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The Kronecker Product of Schur Functions Indexed by Two-Row Shapes or Hook Shapes

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Abstract. The Kronecker product of two Schur functions s_μ and s_ν, denoted by $s_\mu * s_\nu$, is the Frobenius characteristic of the tensor product of the irreducible representations of the symmetric group corresponding to the partitions μ and ν. The coefficient of s_λ in this product is denoted by $\gamma^\lambda_{\mu\nu}$, and corresponds to the multiplicity of the irreducible character χ^λ in $\chi^\mu\chi^\nu$.

We use Sergeev's Formula for a Schur function of a difference of two alphabets and the comultiplication expansion for $s_\lambda[XY]$ to find closed formulas for the Kronecker coefficients $\gamma^\lambda_{\mu\nu}$ when λ is an arbitrary shape and μ and ν are hook shapes or two-row shapes.

Remmel (J.B. Remmel, *J. Algebra* **120** (1989), 100–118; *Discrete Math.* **99** (1992), 265–287) and Remmel and Whitehead (J.B. Remmel and T. Whitehead, *Bull. Belg. Math. Soc. Simon Stiven* **1** (1994), 649–683) derived some closed formulas for the Kronecker product of Schur functions indexed by two-row shapes or hook shapes using a different approach. We believe that the approach of this paper is more natural. The formulas obtained are simpler and reflect the symmetry of the Kronecker product.

Keywords: Kronecker product internal product, Sergeev's formula

1. Introduction

The aim of this paper is to derive an explicit formula for the Kronecker coefficients corresponding to partitions of certain shapes. The Kronecker coefficients, $\gamma^\lambda_{\mu\nu}$, arise when expressing a Kronecker product (also called inner or internal product), $s_\mu * s_\nu$, of Schur functions in the Schur basis,

$$s_\mu * s_\nu = \sum_\lambda \gamma^\lambda_{\mu\nu} s_\lambda. \tag{1}$$

These coefficients can also be defined as the multiplicities of the irreducible representations in the tensor product of two irreducible representations of the symmetric group. A third way to define them is by the comultiplication expansion. Given two alphabets $X = x_1 + x_2 + \cdots$ and $Y = y_1 + y_2 + \cdots$, expressed as the sum of its elements, the comultiplication expansion is given by

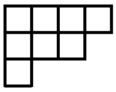
$$s_\lambda[XY] = \sum_{\mu,\nu} \gamma^\lambda_{\mu\nu} s_\mu[X] s_\nu[Y], \tag{2}$$

where $s_\lambda[X]$ means $s_\lambda(x_1, x_1, \ldots)$ and $s_\lambda[XY]$ means $s_\lambda(x_1 y_1, x_1 y_2, \ldots, x_i y_j, \ldots)$. Remmel [9, 10] and Remmel and Whitehead [11] have studied the Kronecker product of Schur functions corresponding to two two-row shapes, two hook shapes, and a hook shape and a two-row shape. We will use the comultiplication expansion (2) for the Kronecker coefficients, and a formula for expanding a Schur function of a difference of two alphabets due to Sergeev [1, 14] to obtain similar results in a simpler way. We believe that the formulas obtained using this approach are elegant and reflect the symmetry of the Kronecker product. In the three cases we found a way to express the Kronecker coefficients in terms of regions and paths in $\mathbf{N}^2$.

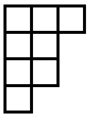
2. Basic definitions

A partition λ of a positive integer n, written as $\lambda \vdash n$, is an unordered sequence of natural numbers adding to n. We write λ as $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, where $\lambda_1 \geq \lambda_2 \geq \ldots$, and consider two such strings equal if they differ by a string of trailing zeroes. The nonzero numbers λ_i are called the parts of λ, and the number of parts is called the length of λ, denoted by $l(\lambda)$. In some cases, it is convenient to write $\lambda = (1^{d_1} 2^{d_2} \cdots n^{d_n})$ for the partition of n that has d_i equals to i. Using this notation, we define the integer z_λ to be $1^{d_1} d_1!\, 2^{d_2} d_2! \cdots n^{d_n} d_n!$.

We identify λ with the set of points (i, j) in $\mathbf{N}^2$ defined by $1 \leq j \leq \lambda_i$, and refer to them as the Young diagram of λ. The Young diagram of a partition λ is thought of as a collection of boxes arranged using matrix coordinates. For instance, the Young diagram corrresponding to $\lambda = (4, 3, 1)$ is



To any partition λ we associate the partition λ', its conjugate partition, defined by $\lambda'_i = |\{j : \lambda_j \geq i\}|$. Geometrically, λ' can be obtained from λ by flipping the Young diagram of λ around its main diagonal. For instance, the conjugate partition of $\lambda = (4, 3, 1)$ is $\lambda' = (3, 2, 2, 1)$, and the corresponding Young diagram is



We recall some facts about the theory of representations of the symmetric group, and about symmetric functions. See [7] or [13] for proofs and details.

Let $R(S_n)$ be the space of class function in S_n, the symmetric group on n letters, and let Λ^n be the space of homogeneous symmetric functions of degree n. A basis for $R(S_n)$ is given by the characters of the irreducible representations of S_n. Let χ^μ be the irreducible character of S_n corresponding to the partition μ. There is a scalar product $\langle\, , \rangle_{S_n}$ on $R(S_n)$

defined by

$$\langle \chi^{\mu}, \chi^{\nu} \rangle_{S_n} = \frac{1}{n!} \sum_{\sigma \in S_n} \chi^{\mu}(\sigma) \chi^{\nu}(\sigma),$$

and extended by linearity.

A basis for the space of symmetric functions is given by the Schur functions. There exists a scalar product $\langle\, , \rangle_{\Lambda^n}$ on Λ^n defined by

$$\langle s_\lambda, s_\mu \rangle_{\Lambda^n} = \delta_{\lambda\mu},$$

where $\delta_{\lambda\mu}$ is the Kronecker delta, and extended by linearity.

Let p_μ be the power sum symmetric function corresponding to μ, where μ is a partition of n. There is an isometry $\mathrm{ch}^n : R(S_n) \mapsto \Lambda^n$, given by the characteristic map,

$$\mathrm{ch}^n(\chi) = \sum_{\mu \vdash n} z_\mu^{-1} \chi(\mu) p_\mu.$$

This map has the remarkable property that if χ^λ is the irreducible character of S_n indexed by λ, then $\mathrm{ch}^n(\chi^\lambda) = s_\lambda$, the Schur function corresponding to λ. In particular, we obtain that $s_\lambda = \sum_{\mu \vdash n} z_\mu^{-1} \chi^\lambda(\mu) p_\mu$.

Finally, we use the fact that the power sum symmetric functions form an orthogonal basis satisfying that $\langle p_\lambda, p_\mu \rangle_{\Lambda^n} = z_\mu \delta_{\lambda\mu}$ to obtain

$$\chi^\lambda(\mu) = \langle s_\lambda, p_\mu \rangle. \tag{3}$$

Let λ, μ, and ν be partitions of n. The Kronecker coefficients $\gamma^{\lambda}_{\mu\nu}$ are defined by

$$\gamma^{\lambda}_{\mu\nu} = \langle \chi^\lambda, \chi^\mu \chi^\nu \rangle_{S_n} = \frac{1}{n!} \sum_{\sigma \in S_n} \chi^\lambda(\sigma) \chi^\mu(\sigma) \chi^\nu(\sigma). \tag{4}$$

Equation (4) shows that the Kronecker coefficients $\gamma^{\lambda}_{\mu\nu}$ are symmetric in λ, μ, and ν. The relevance of the Kronecker coefficients comes from the following fact: Let X^μ be the representation of the symmetric group corresponding to the character χ^μ. Then $\chi^\mu \chi^\nu$ is the character of $X^\mu \otimes X^\nu$, the representation obtained by taking the tensor product of X^μ and X^ν. Moreover, $\gamma^{\lambda}_{\mu\nu}$ is the multiplicity of X^λ in $X^\mu \otimes X^\nu$.

Let f and g be homogeneous symmetric functions of degree n. The Kronecker product, $f * g$, is defined by

$$f * g = \mathrm{ch}^n(uv), \tag{5}$$

where $u = (\mathrm{ch}^n)^{-1}(f)$, and $v = (\mathrm{ch}^n)^{-1}(g)$, and $uv(\sigma) = u(\sigma)v(\sigma)$. To obtain (1) from this definition, we set $f = s_\mu$, $g = s_\nu$, $u = \chi^\mu$, and $v = \chi^\nu$ in (5).

