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THE KAPLANSKY CONDITION AND RINGS OF ALMOST STABLE RANGE 1

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Abstract. We present some variants of the Kaplansky condition for a K-Hermite ring R to be an elementary divisor ring. For example, a commutative K-Hermite ring R is an EDR iff for any elements $x, y, z \in R$ such that $(x, y) = R$ there exists an element $\lambda \in R$ such that $x + \lambda y = uv$, where $(u, z) = (v, 1 - z) = R$.

We present an example of a Bézout domain that is an elementary divisor ring but does not have almost stable range 1, thus answering a question of Warren Wm. McGovern.

1. Introduction

First we recall some basic definitions and known results.

All rings here are commutative with unity. A ring R is *Bézout* if each finitely generated ideal of R is principal.

Two rectangular matrices A and B in $\mathrm{M}_{m,n}(R)$ are *equivalent* if there exist invertible matrices $P \in \mathrm{M}_{m,m}(R)$ and $Q \in \mathrm{M}_{n,n}(R)$ such that $B = PAQ$.

The ring R is *K-Hermite* if every rectangular matrix A over R is equivalent to an upper or a lower triangular matrix (following [9, Appendix to §4] we use the term 'K-Hermite' rather than 'Hermite' as in [8]). From [8] it follows that this definition is equivalent to the definition given there. See also [5, Theorem 3]: by this theorem, a ring is K-Hermite iff for every two elements $a, b \in R$, there are elements $a_1, b_1, d \in R$ such that $(a, b) = (a_1 d + b_1 d)$ and $(a_1, b_1) = R$. Parentheses are used to denote the ideal generated by the specified elements.

A ring R is an *elementary divisor ring (EDR)* iff every rectangular matrix A over R is equivalent to a diagonal matrix. It follows from [8] that this definition is equivalent to the definition given there.

An EDR is K-Hermite, and a K-Hermite ring is Bézout. An integral domain is Bézout iff it is K-Hermite.

By [5, Theorem 6] a ring R is an EDR iff it satisfies the following two conditions:

(1) R is K-Hermite;
(2) R satisfies Kaplansky's condition (see §2, condition (K) below).

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By [4, Example 4.11], (1) $\not\Longrightarrow$ (2). The question in [6] whether a Bézout domain is an EDR (equivalently, whether it satisfies Kaplansky's condition) is still open. On the other hand, (2) $\not\Longrightarrow$ (1) (Remark 2.1 below).

In section 2, we elaborate on the Kaplansky condition.

A row $[r_1, \ldots, r_n]$ over a ring R is *unimodular* if the elements $r_1, \ldots, r_n$ generate the ideal R. The *stable range* $\mathrm{sr}(R)$ of a ring R is the least integer $n \geq 1$ (if it exists) such that for any unimodular row $[r_1, ..., r_{n+1}]$ over R, there exist $t_1, ..., t_n \in R$ such that the row

$$[r_1 + t_1 r_{n+1}, ..., r_n + t_n r_{n+1}]$$

is unimodular (see comments on [9, Theorem 5.2, Ch. VIII]). For background on stable range see [1, §3, Ch. V].

The ring R has *almost stable range* 1 if every proper homomorphic image of R has stable range 1 (see [10]). By [10, Theorem 3.7] a Bézout ring with almost stable range 1 is an EDR. We elaborate on the almost stable range 1 condition in §3. In particular, we present an elementary divisor domain (so Bézout) that does not have almost stable range 1, thus answering the question of Warren Wm. McGovern in [10] (Example 3.3). By Remark 3.2 below, a ring of stable range 1 is of almost stable range 1. On the other hand, $\mathbb{Z}$ is of almost stable range 1 but is not of stable range 1. Indeed, the stable range of $\mathbb{Z}$ is 2: clearly, there is no integer m such that $2 + 5m = \pm 1$. Thus $\mathrm{sr}\,\mathbb{Z} > 1$. On the other hand, the stable range of any Bézout domain is ≤ 2; hence $\mathrm{sr}\,\mathbb{Z} = 2$. More generally, the stable range of any K-Hermite ring is ≤ 2 [11, Proposition 8]. Also by [12, Theorem 1], a Bézout ring is K-Hermite iff it is of stable range ≤ 2.

For general background see [8], [3, §6, Ch. 3] and [10].

