary 4.6. *Let us assume the assumptions of Theorem 1.2. Then, for any $T \in (0,\infty)$ the restriction operator $R_T : A(\Lambda,\eta,\infty) \to A(\Lambda,\eta,T)$ defined by*

$$R_T\varphi = \varphi|_{(0,T)}, \quad \forall \varphi \in A(\Lambda,\eta,\infty)$$

is a topological isomorphism. In particular, there exists a constant $C(T) > 0$ such that

$$\|\varphi\|_{L^2(0,\infty;\mathbb{C})} \leq C(T) \|R_T\varphi\|_{L^2(0,T;\mathbb{C})}, \quad \forall \varphi \in A(\Lambda,\eta,\infty). \qquad \blacksquare$$

This result can be proved following the ideas in [13], [22] or [15]. For the sake of completeness, we include the proof in an appendix at the end of the paper.

Let us now go back to the proof of Theorem 1.2 and so, let us assume that $T \in (0,\infty)$. If we apply the previous case, we deduce the existence of a family $\{\widetilde{\varphi}_{k,j}\}_{k\geq 1, 0\leq j\leq \eta-1} \subset A(\Lambda,\eta,\infty)$ biorthogonal to $\left\{t^j e^{-\Lambda_k t}\right\}_{k\geq 1, 0\leq j\leq \eta-1}$ in $L^2(0,\infty;\mathbb{C})$ which satisfies (12).

Let us set

$$\varphi_{k,j} = \left(R_T^{-1}\right)^* \widetilde{\varphi}_{k,j} \in A(\Lambda,\eta,T), \quad \forall (k,j) : k\geq 1,\ 0\leq j\leq \eta-1.$$

From Corollary 4.6 and the properties of the family $\{\widetilde{\varphi}_{k,j}\}_{k\geq 1, 0\leq j\leq \eta-1}$, it is clear that $\varphi_{k,j}$ satisfies (12) for any (k,j).

On the other hand, with the notation $e_{k,j}(t) = t^j e^{-\Lambda_k t}$, we can write

$$\left\{ \begin{aligned} \delta_{kl}\delta_{ij} &= (e_{k,j}\,,\,\widetilde{\varphi}_{l,i})_{L^2(0,\infty;\mathbb{C})} = (R_T^{-1}R_T e_{k,j}\,,\,\widetilde{\varphi}_{l,i})_{L^2(0,\infty;\mathbb{C})} \\ &= (R_T e_{k,j}\,,\,\left(R_T^{-1}\right)^* \widetilde{\varphi}_{l,i})_{L^2(0,T;\mathbb{C})} = (e_{k,j}\,,\,\varphi_{l,i})_{L^2(0,T;\mathbb{C})}, \quad \forall (k,j),(l,i), \end{aligned} \right.$$

i.e., $\{\varphi_{k,j}\}_{k\geq 1, 0\leq j\leq \eta-1} \subset A(\Lambda,\eta,T)$ is a biorthogonal family to $\{e_{k,j}\}_{k\geq 1, 0\leq j\leq \eta-1}$ in $L^2(0,T;\mathbb{C})$ which also satisfies estimate (12). This ends the proof of Theorem 1.2. ■

Remark 4.1. *In Corollary 4.3 we have proved that, under assumption* (21)*,* $A(\Lambda,\eta,\infty)$ *is a closed proper subspace of* $L^2(0,\infty;\mathbb{C})$*. In fact, from the results proved in Proposition 4.1 it is clear that* $\{e_{k,j} : k\geq 1,\ 1\leq j\leq \eta\}$ *form a strongly independent set, i.e., each element* $e_{k,j}$ *of this set is outside the closure of the space spanned by the other functions of the set. This two results can be easily generalized to the case* $T\in(0,\infty)$*:*

"Let us assume that hypotheses in Proposition 4.1 holds. Then $A(\Lambda,\eta,T)$ *is a closed proper subspace of* $L^2(0,T;\mathbb{C})$*. In addition, the set* $\{e_{k,j} : k\geq 1,\ 0\leq j\leq \eta-1\}$ *forms a strongly independent set.* ■

5. Exact controllability to trajectories. Proof of Theorem 1.1

We will devote this section to proving Theorem 1.1. Using Proposition 2.4, we will just prove the null controllability at time T of the system. To this end, we will follow the method used by Fattorini and Russell in [12] and [13] for the study of the null controllability of a scalar heat equation. By means of this method we will reduce the controllability problem for system (1) to a moment problem.

In subsection 5.1, we explain the moment method and derive the moment problems that must be satisfied by the components of the control v. We end this section with the proof of Theorem 1.1. All along this section we will assume that the coupling matrices A and B fulfill conditions (9).

Recall that $\{\mu_l\}_{1\leq l\leq p}\subset\mathbb{C}$ is the set of distinct eigenvalues of A^*. In this section we will use the notation introduced in Section 3 (see p. 8).

5.1. The moment problem

In this subsection we will see that, under assumption (9), the null controllability problem for system (1) is equivalent to a problem (the moment problem) for the unknown control v.

Let us now fix $y_0\in H^{-1}(0,\pi;\mathbb{C}^n)$. Using formula (15) for $t=T$, we deduce that, if φ is the solution of the adjoint system (14) corresponding to $\varphi_0\in H_0^1(0,\pi;\mathbb{C}^n)$, then the null controllability problem for system (1) is equivalent to the problem

$$\left\{ \begin{array}{l} \text{Find } v\in L^2\left(0,T;\mathbb{C}^m\right) \text{ such that} \\ -\langle y_0,\varphi(\cdot,0)\rangle = \displaystyle\int_0^T (v(t)\,,\,B^*\varphi_x(0,t))_{\mathbb{C}^m}\,dt, \quad \forall\varphi_0\in H_0^1\left(0,\pi;\mathbb{C}^n\right). \end{array} \right. \tag{33}$$

If $\varphi_0\in H_0^1\left(0,\pi;\mathbb{C}^n\right)$ is given, then the corresponding solution to (14) is given by

$$\varphi(x,t) = \sum_{k\geq 1} e^{(-\lambda_k I_d + A^*)(T-t)}\phi_k(x)\varphi_{0,k}, \text{ with } \varphi_{0,k} = \int_0^\pi \varphi_0(x)\phi_k(x)\,dx\in\mathbb{C}^n, \tag{34}$$

(ϕ_k given in (5)).

Let us fix $k_0\geq 1$ as in Proposition 3.2 and let us consider the finite-dimensional space

$$X_0 = \{w : w = \sum_{1\leq k\leq k_0} w_k\phi_k \text{ with } w_k\in\mathbb{C}^n\}.$$

In general, given $y\in H^{-1}(0,\pi;\mathbb{C}^n)$ (resp. $y\in L^2(0,\pi;\mathbb{C}^n)$), we will use the notation

$$y_k = \langle y,\phi_k\rangle\in\mathbb{C}^n, \quad (\text{resp. } y_k = (y,\phi_k)_{L^2(0,\pi)}),$$

where $\langle\cdot,\cdot\rangle$ stands for the usual duality pairing between $H^{-1}(0,\pi)$ and $H_0^1(0,\pi)$. With this notation, we consider

$$Y_0 = \left(\frac{\sqrt{\pi}}{k\sqrt{2}} y_{0,k}\right)_{1\le k\le k_0} \in \mathbb{C}^{nk_0}, \text{ with } y_{0,k} = \langle y_0, \phi_k\rangle \in \mathbb{C}^n, \tag{35}$$

and $\Phi_0 = \left(k\sqrt{\frac{2}{\pi}}\varphi_{0,k}\right)_{1\le k\le k_0} \in \mathbb{C}^{nk_0}$. Then, it is not difficult to see that

$$\begin{cases} B^*\varphi_x(0,t) = B_{k_0}^* e^{\mathcal{L}_{k_0}^*(T-t)}\Phi_0 + \displaystyle\sum_{k>k_0} k\sqrt{\frac{2}{\pi}} B^* e^{(-\lambda_k I_d + A^*)(T-t)}\varphi_{0,k}\,, \quad t\in(0,T), \\ -\langle y_0, \varphi(\cdot,0)\rangle = -(Y_0\,,\, e^{\mathcal{L}_{k_0}^* T}\Phi_0)_{\mathbb{C}^{nk_0}} - \displaystyle\sum_{k>k_0} (y_{0,k}\,,\, e^{(-\lambda_k I_d + A^*)T}\varphi_{0,k})_{\mathbb{C}^n}, \end{cases}$$

with $(B_{k_0}, \mathcal{L}_{k_0})$ given by (7).

Taking first φ_0 arbitrary in X_0 and then $\varphi_0 = a\phi_k$, with $a\in\mathbb{C}^n$ and $k>k_0$, and using that $\{\phi_k\}_{k\ge1}$ is an orthonormal basis of $L^2(0,\pi)$, (33) transforms into the problem

$$\begin{cases} \text{Find } v\in L^2(0,T;\mathbb{C}^m) \text{ such that} \\ \displaystyle\int_0^T (v(T-t)\,,\, B_{k_0}^* e^{\mathcal{L}_{k_0}^* t}\Phi_0)_{\mathbb{C}^m}\,dt = F(Y_0,\Phi_0), \quad \forall\Phi_0\in\mathbb{C}^{nk_0}\,, \\ \displaystyle\int_0^T (v(T-t)\,,\, B^* e^{(-\lambda_k I_d + A^*)t}a)_{\mathbb{C}^m}\,dt = f_k(y_0,a), \quad \forall a\in\mathbb{C}^n,\ \forall k>k_0, \end{cases} \tag{36}$$

where we have introduced the bilinear forms $F:\mathbb{C}^{nk_0}\times\mathbb{C}^{nk_0}\to\mathbb{C}$ and $f_k: H^{-1}(0,\pi;\mathbb{C}^n)\times\mathbb{C}^n\to\mathbb{C}$ given by

$$\begin{cases} F(Y_0,\Phi_0) = -(Y_0\,,\, e^{\mathcal{L}_{k_0}^* T}\Phi_0)_{\mathbb{C}^{nk_0}}\,, \quad \forall (Y_0,\Phi_0)\in\mathbb{C}^{nk_0}\times\mathbb{C}^{nk_0}, \\ f_k(y_0,a) = -\dfrac{1}{k}\sqrt{\dfrac{\pi}{2}}\,(y_{0,k}\,,\, e^{(-\lambda_k I_d + A^*)T}a)_{\mathbb{C}^n}, \quad \forall (y_0,a)\in H^{-1}(0,\pi;\mathbb{C}^n)\times\mathbb{C}^n. \end{cases} \tag{37}$$

So we have reduced our control problem to a vector moment problem. In order to analyze this moment problem, let us first introduce the Jordan structure of the matrix $\mathcal{L}_{k_0}^*$: Let $\{\gamma_\ell\}_{1\le\ell\le\widetilde{p}}\subset\mathbb{C}$ be the set of distinct eigenvalues of $\mathcal{L}_{k_0}^*$. Following the notation that we have introduced for the matrix A^* (see p. 8), for $\ell: 1\le\ell\le\widetilde{p}$, we denote by N_ℓ the geometric multiplicity of γ_ℓ and we assume that we have numbered the eigenvalues in such a way that

$$N_1 \ge N_\ell, \quad 2\le\ell\le\widetilde{p}.$$

Also, the sequence $\{V_{\ell,j}\}_{1\le j\le N_\ell}$ will denote a basis of eigenvectors of $\mathcal{L}_{k_0}^*$ associated to γ_ℓ. To each eigenvector $V_{\ell,j}$ we associate its Jordan chain (of dimension $\widetilde{\tau}_{\ell,j}$) and the corresponding set of generalized eigenvectors $\{V_{\ell,j}^i\}_{1\le i\le\widetilde{\tau}_{\ell,j}}$ defined by:

$$\begin{cases} \mathcal{L}_{k_0}^* V_{\ell,j}^i = \gamma_\ell V_\ell^i + V_\ell^{i+1}, \quad 1\le i<\widetilde{\tau}_{\ell,j}\,, \\ \mathcal{L}_{k_0}^* V_{\ell,j}^{\widetilde{\tau}_{\ell,j}} = \gamma_\ell V_{\ell,j}^{\widetilde{\tau}_{\ell,j}} \end{cases}$$

(so that $V_{\ell,j}^{\widetilde{\tau}_{\ell,j}} = V_{\ell,j}$).