The Kronecker product has the following symmetries:

$$s_\mu * s_\nu = s_\nu * s_\mu.$$
$$s_\mu * s_\nu = s_{\mu'} * s_{\nu'}.$$

Moreover, if λ is a one-row shape

$$\gamma^\lambda_{\mu\nu} = \delta_{\mu,\nu}.$$

We introduce the operation of substitution or plethysm into a symmetric function. Let f be a symmetric function, and let $X = x_1 + x_2 + \cdots$ be an alphabet expressed as the sum of its elements. We define $f[X]$ by

$$f[X] = f(x_1, x_2, \ldots).$$

In general, if u is any element of $\mathbf{Q}[[x_1, x_2, \ldots]]$, we write u as $\sum_\alpha c_\alpha u_\alpha$ where u_α is a monomial with coefficient 1. Then $p_\lambda[u]$ is defined by setting

$$\begin{aligned} p_n[u] &= \sum_\alpha c_\alpha u_\alpha^n \\ p_\lambda[u] &= p_{\lambda_1}[u] \cdots p_{\lambda_n}[u] \end{aligned}$$

for $\lambda = (\lambda_1, \ldots, \lambda_n)$. We define $f[u]$ for all symmetric functions f by saying that $f[u]$ is linear in f.

Let $X = x_1 + x_2 + \cdots$ and $Y = y_1 + y_2 + \cdots$ be two alphabets as the sum of their elements. We define their sum by $X + Y = x_1 + x_2 + \cdots + y_1 + y_2 + \cdots$, and the product by $XY = x_1 y_1 + \cdots + x_i y_j + \cdots$. Then

$$\begin{aligned} p_n[X+Y] &= p_n[X] + p_n[Y], \\ p_n[XY] &= p_n[X] p_n[Y]. \end{aligned} \tag{6}$$

The inner product of function in the space of symmetric functions in two infinite alphabets is defined by

$$\langle\, , \rangle_{XY} = \langle\, , \rangle_X \langle\, , \rangle_Y,$$

where for any given alphabet Z, $\langle\, , \rangle_Z$ denotes the inner product of the space of symmetric functions in Z.

For all partitions ρ, we have that $p_\rho[XY] = p_\rho[X] p_\rho[Y]$. If we rewrite (3) as $p_\rho = \sum_\lambda \chi^\lambda(\rho) s_\lambda$, then

$$\sum_\lambda \chi^\lambda s_\lambda[XY] = \sum_{\mu,\nu} \chi^\mu \chi^\nu s_\mu[X] s_\nu[Y]. \tag{7}$$

Taking the coefficient of χ^λ on both sides of the previous equation we obtain

$$s_\lambda[XY] = \sum \langle \chi^\lambda, \chi^\mu \chi^\nu \rangle s_\mu[X] s_\nu[Y].$$

Finally, using the definition of Kronecker coefficients (4) we obtain the comultiplication expansion (2).

Notation 1 Let p be a point in $\mathbf{N}^2$. We say that (i, j) can be reached from p, written $p \rightsquigarrow (i, j)$, if (i, j) can be reached from p by moving any number of steps south west or north west, when we use the coordinate axes as it is usually done in the cartesian plane. We define the weight function ω by

$$\omega_p(i, j) = \begin{cases} x^i y^j, & \text{if } p \rightsquigarrow (i, j), \\ 0, & \text{otherwise.} \end{cases}$$

Notation 2 We denote by $\lfloor x \rfloor$ the largest integer less than or equal to x and by $\lceil x \rceil$ the smallest integer greater than or equal to x.

If f is a formal power series, then $[x^\alpha]\, f$ denotes the coefficient of x^α in f.

Following Donald Knuth we denote the characteristic function applied to a proposition P by enclosing P with brackets,

$$((P)) = \begin{cases} 1, & \text{if proposition } P \text{ is true,} \\ 0, & \text{otherwise.} \end{cases}$$

We use double brackets to distinguish between the Knuth's brackets and the standard ones.

3. The case of two two-row shapes

The object of this section is to find a closed formula for the Kronecker coefficients when $\mu = (\mu_1, \mu_2)$ and $\nu = (\nu_1, \nu_2)$ are two-row shapes, and when we do not have any restriction on the partition λ. We describe the Kronecker coefficients $\gamma^\lambda_{\mu\nu}$ in terms of paths in $\mathbf{N}^2$. More precisely, we define two rectangular regions in $\mathbf{N}^2$ using the parts of λ. Then we count the number of points in $\mathbf{N}^2$ inside each of these rectangles that can be reached from $(\nu_2, \mu_2 + 1)$, if we are allowed to move any number of steps south west or north west. Finally, we subtract these two numbers.

We begin by introducing two lemmas that allow us to state Theorem 1 in a concise form. Note that we use the coordinate axes as it is usually done in the cartesian plane.

Lemma 1 *Let k and l be positive numbers. Let R be the rectangle with width k, height l, and lower–left square* $(0, 0)$. *Define*

$$\sigma_{k,l}(h) = |\{(u, v) \in R \cap \mathbf{N}^2 : (h, 0) \rightsquigarrow (u, v)\}|$$

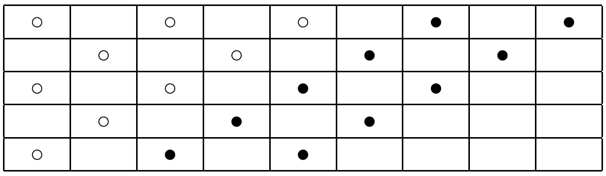


Figure 1. The definition of σ.

Then

$$\sigma_{k,l}(h) = \begin{cases} 0, & \textit{if } h < 0 \\ \left\lfloor \left(\dfrac{h}{2}+1\right)^2 \right\rfloor, & \textit{if } 0 \le h < \min(k,l) \\ \sigma_{k,l}(s) + \left(\dfrac{h-s}{2}\right)\min(k,l), & \textit{if } \min(k,l) \le h < \max(k,l) \\ \left\lceil \dfrac{kl}{2} \right\rceil - \sigma_{k,l}(k+l-h-4), & \textit{if } h \textit{ is even and } \max(k,l) \le h \\ \left\lfloor \dfrac{kl}{2} \right\rfloor - \sigma_{k,l}(k+l-h-4), & \textit{if } h \textit{ is odd and } \max(k,l) \le h \end{cases}$$

where s is defined as follows: If $h - \min(k,l)$ is even, then $s = \min(k,l) - 2$; otherwise $s = \min(k,l) - 1$.

Example 1

By definition $\sigma_{9,5}(4)$ counts the points in $\mathbf{N}^2$ in figure 1 marked with $\circ$. Then $\sigma_{9,5}(4) = 9$. Similarly, $\sigma_{9,5}(8)$ counts the points in $\mathbf{N}^2$ in figure 1 marked either with the symbol $\circ$ or with the symbol $\bullet$. Then $\sigma_{9,5}(8) = 19$.

Proof: If h is to the left of the 0th column, then we cannot reach any of the points in $\mathbf{N}^2$ inside R. Hence, $\sigma_{k,l}(h)$ should be equal to zero.

If $0 \le h \le \min(k,l)$, then we are counting the number of points of $\mathbf{N}^2$ that can be reached from $(h,0)$ inside the square S of side $\min(k,l)$. We have to consider two cases. If h is odd, then we are summing $2 + 4 + \cdots + (h+1) = \lfloor(\frac{h}{2}+1)^2\rfloor$. On the other hand, if h is even, then we are summing $1 + 3 + \cdots + (h+1) = (\frac{h}{2}+1)^2$.

If $\min(k,l) \le h < \max(k,l)$, then we subdivide our problem into two parts. First, we count the number of points of $\mathbf{N}^2$ that can be reached from $(h,0)$ inside the square S by $\sigma_{k,l}(s)$. Then we count those points of $\mathbf{N}^2$ that are in R but not in S. Since $h < \max(k,l)$ all diagonals have length $\min(k,l)$ and there are $\frac{h-s}{2}$ of them. See figure 1 for an example.

If $\max(k,l) \le h$, then it is easier to count the total number of points of $\mathbf{N}^2$ that can be reached from $(h,0)$ inside R by choosing another parameter $\hat{h}$ big enough and with the same parity as h. Then we subtract those points of $\mathbf{N}^2$ in R that are not reachable from $(h,0)$ because h is too close.