2. On the Kaplansky condition

By [5] a K-Hermite ring R is an elementary divisor ring iff it satisfies Kaplansky's condition (see [8, Theorem 5.2]):

(K) *For any three elements a, b, c in R that generate the ideal R, there exist elements $p, q \in R$ so that $(pa, pb + qc) = R$.*

Remark 2.1. A local ring R is of stable range 1; thus R satisfies Kaplansky's condition with $p = 1$. If R is also a Noetherian domain, then R is K-Hermite iff R is a principal ideal ring. Hence a Noetherian local domain that is not a principal ideal ring is of stable range 1 but is not K-Hermite.

In the proof of Lemma 2.3 below, we will use the following well-known fact:

Remark 2.2. Let R be a ring, let A be a matrix in $\mathrm{M}_{m,n}(R)$, let $\mathbf{r}$ be a row in $\mathrm{M}_{1,n}(R)$, and let $1 \leq k \leq n$. Then $\mathbf{r}$ belongs to the submodule of R^n generated by the rows of the matrix A iff there exists a matrix $C \in \mathrm{M}_{k,m}(R)$ such that $\mathbf{r}$ is the first row of the matrix CA.

Lemma 2.3. *Let A be a 2×2-matrix over a ring R, and let $\mathbf{u}$ be a unimodular row of length 2 over R. Then $\mathbf{u}$ belongs to the submodule of R^2 generated by the rows of A $\iff$ there exists an invertible matrix P so that $\mathbf{u}$ is the first row of PA.*

Proof. ($\Longrightarrow$) : By Remark 2.2, there exists a 2-row $\mathbf{r}$ over R so that $\mathbf{u} = \mathbf{r}A$. Since the row $\mathbf{u}$ is unimodular, the row $\mathbf{r}$ is also unimodular. Since $\mathbf{r}$ is unimodular of length 2, there exists an invertible matrix P with its first row equal to $\mathbf{r}$. Thus $\mathbf{u}$ is the first row of the matrix PA.

($\Longleftarrow$) : This follows from Remark 2.2. □

Lemma 2.4. *Let A be a 2×2 matrix over a ring R so that its entries generate the ideal R. Then A is equivalent to a diagonal matrix $\Longleftrightarrow$ the submodule of R^2 generated by the rows of A contains a unimodular row.*

In this case A is equivalent to a matrix of the form $\left(\begin{smallmatrix}1&0\\0&*\end{smallmatrix}\right)$.

Proof. ($\Longrightarrow$) : By assumption, A is equivalent to a diagonal matrix $D=\left(\begin{smallmatrix}d_1&0\\0&d_2\end{smallmatrix}\right)$, where $d_1,d_2\in R$. Since the entries of A generate the ideal R, d_1,d_2 also generate the ideal R. The sum of the rows of D, namely $[d_1,d_2]$, is unimodular.

($\Longleftarrow$) : By Lemma 2.3, the matrix A is equivalent over R to a matrix B with first row unimodular. Hence the submodule generated by the columns of B contains a column of the form $\left(\begin{smallmatrix}1\\ *\end{smallmatrix}\right)$. By Lemma 2.3 again (for columns), we obtain that A is equivalent to a matrix $\left(\begin{smallmatrix}1&r\\ *&*\end{smallmatrix}\right)$. By subtracting the first column of the matrix $\left(\begin{smallmatrix}1&r\\ *&*\end{smallmatrix}\right)$ multiplied by r from its second column and by a similar elementary row transformation, we obtain a diagonal matrix of the form $\left(\begin{smallmatrix}1&0\\0&*\end{smallmatrix}\right)$. □

Theorem 2.5 (see [8, Theorem 5.2] and [5, Corollary 5])**.** *Let R be a ring. Let $A=\left(\begin{smallmatrix}a&b\\0&c\end{smallmatrix}\right)$ a triangular 2×2-matrix over R so that $(a,b,c)=R$. Then A is equivalent to a diagonal matrix over R iff there exist elements p,q in R so that $(pa,pb+qc)=R$.*

Proof. Since $p[a,b]+q[0,c]=[pa,pb+qc]$ for any elements $p,q\in R$, the theorem follows from Lemma 2.4. □

Remark 2.6. Let R be any ring. If Kaplansky's condition $(pb+qc,pa)=R$ holds for elements $a,b,c,p,q\in R$, then

$$(pb+qc,a)=(p,c)=R.$$

Indeed, Kaplansky's condition implies that

$$(pb+qc,a)=(pb+qc,p)=R,$$

so $(p,c)=R$. Cf. the next proposition. □

Proposition 2.7. *Let R be a K-Hermite ring, and let a,b and c be elements of R that generate the ideal R. Then the following four conditions are equivalent:*

(1) *The matrix $\left(\begin{smallmatrix}a&b\\0&c\end{smallmatrix}\right)$ is equivalent to a diagonal matrix.*
(2) *There exist elements p,q in R so that $(pa,pb+qc)=R$.*
(3) *There exist elements p and q in R so that $(pb+qc,a)=(p,c)=R$.*
(4) *For some elements $\lambda,u,v\in R$ we have $b+\lambda c=uv$, and $(u,a)=(v,c)=R$.*

Moreover, in (4) we may choose the elements u and v such that $(u,v)=R$.