In fact, the eigenvalues, eigenvectors and the Jordan canonical form of the matrix $\mathcal{L}_{k_0}^*$ is determined by the eigenvalues, eigenvectors and the Jordan canonical form of A^*. Thus, γ_ℓ, with $1\le\ell\le\widetilde{p}$, is an eigenvalue of $\mathcal{L}_{k_0}^*$ if and only if for $k: 1\le k\le k_0$ and $l: 1\le l\le p$, one has $\gamma_\ell = -\lambda_k + \mu_l$. In this last case, the vector $V = (V_i)_{1\le i\le k_0}\in\mathbb{C}^{nk_0}$ is an eigenvector of $\mathcal{L}_{k_0}^*$ associated to γ_ℓ if and only if

$$V_i = 0, \quad \forall i\ne k, \text{ and } V_k = v_{l,j} \text{ with } j: 1\le j\le n_l.$$

On the other hand, it is also clear that the following properties hold:

$$p \leq \widetilde{p} \leq pk_0, \quad N_1 = \max\{N_\ell : 1 \leq \ell \leq \widetilde{p}\} \geq n_1 = \max\{n_l : 1 \leq l \leq p\}. \tag{38}$$

With the previous notation, if $t \in \mathbb{R}$, we can write:

$$\begin{cases} e^{t\mathcal{L}_{k_0}^*} V_{\ell,\jmath}^{\imath} = e^{\gamma_\ell t} \displaystyle\sum_{\sigma=0}^{\widetilde{\tau}_{\ell,\jmath}-\imath} \frac{t^\sigma}{\sigma!} V_{\ell,\jmath}^{\imath+\sigma}, \quad \forall (\ell,\jmath,\imath) : 1 \leq \ell \leq \widetilde{p},\ 1 \leq \jmath \leq N_\ell,\ 1 \leq \imath \leq \widetilde{\tau}_{\ell,\jmath}\,, \\ e^{tA^*} v_{l,j}^i = e^{\mu_l t} \displaystyle\sum_{\sigma=0}^{\tau_{l,j}-i} \frac{t^\sigma}{\sigma!} v_{l,j}^{i+\sigma}, \quad \forall (l,j,i) : 1 \leq l \leq p,\ 1 \leq j \leq n_l,\ 1 \leq i \leq \tau_{l,j}\,. \end{cases} \tag{39}$$

We will present the moment method under the assumption:

$$\tau_{l,j} = \tau_l, \quad \forall l,j : 1 \leq l \leq p, \quad 1 \leq j \leq n_l\,,$$

i.e., we will suppose that the Jordan blocks associated to the eigenvalues of A^* have the same dimension. This assumption will not be a restriction because we will only apply the moment method in this case. Observe that it also implies $\widetilde{\tau}_{\ell,\jmath} = \widetilde{\tau}_\ell$ for every $\ell, \jmath$, with $1 \leq \ell \leq \widetilde{p}$ and $1 \leq \jmath \leq N_\ell$.

Now, let us fix $(\ell,\jmath,\imath)$ and (l,j,i) with $1 \leq \ell \leq \widetilde{p}$, $1 \leq \jmath \leq N_\ell$, $1 \leq \imath \leq \widetilde{\tau}_{\ell,\jmath} \equiv \widetilde{\tau}_\ell$, $1 \leq l \leq p$, $1 \leq j \leq n_l$ and $1 \leq i \leq \tau_{l,j} \equiv \tau_l$, and let us take $\Phi_0 = V_{\ell,\jmath}^{\imath}$ and $a = v_{l,j}^i$ in (36). So, from (39) and taking into account that $\{V_{\ell,\jmath}^{\imath} : 1 \leq \ell \leq \widetilde{p},\ 1 \leq \jmath \leq N_\ell,\ 1 \leq \imath \leq \widetilde{\tau}_\ell\}$ and $\{v_{l,j}^i : 1 \leq l \leq p,\ 1 \leq j \leq n_l,\ 1 \leq i \leq \tau_l\}$ are basis of $\mathbb{C}^{nk_0}$ and $\mathbb{C}^n$ (resp.), we deduce that (36) is equivalent to

$$\begin{cases} \text{Find } \widetilde{v} \in L^2(0,T;\mathbb{C}^m) \text{ such that} \\ \displaystyle\sum_{\sigma=0}^{\widetilde{\tau}_\ell-\imath} \Big(\int_0^T \frac{t^\sigma}{\sigma!} e^{\gamma_\ell^* t}\, \widetilde{v}(t)\, dt\,,\, B_{k_0}^* V_{\ell,\jmath}^{\imath+\sigma} \Big)_{\mathbb{C}^m} = F(Y_0, V_{\ell,\jmath}^{\imath}), \quad \forall (\ell,\jmath,\imath) : \\ \qquad 1 \leq \ell \leq \widetilde{p},\ 1 \leq \jmath \leq N_\ell \text{ and } 1 \leq \imath \leq \widetilde{\tau}_\ell\,, \\ \displaystyle\sum_{\sigma=0}^{\tau_l-i} \Big(\int_0^T \frac{t^\sigma}{\sigma!} e^{(-\lambda_k+\mu_l^*)t}\, \widetilde{v}(t)\, dt\,,\, B^* v_{l,j}^{i+\sigma} \Big)_{\mathbb{C}^m} = f_k(y_0, v_{l,j}^i), \quad \forall (k,l,j,i) : \\ \qquad k > k_0,\ 1 \leq l \leq p,\ 1 \leq j \leq n_l \text{ and } 1 \leq i \leq \tau_l\,, \end{cases} \tag{40}$$

where $\widetilde{v}(t) = v(T-t)$, for $t \in [0,T]$.

From Proposition 3.1 applied to $C \equiv \mathcal{L}_{k_0}$ and $D \equiv B_{k_0}$, we know that condition (9) (with $k = k_0$) is equivalent to

$$\operatorname{rank}\,[B_{k_0}^* V_{\ell,1}\,,\, B_{k_0}^* V_{\ell,2}\,,\, \cdots\,,\, B_{k_0}^* V_{\ell,N_\ell}] = N_\ell, \quad \forall \ell : 1 \leq \ell \leq \widetilde{p}. \tag{41}$$

In particular, we infer that $m \geq N_\ell$ for every $\ell : 1 \leq \ell \leq \widetilde{p}$ and from (38) also $m \geq n_l$ for all $l : 1 \leq l \leq p$. Thus, the set $\{B_{k_0}^* V_{1,j}\}_{1 \leq j \leq N_1} \subset \mathbb{C}^m$ is linearly independent. We complete the previous set with the vectors $\{\widehat{V}_j\}_{1 \leq j \leq m-N_1} \subset \mathbb{C}^m$ in order to have a basis of $\mathbb{C}^m$:

$$\mathcal{B} \equiv \{B_{k_0}^* V_{1,j}\}_{1 \leq j \leq N_1} \cup \{\widehat{V}_j\}_{1 \leq j \leq m-N_1}. \tag{42}$$

We can associate with each vector $B_{k_0}^* V_{\ell,\jmath}^{\imath}$ and $B^* v_{l,j}^i$ of $\mathbb{C}^m$, its coordinates in this basis:

$$\begin{cases} B_{k_0}^* V_{\ell,\jmath}^{\imath} = \displaystyle\sum_{q=1}^{N_1} \alpha_{\ell,\jmath,\imath}^q B_{k_0}^* V_{1,q} + \sum_{q=1}^{m-N_1} \widehat{\alpha}_{\ell,\jmath,\imath}^q \widehat{V}_q, \quad 1 \leq \ell \leq \widetilde{p},\ 1 \leq \jmath \leq N_\ell,\ 1 \leq \imath \leq \widetilde{\tau}_\ell\,, \\ B^* v_{l,j}^i = \displaystyle\sum_{q=1}^{N_1} \beta_{l,j,i}^q B_{k_0}^* V_{1,q} + \sum_{q=1}^{m-N_1} \widehat{\beta}_{l,j,i}^q \widehat{V}_q, \quad 1 \leq l \leq p,\ 1 \leq j \leq n_l,\ 1 \leq i \leq \tau_l\,. \end{cases} \tag{43}$$

Thus, coming back to (40), we obtain that this problem is equivalent to

$$\begin{cases} \text{Find } \widetilde{v} \in L^2(0,T;\mathbb{C}^m) \text{ such that} \\ \displaystyle\sum_{\sigma=0}^{\widetilde{\tau}_\ell-\imath}\sum_{q=1}^{N_1} \alpha^{q,*}_{\ell,\jmath,\imath+\sigma}\Big(\int_0^T \frac{t^\sigma}{\sigma!}e^{\gamma^*_\ell t}\,\widetilde{v}(t)\,dt\,,\, B^*_{k_0}V_{1,q}\Big)_{\mathbb{C}^m} \\ \qquad\qquad \displaystyle + \sum_{\sigma=0}^{\widetilde{\tau}_\ell-\imath}\sum_{q=1}^{m-N_1} \widehat{\alpha}^{q,*}_{l,\jmath,\imath+\sigma}\Big(\int_0^T \frac{t^\sigma}{\sigma!}e^{\gamma^*_\ell t}\,\widetilde{v}(t)\,dt\,,\, \widehat{V}_q\Big)_{\mathbb{C}^m} = F(Y_0, V^\imath_{\ell,\jmath}), \\ \forall(\ell,\jmath,\imath) \text{ with } 1\le \ell\le \widetilde{p},\ 1\le \jmath\le N_\ell \text{ and } 1\le\imath\le\widetilde{\tau}_\ell\,, \\ \displaystyle\sum_{\sigma=0}^{\tau_l-i}\sum_{q=1}^{N_1} \beta^{q,*}_{l,j,i+\sigma}\Big(\int_0^T \frac{t^\sigma}{\sigma!}e^{(-\lambda_k+\mu^*_l)t}\,\widetilde{v}(t)\,dt\,,\, B^*_{k_0}V_{1,q}\Big)_{\mathbb{C}^m} \\ \qquad\qquad \displaystyle + \sum_{\sigma=0}^{\tau_l-i}\sum_{q=1}^{m-N_1} \widehat{\beta}^{q,*}_{l,j,i+\sigma}\Big(\int_0^T \frac{t^\sigma}{\sigma!}e^{(-\lambda_k+\mu^*_l)t}\,\widetilde{v}(t)\,dt\,,\, \widehat{V}_q\Big)_{\mathbb{C}^m} = f_k(y_0, v^i_{l,j}), \\ \forall(k,l,j,i) \text{ with } k>k_0,\ 1\le l\le p,\ 1\le j\le n_l \text{ and } 1\le i\le \tau_l\,. \end{cases}$$

Let us now consider $\{\Phi_{1,q}\}_{1\le q\le N_1}\cup\{\widehat{\Phi}_{1,q}\}_{1\le q\le m-N_1}$ a biorthogonal basis associated to the previous basis $\mathcal{B}$ of $\mathbb{C}^m$. We will look for a control $\widetilde{v}$ given by

$$\widetilde{v}(t) = \sum_{q=1}^{N_1} u_q(t)\Phi_{1,q}\,, \tag{44}$$

with $u_q \in L^2(0,T;\mathbb{C})$ for $q : 1\le q\le N_1$.

Using the equalities $(\Phi_{1,q}\,,\,B^*_{k_0}V_{1,i})_{\mathbb{C}^m} = \delta_{qi}$ and $(\Phi_{1,q}\,,\,\widehat{V}_j)_{\mathbb{C}^m} = 0$, valid for every (q,i,j) : $1\le q,i\le N_1$ and $1\le j\le m-N_1$, we can rewrite the previous moment problem as

$$\begin{cases} \text{Find } u_q \in L^2(0,T;\mathbb{C}), \text{ with } 1\le q\le N_1, \text{ such that} \\ \displaystyle\sum_{\sigma=0}^{\widetilde{\tau}_\ell-\imath}\sum_{q=1}^{N_1} \alpha^{q,*}_{\ell,\jmath,\imath+\sigma}\int_0^T \frac{t^\sigma}{\sigma!}e^{\gamma^*_\ell t}\,u_q(t)\,dt = F(Y_0,V^\imath_{\ell,\jmath}), \\ \qquad\qquad \forall(\ell,\jmath,\imath) \text{ with } 1\le\ell\le\widetilde{p},\ 1\le\jmath\le N_\ell \text{ and } 1\le\imath\le\widetilde{\tau}_\ell\,, \\ \displaystyle\sum_{\sigma=0}^{\tau_l-i}\sum_{q=1}^{N_1} \beta^{q,*}_{l,j,i+\sigma}\int_0^T \frac{t^\sigma}{\sigma!}e^{(-\lambda_k+\mu^*_l)t}\,u_q(t)\,dt = f_k(y_0,v^i_{l,j}), \\ \qquad\qquad \forall(k,l,j,i) \text{ with } k>k_0,\ 1\le l\le p,\ 1\le j\le n_l \text{ and } 1\le i\le\tau_l\,. \end{cases} \tag{45}$$

Our aim is to prove that the previous moment problem admits, for every q, a solution u_q which lies in $L^2(0,T;\mathbb{C})$. The previous reasoning shows that the corresponding control $v(t)=\widetilde{v}(T-t)$, with $\widetilde{v}$ given by (44), is in $L^2(0,T;\mathbb{C}^m)$ and solves the null controllability problem for system (1).