So, if $\hat{h}$ is even this number is $\lceil kl/2 \rceil$. If $\hat{h}$ is odd this number is $\lfloor kl/2 \rfloor$. Then we subtract those points that we should not have counted. We express this number in terms of the function σ. The line $y = -x + h + 2$ intersects the line $y = l - 1$ at $x = h - l + 3$. This is the x coordinate of the first point on the last row that is not reachable from $(h, 0)$. Then to obtain the number of points that can be reached from this point by moving south west or north west, but that were not supposed to be counted, we subtract $h - l + 3$ to $k - 1$. We have obtained that are $\sigma_{k,l}(k + l - h - 4)$ points that we should not have counted. $\square$

Note that $\sigma_{k,l}$ is symmetrical on k and l

Lemma 2 *Let a, b, c, and d be nonnegative integers. Let R be the rectangle with vertices (a, c), $(a + b, c)$, $(a, c + d)$, and $(a + b, c + d)$. We define*

$$\Gamma(a, b, c, d)(x, y) = |\{(u, v) \in R \cap \mathbf{N}^2 : (x, y) \rightsquigarrow (u, v)\}|.$$

Then

$$\Gamma(a, b, c, d)(x, y) = \begin{cases} \sigma_{b+1,d+1}(x + y - a - c), & 0 \le y \le c \\ \sigma_{b+1,y-c+1}(x - a) + \sigma_{b+1,c+d-y+1}(x - a) - \delta, & c < y < c + d \\ \sigma_{b+1,d+1}(x - y + c + d - a), & c + d \le y \end{cases}$$

where δ is defined as follows If $x < a$, then $\delta = 0$. If $a \le x \le a + b$, then $\delta = \lceil \frac{x-a+1}{2} \rceil$. Finally, if $x > a + b$ then we consider two cases: If $x - a - b$ is even then $\delta = \lceil \frac{b+1}{2} \rceil$; otherwise, $\delta = \lfloor \frac{b+1}{2} \rfloor$.

Proof: We consider three cases. Note that the letter c indicates the height of the base of the rectangle. If $0 \le y \le c$ then the first position inside R that we reach is $(x + y - a - c, c)$. Therefore, we assume that we are starting at this point. Similarly, if $y \ge c + d$, then the first position inside R that we reach is $(x - y + c + d - a, c)$. Again, we can assume that we are starting at this point.

On the other hand, if $c < y < c + d$, we are at a point whose height meets the rectangle. We subdevide the problem in two parts. The number of positions to the north of us is counted by $\sigma_{b+1,y-c+1}(x - a)$. The number of positions to the south of us is counted by $\sigma_{b+1,c+d-y+1}(x - a)$. We define δ to be the number of points of $\mathbf{N}^2$ that we counted twice during this process. Then it is easy to see that δ is given by the previous definition. $\square$

To compute the coefficient u_ν in the expansion $f[X] = \sum_\eta u_\eta s_\eta[X]$ for $f \in \Lambda$, it is enough to expand $f[x_1 + \cdots + x_n] = \sum_\eta u_\eta s_\eta[x_1 + \cdots + x_n]$ for any $n \ge l(\nu)$. (See [7, Section I.3], for proofs and details.) Therefore, in this section we work with symmetric functions in a finite number of variables.

Jacobi's definition of a Schur function on a finite alphabet $X = x_1 + x_2 + \cdots + x_n$ as a quotient of alternants says that

$$s_\lambda[X] = s_\lambda(x_1, \ldots, x_n) = \frac{\det\left(x_i^{\lambda_j+n-j}\right)_{1 \le i,j \le n}}{\prod_{i<j}(x_i - x_j)}. \tag{8}$$

By the symmetry properties of the Kronecker product it is enough to compute the Kronecker coefficients $\gamma^{\lambda}_{\mu\nu}$ when $\nu_2 \leq \mu_2$.

Theorem 1 *Let μ, ν, and λ be partitions of n, where $\mu = (\mu_1, \mu_2)$ and $\nu = (\nu_1, \nu_2)$ are two two-row partitions and let $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ be a partition of length less than or equal to 4. Assume that $\nu_2 \leq \mu_2$. Then*

$$\gamma^{\lambda}_{\mu\nu} = (\Gamma(a, b, a+b+1, c) - \Gamma(a, b, a+b+c+d+2, c))(\nu_2, \mu_2+1).$$

where $a = \lambda_3 + \lambda_4$, $b = \lambda_2 - \lambda_3$, $c = \min(\lambda_1 - \lambda_2, \lambda_3 - \lambda_4)$ and $d = |\lambda_1 + \lambda_4 - \lambda_2 - \lambda_3|$.

Proof: Set $X = 1 + x$ and $Y = 1 + y$ in the comultiplication expansion (2) to obtain

$$s_{\lambda}[(1+y)(1+x)] = \sum \gamma^{\lambda}_{\mu\nu} s_{\mu}[1+y] s_{\nu}[1+x]. \tag{9}$$

Note that the Kronecker coefficients are zero when $l(\lambda) > 4$.

The idea of the proof is to use Jacobi's definition of a Schur function as a quotient of alternants to expand both sides of the previous equation, and then get the Kronecker coefficients by looking at the resulting expansions.

Let φ be the polynomial defined by $\varphi = (1-x)(1-y)s_{\lambda}[(1+y)(1+x)] = (1-x)(1-y)s_{\lambda}(1, y, x, xy)$. Using Jacobi's definition of a Schur function we obtain

$$\varphi = \frac{\begin{vmatrix} 1 & 1 & 1 & 1 \\ y^{\lambda_1+3} & y^{\lambda_2+2} & y^{\lambda_3+1} & y^{\lambda_4} \\ x^{\lambda_1+3} & x^{\lambda_2+2} & x^{\lambda_3+1} & x^{\lambda_4} \\ (xy)^{\lambda_1+3} & (xy)^{\lambda_2+2} & (xy)^{\lambda_3+1} & (xy)^{\lambda_4} \end{vmatrix}}{xy(1-xy)(y-x)(1-x)(1-y)}. \tag{10}$$

On the other hand, we may use Jacobi's definition to expand $s_{\mu}[1+y]$ and $s_{\nu}[1+x]$. Substitute this results into (9):

$$\begin{aligned} s_{\lambda}[(1+y)(1+x)] &= \sum_{\substack{\mu=(\mu_1,\mu_2)\\ \nu=(\nu_1,\nu_2)}} \gamma^{\lambda}_{\mu\nu} \left(\frac{y^{\mu_2} - y^{\mu_1+1}}{1-y}\right)\left(\frac{x^{\nu_2} - x^{\nu_1+1}}{1-x}\right) \\ &= \sum_{\substack{\mu=(\mu_1,\mu_2)\\ \nu=(\nu_1,\nu_2)}} \gamma^{\lambda}_{\mu\nu} \frac{x^{\nu_2}y^{\mu_2} - x^{\nu_2}y^{\mu_1+1} - x^{\nu_1+1}y^{\mu_2} + x^{\nu_1+1}y^{\mu_1+1}}{(1-x)(1-y)}. \end{aligned} \tag{11}$$

Since $\nu_1 + 1$ and $\mu_1 + 1$ are both greater than $\lfloor \frac{n}{2} \rfloor$, Eq. (11) implies that the coefficient of $x^{\nu_2}y^{\mu_2}$ in φ is $\gamma^{\lambda}_{\mu\nu}$.

It is convenient to define an auxiliary polynomial by

$$\zeta = (1-xy)(y-x)\varphi. \tag{12}$$

Let ξ be the polynomial defined by expanding the determinant appearing in (10). Equations (10) and (12) imply

$$\zeta = \frac{\xi}{xy(1-x)(1-y)}.$$

Let $\xi_{i,j}$ be the coefficient of $x^i y^j$ in ξ. (Note that $\xi_{i,j}$ is zero if $i < 1$ or $j < 1$, because ξ is a polynomial divisible by xy.) Let $\zeta_{i,j}$ be the coefficient of $x^i y^j$ in ζ. Then

$$\sum_{i,j\geq 0} \zeta_{i,j} x^i y^j = \frac{1}{xy(1-x)(1-y)} \sum_{i,j\geq 0} \xi_{i,j} x^i y^j = \sum_{i,j,k,l\geq 0} \xi_{i-k,j-l} x^{i-1} y^{j-1}. \tag{13}$$

Comparing the coefficient of $x^i y^j$ on both sides of Eq. (13) we obtain that

$$\zeta_{i,j} = \sum_{k,l\geq 0} \xi_{i+1-k,j+1-l} = \sum_{k=0}^{i} \sum_{l=0}^{j} \xi_{k+1,l+1} \tag{14}$$

We compute $\zeta_{i,j}$ from (14) by expanding the determinant appearing on (10). We consider two cases.