Proof. (1) $\Longleftrightarrow$ (2) : This follows from Theorem 2.5.

(2) $\Longrightarrow$ (4) : Since $(pa,pb+qc)=R$ we obtain $R=(p,pb+qc)=(p,qc)$, so $(p,(pb+qc)c)=R$. Let v be an element of R so that

$$vp\equiv 1\,(\mathrm{mod}\ (pb+qc)c);$$

thus $vp\equiv 1 \pmod c$. We have $v(pb+qc)\equiv b \pmod c$, so $v(pb+qc)=b+\lambda c$ for some element $\lambda\in R$. Hence $b+\lambda c=uv$, where $u=pb+qc$; thus $(u,a)=(v,c)=(u,v)=R$.

(4) $\implies$ (3) : We have $b \equiv uv \pmod{c}$. Let $p \in R$ so that $pv \equiv 1 \pmod{c}$. Hence $pb \equiv u \pmod{c}$, so there exists an element $q \in R$ such that $pb + qc = u$. Thus (3) holds.

(3) $\implies$ (2) : Since R is a K-Hermite ring, we may write $(d) = (p, q)$ and $d = p_1 p + q_1 q$ with $(p_1, q_1) = R$. Hence

$$(p_1, p_1 b + q_1 c) = (p_1, q_1 c) = R,$$

so $(p_1 a, p_1 b + q_1 c) = (p_1, c) = R$. Condition (2) holds with p and q replaced by p_1 and q_1, respectively. $\square$

In the proof of Proposition 2.7, we used the assumption that R is K-Hermite just for the implication (3) $\implies$ (2).

Remark 2.8. If R is a Bézout domain, then the following condition is equivalent to the conditions of Proposition 2.7:

($*$) For some elements $\lambda, a_1, c_1 \in R$ we have

$$b + \lambda c \mid (1 - a_1 a)(1 - c_1 c).$$

Indeed, assume condition ($*$). Let $u \in R$ so that

$$(u) = (b + \lambda c, 1 - a_1 a);$$

thus $(u, a) = R$ and $\frac{b+\lambda c}{u} \mid \left(\frac{1-a_1 a}{u}\right)(1 - c_1 c)$. Since $\left(\frac{b+\lambda c}{u}, \frac{1-a_1 a}{u}\right) = R$, we see that $v := \frac{b+\lambda c}{u}$ divides $1 - c_1 c$, so $(v, c) = R$. Thus condition ($*$) implies condition (4) of Proposition 2.7. The converse implication is obvious. $\square$

Since a K-Hermite ring is an EDR iff each matrix of the form $\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)$ with $(a, b, c) = (1)$ has a diagonal reduction [8], Proposition 2.7 provides necessary and sufficient conditions for a K-Hermite ring to be an EDR. We present an additional condition in the next proposition.

Theorem 2.9. *Let R be a K-Hermite ring. The following two conditions are equivalent:*

(1) *R is an elementary divisor ring.*
(2) *For any elements $x, y, z \in R$ such that $(x, y) = R$, there exists an element $\lambda \in R$ such that $x + \lambda y = uv$, where $(u, z) = (v, 1 - z) = R$.*

Moreover, the elements u and v can be chosen such that $(u, v) = R$.

Proof. (1) $\implies$ (2) [including the requirement that $(u, v) = (1)$]: We apply condition (4) of Proposition 2.7 to the elements $a = z, b = x, c = y(1 - z)$.

(2) $\implies$ (1) : We verify condition (4) of Proposition 2.7. Let $(a, b, c) = R$. Let $(d) = (b, c)$; thus $(d, a) = (b, c, a) = R$. Hence $a \mid 1 - d_1 d$ for some element $d_1 \in R$. Also $b = b_1 d, c = c_1 d$, where $(b_1, c_1) = R$. We apply condition (2) of the present proposition to the elements

$$x = b_1, y = c_1, z = d_1 d.$$

Thus there are elements $\lambda_1, u_1, v \in R$ so that $b_1 + \lambda_1 c_1 = u_1 v$, where $(u_1, 1 - d_1 d) = (v, d_1 d) = R$. Let $u = d u_1$; thus $(u, a) = 1$. Let $\lambda = \lambda_1 d$. Hence $b + \lambda c = d(b_1 + \lambda_1 c_1) = uv$ and $(u, a) = R$. We have $(v, c) = (v, d c_1) = (v, c_1)$ since $(v, d) = R$. Since v divides $b_1 + \lambda_1 c_1$, it follows that $(v, c_1) \mid b_1$, so $(v, c_1) = R$. Thus $(v, c) = R$, as required. $\square$