For (ℓ,ν) and (l,σ) such that $1\le\ell\le\widetilde{p}$, $0\le\nu\le\widetilde{\tau}_\ell-1$, $1\le l\le p$ and $0\le\sigma\le\tau_l-1$, let us set

$$\begin{cases} X_{\ell,\nu} = \left(\displaystyle\int_0^T \frac{t^\nu}{\nu!}e^{\gamma^*_\ell t}\,u_q(t)\,dt\right)_{1\le q\le N_1}, \quad Y^k_{l,\sigma} = \left(\displaystyle\int_0^T \frac{t^\sigma}{\sigma!}e^{(-\lambda_k+\mu^*_l)t}\,u_q(t)\,dt\right)_{1\le q\le N_1} \in\mathbb{C}^{N_1}, \\ F_{\ell,\nu}(Y_0) = \left(F(Y_0,V^{\widetilde{\tau}_\ell-\nu}_{\ell,j})\right)_{1\le j\le N_\ell}\in\mathbb{C}^{N_\ell},\ F^k_{l,\sigma}(y_0) = \left(f_k(y_0,v^{\tau_l-\sigma}_{l,j})\right)_{1\le j\le n_l}\in\mathbb{C}^{n_l} \text{ and} \\ A_{\ell,\nu} = \left(\alpha^{q,*}_{\ell,j,\widetilde{\tau}_\ell-\nu}\right)_{\substack{1\le j\le N_\ell\\1\le q\le N_1}}\in\mathcal{L}(\mathbb{C}^{N_1};\mathbb{C}^{N_\ell}),\ B_{l,\sigma} = \left(\beta^{q,*}_{l,j,\tau_l-\sigma}\right)_{\substack{1\le j\le n_l\\1\le q\le N_1}}\in\mathcal{L}(\mathbb{C}^{N_1};\mathbb{C}^{n_l}), \end{cases} \tag{46}$$

where $F(Y_0,\Phi_0)$ and $f_k(y_0,a)$ are given in (37).

Step 1: Let us first consider the case $\ell = 1$. Let us recall that $\left(\alpha^q_{\ell,j,i}\right)_{1\le q\le N_1}$ are the coordinates of $B^*_{k_0}V^i_{\ell,j}$ with respect to the basis $\mathcal{B} = \{B^*_{k_0}V_{1,j}\}_{1\le j\le N_1} \cup \{\widehat{V}_j\}_{1\le j\le m-N_1}$ of $\mathbb{C}^m$. In particular,

$$B^*_{k_0}V^{\widetilde{\tau}_1}_{1,\jmath} = \sum_{q=1}^{N_1} \alpha^q_{1,\jmath,\widetilde{\tau}_1} B^*_{k_0}V_{1,q} + \sum_{q=1}^{m-N_1} \widehat{\alpha}^q_{1,\jmath,\widetilde{\tau}_1} \widehat{V}_q, \quad \forall \jmath : 1 \le \jmath \le N_1 .$$

Since $V^{\widetilde{\tau}_\ell}_{\ell,j} = V_{\ell,j}$ for every ℓ, j, if $\ell = 1$ we get

$$\begin{cases} \alpha^q_{1,j,\widetilde{\tau}_1} = \delta_{qj}, & \forall j,q : 1 \le j,q \le N_1 \text{ and} \\ \widehat{\alpha}^q_{1,j,\widetilde{\tau}_1} = 0, & \forall j,q : 1 \le j,q \le m - N_1 . \end{cases}$$

From (45) and using the previous equalities ($\ell = 1$) we obtain:

$$\begin{array}{ll} \imath = \widetilde{\tau}_1, & \displaystyle\int_0^T e^{\gamma_1^* t} u_\jmath(t)\,dt = F(Y_0, V^{\widetilde{\tau}_1}_{1,\jmath}), \\ \imath = \widetilde{\tau}_1 - 1, & \displaystyle\sum_{q=1}^{N_1} \alpha^{q,*}_{1,\jmath,\widetilde{\tau}_1-1} \int_0^T e^{\gamma_1^* t} u_q(t)\,dt + \int_0^T t e^{\gamma_1^* t} u_\jmath(t)\,dt = F(Y_0, V^{\widetilde{\tau}_1-1}_{1,\jmath}), \\ \vdots & \vdots \\ \imath = 1, & \displaystyle\sum_{\sigma=0}^{\widetilde{\tau}_1-1} \sum_{q=1}^{N_1} \alpha^{q,*}_{1,\jmath,1+\sigma} \int_0^T \frac{t^\sigma}{\sigma!} e^{\gamma_1^* t} u_q(t)\,dt = F(Y_0, V^1_{1,\jmath}). \end{array}$$

Using (46), we infer that, for $\ell = 1$, the first part of system (45) is equivalent to the linear algebraic system

$$\begin{cases} X_{1,0} = F_{1,0}(Y_0), \\ A_{1,1}X_{1,0} + X_{1,1} = F_{1,1}(Y_0), \\ \vdots \\ A_{1,\widetilde{\tau}_1-1}X_{1,0} + A_{1,\widetilde{\tau}_1-2}X_{1,1} + \cdots + X_{1,\widetilde{\tau}_1-1} = F_{1,\widetilde{\tau}_1-1}(Y_0), \end{cases}$$

or in matrix form:

$$\underbrace{\begin{pmatrix} I_d & 0 & 0 & \cdots & 0 \\ A_{1,1} & I_d & 0 & \cdots & 0 \\ A_{1,2} & A_{1,1} & I_d & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{1,\widetilde{\tau}_1-1} & A_{1,\widetilde{\tau}_1-2} & A_{1,\widetilde{\tau}_1-3} & \cdots & I_d \end{pmatrix}}_{=M_1} \begin{pmatrix} X_{1,0} \\ X_{1,1} \\ X_{1,2} \\ \vdots \\ X_{1,\widetilde{\tau}_1-1} \end{pmatrix} = \begin{pmatrix} F_{1,0}(Y_0) \\ F_{1,1}(Y_0) \\ F_{1,2}(Y_0) \\ \vdots \\ F_{1,\widetilde{\tau}_1-1}(Y_0) \end{pmatrix}.$$

Clearly this system possesses a unique solution given by

$$\begin{pmatrix} X_{1,0} \\ X_{1,1} \\ \vdots \\ X_{1,\widetilde{\tau}_1-1} \end{pmatrix} = M_1^{-1} \begin{pmatrix} F_{1,0}(Y_0) \\ F_{1,1}(Y_0) \\ \vdots \\ F_{1,\widetilde{\tau}_1-1}(Y_0) \end{pmatrix} \equiv \begin{pmatrix} C_{1,0}(y_0) \\ C_{1,1}(y_0) \\ \vdots \\ C_{1,\widetilde{\tau}_1-1}(y_0) \end{pmatrix},$$

(Y_0 given in (35)). Observe that the matrix $M_1 \in \mathcal{L}(\mathbb{C}^{\widetilde{\tau}_1 N_1})$ only depends on the coupling matrices A and B and can be computed independently of the initial datum y_0.

In other words, if we write $C_{1,\sigma}(y_0) = (c_{1,\sigma,q}(y_0))_{1\le q\le N_1} \in \mathbb{C}^{N_1}$, we have established that the components of u_q of the control $\widetilde{v}$ with respect to the set $\{\Phi_{1,q}\}_{1\le q\le N_1}$ must solve the following first family of moment problems:

$$\int_0^T \frac{t^\sigma}{\sigma!} e^{\gamma_1^* t} u_q(t)\,dt = c_{1,\sigma,q}(y_0), \quad \forall(\sigma,q) : 0 \le \sigma \le \widetilde{\tau}_1 - 1,\ 1 \le q \le N_1 .$$

Finally, taking into account the expressions of $C_{1,\sigma}(y_0)$, $F_{1,\sigma}(Y_0)$ (see (46)) and $F(Y_0,\Phi_0)$ (see (37)), we deduce the existence of a positive constant $C = C(A,B)$ for which

$$|c_{1,\sigma,q}(y_0)| \le C\|e^{\mathcal{L}^*_{k_0}T}\|_{\mathcal{L}(\mathbb{C}^{nk_0})}\|y_0\|_{H^{-1}(0,\pi;\mathbb{C}^n)}\,, \quad \forall(\sigma,q): 0\le\sigma\le\widetilde{\tau}_1-1,\ 1\le q\le N_1.$$

Step 2: Consider now the case $2\le\ell\le\widetilde{p}$. Following the same reasoning as before, we find that the first equation of system (45) can be equivalently rewritten as:

$$\begin{cases} A_{\ell,0}X_{\ell,0} = F_{\ell,0}(Y_0), \\ A_{\ell,1}X_{\ell,0} + A_{\ell,0}X_{\ell,1} = F_{\ell,1}(Y_0), \\ \vdots \\ A_{\ell,\widetilde{\tau}_\ell-1}X_{\ell,0} + A_{\ell,\widetilde{\tau}_\ell-2}X_{\ell,1} + \cdots + A_{\ell,0}X_{\ell,\widetilde{\tau}_\ell-1} = F_{\ell,\widetilde{\tau}_\ell-1}(Y_0). \end{cases} \tag{47}$$

Let us show that this linear system (which has $\widetilde{\tau}_\ell N_1$ unknowns and $\widetilde{\tau}_\ell N_\ell \le \widetilde{\tau}_\ell N_1$ equations) is compatible.

Indeed, remember that $A_{\ell,0}\in\mathcal{L}(\mathbb{C}^{N_1};\mathbb{C}^{N_\ell})$ (see (46)) and its components $\alpha^q_{\ell,j,\widetilde{\tau}_\ell}$ are the coordinates of the vectors $B^*_{k_0}V^{\widetilde{\tau}_\ell}_{\ell,j} = B^*_{k_0}V_{\ell,j}$ with respect to the basis $\mathcal{B}$ (see (43)). Again condition (41) implies that the set $\{B^*_{k_0}V_{\ell,j} : 1\le j\le N_\ell\}\subset\mathbb{C}^m$ is linearly independent and, evidently, also $\operatorname{rank} A_{\ell,0} = N_\ell$. Then, there exists a permutation matrix $P_\ell\in\mathcal{L}(\mathbb{R}^{N_1})$ (only depending on $A_{\ell,0}$, i.e., on ℓ, A and B) such that $A_{\ell,0}P_\ell = \left[\widetilde{A}_{\ell,0}\,|\,\widetilde{D}_{\ell,0}\right]$ with $\widetilde{A}_{\ell,0}\in\mathcal{L}(\mathbb{C}^{N_\ell})$ a squared matrix and $\operatorname{rank}\widetilde{A}_{\ell,0} = \operatorname{rank} A_{\ell,0} = N_\ell$. For each σ with $0\le\sigma\le\widetilde{\tau}_\ell-1$, we also set $A_{\ell,\sigma}P_\ell = \left[\widetilde{A}_{\ell,\sigma}\,|\,\widetilde{D}_{\ell,\sigma}\right]$ with $\widetilde{A}_{\ell,\sigma}\in\mathcal{L}(\mathbb{C}^{n_\ell})$. If we look for a solution under the form $X_{\ell,\sigma} = P_\ell\begin{bmatrix}\widetilde{X}_{\ell,\sigma}\\ 0\end{bmatrix}$, the previous system transforms into:

$$\begin{cases} \widetilde{A}_{\ell,0}\widetilde{X}_{\ell,0} = F_{\ell,0}(Y_0), \\ \widetilde{A}_{\ell,1}\widetilde{X}_{\ell,0} + \widetilde{A}_{\ell,0}\widetilde{X}_{\ell,1} = F_{\ell,1}(Y_0), \\ \vdots \\ \widetilde{A}_{\ell,\widetilde{\tau}_\ell-1}\widetilde{X}_{\ell,0} + \widetilde{A}_{\ell,\widetilde{\tau}_\ell-2}\widetilde{X}_{\ell,1} + \cdots + \widetilde{A}_{\ell,0}\widetilde{X}_{l,\widetilde{\tau}_\ell-1} = F_{\ell,\widetilde{\tau}_\ell-1}(Y_0). \end{cases}$$

This system has a unique solution which can be written as $\widetilde{X}_{\ell,\sigma} = \widetilde{C}_{\ell,\sigma}(y_0)$ $(0\le\sigma\le\widetilde{\tau}_\ell-1)$ with

$$\begin{pmatrix} \widetilde{C}_{\ell,0}(y_0) \\ \widetilde{C}_{\ell,1}(y_0) \\ \vdots \\ \widetilde{C}_{\ell,\widetilde{\tau}_\ell-1}(y_0) \end{pmatrix} = M_\ell^{-1}\begin{pmatrix} F_{\ell,0}(Y_0) \\ F_{\ell,1}(Y_0) \\ \vdots \\ F_{\ell,\widetilde{\tau}_\ell-1}(Y_0) \end{pmatrix}, \tag{48}$$

where $M_\ell\in\mathcal{L}(\mathbb{C}^{\widetilde{\tau}_\ell N_\ell})$ is the coefficient matrix of the linear system (which, once again, only depends on ℓ and the coupling matrices A and B) and $F_{\ell,\sigma}(Y_0)\in\mathbb{C}^{N_\ell}$ and Y_0 are given in (46) and (35). This proves that the system (47) is compatible and, in fact, we have obtained a particular solution.