Case 1. Suppose that $\lambda_1 + \lambda_4 > \lambda_2 + \lambda_3$. Then

$$\lambda_1+\lambda_2+4 > \lambda_1+\lambda_3+3 > \lambda_1+\lambda_4+2 \geq \lambda_2+\lambda_3+2 > \lambda_2+\lambda_4+1 > \lambda_3+\lambda_4.$$

We record the values of $\xi_{j+1,i+1}$ in Table 1. We use the convention that $\xi_{i+1,j+1}$ is zero whenever the (i, j) entry is not in Table 1.

Equation (14) shows that the value of $\zeta_{i,j}$ can be obtained by adding the entries north west of the point (i, j) in Table 1. In Table 2 we record the values of $\zeta_{i,j}$.

Table 1. The values of $\xi_{j+1,i+1}$ when $\lambda_1 + \lambda_4 \geq \lambda_2 + \lambda_3$.

$i\backslash j$	$\lambda_3+\lambda_4$	$\lambda_2+\lambda_4+1$	$\lambda_2+\lambda_3+2$	$\lambda_1+\lambda_4+2$	$\lambda_1+\lambda_3+3$	$\lambda_1+\lambda_2+4$
$\lambda_3+\lambda_4$	0	-1	$+1$	$+1$	-1	0
$\lambda_2+\lambda_4+1$	$+1$	0	-1	-1	0	$+1$
$\lambda_2+\lambda_3+2$	-1	$+1$	0	0	$+1$	-1
$\lambda_1+\lambda_4+2$	-1	$+1$	0	0	$+1$	-1
$\lambda_1+\lambda_3+3$	$+1$	0	-1	-1	0	$+1$
$\lambda_1+\lambda_2+4$	0	-1	$+1$	$+1$	-1	0

Table 2. The values of $\zeta_{i,j}$ when $\lambda_1 + \lambda_4 \geq \lambda_2 + \lambda_3$.

$i\backslash j$	I_1	I_2	I_3	I_4	I_5	I_6	I_7
I_1	0	0	0	0	0	0	0
I_2	0	0	-1	0	$+1$	0	0
I_3	0	$+1$	0	0	0	-1	0
I_4	0	0	0	0	0	0	0
I_5	0	-1	0	0	0	$+1$	0
I_6	0	0	$+1$	0	-1	0	0
I_7	0	0	0	0	0	0	0

Where

$$\begin{aligned}
I_1 &= [0, \lambda_3 + \lambda_4),\\
I_2 &= [\lambda_3 + \lambda_4, \lambda_2 + \lambda_4],\\
I_3 &= [\lambda_2 + \lambda_4 + 1, \lambda_2 + \lambda_3 + 1],\\
I_4 &= [\lambda_2 + \lambda_3 + 2, \lambda_1 + \lambda_4 + 1],\\
I_5 &= [\lambda_1 + \lambda_4 + 2, \lambda_1 + \lambda_3 + 2],\\
I_6 &= [\lambda_1 + \lambda_3 + 3, \lambda_1 + \lambda_2 + 3],\\
I_7 &= [\lambda_1 + \lambda_2 + 4, \infty].
\end{aligned}$$

Case 2. Suppose that $\lambda_1 + \lambda_4 \leq \lambda_2 + \lambda_3$. Then

$$\lambda_1 + \lambda_2 + 4 > \lambda_1 + \lambda_3 + 3 > \lambda_2 + \lambda_3 + 2 > \lambda_1 + \lambda_4 + 2 > \lambda_2 + \lambda_4 + 1 > \lambda_3 + \lambda_4.$$

Note that in Table 2, the rows and columns corresponding to $\lambda_1 + \lambda_4 + 2$ and $\lambda_2 + \lambda_3 + 2$ are the same. Therefore, the values of $\xi_{i,j}$ for $\lambda_1 + \lambda_4 \leq \lambda_2 + \lambda_3$ are recorded in Table 2, if we set

$$\begin{aligned}
I_3 &= [\lambda_2 + \lambda_4 + 1, \lambda_1 + \lambda_4 + 1]\\
I_4 &= [\lambda_2 + \lambda_4 + 2, \lambda_2 + \lambda_3 + 1]\\
I_5 &= [\lambda_2 + \lambda_3 + 2, \lambda_1 + \lambda_3 + 2],
\end{aligned}$$

and define the other intervals as before.

In both cases, let $\varphi_{i,j}$ be the coefficient of $x^i y^j$ in φ. Using (12) we obtain that

$$\begin{aligned}
\varphi &= \frac{1}{(1 - xy)(y - x)} \sum_{i,j \geq 0} \zeta_{i,j} x^i y^j\\
&= \frac{1}{y - x} \sum_{i,j,l \geq 0} \zeta_{i-l,j-l} x^i y^j\\
&= \sum_{i,j,k,l \geq 0} \zeta_{i-k-l,j+k-l+1} x^i y^j. \qquad (15)
\end{aligned}$$

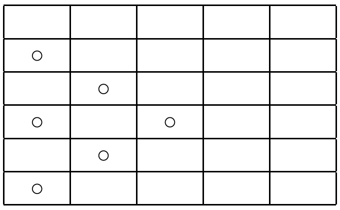


Figure 2. The right-most point in figure 2 has coordinates (2, 2).

(Note: We can divide by $y - x$ because $\varphi = 0$ when $x = y$.) Comparing the coefficients of $x^i y^j$ on both sides of Eq. (15), we obtain $\varphi_{i,j} = \sum_{k,l\geq 0} \zeta_{i-k-l,j+k-l+1}$. Therefore,

$$\varphi_{\nu_2,\mu_2} = \sum_{i,j=0}^{\nu_2} \zeta_{\nu_2-i-j,\mu_2+i-j+1}. \tag{16}$$

We have concluded that the coefficient of $x^{\nu_2} y^{\mu_2}$ in φ is $\gamma^{\lambda}_{\mu,\nu}$. Hence $\gamma^{\lambda}_{\mu\nu} = \varphi_{\nu_2,\mu_2}$ can be obtained by adding the entries in Table 1 at all points of $\mathbf{N}^2$ that can be reached from $(\nu_2, \mu_2 + 1)$. after we flip Table 2 around the horizontal axis. See figure 2.

By hypothesis $\nu_2 \leq \mu_2 \leq \lfloor n/2 \rfloor$. Then, if we start at $(\nu_2, \mu_2 + 1)$ and move as previously described, the only points of $\mathbf{N}^2$ that we can possibly reach and that are nonzero in Table 2 are those in $I_2 \times I_3$ or $I_2 \times I_5$. Hence, we have that φ_{ν_2,μ_2} is the number of points of $\mathbf{N}^2$ inside $I_2 \times I_3$ that can be reached from $(\nu_2, \mu_2 + 1)$ minus the ones that can be reached in $I_2 \times I_5$. All other entries that are reachable from $(\nu_2, \mu_2 + 1)$ are equal to zero.

Case 1. The inequality $\lambda_1 + \lambda_4 > \lambda_2 + \lambda_3$ implies that $\lambda_1 + \lambda_4 + 1 > t\lfloor \frac{n}{2} \rfloor$. Moreover, $\mu_2 \geq \nu_2$ implies that we are only considering the region of $\mathbf{N}^2$ given by $0 \leq i \leq j \leq \lfloor \frac{n}{2} \rfloor$. The number of points of $\mathbf{N}^2$ that can be reached from $(\nu_2, \mu_2 + 1)$ inside $I_2 \times I_3$ is given by $\Gamma(\lambda_3 + \lambda_4, \lambda_2 - \lambda_3, \lambda_2 + \lambda_4 + 1, \lambda_3 - \lambda_4)$. Similarly, the number of points of $\mathbf{N}^2$ that can be reached from $(\nu_2, \mu_2 + 1)$ inside $I_2 \times I_5$ is given by $\Gamma(\lambda_3 + \lambda_4, \lambda_2 - \lambda_3, \lambda_1 + \lambda_4 + 2, \lambda_3 - \lambda_4)$.