Proposition 2.10. *Let R be a Bézout domain. The following two conditions are equivalent:*

(1) *R is an elementary divisor ring.*
(2) *For any nonzero elements $x, y, z \in R$, there exist elements $\lambda, a, b \in R$ such that $x + \lambda y \mid y(1 - az)(1 - b(1 - z))$ in R.*

Proof. (1) $\implies$ (2) : Let $(d) = (x, y)$; thus $\frac{x}{d}$ and $\frac{y}{d}$ are comaximal. By Theorem 2.9, there are elements $\lambda, a, b \in R$ so that $(\frac{x}{d} + \lambda \frac{y}{d}) \mid (1 - az)(1 - bz(1 - z))$. Hence $x + \lambda y \mid d(1 - az)(1 - b(1 - z))$, so $x + \lambda y \mid y(1 - az)(1 - b(1 - z))$.

(2) $\implies$ (1) : Let x_0 and y_0 be comaximal elements in R, and let $z \in R$. Thus $(x_0 + \lambda y_0) \mid y_0(1 - az)(1 - b(1 - z))$ for some elements $\lambda, a, b \in R$. Since the elements $x_0 + \lambda y_0$ and y_0 are comaximal, we obtain that $(x_0 + \lambda y_0) \mid (1 - az)(1 - b(1 - z))$, so R is an EDR by Remark 2.8. ☐

3. On rings of almost stable range 1

Proposition 3.1. *Let R be any ring. The following conditions are equivalent:*

(1) *R is of almost stable range 1.*
(2) *For each nonzero element $z \in R$, the ring R/zR is of stable range 1.*
(3) *For every three elements $x, y, z \in R$ such that $(x, y) = R$ and $z \neq 0$, there exists an element $\lambda \in R$ such that $(x + \lambda y, z) = R$.*

Proof (Cf. [1, Proposition 3.2, Ch. V]).

(1) $\implies$ (2) $\implies$ (3): Clear.

(3) $\implies$ (1) : Let I be a nonzero ideal of R and let z be a nonzero element in I. Let $x + I, y + I$ be two comaximal elements in R/I. Hence there exist elements $r, s \in R$ such that $1 - rx - sy \in I$. By assumption, there exists an element $\lambda \in R$ such that $(x + \lambda(1 - rx), z)R = R$. Thus $x + \lambda sy$ is invertible modulo the ideal I. It follows that R/I is of stable range 1, so R is almost of stable range 1. ☐

Remark 3.2. The implication (3) $\implies$ (1) in the previous proposition is clear since if T is a homomorphic image of a ring R with finite stable range, then $\mathrm{sr}(T) \leq \mathrm{sr}(R)$ [1, Proposition 3.2, Ch. V], although this fact was not used explicitly but rather its proof (in the above proof of the implication (2) $\implies$ (3) we have $\mathrm{sr}(R/I) \leq \mathrm{sr}(R/(z)) = 1$). This fact implies that if R is an arbitrary ring of stable range 1, then R is of almost stable range 1, thus answering the question in [10, Remark 3.3]. See also [10, Proposition 3.2].

As we have seen in §2, the stable range 1 property implies Kaplansky's condition for an arbitrary ring. The converse is false since even if R is an elementary divisor domain so that R satisfies Kaplansky's condition, R does not necessarily have almost stable range 1.

Example 3.3. An elementary divisor domain (and so Bézout) that does not have almost stable range 1 (this example answers the question in [10, Remark 4.7]).

We use a well-known example of a Bézout domain, namely, $R = \mathbb{Z} + X\mathbb{Q}[X]$ (for a general theorem on pullbacks of Bézout domains, see [7, Theorem 1.9]). R is an elementary divisor ring by [2, Theorem 4.61]. However, $R/X\mathbb{Q}[X]$ is isomorphic to $\mathbb{Z}$ and $\mathrm{sr}\,\mathbb{Z} = 2$. Hence R does not have almost stable range 1. ☐

We conjecture that a Bézout domain that is a pullback of type □ (as defined in [7]) of elementary divisor domains is again an EDR. In this case the conditions of [7, Theorem 1.9] must be satisfied. If this conjecture proves to be false, this will yield a negative answer to the question in [6] as to whether a Bézout domain is an EDR.

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