We can repeat the arguments shown in the case $\ell = 1$ and deduce that the components of the previous solution of system (47) satisfy the family of moment problems

$$\int_0^T \frac{t^\sigma}{\sigma!}e^{\gamma^*_\ell t}\,u_q(t)\,dt = c_{\ell,\sigma,q}(y_0), \quad \forall(\ell,\sigma,q): 2\le\ell\le\widetilde{p},\ 0\le\sigma\le\widetilde{\tau}_\ell-1,\ 1\le q\le N_1\,,$$

where the coefficients $c_{\ell,\sigma,q}(y_0)$ are given by $(c_{\ell,\sigma,q}(y_0))_{1\le q\le N_1} = P_\ell\begin{bmatrix}\widetilde{C}_{\ell,\sigma}(y_0)\\ 0\end{bmatrix}$.

Again, taking into account the expressions of $F_{\ell,\sigma}(Y_0)$ and $F(Y_0,\Phi_0)$ (see (46) and (37)), we deduce again the existence of a positive constant $C = C(A,B)$ such that, for every (ℓ,σ,q) with $2 \le \ell \le \widetilde{p}$, $0 \le \sigma \le \widetilde{\tau}_\ell - 1$ and $1 \le q \le N_1$, one has

$$|c_{\ell,\sigma,q}(y_0)| \le C \|e^{\mathcal{L}^*_{k_0}T}\|_{\mathcal{L}(\mathbb{C}^{nk_0})} \|y_0\|_{H^{-1}(0,\pi;\mathbb{C}^n)}. \tag{49}$$

Step 3: Now, we are going to obtain an infinite family of moment problems using the second identity in (45). We fix $k > k_0$ and $l : 1 \le l \le p$. Following the same ideas as before, we obtain that the second identity in (45) is equivalent to

$$\begin{cases} B_{l,0}Y^k_{l,0} = F^k_{l,0}(y_0), \\ B_{l,1}Y^k_{l,0} + B_{l,0}Y^k_{l,1} = F^k_{l,1}(y_0), \\ \vdots \\ B_{l,\tau_l-1}Y^k_{l,0} + B_{l,\tau_l-2}Y^k_{l,1} + \cdots + B_{l,0}Y^k_{l,\tau_l-1} = F^k_{l,\tau_l-1}(y_0), \end{cases} \tag{50}$$

where $B_{l,\sigma}$ and $F^k_{l,\sigma}$ are given in (46).

Again, we have a compatible system of dimension $(\tau_l n_l) \times (\tau_l N_1)$ ($n_l \le N_1$, see (38)). Indeed, from the Kalman rank condition (2) we deduce

$$\operatorname{rank}\left[B^*v_{l,1}\,,\, B^*v_{l,2}\,,\, \cdots\,,\, B^*v_{l,n_l}\right] = n_l, \quad \forall l : 1 \le l \le p,$$

and also $\operatorname{rank} B_{l,0} = n_l$. Therefore, for a permutation matrix $Q_l \in \mathcal{L}(\mathbb{R}^{N_1})$ (only depending on l, A and B), we can write $B_{l,0}Q_l = \left[\widetilde{B}_{l,0} \,|\, \widehat{D}_{l,0}\right]$ with $\widetilde{B}_{l,0} \in \mathcal{L}(\mathbb{C}^{n_l})$ and $\operatorname{rank} \widetilde{B}_{l,0} = n_l$. We also write $B_{l,\sigma}Q_l = \left[\widetilde{B}_{l,\sigma} \,|\, \widehat{D}_{l,\sigma}\right]$, with $\widetilde{B}_{l,\sigma} \in \mathcal{L}(\mathbb{C}^{n_l})$ $(0 \le \sigma \le \tau_l - 1)$, and we obtain a solution to (50) as $Y^k_{l,\sigma} = Q_l \begin{bmatrix} \widetilde{Y}^k_{l,\sigma} \\ 0 \end{bmatrix}$ with $\widetilde{Y}^k_{l,\sigma} \in \mathbb{C}^{n_l}$ solution to

$$\begin{cases} \widetilde{B}_{l,0}\widetilde{Y}^k_{l,0} = F^k_{l,0}(y_0), \\ \widetilde{B}_{l,1}\widetilde{Y}^k_{l,0} + \widetilde{B}_{l,0}\widetilde{Y}^k_{l,1} = F^k_{l,1}(y_0), \\ \vdots \\ \widetilde{B}_{l,\tau_l-1}\widetilde{Y}^k_{l,0} + \widetilde{B}_{l,\tau_l-2}\widetilde{Y}^k_{l,1} + \cdots + \widetilde{B}_{l,0}\widetilde{Y}^k_{l,\tau_l-1} = F^k_{l,\tau_l-1}(y_0). \end{cases}$$

If we denote by means of $\widetilde{M}_l \in \mathcal{L}(\mathbb{C}^{\tau_l n_l})$ the previous coefficient matrix (which only depends on l, A and B), then we have managed to obtain a solution of the system (50) under the form $Y^k_{l,\sigma} = D^k_{l,\sigma}(y_0)$ with $D^k_{l,\sigma}(y_0) = Q_l \begin{bmatrix} \widetilde{D}^k_{l,\sigma}(y_0) \\ 0 \end{bmatrix}$ and

$$\left(\widetilde{D}^k_{l,\sigma}(y_0)\right)_{1 \le \sigma \le \tau_l - 1}(y_0) = \widetilde{M}_l^{-1}\left(F^k_{l,\sigma}(y_0)\right)_{1 \le \sigma \le \tau_l - 1}. \tag{51}$$

Finally, let us remark that, from the expressions of $F^k_{l,\sigma}(y_0)$ and $f_k(y_0,a)$ (see (46) and (37)), we have the existence of a positive constant $C = C(A,B)$ such that the components $d^k_{1,\sigma,q}(y_0)$ $(q : 1 \le q \le N_1)$ of $D^k_{l,\sigma}(y_0)$ satisfy

$$|d^k_{1,\sigma,q}(y_0)| \le \frac{C}{k} \|e^{(-\lambda_k I_d + A^*)T}\|_{\mathcal{L}(\mathbb{C}^n)} |y_{0,k}| \tag{52}$$

for every (k,l,σ,q) with $k > k_0$, $1 \le l \le p$, $0 \le \sigma \le \tau_l - 1$ and $1 \le q \le N_1$.

Summarizing, with the previous notation and assuming that $\tau_{l,j} = \tau_l$, for all $l,j : 1 \le l \le p$ and $1 \le j \le n_l$, we have proved:

Proposition 5.1. *Assume that condition* (9) *holds and let us consider the integer* k_0 *provided by Proposition 3.2. Let us also fix* $y_0 \in H^{-1}(0,\pi;\mathbb{C}^n)$. *Then, for every* (ℓ,l) *with* $1 \le \ell \le \widetilde{p}$ *and* $1 \le l \le p$, *there exist matrices* $M_\ell \in \mathcal{L}(\mathbb{C}^{\widetilde{\tau}_\ell N_\ell})$, $\widetilde{M}_l \in \mathcal{L}(\mathbb{C}^{\tau_l n_l})$ *and permutation matrices* $P_\ell, Q_l \in \mathcal{L}(\mathbb{R}^{N_1})$, *which only depend on the coupling matrices* A *and* B, *such that, if for every* q, *with* $1 \le q \le N_1$, *the function* $u_q \in L^2(0,T;\mathbb{C})$ *satisfies the family of moment problems:*

$$\left\{ \begin{array}{l} \displaystyle\int_0^T \frac{t^\nu}{\nu!} e^{\gamma_\ell^* t} u_q(t)\,dt = c_{\ell,\nu,q}(y_0), \quad \forall(\ell,\nu): 1 \le \ell \le \widetilde{p},\ 0 \le \nu \le \widetilde{\tau}_\ell - 1, \\ \displaystyle\int_0^T \frac{t^\sigma}{\sigma!} e^{(-\lambda_k+\mu_l^*)t} u_q(t)\,dt = d^k_{l,\sigma,q}(y_0), \quad \forall(k,l,\sigma): k > k_0,\ 1 \le l \le p,\ 0 \le \sigma \le \tau_l - 1, \end{array} \right. \tag{53}$$

then the control v *given by* $v(t) = \widetilde{v}(T-t)$ $(t \in (0,T))$, *with* $\widetilde{v}$ *given by* (44), *is in* $L^2(0,T;\mathbb{C}^m)$ *and solves the null controllability problem for system* (1). *In* (53) *the coefficients* $c_{\ell,\nu,q}$ *and* $d^k_{l,\sigma,q}(y_0)$ *are respectively given by*

$$(c_{\ell,\nu,q}(y_0))_{1\le q\le N_1} = P_\ell \begin{bmatrix} \widetilde{C}_{\ell,\nu}(y_0) \\ 0 \end{bmatrix}, \quad (d^k_{l,\sigma,q}(y_0))_{1\le q\le N_1} = Q_l \begin{bmatrix} \widetilde{D}^k_{l,\sigma}(y_0) \\ 0 \end{bmatrix}$$

and $\widetilde{D}^k_{l,\sigma}(y_0)$ *and* $\widetilde{C}_{\ell,\nu}(y_0)$ *by* (48) *and* (51). *Finally, there exists a positive constant* C, *only depending on* A *and* B, *such that* (49) *and* (52) *hold.* ■

Remark 5.1. Let us now assume that $y_0 \in X_0^\perp$ with

$$X_0 = \{w : w = \sum_{1\le k\le k_0} w_k\phi_k \text{ with } w_k \in \mathbb{C}^n\}.$$

and k_0 given by Proposition 3.2. In this case the moment problem reduces to (40) with $Y_0 \equiv 0$ (see (35)) and, evidently, $F(Y_0, V^\imath_{\ell,\jmath}) \equiv 0$ for every $(\ell,\jmath,\imath)$. If we assume the algebraic Kalman condition (2) then, using Proposition 3.1, we conclude that the set $\{B^*v_{1,j} : 1 \le j \le n_1\} \subset \mathbb{C}^m$ is linearly independent. This set can be completed in order to have a basis of $\mathbb{C}^m$: $\widetilde{\mathcal{B}} = \{B^*v_{1,j}\}_{1\le j\le n_1} \cup \{\widehat{v}_j\}_{1\le j\le m-n_1}$. We can work with this basis $\widetilde{\mathcal{B}}$ and its biorthogonal basis $\{\widetilde{\Phi}_{1,q}\}_{1\le q\le n_1} \cup \{\overline{\Phi}_{1,q}\}_{1\le q\le m-n_1}$ instead of $\mathcal{B}$ (see (42)) and its biorthogonal basis $\{\Phi_{1,q}\}_{1\le q\le N_1} \cup \{\widehat{\Phi}_{1,q}\}_{1\le q\le m-N_1}$. If we follow the previous arguments, it is possible to rewrite the moment problem as (45) with n_1, $\widetilde{\alpha}^{q,*}_{\ell,\jmath,\imath+\sigma}$, $\widetilde{\beta}^{q,*}_{l,j,i+\sigma}$ and $F(Y_0, V^\imath_{\ell,\jmath}) \equiv 0$ instead of N_1, $\alpha^{q,*}_{\ell,\jmath,\imath+\sigma}$ and $\beta^{q,*}_{l,j,i+\sigma}$. The first family of equalities can be always solved. Adapting all the previous arguments, we can obtain the property: If $u_q \in L^2(0,T;\mathbb{C})$ satisfies the family of moment problems (compare with (53))

$$\left\{ \begin{array}{l} \displaystyle\int_0^T \frac{t^\nu}{\nu!} e^{\gamma_\ell^* t} u_q(t)\,dt = 0, \quad \forall(\ell,\nu): 1 \le \ell \le \widetilde{p},\ 0 \le \nu \le \widetilde{\tau}_\ell - 1, \\ \displaystyle\int_0^T \frac{t^\sigma}{\sigma!} e^{(-\lambda_k+\mu_l^*)t} u_q(t)\,dt = \widetilde{d}^k_{l,\sigma,q}(y_0), \quad \forall(k,l,\sigma): k > k_0,\ 1 \le l \le p,\ 0 \le \sigma \le \tau_l - 1, \end{array} \right.$$

then the control v given by $v(t) = \sum_{q=1}^{n_1} u_q(T-t)\widetilde{\Phi}_{1,q}$ is in $L^2(0,T;\mathbb{C}^m)$ and solves the null controllability problem for system (1). ■

5.2. Proof of Theorem 1.1

In this subsection we are going to prove Theorem 1.1.