Case 2. The inequality $\lambda_2 + \lambda_3 \geq \lambda_1 + \lambda_4$, implies that $\lambda_1 + \lambda_4 + 1 > \lfloor \frac{n}{2} \rfloor$. Moreover, $\mu_2 \geq \nu_2$ implies that we are only considering the region of $\mathbf{N}^2$ given by $0 \leq i \leq j \leq \lfloor \frac{n}{2} \rfloor$. The number of points of $\mathbf{N}^2$ that can be reached from $(\nu_2, \mu_2 + 1)$ inside $I_2 \times I_3$ is given by $\Gamma(\lambda_3 + \lambda_4, \lambda_2 - \lambda_3, \lambda_2 + \lambda_4 + 1, \lambda_1 - \lambda_2)$. Similarly, the number of points of $\mathbf{N}^2$ that can be reached from $(\nu_2, \mu_2 + 1)$ inside $I_2 \times I_5$ is given by $\Gamma(\lambda_3 + \lambda_4, \lambda_2 - \lambda_3, \lambda_2 + \lambda_3 + 2, \lambda_1 - \lambda_2)$. □

Corollary 1 *Let* $\mu = (\mu_1, \mu_2)$, $\nu = (\nu_1, \nu_2)$, *and* $\lambda = (\lambda_1, \lambda_2)$ *be partitions of* n. *Assume that* $\nu_2 \leq \mu_2 \leq \lambda_2$. *Then*

$$\gamma^{\lambda}_{\mu\nu} = (y - x)(y \geq x),$$

where $x = \max(0, \lceil \frac{\mu_2+\nu_2+\lambda_2-n}{2} \rceil)$ *and* $y = \lceil \frac{\mu_2+\nu_2-\lambda_2+1}{2} \rceil$.

Proof: Set $\lambda_3 = \lambda_4 = 0$ in Theorem 1. Then we notice that the second possibility in the definition of Γ, that is, when $c < y < c + d$, never occurs. Note that $\nu_2 + \mu_2 - \lambda_2 \geq \nu_2 + \mu_2 - \lambda_1 - 1$ for all partitions μ, ν, and λ. Therefore,

$$\gamma^{\lambda}_{\mu\nu} = \sigma_{\lambda_2+1,1}(\nu_2 + \mu_2 - \lambda_2) - \sigma_{\lambda_2+1,1}(\nu_2 + \mu_2 - \lambda_1 - 1).$$

Suppose that $\nu_2 + \mu_2 - \lambda_2 < 0$. By Lemma 1, $\sigma_{\lambda_2+1,1}(h) = 0$ when $h < 0$. Hence, we obtain that $\gamma^{\lambda}_{\mu\nu} = 0$. Therefore, in order to have $\gamma^{\lambda}_{\mu\nu}$ not equal to zero, we should assume that $\nu_2 + \mu_2 - \lambda_2 \geq 0$.

If $0 \leq \nu_2 + \mu_2 - \lambda_2 < \lambda_2 + 1$, then

$$\sigma_{\lambda_2+1,1}(\nu_2 + \mu_2 - \lambda_2) = \left\lceil \frac{\nu_2 + \mu_2 - \lambda_2 + 1}{2} \right\rceil$$

Similarly, if $0 \leq \nu_2 + \mu_2 - \lambda_2 < \lambda_2 + 1$, then

$$\sigma_{\lambda_2+1,1}(\nu_2 + \mu_2 - \lambda_1 - 1) = \left\lceil \frac{\nu_2 + \mu_2 + \lambda_2 - n}{2} \right\rceil$$

It is easy to see that all other cases obtained in Lemma 1 for the computation of $\sigma_{k,l}$ can not occur. Therefore, defining x and y as above, we obtain the desired result. □

Example 2 If $\mu = \nu = \lambda = (l, l)$ or $\mu = \nu = (2l, 2l)$ and $\lambda = (3l, l)$, then from the previous corollary, we obtain that

$$\gamma^{\lambda}_{\mu\nu} = \left\lceil \frac{l+1}{2} \right\rceil - \left\lceil \frac{l}{2} \right\rceil = ((l \text{ is even}))$$

Note that to apply Corollary 1 to the second family of shapes, we should first use the symmetries of the Kronecker product.

Corollary 2 *The Kronecker coefficients $\gamma^{\lambda}_{\mu\nu}$, where μ and ν are two-row partitions, are unbounded.*

Proof: It is enough to construct an unbounded family of Kronecker coefficients. Assume that $\mu = \nu = \lambda = (3l, l)$. Then from the previous corollary we obtain that

$$\gamma^{\lambda}_{\mu\nu} = \left\lceil \frac{l+1}{2} \right\rceil$$

□

4. Sergeev's formula

The fundamental tool for the study of the Kronecker product on the remaining two cases is Sergeev's formula for the difference of two alphabets. See [1, 14], or [7, section I.3] for proofs and comments. In order to state Sergeev's formula we need to introduce some definitions.

Definition 1 Let $X_m = x_1 + \cdots + x_m$ be a finite alphabet, and let $\delta_m = (m-1, m-2, \ldots, 1, 0)$. We define $X_m^{\delta_m}$ by $X_m^{\delta_m} = x_1^{m-1} \cdots x_{m-1}$.

Definition 2 An inversion of a permutation $\alpha_1\alpha_2 \cdots \alpha_n$ is a pair (i, j), with $1 \leq i < j \leq n$, such that $\alpha_i > \alpha_j$. Let $i(\alpha)$ be the number of inversions of α. We define the alternant to be

$$A_m^x P = \sum_{\alpha \in S_m} (-1)^{i(\alpha)} P\big(x_{\alpha(1)}, \ldots, x_{\alpha(m)}\big),$$

for any polynomial $P(x_1, \ldots, x_n)$.

Definition 3 Let Δ be the operation of taking the Vandermonde determinant of an alphabet, i.e.,

$$\Delta(X_m) = \det\big(x_i^{m-j}\big)_{i,j=1}^{m}.$$

Theorem 2 (Sergeev's Formula) *Let $X_m = x_1 + \cdots + x_m$, and $Y_n = y_1 + \cdots + y_n$ be two alphabets. Then*

$$s_\lambda[X_m - Y_n] = \frac{1}{\Delta(X_m)\Delta(Y_n)} A_m^x A_n^y X_m^{\delta_m} Y_n^{\delta_n} \prod_{(i,j)\in\lambda} (x_i - y_j)$$

The notation $(i, j) \in \lambda$ means that the point (i, j) belongs to the diagram of λ. We set $x_i = 0$ for $i > m$ and $y_j = 0$ for $j > n$.

We use Sergeev's formula as a tool for making some calculations we need for the next two sections.

1. Let $\mu = (1^{e_1} m_2)$ be a hook. (We are assuming that $e_1 \geq 1$ and $m_2 \geq 2$.) Let $X^1 = \{x_1\}$ and $X^2 = \{x_2\}$.

$$s_\mu[x_1 - x_2] = (-1)^{e_1} x_1^{m_2 - 1} x_2^{e_1} (x_1 - x_2). \tag{17}$$

2. Let $\nu = (\nu_1, \nu_2)$ be a two-row partition. Let $Y = \{y_1, y_2\}$. Then

$$s_\nu[y_1 + y_2] = \frac{(y_1 y_2)^{\nu_2}\big(y_1^{\nu_1 - \nu_2 + 1} - y_2^{\nu_1 - \nu_2 + 1}\big)}{y_1 - y_2}. \tag{18}$$

3. We say that a partition λ is a double hook if $(2, 2) \in \lambda$ and it has the form $\lambda = (1^{d_1} 2^{d_2} n_3\, n_4)$. In particular any two-row shape is a double hook.

Let λ be a double hook. Let $U=\{u_1,u_2\}$ and $V=\{v_1,v_2\}$. If $n_4\neq 0$ then $s_\lambda[u_1+u_2-v_1-v_2]$ equals

$$\begin{aligned}&\frac{(u_1-v_1)(u_2-v_1)(u_1-v_2)(u_2-v_2)}{(u_1-u_2)(v_1-v_2)}(-1)^{d_1}(u_1u_2)^{n_3-2}(v_1v_2)^{d_2}\\&\times\left(u_2^{n_4-n_3+1}-u_1^{n_4-n_3+1}\right)\left(v_2^{d_1+1}-v_1^{d_1+1}\right).\end{aligned}\tag{19}$$

On the other hand, if $n_4=0$ then to compute $s_\lambda[u_1+u_2-v_1-v_2]$ we should write λ as $(1^{d_1}2^{d_2-1}2n_3)$.

4. Let λ be a hook shape, $\lambda=(1^{d_1}n_2)$. (We are assuming that $d_1\geq 1$ and $n_2\geq 2$.) Let $U=\{u_1,u_2\}$ and $V=\{v_1,v_2\}$. Then $s_\lambda[u_1+u_2-v_1-v_2]$ equals

$$\begin{aligned}&(-1)^{d_1-1}\frac{1}{(u_1-u_2)}\frac{1}{(v_1-v_2)}\\&\times\big\{u_1v_1(u_1-v_1)(u_1-v_2)(u_2-v_1)u_1^{n_2-2}v_1^{d_1-1}\\&\quad-u_1v_2(u_1-v_2)(u_1-v_1)(u_2-v_2)u_1^{n_2-2}v_2^{d_1-1}\\&\quad-u_2v_1(u_2-v_1)(u_2-v_2)(u_1-v_1)u_2^{n_2-2}v_1^{d_1-1}\\&\quad+u_2v_2(u_2-v_2)(u_2-v_1)(u_1-v_2)u_2^{n_2-2}v_2^{d_1-1}.\big\}\end{aligned}\tag{20}$$

5. The case of two hook shapes

In this section we derive an explicit formula for the Kronecker coefficients $\gamma^\lambda_{\mu\nu}$ in the case in which $\mu=(1^eu)$, and $\nu=(1^fv)$ are both hook shapes. Given a partition λ the Kronecker coefficient $\gamma^\lambda_{\mu\nu}$ tells us whether point (u,v) belongs to some regions in $\mathbf{N}^2$ determined by μ, ν and λ.

Lemma 3 *Let $(u,v)\in\mathbf{N}^2$ and let R be the rectangle with vertices (a,b), (b,a), (c,d), and (d,c), with $a\geq b$, $c\geq d$, $c\geq a$ and $d\geq b$. (Sometimes, when $c=d=e$, we denote this rectangle as $(a,b;e)$.)*

Then $(u,v)\in R$ if and only if $|v-u|\leq a-b$ and $a+b\leq u+v\leq c+d$

Proof: Each of the four inequalities corresponds to whether the point (u,v) is in the proper half-plane formed by two of four edges of the rectangle. □

Theorem 3 *Let λ, μ and ν be partitions of n, where $\mu=(1^eu)$ and $\nu=(1^fv)$ are hook shapes. Then the Kronecker coefficients $\gamma^\lambda_{\mu\nu}$ are given by the following:*

1. *If λ is a one-row shape, then $\gamma^\lambda_{\mu\nu}=\delta_{\mu,\nu}$.*
2. *If λ is not contained in a double hook shape, then $\gamma^\lambda_{\mu\nu}=0$.*
3. *Let $\lambda=(1^{d_1}2^{d_2}n_3n_4)$ be a double hook. Let $x=2d_2+d_1$. Then*

$$\begin{aligned}\gamma^\lambda_{\mu\nu}=&\left(\left(n_3-1\leq\frac{e+f-x}{2}\leq n_4\right)\right)((|f-e|\leq d_1))\\&+\left(\left(n_3\leq\frac{e+f-x+1}{2}\leq n_4\right)\right)((|f-e|\leq d_1+1)).\end{aligned}$$

Note that if $n_4 = 0$, *then we shall rewrite* $\lambda = (1^{d_1}2^{d_2-1}2\,n_3)$ *before using the previous formula.*

4. *Let* $\lambda = (1^d w)$ *be a hook shape. Suppose that* $e \le u$, $f \le v$, *and* $d \le w$. *Then*

$$\gamma^{\lambda}_{\mu\nu} = ((e \le d + f))((d \le e + f))((f \le e + d)).$$

Proof: Set $X = \{1, x\}$ and $Y = \{1, y\}$ in the comultiplication expansion (2) to obtain

$$s_\lambda[(1-x)(1-y)] = \sum_{\mu,\nu} \gamma^{\lambda}_{\mu\nu} s_\mu[1-x] s_\nu[1-y], \tag{21}$$

We use Eq. (17) to replace s_μ and s_ν in the right hand side of (21). Then we divide the resulting equation by $(1-x)(1-y)$ to get

$$\frac{s_\lambda[1-y-x+xy]}{(1-x)(1-y)} = \sum_{\mu,\nu} \gamma^{\lambda}_{\mu\nu}(-x)^e(-y)^f.$$

Therefore,

$$\gamma^{\lambda}_{\mu\nu} = [(-x)^e(-y)^f]\frac{s_\lambda[1-y-x+xy]}{(1-x)(1-y)},$$

when μ and ν are hook shapes.

Case 1. If λ is not contained in any double hook, then the point $(3, 3)$ is in λ, and by Sergeev's formula, $s_\lambda[1-y-x+xy]$ equals zero.

Case 2. Let $\lambda = (1^{d_1}2^{d_2}n_3\,n_4)$ be a double hook. Set $u_1 = 1$, $u_2 = xy$, $v_1 = x$, and $v_2 = y$ in (19). Then we divide by $(1-x)(1-y)$ on both sides of the resulting equation to obtain

$$\begin{aligned}\frac{s_\lambda[1-y-x+xy]}{(1-x)(1-y)} &= (-1)^{d_1}(xy)^{n_3+d_2-1}\\ &\quad\times(1-x)(1-y)\left(\frac{1-(xy)^{n_4-n_3+1}}{1-xy}\right)\left(\frac{x^{d_1+1}-y^{d_1+1}}{x-y}\right).\end{aligned} \tag{22}$$

Note: If $n_4 = 0$ then we should write $\lambda = (1^{d_1}2^{d_2-1}2\,n_4)$ in order to use (19).

Let $\omega_p(T) = \sum_{(i,j)\in T}\omega_p(i,j)$ be the generating function of a region T in $\mathbf{N}^2$. Let R be the rectangle with vertices $(0, d_1)$, $(d_1, 0)$, $(d_1+n_4-n_3, n_4-n_3)$ and $(n_4-n_3, d_1+n_4-n_3)$. Then

$$\begin{aligned}\omega_{(d_1+n_4-n_3,n_4-n_3)}(R) &= \left(\frac{1-(xy)^{n_4-n_3+1}}{1-xy}\right)\left(\frac{x^{d_1+1}-y^{d_1+1}}{x-y}\right)\\ &= \sum_{k=0}^{n_4-n_3}\sum_{i+j=d_1}(xy)^k x^i y^j.\end{aligned}$$

See figure 3.

				1				
			1		1			
		1		1		1		
	1		1		1		1	
1		1		1		1		1
	1		1		1		1	
		1		1		1		
			1		1			
				1				

Figure 3. $d_1 = 4$ and $n_3 - n_4 = 4$.

The four vertices of R in Figure 3 are $(0, 4)$, $(4, 0)$, $(8, 4)$, and $(4, 8)$.

We interpret the right-hand side of (22) as the sum of four different generating functions. To be more precise, the right-hand side of (22) can be written as $\sum_{i=1}^{4} \omega_{p_i}(r_i)$ where $p_1 = (n_4 + d_2 - 1, n_4 + d_2 + d_1 - 1)$ and $R_1 = \{n_3 + d_2 + d_1 - 1, n_3 + d_2 - 1; n_4 - n_3\}$, $p_2 = (n_4 + d_2, n_34 + d_2 + d_1 - 1)$ and $R_2 = \{n_3 + d_2 + d_1, n_3 + d_2 - 1; n_4 - n_3\}$, $p_3 = (n_4 + d_2 - 1, n_4 + d_2 + d_1)$ and $R_3 = \{n_3 + d_2 + d_1 - 1, n_3 + d_2; n_4 - n_3\}$, and $p_4 = (n_4 + d_2 + d_1, n_4 + d_2 + d_1)$ and $R_4 = \{n_3 + d_2 + d_1, n_3 + d_2 + d_1; n_4 - n_3\}$.