Necessary condition: Let us show that condition (9) is necessary in order to get the exact controllability to the trajectories of system (1). To this end, we will use Proposition 2.4. To be precise, let us assume that, for $k_0 \ge 1$, one has

$$\operatorname{rank} \mathcal{K}_{k_0} = \operatorname{rank}\,[B_{k_0}\,,\ \mathcal{L}_{k_0}B_{k_0}\,,\ \mathcal{L}^2_{k_0}B_{k_0}\,,\ \cdots\,,\ \mathcal{L}^{nk_0-2}_{k_0}B_{k_0}\,,\ \mathcal{L}^{nk_0-1}_{k_0}B_k] < nk_0\,,$$

and let us prove that the observability inequality (16) fails ($\mathcal{L}_k$ and B_k are given in (7)).

Indeed, using the Kalman rank condition for ordinary differential systems, we deduce that the pair $(\mathcal{L}_{k_0}^*, B_{k_0}^*)$ is not observable, i.e., there exists $\Phi_0 \in \mathbb{C}^{nk_0}$ with $\Phi_0 \not\equiv 0$ such that the solution Φ to the system

$$\begin{cases} -\Phi' = \mathcal{L}_{k_0}^* \Phi \text{ in } (0,T), \\ \Phi(T) = \Phi_0 \in \mathbb{C}^{nk_0}, \end{cases}$$

satisfies

$$B_{k_0}^* \Phi(t) = 0, \quad \forall t \in (0,T).$$

Now if we do $\Phi_0 = \left(k\sqrt{\frac{2}{\pi}}\varphi_{0,k} \right)_{1\le k\le k_0}$ (with $\varphi_{0,k} \in \mathbb{C}^n$ for every $k : 1 \le k \le k_0$) and we take

$$\varphi_0(x) = \sum_{k=1}^{k_0} \varphi_{0,k}\phi_k(x),$$

then, $\varphi_0 \in H_0^1(0,\pi;\mathbb{C}^n)$, $\varphi_0 \not\equiv 0$ and the corresponding solution φ to (14) is given by

$$\varphi(x,t) = \sqrt{\frac{2}{\pi}} \sum_{k=1}^{k_0} e^{(-\lambda_k + A^*)(T-t)} \varphi_{0,k} \sin(kx), \quad \forall (x,t) \in Q$$

and satisfies

$$B^*\varphi_x(0,t) = \sqrt{\frac{2}{\pi}} \sum_{k=1}^{k_0} B^* e^{(-\lambda_k + A^*)(T-t)} k\varphi_{0,k} = B_{k_0}^* e^{\mathcal{L}_{k_0}^*(T-t)} \Phi_0 = B_{k_0}^* \Phi(t) = 0 \text{ on } (0,T).$$

Evidently, this proves that inequality (16) fails.

Sufficient condition: Let us assume that the pair (A,B) satisfies condition (9) and let us prove that system (1) is exactly controllable to trajectories at time T or, equivalently, is null controllable at time T ($T \in (0,\infty)$ is given). To this end, let us fix $y_0 \in H^{-1}(0,\pi;\mathbb{C}^n)$.

As said before, we will follow the technique from [12] and we will prove the result as a consequence of Proposition 5.1 and Theorem 1.2.

Case 1: With the notation of Section 3 and Subsection 5.1, let us first assume that the matrix A is such that $\tau_{l,j} = \tau_l$, for all $l,j : 1 \le l \le p$ and $1 \le j \le n_l$.

Let us also take $k_0 \ge 1$ provided by Proposition 3.2. Recall that $\{\gamma_\ell\}_{1\le\ell\le\widetilde{p}} \subset \mathbb{C}$ and $\{\mu_l\}_{1\le l\le p} \subset \mathbb{C}$ are, resp., the set of distinct eigenvalues of the matrices $\mathcal{L}_{k_0}^*$ and A^*. Let us fix $\mu > 0$ such that $\Re(\lambda_1 - \mu_l + \mu) > 0$ for every $l : 1 \le l \le p$. With this notation, let us set

$$\begin{cases} \Lambda_\ell = -\gamma_\ell + \mu, \quad \text{for } 1 \le \ell \le \widetilde{p}, \\ \Lambda_{\widetilde{p}+(i-1)p+l} = \lambda_{i+k_0} - \mu_l + \mu, \quad \text{for } i \ge 1,\ 1 \le l \le p. \end{cases}$$

Recall that, given $\ell : 1 \le \ell \le \widetilde{p}$, one has $\gamma_\ell = -\lambda_k + \mu_l$, with $1 \le k \le k_0$ and $1 \le l \le p$. Thanks to Proposition 3.2 we also have that the sequence $\Lambda = \{\Lambda_k\}_{k\ge1}$ satisfies (20). On the other hand, from the property satisfied by μ, we deduce that $\Re\Lambda_k > 0$, i.e.,

$$\{\Lambda_k\}_{k\ge1} \subset \mathbb{C}_+ .$$

Our next task will be to prove that the sequence $\{\Lambda_k\}_{k\ge1}$ satisfies (11):

(a) $\Re\Lambda_k \ge \delta|\Lambda_k|$ for a positive δ wich only depends on A. Indeed,

$$\lim_{k\to\infty} \frac{\Re\Lambda_k}{|\Lambda_k|} = 1,$$

whence we deduce the existence of $K_0 \geq 1$ (only depending on $\{\mu_l\}_{1\leq l\leq p}$) such that $\Re\Lambda_k > \frac{1}{2}|\Lambda_k|$ for every $k \geq K_0$. Taking

$$\delta = \min\left\{\frac{1}{2}, \frac{\Re\Lambda_k}{|\Lambda_k|} : 1 \leq k \leq K_0\right\},$$

we deduce this first property.

(b) Let us recall that $\lambda_k = k^2$ (see (5)). Therefore, the second property

$$\sum_{k\geq 1} \frac{1}{|\Lambda_k|} < \infty,$$

can be easily deduced.

(c) Finally, let us prove that there exists $\rho > 0$ (which only depends on A) such that $|\Lambda_k - \Lambda_l| \geq \rho|k-l|$ for every $k > l \geq 1$. Let us first consider $k \geq \widetilde{p}+1$ and $l < k$. Then, for some $\nu : 1 \leq \nu \leq p$, we can write $k = \widetilde{p} + (i-1)p + \nu$ and $\Lambda_k = \lambda_{i+k_0^2} - \mu_\nu + \mu$. Let us take,

$$M_0 = \max_{1\leq\nu,\sigma\leq p} |\mu_\nu - \mu_\sigma|, \quad M_1 = \max_{1\leq\nu\leq p, 1\leq\ell\leq\widetilde{p}} |\gamma_\ell - \mu_\nu| \geq \lambda_{k_0},$$

and

$$\begin{cases} I_0 = \max\{0, [M_0 + 2(p - k_0 - 1)]\} + 1, \\ I_1 = \max\left\{0, \left[\frac{1}{2}\left(p - 2k_0 + \sqrt{(p-2k_0)^2 + 4(M_1 - k_0^2 + \widetilde{p} - 1)}\right)\right]\right\} + 1, \\ K_0 = \widetilde{p} + I_0 p \text{ and } K_1 = \widetilde{p} + I_1 p. \end{cases}$$

Observe that $M_1 \geq k_0^2$ and, thus, I_1 is well defined. It is also clear that the constants M_0, M_1, K_0 and K_1 only depend on the matrix A.

Then, if $k \geq K_0$ and $\widetilde{p} < l < k$, that is to say, if $k \geq K_0$ and $l = \widetilde{p} + (j-1)p + \sigma$ with $1 \leq j < i$ and $\sigma : 1 \leq \sigma \leq p$, we have $i \geq I_0$ and $\Lambda_l = \lambda_{j+k_0^2} - \mu_\sigma + \mu$. From the definition of I_0, we get

$$(i + j + 2k_0 - p)(i - j) \geq i + 1 + 2k_0 - p \geq M_0 + p - 1,$$

i.e., $(i-j)p + p - 1 \leq (i + j + 2k_0)(i-j) - M_0$. So,

$$\begin{cases} |k - l| = (i-j)p + \nu - \sigma \leq (i-j)p + p - 1 \leq (i + j + 2k_0)(i-j) - M_0 \\ \qquad \leq (i + k_0)^2 - (j + k_0)^2 - |\mu_\sigma - \mu_\nu| \leq |\Lambda_k - \Lambda_l|. \end{cases}$$

Let us now assume that $k \geq K_1$ and $1 \leq l \leq \widetilde{p}$. In particular, $i \geq I_1$, $\Lambda_l = -\gamma_l + \mu$. From the definition of I_1 we can write

$$i^2 + (2k_0 - p)i - (M_1 - k_0^2 + \widetilde{p} - 1) \geq 0,$$

i.e., $\nu - l \leq \widetilde{p} + ip - 1 \leq (i + k_0)^2 - M_1$. Therefore,

$$\begin{cases} |k - l| = \widetilde{p} + (i-1)p + \nu - l \leq \widetilde{p} + ip - 1 \leq (i + k_0)^2 - M_1 \\ \qquad \leq (i + k_0)^2 - |\gamma_l - \mu_\nu| \leq |\Lambda_k - \Lambda_l|. \end{cases}$$

Summarizing, if $k \geq K = \max\{K_0, K_1\}$, we have proved

$$|\Lambda_k - \Lambda_l| \geq |k - l|, \quad \forall l : 1 \leq l < k.$$

Finally, let us set

$$\rho_0 = \frac{m_0}{K-2}, \quad \text{with } m_0 = \min_{1\leq l<k<K} |\Lambda_k - \Lambda_l|.$$

Thus, $\rho_0 > 0$ (thanks to (20)), ρ_0 only depends on the matrix A and $|\Lambda_k - \Lambda_l| \geq \rho_0|k - l|$ for every k, l with $1 \leq l < k < K$. Thus, we have obtained the second inequality in condition (11) for $\rho = \min\{1, \rho_0\}$. This finishes the proof of this condition.

Let us take $\eta = \max\{\tau_l, \widetilde{\tau}_\ell : 1 \le l \le p,\ 1 \le \ell \le \widetilde{p}\}$ (we are following the notations introduced in Subsection 5.1). With η and the sequence $\{\Lambda_k\}_{k\ge1}$ we can apply Theorem 1.2 and deduce the existence of a family $\mathcal{F} \equiv \{\varphi_{k,j}\}_{k\ge1, 0\le j\le \eta-1} \subset L^2(0,T;\mathbb{C})$ biorthogonal to $\left\{t^j e^{-\Lambda_k t}\right\}_{k\ge1,0\le j\le\eta-1}$ which satisfies (12).