We observe that $R_1 \cup R_2$ (and $R_3 \cup R_4$) are rectangles in $\mathbf{N}^2$. Moreover,

$$\gamma_{\mu\nu}^{\lambda} = (((e, f) \in R_1 \cup R_2)) + (((e, f) \in R_3 \cup R_4)). \tag{23}$$

The vertices of rectangle $R_1 \cup R_4$ are given (using the notation of Lemma 3) by

$$\begin{aligned} a &= n_3 + d_2 + d_1 - 1 & b &= n_3 + d_2 - 1 \\ c &= n_4 + d_2 + d_1 & d &= n_4 + d_2 \end{aligned}$$

Similarly, the vertices of rectangle $R_2 \cup R_3$ are given by

$$\begin{aligned} a &= n_3 + d_2 + d_1 & b &= n_3 + d_2 - 1 \\ c &= n_4 + d_2 + d_1 & d &= n_4 + d_2 - 1 \end{aligned}$$

Applying Lemma 3 to (23) we obtain

$$\begin{aligned} \gamma_{\mu\nu}^{\lambda} = {} & \left(\left(n_3 - 1 \le \frac{e + f - x}{2} \le n_4\right)\right)((|f - e| \le d_1)) \\ & + \left(\left(n_3 \le \frac{e + f - x + 1}{2} \le n_4\right)\right)((|f - e| \le d_1 + 1)). \end{aligned}$$

Case 3 (λ is a hook). Suppose that λ is a hook, $\lambda = (1^d w)$. Set $u_1 = 1$, $u_2 = xy$, $v_1 = x$, and $v_2 = y$ in (20). Then we divide by $(1 - x)(1 - y)$ on both sides of the resulting equation to obtain

$$\frac{s_\lambda[1-y-x+xy]}{(1-x)(1-y)} = (-1)^d\left(\frac{x^{d+1}-y^{d+1}}{x-y}\right)\left(\frac{1-(xy)^w}{1-xy}\right) + (-1)^{d-1}xy\left(\frac{x^d-y^d}{x-y}\right)\left(\frac{1-(xy)^{w-1}}{1-xy}\right). \tag{24}$$

We want to interpret this equation as a generating function for a region T using the weight ω. We proceed as follows:

Let R_1 be the rectangle with vertices $(d, 0)$, $(0, d)$, $(d+w-1, w-1)$, and $(w-1, d+w-1)$. Then

$$\omega_{(w-1,d+w-1)}(R_1) = \left(\frac{1-(xy)^w}{1-xy}\right)\left(\frac{x^{d+1}-y^{d+1}}{x-y}\right) = \sum_{k=0}^{w-1}\sum_{i+j=d}(xy)^k x^i y^j. \tag{25}$$

(See figure 3.) Similarly, let R_2 be the rectangle with vertices $(d, 1)$, $(1, d)$, $(d+w-2, w-1)$, and $(w-1, d+w-2)$. Then

$$\begin{aligned}\omega_{(w-1,d+w-2)}(R_2) &= xy\left(\frac{1-(xy)^{w-1}}{1-xy}\right)\left(\frac{x^d-y^d}{x-y}\right)\\ &= xy\sum_{k=0}^{w-2}\sum_{i+j=d-1}(xy)^k x^i y^j.\end{aligned} \tag{26}$$

Observe that the points of $\mathbf{N}^2$ that can be reached from $(0, d)$ in R_1 and the points of $\mathbf{N}^2$ that can be reached from $(1, d)$ in R_2 are disjoint. Moreover, they completely fill the rectangle $R_1 \cup R_2$. See figure 4.

Note that R_2 is contained in R_1. We obtain that

$$\omega_{(w-1,d+w-1)}(R_1) + \omega_{(w-1,d+w-2)}(R_2) = (((e, f) \in R_1))$$

				1					
			1	−1	1				
		1	−1	1	−1	1			
	1	−1	1	−1	1	−1	1		
1	−1	1	−1	1	−1	1	−1	1	
	1	−1	1	−1	1	−1	1	−1	1
		1	−1	1	−1	1	−1	1	
			1	−1	1	−1	1		
				1	−1	1			
					1				

Figure 4. $d = 4$, $w = 6$.

We use apply Lemma 3 to the previous equation to obtain:

$$((|e - f| \leq d))((d \leq e + f \leq d + 2w - 2)).$$

But, by hypothesis, $e \leq u$, $f \leq v$, and $d \leq w$. Therefore, this system is equivalent to $((d \leq e + f))((f \leq e + d))((e \leq d + f))$, as desired. □

Corollary 3 *Let λ, μ, and ν be partitions of n, where $\mu = (1^e u)$ and $\nu = (1^f v)$ are hook shapes and $\lambda = (\lambda_1, \lambda_2)$ is a two-row shape. Then the Kronecker coefficients $\gamma^{\lambda}_{\mu\nu}$ are given by*

$$\gamma^{\lambda}_{\mu\nu} = ((\lambda_2 - 1 \leq e \leq \lambda_1))((e = f)) + \left(\left(\lambda_2 \leq k\frac{e + f + 1}{2} \leq \lambda_1\right)\right)((|e - f| \leq 1)).$$

Proof: In Theorem 3, set $d_1 = d_2 = 0$, $n_3 = \lambda_2$ and $n_4 = \lambda_1$. □

Corollary 4 *Let λ, μ and ν be partitions of n, where μ and ν are hook shapes. Then the Kronecker coefficients are bounded. Moreover, the only possible values for the Kronecker coefficients are* 0, 1 *or* 2.

6. The case of a hook shape and a two-row shape

In this section we derive an explicit formula for the Kronecker coefficients in the case $\mu = (1^{e_1} m_2)$ is a hook and $\nu = (\nu_1, \nu_2)$ is a two-row shape.

Using the symmetry properties of the Kronecker product, we may assume that if $\lambda = (1^{d_1} 2^{d_2} n_3 n_4)$ then $n_4 - n_3 \leq d_1$. (If $n_4 = 0$ then we should rewrite λ as $(1^{d_1} 2^{d_2 - 1} 2 n_3)$. Moreover, our hypothesis becomes $n_3 - 2 \leq d_1$.)

Theorem 4 *Let λ, μ and ν be partitions of n, where $\mu = (1^{e_1} m_2)$ is a hook and $\nu = (\nu_1, \nu_2)$ is a two-row shape. Then the Kronecker coefficients $\gamma^{\lambda}_{\mu\nu}$ are given by the following*:

1. *If λ is a one-row shape, then $\gamma^{\lambda}_{\mu\nu} = \delta_{\mu,\nu}$.*
2. *If λ is not contained in any double hook, then $\gamma^{\lambda}_{\mu\nu} = 0$.*
3. *Suppose $\lambda = (1^{d_1} 2^{d_2} n_3 n_4)$ is a double hook. Assume that $n_4 - n_3 \leq d_1$. (If $n_4 = 0$, then we should write $\lambda = (1^{d_1} 2^{d_2 - 1} 2 n_3)$.) Then*

$$\begin{aligned}\gamma^{\lambda}_{\mu\nu} &= ((n_3 \leq \nu_2 - d_2 - 1 \leq n_4))((d_1 + 2d_2 < e_1 < d_1 + 2d_2 + 3)) \\ &\quad + ((n_3 \leq \nu_2 - d_2 \leq n_4))((d_1 + 2d_2 \leq e_1 \leq d_1 + 2d_2 + 3)) \\ &\quad + ((n_3 \leq \nu_2 - d_2 + 1 \leq n_4))((d_1 + 2d_2 < e_1 < d_1 + 2d_2 + 3)) \\ &\quad - ((n_3 + d_2 + d_1 = \nu_2))((d_1 + 2d_2 + 1 \leq e_1 \leq d_1 + 2d_2 + 2)).\end{aligned}$$

4. *If λ is a hook, see Corollary* 3.

Proof: Set $X = 1 + x$ and $Y = 1 + y$ in the comultiplication expansion (2) to obtain

$$s_{\lambda}[(1 - x)(1 + y)] = \sum_{\mu,\nu} \gamma^{\lambda}_{\mu\nu} s_{\mu}[1 - x] s_{\nu}[1 + y]. \tag{27}$$

Use (17) and (18) to replace s_μ and s_ν in the right-hand side of (27), and divide by $(1-x)$ to obtain

$$\frac{s_\lambda[(1-x)(1+y)]}{1-x} = \sum_{\substack{\mu=(1^{e_1}m_2)\\ \nu=(\nu_1,\nu_2)}} \gamma^\lambda_{\mu\nu}(-x)^{e_1}y^{\nu_2}\left(\frac{1-y^{\nu_1-\nu_2+1}}{1-y}\right). \tag{28}$$

If λ is not contained in any double hook, then the point $(3,3)$ is in λ, and by Sergeev's formula, $s_\lambda[(1-x)(1+y)]$ equals zero.

Since we already computed the Kronecker coefficients when λ is contained in a hook, we can assume for the rest of this proof that λ is a double hook. Let $\lambda = (1^{d_1}2^{d_2}n_3\,n_4)$. (Note: If $n_4 = 0$ then we should write $\lambda = (1^{d_1}2^{d_2-1}2\,n_4)$.)