Our objective is to apply Proposition 5.1 and, in particular, to solve the family of moment problems (53). Given $y_0 \in H^{-1}(0,\pi;\mathbb{C}^n)$, we will take as control in system (1) the function $v(t) = \widetilde{v}(T-t)$ for every $t \in (0,T)$, with $\widetilde{v}$ given by (44) and u_q $(1 \le q \le N_1)$ defined on the interval $(0,T)$ by

$$u_q(t) = \sum_{\ell=1}^{\widetilde{p}} \sum_{\nu=0}^{\widetilde{\tau}_\ell - 1} c_{\ell,\nu,q}(y_0) e^{-\mu t} \varphi_{\ell,\nu}(t) + \sum_{k>k_0} \sum_{l=1}^{p} \sum_{\sigma=0}^{\tau_l-1} d^k_{l,\sigma,q}(y_0) e^{-\mu t} \varphi_{\widetilde{p}+(k-k_0-1)+l,\sigma}(t),$$

where the coefficients $c_{\ell,\nu,q}(y_0)$ and $d^k_{l,\sigma,q}(y_0)$ are provided by Proposition 5.1 and satisfy (49) and (52).

Using the orthogonality properties of the family $\mathcal{F}$ we deduce that v and u_q solve the moment problems (53). Therefore, if we prove that $u_q \in L^2(0,T;\mathbb{C})$, we could apply Proposition 5.1 and conclude that the control $v \in L^2(0,T;\mathbb{C}^m)$ solves the null controllability problem for the coupled parabolic system (1).

Let us take $\varepsilon > 0$ (which will be chosen later). Using (12), (49) and (52) we get

$$\left\{\begin{aligned} \|u_q\|_{L^2(0,T;\mathbb{C})} &\le C(\varepsilon,T,A,B) \|e^{\mathcal{L}^*_{k_0} T}\|_{\mathcal{L}(\mathbb{C}^{nk_0})} \|y_0\|_{H^{-1}(0,\pi;\mathbb{C}^n)} \sum_{\ell=1}^{\widetilde{p}} e^{-\varepsilon \Re \gamma_\ell} \\ &\quad + C(\varepsilon,T,A,B) \sum_{k>k_0} \sum_{l=1}^{p} \frac{1}{k} \|e^{(-\lambda_k I_d + A^*)T}\|_{\mathcal{L}(\mathbb{C}^n)} e^{\varepsilon \Re(\lambda_k - \mu_l)} |y_{0,k}| \\ &\le C(\varepsilon,T,A,B) \left[\|y_0\|_{H^{-1}(0,\pi;\mathbb{C}^n)} + \sum_{k>k_0} \frac{1}{k} e^{-(T-\varepsilon)\lambda_k} |y_{0,k}| \right] \\ &\le C(\varepsilon,T,A,B) \left[1 + \sum_{k>k_0} \frac{1}{k} e^{-2(T-\varepsilon)\lambda_k} \right] \|y_0\|_{H^{-1}(0,\pi;\mathbb{C}^n)}, \end{aligned}\right. \tag{54}$$

where $C(\varepsilon,T,A,B)$ is a positive constant. Taking $\varepsilon \in (0,T)$, for example $\varepsilon = T/2$, we obtain that the series in the definition of u_q converges absolutely in $L^2(0,T;\mathbb{C})$. As a consequence $u_q \in L^2(0,T;\mathbb{C})$, for every $q : 1 \le q \le N_1$, and for a positive constant $C(T,A,B)$ the control v satisfies

$$\|v\|_{L^2(0,T;\mathbb{C}^m)} \le C \|y_0\|_{H^{-1}(0,\pi;\mathbb{C}^n)}.$$

This proves the sufficient implication of Theorem 1.1 under the hypothesis $\tau_{l,j} = \tau_l$, for all l, j : $1 \le l \le p$ and $1 \le j \le n_l$.

Case 2: Let us now prove that system (1) is null controllable at time T in the general case. Thanks to Proposition 2.4, this null controllability property is equivalent to the observability inequality (16) for the solutions φ of the adjoint system (14). Thus, let us show this observability inequality.

Following the notations introduced in Subsection 3 (see p. 8), we can write $A^* = PJ^*P^{-1}$ with $P \in \mathcal{L}(\mathbb{C}^n)$ a regular matrix and J^* the Jordan canonical form of A^* which is given by

$$J^* = \operatorname{diag}(J_1(\mu_1), J_2(\mu_2), \cdots, J_p(\mu_p)) \in \mathcal{L}(\mathbb{C}^n),$$

where

$$J_l(\mu_l) = \operatorname{diag}(J_{l,1}(\mu_l), J_{l,2}(\mu_l), \cdots, J_{l,n_l}(\mu_l)) \in \mathcal{L}(\mathbb{C}^{m_l}), \quad 1 \le l \le p,$$

$m_l = \sum_{j=1}^{n_l} \tau_{l,j}$ is the algebraic multiplicity of μ_l and $J_{l,j}(\mu_l) \in \mathcal{L}(\mathbb{C}^{\tau_{l,j}})$ $(1 \le j \le n_l)$ is the Jordan block associated to the eigenvector $v_{l,j}$ of A^*, i.e.,

$$J_{l,j}(\mu_l) = \begin{pmatrix} \mu_l & 0 & \cdots & 0 \\ 1 & \mu_l & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & \mu_l \end{pmatrix} \in \mathcal{L}(\mathbb{C}^{\tau_{l,j}}).$$

Let us set $\widetilde{B} = P^* B$. If we perform the change of variables $\varphi = P\psi$, with φ the solution to (14) associated to $\varphi_0 \in H_0^1(0,\pi;\mathbb{C}^n)$, then the observability inequality (16) is equivalent to the existence of a positive constant C_1 such that

$$\|\psi(\cdot,0)\|^2_{H_0^1(0,\pi;\mathbb{C}^n)} \le C_1 \int_0^T |\widetilde{B}^* \psi_x(0,t)|^2\, dt, \tag{55}$$

for every $\psi_0 \in H_0^1(0,\pi;\mathbb{C}^n)$, with ψ the solution to

$$\begin{cases} -\psi_t = \psi_{xx} + J^*\psi & \text{in } Q, \\ \psi(0,\cdot) = 0, \quad \psi(\pi,\cdot) = 0 & \text{on } (0,T), \\ \psi(\cdot,T) = \psi_0 & \text{in } (0,\pi). \end{cases} \tag{56}$$

Let us fix a positive integer k. With the new matrices $(J, \widetilde{B})$, we can introduce $(\widetilde{\mathcal{L}}_k, \widetilde{B}_k)$ as in (7), i.e.,

$$\begin{cases} \widetilde{B}_k = \begin{pmatrix} \widetilde{B} \\ \vdots^{k)} \\ \widetilde{B} \end{pmatrix} \in \mathcal{L}(\mathbb{C}^m;\mathbb{C}^{nk}), \quad \widetilde{\mathcal{L}}_k = \operatorname{diag}(\widetilde{L}_1, \cdots, \widetilde{L}_k) \in \mathcal{L}(\mathbb{C}^{nk}) \text{ and} \\ \widetilde{\mathcal{K}}_k = [\widetilde{\mathcal{L}}_k \,|\, \widetilde{B}_k] = [\widetilde{B}_k\,,\, \widetilde{\mathcal{L}}_k\widetilde{B}_k\,,\, \widetilde{\mathcal{L}}_k^2\widetilde{B}_k\,,\, \cdots\,,\, \widetilde{\mathcal{L}}_k^{nk-2}\widetilde{B}_k\,,\, \widetilde{\mathcal{L}}_k^{nk-1}\widetilde{B}_k] \in \mathcal{L}(\mathbb{C}^{mnk};\mathbb{C}^{nk}), \end{cases}$$

where $\widetilde{L}_i = -\lambda_i I_d + J$.

One has:

$$\operatorname{rank} \widetilde{\mathcal{K}}_k = nk, \quad \forall k \ge 1. \tag{57}$$

Indeed, if we set $\widetilde{P}_k = \operatorname{diag}(P, P, \overset{k)}{\cdots}, P) \in \mathcal{L}(\mathbb{C}^{nk})$, then $\widetilde{P}_k$ is a regular matrix, $\widetilde{B}_k = \widetilde{P}_k^* B_k$, $\widetilde{\mathcal{L}}_k = \widetilde{P}_k^* \mathcal{L}_k (\widetilde{P}_k^*)^{-1}$ and $\widetilde{\mathcal{K}}_k = \widetilde{P}_k^* \mathcal{K}_k$. From (9) we infer (57).

From the Jordan canonical form of A^* we can obtain a decomposition of $\mathbb{C}^n$ as follows: if $z \in \mathbb{C}^n$, then

$$z = \begin{pmatrix} P_1(z) \\ \vdots \\ P_p(z) \end{pmatrix} \text{ and } P_l(z) = \begin{pmatrix} P_{l,1}(z) \\ \vdots \\ P_{l,n_l}(z) \end{pmatrix},$$

with $P_l : z \in \mathbb{C}^n \mapsto P_l(z) \in \mathbb{C}^{m_l}$ and $P_{l,j} : z \in \mathbb{C}^n \mapsto P_{l,j}(z) \in \mathbb{C}^{\tau_{l,j}}$ $(1 \le l \le p,\ 1 \le j \le n_l)$.

Our next objective is to change the matrices J and $\widetilde{B}$ in order to get new matrices $\widehat{J}$ and $\widehat{B}$ such that the set $\{\mu_l\}_{1 \le l \le p}$ is also the set of distinct eigenvalues of $\widehat{J}^*$ (with the same geometric multiplicity n_l), with the property: "*for every $l : 1 \le l \le p$, the Jordan blocks of $\widehat{J}$ associated to μ_l have the same dimension $\widehat{\tau}_l$*" and for which the previous case could be applied.

To this end, let us take

$$\widehat{\tau}_l = \max_{1 \le j \le n_l} \tau_{l,j}, \quad \widehat{m}_l = n_l \widehat{\tau}_l, \quad \widehat{n} = \sum_{l=1}^{p} \widehat{m}_l$$

and

$$\widehat{J}^* = \operatorname{diag}(\widehat{J}_1(\mu_1), \widehat{J}_2(\mu_2), \cdots, \widehat{J}_p(\mu_p)) \in \mathcal{L}(\mathbb{C}^{\widehat{n}}),$$

where

$$\begin{cases} \widehat{J}_l(\mu_l) = \operatorname{diag}\left(\widehat{J}_{l,1}(\mu_l), \widehat{J}_{l,2}(\mu_l), \cdots, \widehat{J}_{l,n_l}(\mu_l)\right) \in \mathcal{L}(\mathbb{C}^{\widehat{m}_l}), \quad 1 \le l \le p, \text{ and} \\ \widehat{J}_{l,j}(\mu_l) = \begin{pmatrix} \mu_l & 0 & \cdots & 0 \\ 1 & \mu_l & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & \mu_l \end{pmatrix} \in \mathcal{L}(\mathbb{C}^{\widehat{\tau}_l}), \quad 1 \le j \le n_l \,. \end{cases}$$

In particular the Jordan blocks of $\widehat{J}^*$ associated to each eigenvalue have the same dimension.

Let us also introduce the operator $\Pi : z \in \mathbb{C}^n \mapsto \Pi z \in \mathbb{C}^{\widehat{n}}$ given by

$$\Pi z = \begin{pmatrix} \Pi_1 z \\ \vdots \\ \Pi_p z \end{pmatrix}, \ \Pi_l z = \begin{pmatrix} \Pi_{l,1} z \\ \vdots \\ \Pi_{l,n_l} z \end{pmatrix} \in \mathbb{C}^{\widehat{m}_l} \text{ and } \Pi_{l,j} : z \in \mathbb{C}^n \mapsto \Pi_{l,j} z = \begin{pmatrix} 0 \\ P_{l,j} z \end{pmatrix} \in \mathbb{C}^{\widehat{\tau}_l}.$$

Finally, if $\widetilde{B} = (\widetilde{b}_1 \,|\, \cdots \,|\, \widetilde{b}_m)$, let us set $\widehat{B} = (\Pi \widehat{b}_1 \,|\, \cdots \,|\, \Pi \widehat{b}_m)$.

With the previous notation and using the pair $(\widehat{J}, \widehat{B})$, we can also construct the corresponding matrices $\widehat{B}_k \in \mathcal{L}(\mathbb{C}^m; \mathbb{C}^{\widehat{n}k})$, $\widehat{\mathcal{L}}_k \in \mathcal{L}(\mathbb{C}^{\widehat{n}k})$ and $\widehat{\mathcal{K}}_k \in \mathcal{L}(\mathbb{C}^{m\widehat{n}k}; \mathbb{C}^{\widehat{n}k})$ as above. Thus, if $k \ge 1$ is given, one has the following properties:

1. $\sigma(\widetilde{\mathcal{L}}_k^*) = \sigma(\widehat{\mathcal{L}}_k^*) = \{-\lambda_i + \mu_l : 1 \le i \le k, \ 1 \le l \le p\}$. Moreover, the geometric multiplicity of $\mu \in \sigma(\widetilde{\mathcal{L}}_k^*) \equiv \sigma(\widehat{\mathcal{L}}_k^*)$ coincides.
2. $V \in \mathbb{C}^{nk}$ is an eigenvector of $\widetilde{\mathcal{L}}_k^*$ associated to μ if and only if $\Pi_k V \in \mathbb{C}^{\widehat{n}k}$ is an eigenvector of $\widehat{\mathcal{L}}_k^*$ associated to μ ($\Pi_k : \mathbb{C}^{nk} \to \mathbb{C}^{\widehat{n}k}$ is the operator defined as follows: if $V = (V_i)_{1 \le i \le k} \in \mathbb{C}^{nk}$, with $V_i \in \mathbb{C}^n$, then $\Pi_k V = (\Pi V_i)_{1 \le i \le k} \in \mathbb{C}^{\widehat{n}k}$). Indeed, the set of eigenvectors of the matrix $\widetilde{\mathcal{L}}_k^*$ (resp. $\widehat{\mathcal{L}}_k^*$) can be easily constructed from the eigenvectors of J^* (resp. $\widehat{J}^*$). On the other hand, it is also easy to check that $v \in \mathbb{C}^n$ is an eigenvector of J^* associated to μ_l if and only if $\Pi v \in \mathbb{C}^{\widehat{n}}$ is an eigenvector of $\widehat{J}^*$ associated to μ_l.
3. $\operatorname{rank} \widehat{\mathcal{K}}_k = \widehat{n}k$. Indeed, condition (57) holds. Using Proposition 3.1, this last condition is equivalent to:

$$\dim \operatorname{span} \{\widetilde{B}_k^* V : V \text{ is an eigenvector of } \widetilde{\mathcal{L}}_k^* \text{ associated to } \mu\} = \text{geometric multiplicity of } \mu,$$

for every $\mu \in \sigma(\widetilde{\mathcal{L}}_k^*)$. From the two previous properties we can clearly deduce:

$$\begin{aligned} &\operatorname{span} \{\widehat{B}_k^* W : W \in \mathbb{C}^{\widehat{n}k} \text{ is an eigenvector of } \widehat{\mathcal{L}}_k^* \text{ associated to } \mu\} \equiv \\ &\operatorname{span} \{\widetilde{B}_k^* V : V \in \mathbb{C}^{nk} \text{ is an eigenvector of } \widetilde{\mathcal{L}}_k^* \text{ associated to } \mu\}, \end{aligned}$$

for all $\mu \in \sigma(\widetilde{\mathcal{L}}_k^*) \equiv \sigma(\widehat{\mathcal{L}}_k^*)$. Therefore, using again Proposition 3.1, we conclude that $\operatorname{rank} \widehat{\mathcal{K}}_k = \widehat{n}k$.
4. We can apply the previous step to system (1) (with coupling matrices $(\widehat{J}, \widehat{B})$ instead of (A, B)) and conclude that this system is exactly controllable to the trajectories at time T. Equivalently, there exists a positive constant C_1 such that the observability inequality

$$\|\widehat{\psi}(\cdot, 0)\|^2_{H_0^1(0,\pi;\mathbb{C}^{\widehat{n}})} \le C_1 \int_0^T |\widehat{B}^* \widehat{\psi}_x(0,t)|^2 \, dt$$

holds for every solution $\widetilde{\psi}$ of

$$\begin{cases} -\widehat{\psi}_t = \widehat{\psi}_{xx} + \widehat{J}^* \widehat{\psi} & \text{in } Q, \\ \widehat{\psi}(0, \cdot) = 0, \quad \widehat{\psi}(\pi, \cdot) = 0 & \text{on } (0,T), \\ \widehat{\psi}(\cdot, T) = \widehat{\psi}_0 & \text{in } (0,\pi), \end{cases} \tag{58}$$

with $\widehat{\psi}_0 \in H_0^1(0, \pi; \mathbb{C}^{\widehat{n}})$.

Let us now finalize the proof of Theorem 1.1. If we fix $\psi_0 \in H_0^1(0,\pi;\mathbb{C}^n)$ and we take $\widehat{\psi}_0 = \Pi\psi_0$, the corresponding solution to (58) is given by $\widehat{\psi} = \Pi\psi$ with ψ the solution to problem (56) associated to ψ_0. The observability inequality (55) is now an easy consequence of the corresponding observability inequality established for the solutions to problem (58).

This ends the proof of Theorem 1.1. ■

The arguments given in the proof of Theorem 1.1, Proposition 3.2 and Remark 5.1 allow us to prove the following consequence:

Corollary 5.2. *Let us fix $A \in \mathcal{L}(\mathbb{C}^n)$ and $B \in \mathcal{L}(\mathbb{C}^m;\mathbb{C}^n)$. Let us assume that the algebraic Kalman condition (2) holds. Then for any $y_0 \in X_0^\perp$ there exists $v \in L^2(0,T;\mathbb{C}^m)$ such that the solution to (1) satisfies $y(\cdot,T) = 0$ in $(0,\pi)$. The space X_0 is given by*

$$X_0 = \{w : w = \sum_{1\le k\le k_0} w_k\phi_k \text{ with } w_k \in \mathbb{C}^n\},$$

where k_0 is provided in Proposition 3.2. ■

6. Further results and open problems

1. In this work we have dealt with the null controllability result for system (1). Taking into account the results in the paper, it is not difficult to prove the following approximate controllability result:

Theorem 6.1. *Let us fix $A \in \mathcal{L}(\mathbb{C}^n)$ and $B \in \mathcal{L}(\mathbb{C}^m;\mathbb{C}^n)$. Then, system (1) is approximately controllable at any time $T > 0$ if and only if*

$$\operatorname{rank} \mathcal{K}_k = nk, \quad \forall k \ge 1.$$ ■

2. Let us assume now that in system (1) $A \in \mathcal{L}(\mathbb{R}^n)$ and $B \in \mathcal{L}(\mathbb{R}^m;\mathbb{R}^n)$. In this case, if $y_0 \in H^{-1}(0,\pi;\mathbb{R}^n)$ the null control for system (1) can be chosen in $L^2(0,T;\mathbb{R}^m)$. Indeed, if $v \in L^2(0,T;\mathbb{C}^m)$ is a control for which the solution y of system (1) satisfies $y(\cdot,T) = 0$ in $(0,\pi)$, then $\Re v$ also gives the null controllability result.

3. For the sake of simplicity, we have presented our controllability result for the Laplacian operator $-\partial_{xx}$ with boundary Dirichlet conditions. It is possible to consider general second order self-adjoint differential operators R given by

$$(Ry)(x) = (p(x)y'(x))' + q(x)y(x), \quad x \in (0,\pi),$$

where $p \in C^2(0,\pi)$, $q \in C^0(0,\pi)$ and for a positive constant c_1 one has

$$0 < c_1 \le p(x), \quad x \in (0,\pi).$$

In this case it is well known that the operator R with homogeneous boundary conditions has a sequence of eigenvalues $\{\lambda_k\}_{k\ge1}$ and eigenfunctions $\{\phi_k\}_{k\ge1}$ such that

$$\lambda_k = (k+\alpha)^2 + O(1), \quad |\phi_k'(0)| = c_2\sqrt{\lambda_k} + O(1), \quad \text{for } k \to \infty,$$

with c_2 a positive constant (for instance, see [12]). The same proof of Theorem 1.1 given in this work can be easily adapted to the operator R to give the same result. Indeed, in this case we have

$$f_k(y_0,a) = -\frac{1}{\phi_k'(0)}\,(y_{0,k}\,,\, e^{(-\lambda_k I_d + A^*)T}a)_{\mathbb{C}^n}, \quad \forall (y_0,a) \in H^{-1}(0,\pi;\mathbb{C}^n)\times\mathbb{C}^n$$

$$|d^k_{1,\sigma,q}(y_0)| \le \frac{C}{|\phi_k'(0)|}\|e^{(-\lambda_k I_d + A^*)T}\|_{\mathcal{L}(\mathbb{C}^n)}|y_{0,k}|$$

instead of (37) and (52). But the asymptotic behavior is the same as before and we can obtain inequality (54).

4. As in [12], one can consider a control that depends only on time

$$\begin{cases} y_t = y_{xx} + Ay + Bgv & \text{in } Q = (0,\pi) \times (0,T), \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases} \tag{59}$$

where $A \in \mathcal{L}(\mathbb{C}^n)$ and $B \in \mathcal{L}(\mathbb{C}^m;\mathbb{C}^n)$ are two given matrices, $y_0 \in L^2(0,\pi;\mathbb{C}^n)$ is the initial datum and $g \in L^2(0,\pi;\mathbb{C})$ is a given function such that for every $\varepsilon > 0$

$$\inf_{k\geq 1} |g_k|\, e^{\varepsilon\lambda_k} > 0, \tag{60}$$

where $g_k = (g, \phi_k)_{L^2(0,T)} \in \mathbb{C}$. In system (59), $v \in L^2(0,T;\mathbb{C}^m)$ is a control function that, of course, only depends on time.

In order to deal with this controllability problem we can reason as before. In this case the control problem is

$$\begin{cases} \text{Find } v \in L^2(0,T;\mathbb{C}^m) \text{ such that} \\ -(y_0, \varphi(\cdot,0))_{L^2(0,\pi;\mathbb{C}^n)} = \displaystyle\int_0^T (v(t)\,,\, B^*(g\,,\,\varphi(\cdot,t))_{L^2(0,\pi)})_{\mathbb{C}^m}\, dt, \quad \forall \varphi_0 \in L^2(0,\pi;\mathbb{C}^n)\,. \end{cases} \tag{61}$$

In this case we can also apply the moment method and obtain the formula (36) with

$$f_k(y_0, a) = -\frac{1}{g_k}\sqrt{\frac{\pi}{2}}\,(y_{0,k}\,,\, e^{(-\lambda_k I_d + A^*)T} a)_{\mathbb{C}^n}, \quad \forall (y_0, a) \in H^{-1}(0,\pi;\mathbb{C}^n) \times \mathbb{C}^n.$$

An inspection of the proof of Theorem 1.1 shows that by using the same arguments one has (compare with (54)):

$$\|u_q\|_{L^2(0,T;\mathbb{C})} \leq C(\varepsilon, T, A, B)\left[1 + \sum_{k>k_0} \frac{1}{|g_k|} e^{-2(T-\varepsilon)\lambda_k}\right] \|y_0\|_{H^{-1}(0,\pi;\mathbb{C}^n)}$$

and therefore we obtain a solution $v \in L^2(0,T;\mathbb{C}^m)$ to (61). We have proved:

Theorem 6.2. *Let us fix $A \in \mathcal{L}(\mathbb{C}^n)$, $B \in \mathcal{L}(\mathbb{C}^m;\mathbb{C}^n)$ and $g \in L^2(0,\pi;\mathbb{C})$ satisfying (60). Then, system (59) is exactly controllable to trajectories at any time T if and only if the pair $(\mathcal{L}_k, B_k)$ is controllable for all $k \geq 1$, i.e., if and only if $\operatorname{rank} \mathcal{K}_k = nk$, for any $k \geq 1$. The matrices $\mathcal{L}_k$, B_k and $\mathcal{K}_k$ are respectively defined in (7) and (8).* ∎

5. A natural question is what happens if we consider the situation where some controls act on $x = 0$ and $x = \pi$.

Let

$$\begin{cases} y_t = y_{xx} + Ay & \text{in } Q = (0,\pi) \times (0,T), \\ y(0,\cdot) = B_1 v_1, \quad y(\pi,\cdot) = B_2 v_2 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases} \tag{62}$$

where $A \in \mathcal{L}(\mathbb{C}^n)$, $B_1 \in \mathcal{L}(\mathbb{C}^{m_1};\mathbb{C}^n)$, $B_2 \in \mathcal{L}(\mathbb{C}^{m_2};\mathbb{C}^n)$ are given matrices and $y_0 \in H^{-1}(0,\pi;\mathbb{C}^n)$ is the initial datum. In system (62), $v_1 \in L^2(0,T;\mathbb{C}^{m_1})$, $v_2 \in L^2(0,T;\mathbb{C}^{m_2})$ are the controls functions which act on the system by means of the Dirichlet boundary condition at points $x = 0$ and $x = \pi$.