Set $u_1 = 1$, $u_2 = y$, $v_1 = x$, and $v_2 = xy$ in (19), and multiply by $\frac{1-y}{1-x}$ on both sides of the resulting equation.

$$\begin{aligned}\sum_{\substack{\mu=(1^{e_1}m_2)\\ \nu=(\nu_1,\nu_2)}} \gamma^\lambda_{\mu\nu}(-x)^{e_1}y^{\nu_2}\left(1-y^{\nu_1-\nu_2+1}\right) &= (y-x)(1-xy)(1-x)\\ &\quad\times(-x)^{d_1+2d_2}y^{n_3+d_2-1}\left(\frac{\left(1-y^{n_4-n_3+1}\right)\left(1-y^{d_1+1}\right)}{1-y}\right).\end{aligned} \tag{29}$$

We have that $(y-x)(1-xy)(1-x) = y - x(1+y+y^2) + x^2(1+y+y^2) - x^3y$. Therefore, looking at the coefficient of x on both sides of the equation, we see that $\gamma^\lambda_{\mu\nu}$ is zero if e_1 is different from d_1+2d_2, d_1+2d_2+1, d_1+2d_2+2, or d_1+2d_2+3.

Let $e_1 = d_1+2d_2$ or $e_1 = d_1+2d_2+3$. Since $\nu_2 \le n/2$, we have that

$$\begin{aligned}\gamma^\lambda_{\mu\nu} &= [y^{\nu_2}]\sum_{\substack{\mu=(1^{e_1}m_2)\\ \nu=(\nu_1,\nu_2)}}\gamma^\lambda_{\mu\nu}y^{\nu_2}\\ &= [y^{\nu_2}]\sum_{\substack{\mu=(1^{e_1}m_2)\\ \nu=(\nu_1,\nu_2)}}\gamma^\lambda_{\mu\nu}y^{\nu_2}\left(1-y^{\nu_1-\nu_2+1}\right) && (\nu_1+1>n/2)\\ &= [y^{\nu_2}]y^{n_3+d_2}\left(1-y^{d_1+1}\right)\left(\frac{1-y^{n_4-n_3+1}}{1-y}\right) && (\text{Eq. 29})\\ &= [y^{\nu_2}]y^{n_3+d_2}\left(1-y^{d_1+1}\right)\sum_{k=0}^{n_4-n_3}y^k\\ &= [y^{\nu_2}]y^{n_3+d_2}\sum_{k=0}^{n_4-n_3}y^k. && (n_3+d_2+d_1\ge n/2)\end{aligned}$$

We have obtained that for $e_1 = d_1+2d_2$ or $e_1 = d_1+2d_2+3$

$$\gamma^\lambda_{\mu\nu} = ((n_3 \le \nu_2 - d_2 \le n_4)).$$

Let $e_1 = d_1 + 2d_2 + 1$ or $e_1 = d_1 + 2d_2 + 2$. Since $\nu_2 \le \lfloor \frac{n}{2} \rfloor$ we have that

$$\begin{aligned}
\gamma^{\lambda}_{\mu\nu} &= [y^{\nu_2}] \sum_{\substack{\mu=(1^{e_1} m_2)\\ \nu=(\nu_1,\nu_2)}} \gamma^{\lambda}_{\mu\nu} y^{\nu_2} \\
&= [y^{\nu_2}] \sum_{\substack{\mu=(1^{e_1} m_2)\\ \nu=(\nu_1,\nu_2)}} \gamma^{\lambda}_{\mu\nu} \left(1 - y^{\nu_1-\nu_2+1}\right) \\
&= [y^{\nu_2}] y^{n_3+d_2-1}(1 + y + y^2)\left(1 - y^{d_1+1}\right)\left(\frac{1 - y^{n_4-n_3+1}}{1 - y}\right) \\
&= \left([y^{\nu_2}] y^{n_3+d_2-1}(1 + y + y^2)\left(\frac{1 - y^{n_4-n_3+1}}{1 - y}\right)\right) - ((n_3 + d_2 + d_1 = \nu_2)) \\
&= \left([y^{\nu_2}] y^{n_3+d_2-1}(1 + y + y^2)\sum_{k=0}^{n_4-n_3} y^k\right) - ((n_3 + d_2 + d_1 = \nu_2))
\end{aligned}$$

We have obtained that for $e_1 = d_1 + 2d_2 + 1$ or $e_1 = d_1 + 2d_2 + 2$

$$\begin{aligned}
\gamma^{\lambda}_{\mu\nu} &= ((n_3 \le \nu_2 - d_2 - 1 \le n_4)) + ((n_3 \le \nu_2 - d_2 \le n_4)) \\
&\quad + ((n_3 \le \nu_2 - d_2 + 1 \le n_4)) - ((n_3 + d_2 + d_1 = \nu_2)). \qquad (30)
\end{aligned}$$

□

Corollary 5 *The Kronecker cofficients,* $\gamma^{\lambda}_{\mu\nu}$, *where* μ *is a hook and* ν *is a two-row shape are always* 0, 1, 2 *or* 3.

7. Final comments

The inner product of symmetric functions was discovered by J.H. Redfield [8] in 1927, together with the scalar product of symmetric functions. He called them cup and cap products, respectively. D.E. Littlewood [5, 6] reinvented the inner product in 1956.

I.M. Gessel [3] and A. Lascoux [4] obtained combinatorial interpretations for the Kronecker coefficients in some restricted cases; Lascoux in the case where μ and ν are hooks, and λ a straight tableaux, and Gessel in the case that μ and ν are zigzag shapes and λ is an arbitrary skew shape. A.M. Garsia and J.B. Remmel [2] founded a way to relate shuffles of permutations and Kronecker coefficients. From here they obtained a combinatorial interpretation for the Kronecker coefficients when λ is a product of homogeneous symmetric functions, and μ and ν are arbitrary skew shapes. They also showed how Gessel's and Lascoux's results are related.

More recently, J.B. Remmel [9, 10], and J.B. Remmel and T. Whitehead [11] have obtained formulas for computing the Kronecker coefficients in the same cases considered in this paper. Their approach was mainly combinatorial. They expanded the Kronecker product $s_\mu * s_\nu$ in terms of Schur functions using the Garsia-Remmel algorithm [2]. By doing this, the problem of computing the Kronecker coefficients was reduced to computing signed sums of certain products of skew Schur functions. In general, it is not obvious how to go from the determination of the Kronecker coefficients $\gamma^{\lambda}_{\mu,\nu}$ when μ and ν are two-row shapes found in this paper, and the one obtained by J.B. Remmel and T. Whitehead [11]. But, in some particular cases this is easy to see. For instance, when λ is also a two-row shape, both formulas are exactly the same.

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References

1. N. Bergeron and A.M. Garsia, "Sergeev's formula and the Littlewood-Richardson Rule," *Linear and Multilinear Algebra* **27** (1990), 79–100.
2. A.M. Garsia and J.B. Remmel, "Shuffles of permutations and Kronecker products," *Graphs Combin.* **1**(3) (1985), 217–263.
3. I.M. Gessel, "Multipartite P-partitions and inner products of Schur functions," *Contemp. Math.* (1984), 289–302.
4. A. Lascoux, "Produit de Kronecker des representations du group symmetrique," *Lecture Notes in Mathematics* Springer Verlag, **795** (1980), 319–329.
5. D.E. Littlewood, "The Kronecker product of symmetric group representations," *J. London Math. Soc.* **31** (1956), 89–93.
6. D.E. Littlewood, "Plethysm and inner product of S-functions," *J. London Math. Soc.* **32** (1957), 18–22.
7. I.G. Macdonald, *Symmetric Functions and Hall Polynomials*, second edition, Oxford: Oxford University Press, 1995.
8. J.H. Redfield, "The theory of group reduced distribution," *Amer. J. Math.* **49** (1927), 433–455.
9. J.B. Remmel, "A formula for the Kronecker product of Schur functions of hook shapes," *J. Algebra* **120** (1989), 100–118.
10. J.B. Remmel, "Formulas for the expansion of the Kronecker products $S_{(m,n)} \otimes S_{(1^{p-r},r)}$ and $S_{(1^k 2^l)} \otimes S_{(1^{p-r},r)}$," *Discrete Math.* **99** (1992), 265–287.
11. J.B. Remmel and T. Whitehead, "On the Kronecker product of Schur functions of two row shapes," *Bull. Belg. Math. Soc. Simon Stevin* **1** (1994), 649–683.
12. M.H. Rosas, "A combinatorial overview of the theory of MacMahon symmetric functions and a study of the Kronecker product of Schur functions," Ph.D. Thesis, Brandeis University (1999).
13. B.E. Sagan, *The Symmetric Group*, Wadsworth & Brooks/Cole, Pacific Grove, California, 1991.
14. A.N. Sergeev, "The tensor algebra of the identity representation as a module over the Lie superalgebras $gl(n, m)$ and $Q(n)$," Math. USSR Sbornik, **51**, pp. 419–427.