We set

$$B = (B_1\,,\, B_2) \in \mathcal{L}(\mathbb{C}^{m_1+m_2};\mathbb{C}^n), \quad \forall k \geq 1.$$

Let $m = m_1 + m_2$. The null controllability problem for system (62) is equivalent to the problem

$$\begin{cases} \text{Find } v = (v_1, v_2) \in L^2(0,T;\mathbb{C}^m) \text{ such that} \\ -\langle y_0, \varphi(\cdot,0)\rangle = \displaystyle\int_0^T (v_1(t)\,,\, B_1^*\varphi_x(0,t))_{\mathbb{C}^{m_1}} - (v_2(t)\,,\, B_2^*\varphi_x(\pi,t))_{\mathbb{C}^{m_2}}\,dt, \quad \forall \varphi_0 \in H_0^1(0,\pi;\mathbb{C}^n)\,. \end{cases}$$

With the notations introduced in (6) and subsection 5.1, after simple computations, the previous problem writes:

$$\begin{cases} \text{Find } v = (v_1, v_2) \in L^2(0,T;\mathbb{C}^m) \text{ such that} \\ -\langle y_0, \varphi(\cdot,0)\rangle = \displaystyle\sum_{k\geq 1}\int_0^T \left(v(t)\,,\, \begin{pmatrix} B_1^* \\ (-1)^{k+1} B_2^* \end{pmatrix} e^{L_k^*(T-t)}\sqrt{\frac{2}{\pi}}k\varphi_{0,k}\right)_{\mathbb{C}^m} dt, \end{cases}$$

for any $\varphi_0 \in H_0^1(0,\pi;\mathbb{C}^n)$. This suggests to introduce the following matrix (compare with (7)):

$$\widetilde{B}_k = \begin{pmatrix} B_1 & B_2 \\ \vdots & \vdots \\ B_1 & (-1)^{k+1} B_2 \end{pmatrix} \in \mathcal{L}(\mathbb{C}^m;\mathbb{C}^{nk}), k \geq 1.$$

As in subsection 5.1, we derive the analogous system to (36):

$$\begin{cases} \text{Find } v \in L^2(0,T;\mathbb{C}^m) \text{ such that} \\ \displaystyle\int_0^T (v(T-t)\,,\, \widetilde{B}_{k_0}^* e^{\mathcal{L}_{k_0}^* t}\Phi_0)_{\mathbb{C}^m}\,dt = F(Y_0,\Phi_0), \quad \forall \Phi_0 \in \mathbb{C}^{nk_0}\,, \\ \displaystyle\int_0^T (v(T-t)\,, \begin{pmatrix} B_1^* \\ (-1)^{k+1} B_2^* \end{pmatrix} e^{L_k^* t} a)_{\mathbb{C}^m}\,dt = f_k(y_0,a), \quad \forall a \in \mathbb{C}^n,\ \forall k > k_0. \end{cases}$$

Following the previous ideas, it is clear that this system has a solution for any initial datum $y_0 \in H^{-1}(0,\pi;\mathbb{C}^n)$ if and only if $\operatorname{rank} \widetilde{\mathcal{K}}_k := \operatorname{rank}[\mathcal{L}_k \,|\, \widetilde{B}_k] = nk$ for every $k \geq 1$. Then we arrive to the following statement:

Theorem 6.3. *For $A \in \mathcal{L}(\mathbb{C}^n)$, $B = (B_1, B_2) \in \mathcal{L}(\mathbb{C}^m;\mathbb{C}^n)$ $(i = 1, 2)$, System (62) is exactly controllable to trajectories at any time T if and only if*

$$\operatorname{rank} \widetilde{\mathcal{K}}_k = nk, \quad \forall k \geq 1. \qquad \blacksquare$$

6. The boundary controllability problem for this kind of parabolic systems in higher dimension of space is widely open except of course in the case where $\operatorname{rank} B = n$. A first result in this direction for 2×2 parabolic systems can be found in [2].

7. Let us consider the system

$$\begin{cases} y_t = Dy_{xx} + Ay & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases} \tag{63}$$

where

$$D = \operatorname{diag}(d_1,\ldots,d_n)\,, \quad A \in \mathcal{L}(\mathbb{C}^n)\,, \quad B \in \mathcal{L}(\mathbb{C}^m;\mathbb{C}^n)\,,$$

with $d_i > 0$ for $1 \leq i \leq n$. The null controllability problem for this system is widely open. When $n = 2$, $m = 1$ and A and B are given by

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

in [15] it is proved that the approximate controllability of system (63) at time $T > 0$ holds if and only if

$$\sqrt{d_1/d_2} \notin \mathbb{Q}.$$

The null controllability problem is much more intricate; in [26] it is also showed that there are matrices D such that $\sqrt{d_1/d_2} \notin \mathbb{Q}$ (and then, system (63) is approximately controllable at any time T) for which system (63) is not null controllable for any time T.

In a forthcoming work [8], we show that we can define a Kalman condition for system (63) by replacing L_k (defined in (6)) by $-\lambda_k D + A$. The approximate controllability of this system is equivalent to the same Kalman rank condition (9). Nevertheless we cannot generalize the proof of exact controllability to trajectories. The main difference between $D = I_d$ and the previous case is that the eigenvalues of $L^* = D\partial_{xx} + A^*$ may not satisfy the separability condition in assumption (11) of Theorem 1.2.

Appendix. Proof of Corollary 4.6

We will devote the appendix to giving a proof of Corollary 4.6. To this end, we follow the arguments in [15] (see the proof of Lemma 3.2, p. 1739). Let us fix $T \in (0, \infty)$ and consider the linear spaces

$$\begin{cases} \mathcal{D}_\infty = \{\varphi : \varphi(t) = \displaystyle\sum_{j=0}^{\eta-1}\sum_{k=1}^{N} a_{kj} t^j e^{-\Lambda_k t} \ \forall t \in (0,\infty), \text{ with } N \in \mathbb{N} \text{ and } a_{kj} \in \mathbb{C}\}, \\ \mathcal{D}_T = \{\varphi : \varphi(t) = \displaystyle\sum_{j=0}^{\eta-1}\sum_{k=1}^{N} a_{kj} t^j e^{-\Lambda_k t} \ \forall t \in (0,T), \text{ with } N \in \mathbb{N} \text{ and } a_{kj} \in \mathbb{C}\}. \end{cases}$$

Evidently, $\mathcal{D}_\infty$ and $\mathcal{D}_T$ are, resp., dense spaces in $A(\Lambda, \eta, \infty)$ and $A(\Lambda, \eta, T)$ and the operator $R_T : \mathcal{D}_\infty \to \mathcal{D}_T$ is bijective. So, the proof of the result is easily obtained if we prove the existence of a positive constant $C(T)$ such that

$$\|\varphi\|_{L^2(0,\infty;\mathbb{C})} \le C(T)\|R_T\varphi\|_{L^2(0,T;\mathbb{C})}, \quad \forall \varphi \in \mathcal{D}_\infty.$$

The main idea is to use Theorem 1.2 when $T = \infty$.

By contradiction, let us assume that for any $m \ge 1$ there exists $\varphi_m \in \mathcal{D}_\infty$ such that

$$\|\varphi_m\|_{L^2(0,\infty;\mathbb{C})} = 1 \quad \text{and} \quad \|R_T\varphi_m\|_{L^2(0,T;\mathbb{C})} < \frac{1}{m}, \quad \forall m \ge 1. \tag{64}$$

Observe that $\varphi_m(t) = \sum_{j=0}^{\eta-1}\sum_{k=1}^{N(m)} a_{kj}^{(m)} t^j e^{-\Lambda_k t}$ in $(0,\infty)$, for some $N(m) \in \mathbb{N}$ and $a_{kj}^{(m)} \in \mathbb{C}$. Using the properties of the biorthogonal family $\{\varphi_{k,j}\}_{k\ge1, 0\le j\le\eta-1}$ to $\{e_{kj}\}_{k\ge1,0\le j\le\eta-1}$ in $L^2(0,\infty;\mathbb{C})$ (see (12)), we deduce that for any $\varepsilon > 0$ there exists a positive constant $C(\varepsilon)$ such that

$$|a_{kj}^{(m)}| = |(\varphi_m, \varphi_{k,j})_{L^2(0,\infty;\mathbb{C})}| \le \|\varphi_m\|_{L^2(0,\infty;\mathbb{C})}\|\varphi_{k,j}\|_{L^2(0,\infty;\mathbb{C})} \le C(\varepsilon)e^{\varepsilon\Re\Lambda_k},$$

for any $(k,j) : 1 \le k \le N(m),\ 0 \le j \le \eta - 1$.

Let us fix $\varepsilon \in (0, T/3)$ and let us define

$$\mathcal{U}_\varepsilon = \{z \in \mathbb{C} : \Re z > 3\varepsilon,\ |\Im z| < \left(\delta^{-2} - 1\right)^{-1/2}\varepsilon\},$$

with $\delta > 0$ given in (11). Observe that thanks to (11) one has $\Im\Lambda_k \le \left(\delta^{-2} - 1\right)^{1/2}\Re\Lambda_k$ and, if $z \in \mathcal{U}_\varepsilon$, we can estimate

$$|e^{-\Lambda_k z}| = e^{\Im\Lambda_k\Im z - \Re\Lambda_k\Re z} \le e^{-(\Re z - \varepsilon)\Re\Lambda_k}, \quad \forall k \ge 1.$$

We will use the previous estimates to bound φ_m in the set $\mathcal{U}_\varepsilon$. Thus, for $z \in \mathcal{U}_\varepsilon$ one has

$$\begin{cases} |\varphi_m(z)| \leq \displaystyle\sum_{j=0}^{\eta-1} \sum_{k=1}^{N(m)} |a_{kj}^{(m)}||z|^j e^{-(\Re z-\varepsilon)\Re\Lambda_k} \leq C(\varepsilon) Q_\eta(|z|) \sum_{k=1}^{N(m)} e^{-(\Re z-2\varepsilon)\Re\Lambda_k} \\ \qquad \leq C(\varepsilon) e^{-m_1(\Re z-2\varepsilon)} Q_\eta(|z|) \displaystyle\sum_{k=1}^{N(m)} e^{-\varepsilon[\Re\Lambda_k - m_1]} \leq \widetilde{C}(\varepsilon) e^{-m_1(\Re z-2\varepsilon)} Q_\eta(|z|), \end{cases}$$

where $m_1 = \min_{k\geq 1} \Re\Lambda_k > 0$ and Q_η is the polynomial given by $G(s) = 1 + s + \cdots + s^{\eta-1}$.

From the last inequality, we infer that the holomorphic function φ_m is uniformly bounded in $\mathcal{U}_\varepsilon$ and has a subsequence (still denoted by φ_m) which converges uniformly on the compact sets of $\mathcal{U}_\varepsilon$ to φ, a holomorphic function in $\mathcal{U}_\varepsilon$. In particular, $\varphi_m(t) \to \varphi(t)$, for every $t \in (3\varepsilon, \infty)$, and

$$|\varphi_m(t)| \leq \widetilde{C}(\varepsilon) e^{-m_1(t-2\varepsilon)} Q_\eta(t), \quad \forall t \in (3\varepsilon, \infty).$$

We can apply the Lebesgue theorem and deduce that $\varphi_m \to \varphi$ in $L^2(3\varepsilon, \infty; \mathbb{C})$. From (64), we also have that the holomorphic function φ satisfies $\varphi(t) = 0$ for any $t \in (3\varepsilon, T)$. Therefore $\varphi \equiv 0$ in $\mathcal{U}_\varepsilon$. Summarizing, we have proved that $\varphi_m \to 0$ in $L^2(3\varepsilon, \infty; \mathbb{C})$ for any $\varepsilon \in (0, T/3)$ and $\|\varphi_m\|_{L^2(0,\infty;\mathbb{C})} = 1$ for every $m \in \mathbb{N}$. This is, evidently, absurd. This ends the proof. ■

Acknowledgements

The authors wish to thank Institut Henri Poincaré (Paris, France) for providing a very stimulating environment during the "Control of Partial and Differential Equations and Applications" program in the Fall 2010. The last draft of this paper was done during the cited trimester.

The authors also wish to thank the referee for his/her interesting comments and suggestions.

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